Electronic Companion to "Optimal Policies for a Dual-Sourcing Inventory Problem with Endogenous Stochastic Lead Times"

For any function $g(x,y,z)$, define

$$\begin{aligned}\mathcal{L}^g(x,y,z) =&\lambda g(x-1,y,z)+\mu_1[\mathbf{I}_{\{z>0\}}g(x,y+1,z-1)+\mathbf{I}_{\{z=0\}}g(x,y,z)]\\&+\mu_2[\mathbf{I}_{\{y>0\}}g(x+1,y-1,z)+\mathbf{I}_{\{y=0\}}g(x,y,z)].\end{aligned}$$

Proof of Lemma 1. For the first inequality in (2), it suffices to show that

$$c_2+\triangle_y(hx^+ + bx^- + h_2 y + h_1 z + \mathcal{T}f(x,y,z)) \geq 0. \tag{EC.1}$$

(EC.1) is true because

$$\begin{aligned}&c_2+\triangle_y(hx^+ + bx^- + h_2 y + h_1 z + \mathcal{T}f(x,y,z))\\&=c_2+h_2+\min_{u\geq y+1, v\geq z}\Big\{(h_2+c_2)(u-(y+1))+(h_1+c_1)(v-z)+\mathcal{L}^f(x,u,v)\Big\}\\&\quad-\min_{u\geq y, v\geq z}\Big\{(h_2+c_2)(u-y)+(h_1+c_1)(v-z)+\mathcal{L}^f(x,u,v)\Big\}\\&=\min_{u\geq y+1, v\geq z}\Big\{(h_2+c_2)(u-y)+(h_1+c_1)(v-z)+\mathcal{L}^f(x,u,v)\Big\}\\&\quad-\min_{u\geq y, v\geq z}\Big\{(h_2+c_2)(u-y)+(h_1+c_1)(v-z)+\mathcal{L}^f(x,u,v)\Big\}\\&\geq 0.\end{aligned}$$

This proves the first inequality in (2). The second inequality follows similarly.

Now we prove that it is not optimal to order from server 2 when $y>0$, nor to order from server 1 when $z>0$. In view of the optimality equation (1), it suffices to show that, for any nonnegative integers $m,n\geq 0$,

$$\mathcal{L}^f(x,y+m,z+n) \leq \mathcal{L}^f(x,y+m+1,z+n+1)+c_1+c_2+h_1+h_2 \text{ if } y>0 \text{ and } z>0, \tag{EC.2}$$

$$\mathcal{L}^f(x,y,z+n) \leq \mathcal{L}^f(x,y,z+n+1)+c_1+h_1 \text{ if } y=0 \text{ and } z>0, \tag{EC.3}$$

$$\mathcal{L}^f(x,y+m,z) \leq \mathcal{L}^f(x,y+m+1,z)+c_2+h_2 \text{ if } y>0 \text{ and } z=0. \tag{EC.4}$$

To verify (EC.2), we have when $y>0$ and $z>0$,

$$\begin{aligned}&\mathcal{L}^f(x,y+m+1,z+n+1)+c_1+c_2+h_1+h_2-\mathcal{L}^f(x,y+m,z+n)\\&=c_1+c_2+h_1+h_2+\lambda[\triangle_y f(x-1,y+m,z+n+1)+\triangle_z f(x-1,y+m,z+n)]\\&\quad+\mu_1[\triangle_y f(x,y+m+1,z+n)+\triangle_z f(x,y+m+1,z+n-1)]\end{aligned}$$

$$
\begin{aligned}
&+\mu_2[\triangle_y f(x+1,y+m-1,z+n+1)+\triangle_z f(x+1,y+m-1,z+n)]\\
&= h_1+h_2+\lambda[c_2+\triangle_y f(x-1,y+m,z+n+1)]+\mu_1[c_2+\triangle_y f(x,y+m+1,z+n)]\\
&\quad +\mu_2[c_2+\triangle_y f(x+1,y+m-1,z+n+1)]+(1-\lambda-\mu_1-\mu_2)c_2\\
&\quad +\lambda[c_1+\triangle_z f(x-1,y+m,z+n)]+\mu_1[c_1+\triangle_z f(x,y+m+1,z+n-1)]\\
&\quad +\mu_2[c_1+\triangle_z f(x+1,y+m-1,z+n)]+(1-\lambda-\mu_1-\mu_2)c_1\\
&\geq h_1+h_2+(1-\lambda-\mu_1-\mu_2)(c_1+c_2)\\
&\geq 0,
\end{aligned}
$$

where the first inequality follows from (2), and the last from $\alpha+\lambda+\mu_1+\mu_2=1$. Thus, (EC.2) holds. Similarly, one can verify (EC.3)-(EC.4). ☐

Proof of Lemma 2. The argument for this lemma is similar to that of Lemma 1. ☐

Proof of Lemma 3. In view of (1), we have

$$
\begin{aligned}
f(x,y,0) &= hx^+ + bx^- + h_2 y\\
&\quad + \min_{u\geq y, v\geq 0}\Big\{c_1 v+(h_2+c_2)(u-y)+\mathcal{L}^f(x,u,v)\Big\}, \qquad \text{(EC.5)}\\
f(x,0,1) &= hx^+ + bx^- + \min_{u\geq 0}\Big\{(h_2+c_2)u+\mathcal{L}^f(x,u,1)\Big\}. \qquad \text{(EC.6)}
\end{aligned}
$$

Similarly, from (3), we have

$$
\begin{aligned}
\hat{f}(x,y,0) &= hx^+ + bx^- + h_2 y\\
&\quad + \min_{u\geq y, v\geq 0}\Big\{(h_2+c_2)(u-y)+\mathcal{L}^{\hat{f}}(x,u,v)+\mu_1\mathbf{I}_{\{v>0\}}\tilde{c}_1\Big\}, \qquad \text{(EC.7)}\\
\hat{f}(x,0,1) &= hx^+ + bx^- + \min_{u\geq 0}\Big\{(h_2+c_2)u+\mathcal{L}^{\hat{f}}(x,u,1)+\mu_1\tilde{c}_1\Big\}. \qquad \text{(EC.8)}
\end{aligned}
$$

To prove that System **O** and System **M** have the same optimal policy, it is sufficient to show that (EC.5) and (EC.7) have the same solution, and (EC.6) and (EC.8) have the same solution. Notice that the right-hand sides of equations (EC.5)-(EC.8) both involve minimum operations. Therefore, for (EC.5) and (EC.7) to have the same solution, it is sufficient that, for any fixed integer x and nonnegative integer y,

$$
\begin{aligned}
&c_1+\mathcal{L}^f(x,y,1)\geq \mathcal{L}^f(x,y,0) \text{ if and only if } \mu_1\tilde{c}_1+\mathcal{L}^{\hat{f}}(x,y,1)\geq \mathcal{L}^{\hat{f}}(x,y,0), \qquad \text{(EC.9)}\\
&h_2+c_2+\mathcal{L}^f(x,y+1,0)\geq \mathcal{L}^f(x,y,0) \text{ if and only if}\\
&\quad h_2+c_2+\mathcal{L}^{\hat{f}}(x,y+1,0)\geq \mathcal{L}^{\hat{f}}(x,y,0), \qquad \text{(EC.10)}\\
&c_1+h_2+c_2+\mathcal{L}^f(x,y+1,1)\geq \mathcal{L}^f(x,y,0) \text{ if and only if}\\
&\quad \mu_1\tilde{c}_1+h_2+c_2+\mathcal{L}^{\hat{f}}(x,y+1,1)\geq \mathcal{L}^{\hat{f}}(x,y,0). \qquad \text{(EC.11)}
\end{aligned}
$$

Also, for (EC.6) and (EC.8) to have the same solution, it is sufficient that, for any fixed nonnegative integer x,

$$\begin{aligned} &h_2+c_2+\mathcal{L}^f(x,1,1)\geq \mathcal{L}^f(x,0,1) \text{ if and only if} \\ &\quad h_2+c_2+\mathcal{L}^{\hat{f}}(x,1,1)\geq \mathcal{L}^{\hat{f}}(x,0,1). \end{aligned} \tag{EC.12}$$

But it directly follows from (5)-(6) that

$$\begin{aligned} &c_1+\mathcal{L}^f(x,y,1)=(\alpha+\mu_1)c_1+\mathcal{L}^{\hat{f}}(x,y,1), \\ &\mathcal{L}^f(x,y,0)=\mathcal{L}^{\hat{f}}(x,y,0), \\ &h_2+c_2+\mathcal{L}^f(x,1,1)-\mathcal{L}^f(x,0,1)=h_2+c_2+\mathcal{L}^{\hat{f}}(x,1,1)-\mathcal{L}^{\hat{f}}(x,0,1). \end{aligned}$$

Consequently, (EC.9)-(EC.12) hold, and the lemma is proven. □

Proof of Lemma 4. It follows from (7) that

$$\begin{aligned} &c_2+\triangle_y\tilde{f}(x,y) \\ &\quad =h_2+c_2+\min_{u\geq y+1}\Big\{(h_2+c_2)(u-y-1)+\lambda\tilde{f}(x-1,u) \\ &\qquad\qquad +\mu_1\min\{\tilde{f}(x,u+1)+\tilde{c}_1,\tilde{f}(x,u)\} \\ &\qquad\qquad +\mu_2[\mathbf{I}_{\{u>0\}}\tilde{f}(x+1,u-1)+\mathbf{I}_{\{u=0\}}\tilde{f}(x,u)]\Big\} \\ &\qquad -\min_{u\geq y}\Big\{(h_2+c_2)(u-y)+\lambda\tilde{f}(x-1,u)+\mu_1\min\{\tilde{f}(x,u+1)+\tilde{c}_1,\tilde{f}(x,u)\} \\ &\qquad\qquad +\mu_2[\mathbf{I}_{\{u>0\}}\tilde{f}(x+1,u-1)+\mathbf{I}_{\{u=0\}}\tilde{f}(x,u)]\Big\} \\ &\quad \geq \min_{u\geq y}\Big\{(h_2+c_2)(u-y)+\lambda\tilde{f}(x-1,u)+\mu_1\min\{\tilde{f}(x,u+1)+\tilde{c}_1,\tilde{f}(x,u)\} \\ &\qquad\qquad +\mu_2[\mathbf{I}_{\{u>0\}}\tilde{f}(x+1,u-1)+\mathbf{I}_{\{u=0\}}\tilde{f}(x,u)]\Big\} \\ &\qquad -\min_{u\geq y}\Big\{(h_2+c_2)(u-y)+\lambda\tilde{f}(x-1,u)+\mu_1\min\{\tilde{f}(x,u+1)+\tilde{c}_1,\tilde{f}(x,u)\} \\ &\qquad\qquad +\mu_2[\mathbf{I}_{\{u>0\}}\tilde{f}(x+1,u-1)+\mathbf{I}_{\{u=0\}}\tilde{f}(x,u)]\Big\} \\ &\quad =0. \end{aligned}$$

Hence, we have (8). The second part of the lemma follows from an argument along the same lines as that of the second part of Lemma 3 (i). □

Proof of Lemma 5. First the uniqueness of $S(y)$ directly follows from its definition. It can be $-\infty$, $+\infty$, or any integer in between. Supermodularity of $\tilde{f}(x,y)$ implies that $\tilde{f}(x,y)+\tilde{c}_1$ is increasing in x. Thus we have that $\tilde{f}(x,y)+\tilde{c}_1<0$ for $x<S(y)$, and $\tilde{f}(x,y)+\tilde{c}_1\geq 0$ for $x\geq S(y)$. Property (P3) of $\tilde{f}(x,y)$ implies that $\tilde{f}(x,y)+\tilde{c}_1$ is increasing in y. Using these two results, we will have $S(y)\leq S(y-1)$, which implies the monotonicity of $S(\cdot)$.

We now, by contradiction, prove that if $S(y)$ and $S(y-1)$ are finite, then $S(y) < S(y-1)$. Suppose there exists y_0 such that

$$S(y_0) = S(y_0+1).$$

By the definition of $S(\cdot)$, we have

$$\triangle_y \tilde{f}(S(y_0), y_0) + \tilde{c}_1 \geq 0$$

and

$$\triangle_y \tilde{f}(S(y_0)-1, y_0+1) + \tilde{c}_1 < 0,$$

which together imply

$$\triangle_y \tilde{f}(S(y_0), y_0) > \triangle_y \tilde{f}(S(y_0)-1, y_0+1).$$

This contradicts Property (P3), $\triangle_{yy} \tilde{f}(S(y_0)-1, y_0) \geq \triangle_{xy} \tilde{f}(S(y_0)-1, y_0)$, which can be rewritten as $\triangle_y \tilde{f}(S(y_0), y_0) \leq \triangle_y \tilde{f}(S(y_0)-1, y_0+1)$. □

Proof of Lemma 6. The uniqueness of R is directly given by its definition. By the supermodularity and convexity of $\tilde{f}(x,y)$ in x, the quantity $h_2 + c_2 - \mu_1\tilde{c}_1 + \lambda\triangle_y \tilde{f}(x-1,0) + \mu_2\triangle_x \tilde{f}(x,0)$ is increasing in x, and $-\mu_1(\triangle_y \tilde{f}(x,1) + \tilde{c}_1)\mathbf{I}_{\{x<S(1)\}} - \mu_1(\triangle_y \tilde{f}(x,0) + \tilde{c}_1)\mathbf{I}_{\{x\geq S(0)\}}$ is decreasing in x. Thus

$$\begin{aligned}
&h_2 + c_2 - \mu_1\tilde{c}_1 + \lambda\triangle_y \tilde{f}(x-1,0) + \mu_2\triangle_x \tilde{f}(x,0) \\
&\quad < -\mu_1(\triangle_y \tilde{f}(x,1) + \tilde{c}_1)\mathbf{I}_{\{x<S(1)\}} - \mu_1(\triangle_y \tilde{f}(x,0) + \tilde{c}_1)\mathbf{I}_{\{x\geq S(0)\}} \quad \text{for } x < R; \\
&h_2 + c_2 - \mu_1\tilde{c}_1 + \lambda\triangle_y \tilde{f}(x-1,0) + \mu_2\triangle_x \tilde{f}(x,0) \\
&\quad \geq -\mu_1(\triangle_y \tilde{f}(x,1) + \tilde{c}_1)\mathbf{I}_{\{x<S(1)\}} - \mu_1(\triangle_y \tilde{f}(x,0) + \tilde{c}_1)\mathbf{I}_{\{x\geq S(0)\}} \quad \text{for } x \geq R.
\end{aligned}$$

This implies that for state $(x,0)$, if $x < R$, it is optimal to feed server 2; if $x \geq R$, it is optimal to do nothing at server 2. □

Proof of Lemma 7. It suffices to show that, for any positive integer y,

$$\begin{aligned}
&\lambda\psi(x-1,y) + \mu_1 \min\{\psi(x,y+1) + \tilde{c}_1, \psi(x,y)\} + \mu_2\psi(x+1,y-1) \\
&\quad \leq (h_2 + c_2) + \lambda\psi(x-1,y+1) \\
&\qquad + \mu_1 \min\{\psi(x,y+2) + \tilde{c}_1, \psi(x,y+1)\} + \mu_2\psi(x+1,y).
\end{aligned} \tag{EC.13}$$

Consider two cases:

$$\min\{\psi(x,y+2) + \tilde{c}_1, \psi(x,y+1)\} = \psi(x,y+1),$$

and

$$\min\{\psi(x,y+2) + \tilde{c}_1, \psi(x,y+1)\} = \psi(x,y+2) + \tilde{c}_1.$$

For the first case,

$$\begin{aligned}
&(h_2+c_2)+\lambda\psi(x-1,y+1)+\mu_1\min\{\psi(x,y+2)+\tilde{c}_1,\psi(x,y+1)\}+\mu_2\psi(x+1,y)\\
&\;-\Big(\lambda\psi(x-1,y)+\mu_1\min\{\psi(x,y+1)+\tilde{c}_1,\psi(x,y)\}+\mu_2\psi(x+1,y-1)\Big)\\
&\quad=(h_2+c_2)+\lambda\psi(x-1,y+1)+\mu_1\psi(x,y+1)+\mu_2\psi(x+1,y)\\
&\quad\;-\Big(\lambda\psi(x-1,y)+\mu_1\min\{\psi(x,y+1)+\tilde{c}_1,\psi(x,y)\}+\mu_2\psi(x+1,y-1)\Big)\\
&\quad\ge(h_2+c_2)+\lambda\psi(x-1,y+1)+\mu_1\psi(x,y+1)+\mu_2\psi(x+1,y)\\
&\quad\;-\Big(\lambda\psi(x-1,y)+\mu_1\psi(x,y)+\mu_2\psi(x+1,y-1)\Big)\\
&\quad=(h_2+c_2)+\lambda\triangle_y\psi(x-1,y)+\mu_1\triangle_y\psi(x,y)+\mu_2\triangle_y\psi(x+1,y-1)\\
&\quad\ge h_2+\lambda(c_2+\lambda\triangle_y\psi(x-1,y))+\mu_1(c_2+\triangle_y\psi(x,y))+\mu_2(c_2+\triangle_y\psi(x+1,y-1))\\
&\quad\ge 0.
\end{aligned}$$

Hence, (EC.13) holds. By a similar argument, (EC.13) holds in the second case. □

Proof of Lemma 8. Define

$$g(x,y)=\min\Big\{\psi(x,y),\psi(x,y+1)+\tilde{c}_1\Big\}. \tag{EC.14}$$

In order to prove the lemma, by the definition of $\tilde{\mathcal{T}}$ and Lemma 7, we need only prove that $g(x,y)$ satisfies Properties (P1)-(P3) for $y>0$, provided that $\psi(x,y)$ satisfies (P1)-(P3) for $y\ge 0$.

First we prove supermodularity of $g(x,y)$:

$$g(x+1,y+1)+g(x,y)\ge g(x+1,y)+g(x,y+1). \tag{EC.15}$$

Define

$$G(x,y,w)=w\cdot\Big(\psi(x,y+1)+\tilde{c}_1\Big)+(1-w)\cdot\psi(x,y).$$

Then, by the definition of $g(\cdot,\cdot)$ in (EC.14),

$$g(x,y)=\min_{w\in\{0,1\}}G(x,y,w). \tag{EC.16}$$

By the supermodularity of $\psi(x,y)$,

$$\triangle_x G(x,y,1)=\triangle_x\psi(x,y+1)\ge\triangle_x\psi(x,y)=\triangle_x G(x,y,0).$$

Hence, $G(x,y,w)$ is supermodular in (x,w). Similarly, one can show that $G(x,y,w)$ is supermodular in (y,w) for $y\ge 0$. Let

$$g(x,y)=G(x,y,w_1)\quad\text{and}\quad g(x+1,y+1)=G(x+1,y+1,w_2).$$

If $w_1 \le w_2$, then

$$\begin{aligned}
&g(x+1,y)+g(x,y+1)\\
&\quad\le G(x+1,y,w_1)+G(x,y+1,w_2)\\
&\quad\le G(x+1,y,w_2)+G(x,y,w_1)-G(x,y,w_2)+G(x,y+1,w_2)\\
&\quad\le G(x,y,w_1)+G(x+1,y+1,w_2)\\
&\quad= g(x,y)+g(x+1,y+1),
\end{aligned}$$

where the first inequality uses (EC.16), the supermodularity of $G(x,y,w)$ in (x,w) implies the second inequality, and the last inequality follows from the supermodularity of $G(x,y,w)$ for fixed w. Equation (EC.15) follows immediately.

If $w_1 > w_2$, then $w_1 = 1$ and $w_2 = 0$. Thus,

$$\begin{aligned}
&[g(x+1,y)+g(x,y+1)]-[g(x+1,y+1)+g(x,y)]\\
&\quad=\min\Big\{\psi(x+1,y+1)+\tilde{c}_1,\psi(x+1,y)\Big\}+\min\Big\{\psi(x,y+2)+\tilde{c}_1,\psi(x,y+1)\Big\}\\
&\qquad-\psi(x+1,y+1)-\psi(x,y+1)-\tilde{c}_1\\
&\quad\le\psi(x+1,y+1)+\tilde{c}_1+\psi(x,y+1)-\psi(x+1,y+1)-\psi(x,y+1)-\tilde{c}_1\\
&\quad=0,
\end{aligned}$$

which again yields (EC.15).

Second, we prove diagonal dominance in x, that is, $\triangle_{xx}g(x,y)\ge\triangle_{xy}g(x,y)$, or

$$g(x+2,y)+g(x,y+1)\ge g(x+1,y+1)+g(x+1,y). \tag{EC.17}$$

Let

$$g(x+2,y)=G(x+2,y,w_2)\quad\text{and}\quad g(x,y+1)=G(x,y+1,w_1).$$

The proof of (EC.17) is divided into two cases.

Case (i): $w_1 \le w_2$. Here,

$$\begin{aligned}
&g(x+1,y+1)+g(x+1,y)\\
&\quad\le G(x+1,y+1,w_1)+G(x+1,y,w_2)\\
&\quad\le G(x+1,y+1,w_1)+G(x+2,y,w_2)+G(x+1,y,w_1)-G(x+2,y,w_1)\\
&\quad\le G(x,y+1,w_1)+G(x+2,y,w_2)\\
&\quad= g(x+2,y)+g(x,y+1).
\end{aligned}$$

The first inequality follows from (EC.16), the second comes from the supermodularity of $G(x,y,w)$ in (x,w), and the third is due to the diagonal dominance of $G(x,y,w)$ for fixed w. Hence, (EC.17) holds.

Case (ii): $w_1 > w_2$, that is, $w_1 = 1$ and $w_2 = 0$. First,

$$g(x+2,y) = \psi(x+2,y) \quad g(x,y+1) = \psi(x,y+2) + \tilde{c}_1,$$

Then,

$$\begin{aligned}
&g(x+1,y+1) + g(x+1,y) - g(x+2,y) - g(x,y+1)\\
&= \min\Big\{\psi(x+1,y+1), \psi(x+1,y+2) + \tilde{c}_1\Big\}\\
&\quad + \min\Big\{\psi(x+1,y), \psi(x+1,y+1) + \tilde{c}_1\Big\}\\
&\quad - \psi(x+2,y) - \psi(x,y+2) - \tilde{c}_1\\
&\le \psi(x+1,y+1) - \psi(x+2,y) - \psi(x,y+2) + \psi(x+1,y+1)\\
&= \psi(x+1,y+1) - \psi(x+1,y) + \psi(x+1,y) - \psi(x+2,y)\\
&\quad - \psi(x,y+2) + \psi(x,y+1) - \psi(x,y+1) + \psi(x+1,y+1)\\
&= \triangle_y\psi(x+1,y) - \triangle_x\psi(x+1,y) - \triangle_y\psi(x,y+1) + \triangle_x\psi(x,y+1)\\
&= \triangle_{xy}\psi(x,y) - \triangle_{yy}\psi(x,y) - \Big(\triangle_{xx}\psi(x,y) - \triangle_{xy}\psi(x,y)\Big)\\
&\le 0.
\end{aligned}$$

This implies (EC.17).

To prove that $g(x,y)$ satisfies Property (P3), we just follow the same procedure as above for property (P2). That is, replace $\triangle_{xx}$ by $\triangle_{yy}$, and keep $\triangle_{xy}$ as is. ☐

Proof of Lemma 9. We need to prove the following three inequalities:

$$\tilde{\mathcal{T}}\psi(x+1,1) + \tilde{\mathcal{T}}\psi(x,0) \ge \tilde{\mathcal{T}}\psi(x+1,0) + \tilde{\mathcal{T}}\psi(x,1), \tag{EC.18}$$

$$\tilde{\mathcal{T}}\psi(x+2,0) + \tilde{\mathcal{T}}\psi(x,1) \ge \tilde{\mathcal{T}}\psi(x+1,1) + \tilde{\mathcal{T}}\psi(x+1,0), \tag{EC.19}$$

$$\tilde{\mathcal{T}}\psi(x,2) + \tilde{\mathcal{T}}\psi(x+1,0) \ge \tilde{\mathcal{T}}\psi(x+1,1) + \tilde{\mathcal{T}}\psi(x,1). \tag{EC.20}$$

Define $H(x,y,w)$ by

$$\begin{aligned}
&\frac{1}{2}w(2-w)(3-w)\Big[(h_2+c_2) + \lambda\psi(x-1,y+1) + \mu_1\psi(x,y+1) + \mu_2\psi(x+1,y)\Big]\\
&+ \frac{1}{6}(1-w)(2-w)(3-w)\Big[(h_2+c_2) + \lambda\psi(x-1,y+1) + \mu_1\psi(x,y+2)\\
&\qquad\qquad + \mu_1\tilde{c}_1 + \mu_2\psi(x+1,y)\Big]\\
&+ \frac{1}{6}w(w-1)(w-2)\Big[\lambda\psi(x-1,y) + \mu_1\psi(x,y) + \mu_2\mathbf{I}_{\{y>0\}}\psi(x+1,y-1)
\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad\qquad + \mu_2 \mathbf{I}_{\{y=0\}} \psi(x,y)\Big] \\
&+ \frac{1}{2} w(w-1)(3-w)\Big[\lambda\psi(x-1,y) + \mu_1\psi(x,y+1) + \mu_1\tilde{c}_1 \\
&\qquad\qquad\qquad + \mu_2 \mathbf{I}_{\{y>0\}}\psi(x+1,y-1) + \mu_2 \mathbf{I}_{\{y=0\}}\psi(x,y)\Big].
\end{aligned}
$$

Based on an argument similar to the discussion of $G(x,y,w)$ in the proof of Lemma 8, $H(x,y,w)$ is submodular in (x,w), i.e.,

$$\triangle_x H(x,y,0) \geq \triangle_x H(x,y,1) \geq \triangle_x H(x,y,2) \geq \triangle_x H(x,y,3).$$

For $y \geq 0$, by (16), $H(x,y,w)$ is submodular in (y,w), i.e.,

$$\triangle_y H(x,y,0) \geq \triangle_y H(x,y,1) \geq \triangle_y H(x,y,2) \geq \triangle_y H(x,y,3).$$

Finally, for fixed w, by (16), $H(x,y,w)$ satisfies Properties (P1)-(P3). We can rewrite

$$\tilde{\mathcal{T}}\psi(x,y) = \min_{w\in\{0,1,2,3\}} H(x,y,w). \tag{EC.21}$$

First, we prove inequality (EC.18). To do this, let

$$\tilde{\mathcal{T}}\psi(x+1,1) = H(x+1,1,w_2) \text{ and } \tilde{\mathcal{T}}\psi(x,0) = H(x,0,w_1).$$

The proof is divided into several cases.

Case a.i: $w_1 \geq w_2$. For this case,

$$
\begin{aligned}
\tilde{\mathcal{T}}\psi(x+1,0) + \tilde{\mathcal{T}}\psi(x,1) &\leq H(x+1,0,w_1) + H(x,1,w_2) \\
&\leq H(x+1,0,w_2) + H(x,0,w_1) - H(x,0,w_2) + H(x,1,w_2) \\
&\leq H(x+1,1,w_2) + H(x,0,w_1) \\
&= \tilde{\mathcal{T}}\psi(x+1,1) + \tilde{\mathcal{T}}\psi(x,0),
\end{aligned}
$$

where the second inequality follows from the submodularity of $H(x,y,w)$ in (x,w), and the last inequality comes from the supermodularity of $H(x,y,w)$ given w. Thus, (EC.18) holds.

Case a.ii: $w_1 < w_2$. Note that, by Properties (P1)-(P3) of $\psi(\cdot,\cdot)$, the minimizer of the right-hand side of the following equation (EC.22) is 1:

$$
\begin{aligned}
\tilde{\mathcal{T}}\psi(x+1,1) = \inf_{u\geq 1}\Big\{&(h_2+c_2)(u-1) + \lambda\psi(x,u) \\
&+\mu_1 \min\{\psi(x+1,u+1)+\tilde{c}_1, \psi(x+1,u)\} + \mu_2\psi(x+2,u-1)\Big\}.
\end{aligned} \tag{EC.22}
$$

Hence, according to the definition of w_2 and (EC.21), w_2 can only take the values 2 or 3. Thus, we divide this case into five subcases, namely, $(w_1,w_2) = (0,2)$, $(w_1,w_2) = (0,3)$, $(w_1,w_2) = (1,2)$, $(w_1,w_2) = (1,3)$, and $(w_1,w_2) = (2,3)$.

Subcase a.ii.1: $(w_1, w_2) = (0, 2)$. By (EC.21),

$$\begin{aligned}
&\tilde{\mathcal{T}}\psi(x+1,0) + \tilde{\mathcal{T}}\psi(x,1) \\
&\quad \le h_2 + c_2 + \lambda\psi(x,1) + \mu_1\psi(x+1,2) + \mu_1\tilde{c}_1 + \mu_2\psi(x+2,0) \\
&\qquad\quad + \lambda\psi(x-1,1) + \mu_1\psi(x,2) + \mu_1\tilde{c}_1 + \mu_2\psi(x+1,0) \\
&\quad = \lambda\psi(x,1) + \mu_1\psi(x+1,2) + \mu_1\tilde{c}_1 + \mu_2\psi(x+2,0) \\
&\qquad\quad + h_2 + c_2 + \lambda\psi(x-1,1) + \mu_1\psi(x,2) + \mu_1\tilde{c}_1 + \mu_2\psi(x+1,0) \\
&\quad = \tilde{\mathcal{T}}\psi(x+1,1) + \tilde{\mathcal{T}}\psi(x,0).
\end{aligned}$$

This yields (EC.18).

Subcase a.ii.2: $(w_1, w_2) = (0, 3)$. Again, by (EC.21),

$$\begin{aligned}
&\tilde{\mathcal{T}}\psi(x+1,0) + \tilde{\mathcal{T}}\psi(x,1) \\
&\quad \le h_2 + c_2 + \lambda\psi(x,1) + \mu_1\psi(x+1,1) + \mu_2\psi(x+2,0) \\
&\qquad\quad + \lambda\psi(x-1,1) + \mu_1\psi(x,2) + \mu_1\tilde{c}_1 + \mu_2\psi(x+1,0) \\
&\quad = \lambda\psi(x,1) + \mu_1\psi(x+1,1) + \mu_2\psi(x+2,0) \\
&\qquad\quad + h_2 + c_2 + \lambda\psi(x-1,1) + \mu_1\psi(x,2) + \mu_1\tilde{c}_1 + \mu_2\psi(x+1,0) \\
&\quad = \tilde{\mathcal{T}}\psi(x+1,1) + \tilde{\mathcal{T}}\psi(x,0).
\end{aligned}$$

Hence, (EC.18) holds.

Subcase a.ii.3: $(w_1, w_2) = (1, 2)$. By (EC.21), the following inequality directly proves (EC.18):

$$\begin{aligned}
&\tilde{\mathcal{T}}\psi(x+1,0) + \tilde{\mathcal{T}}\psi(x,1) \\
&\quad \le h_2 + c_2 + \lambda\psi(x,1) + \mu_1\psi(x+1,2) + \mu_1\tilde{c}_1 + \mu_2\psi(x+2,0) \\
&\qquad\quad + \lambda\psi(x-1,1) + \mu_1\psi(x,1) + \mu_2\psi(x+1,0) \\
&\quad = \lambda\psi(x,1) + \mu_1\psi(x+1,2) + \mu_1\tilde{c}_1 + \mu_2\psi(x+2,0) \\
&\qquad\quad + h_2 + c_2 + \lambda\psi(x-1,1) + \mu_1\psi(x,1) + \mu_2\psi(x+1,0) \\
&\quad = \tilde{\mathcal{T}}\psi(x+1,1) + \tilde{\mathcal{T}}\psi(x,0).
\end{aligned}$$

Subcase a.ii.4: $(w_1, w_2) = (1, 3)$. Similar to the subcases above, (EC.18) follows from the following inequality:

$$\begin{aligned}
&\tilde{\mathcal{T}}\psi(x+1,0) + \tilde{\mathcal{T}}\psi(x,1) \\
&\quad \le h_2 + c_2 + \lambda\psi(x,1) + \mu_1\psi(x+1,1) + \mu_2\psi(x+2,0) \\
&\qquad\quad + \lambda\psi(x-1,1) + \mu_1\psi(x,1) + \mu_2\psi(x+1,0)
\end{aligned}$$

$$\begin{aligned}&= \lambda\psi(x,1) + \mu_1\psi(x+1,1) + \mu_2\psi(x+2,0)\\&\quad + h_2 + c_2 + \lambda\psi(x-1,1) + \mu_1\psi(x,1) + \mu_2\psi(x+1,0)\\&= \tilde{\mathcal{T}}\psi(x+1,1) + \tilde{\mathcal{T}}\psi(x,0).\end{aligned}$$

Subcase a.ii.5: $(w_1,w_2)=(2,3)$. By (EC.21), we have

$$\begin{aligned}&\tilde{\mathcal{T}}\psi(x+1,0) + \tilde{\mathcal{T}}\psi(x,1)\\&\quad\le \lambda\psi(x,0) + \mu_1\psi(x+1,1) + \mu_1\tilde{c}_1 + \mu_2\psi(x+1,0)\\&\qquad + \lambda\psi(x-1,1) + \mu_1\psi(x,1) + \mu_2\psi(x+1,0)\\&\quad\le \lambda\psi(x,1) + \mu_1\psi(x+1,1) + \mu_2\psi(x+2,0)\\&\qquad + \lambda\psi(x-1,0) + \mu_1\psi(x,1) + \mu_1\tilde{c}_1 + \mu_2\psi(x,0)\\&\quad= \tilde{\mathcal{T}}\psi(x+1,1) + \tilde{\mathcal{T}}\psi(x,0).\end{aligned}$$

Here, in the second inequality, we compare the left-hand and right-hand sides term by term. The term on the left with coefficient λ is no greater than its counterpart on the right, because of supermodularity of $\psi(\cdot,\cdot)$. Likewise, the inequality holds for the terms with coefficient μ_2, because of the convexity of $\psi(\cdot,\cdot)$ with respect to its first coordinate. Finally, the terms with coefficient μ_1 on the left and right are identical. Thus, the proof of (EC.18) is complete.

Next, we prove (EC.19). Again, let

$$\tilde{\mathcal{T}}\psi(x+2,0) = H(x+2,0,w_2) \quad\text{and}\quad \tilde{\mathcal{T}}\psi(x,1) = H(x,1,w_1). \tag{EC.23}$$

Again, consider two cases:

Case b.i: $w_1 \ge w_2$. By (EC.21),

$$\begin{aligned}&\tilde{\mathcal{T}}\psi(x+1,1) + \tilde{\mathcal{T}}\psi(x+1,0)\\&\quad\le H(x+1,1,w_1) + H(x+1,0,w_2)\\&\quad\le H(x+1,1,w_2) + H(x,1,w_1) - H(x,1,w_2) + H(x+1,0,w_2)\\&\quad\le H(x+2,0,w_2) + H(x,1,w_1)\\&\quad= \tilde{\mathcal{T}}\psi(x+2,0) + \tilde{\mathcal{T}}\psi(x,1),\end{aligned}$$

where the second inequality comes from the submodularity of $H(x,y,w)$ in (x,w), and the last inequality comes from the fact that $H(x,y,w)$ satisfies Property (P2) for fixed w. (EC.19) directly follows.

Case b.ii: $w_1 < w_2$. Similar to Case a.ii, we have $(w_1, w_2) = (2, 3)$. Then, by (EC.21),

$$\begin{aligned}
&\tilde{\mathcal{T}}\psi(x+1,1) + \tilde{\mathcal{T}}\psi(x+1,0) \\
&\quad \le \lambda\psi(x,1) + \mu_1\psi(x+1,1) + \mu_2\psi(x+2,0) \\
&\qquad + \lambda\psi(x,0) + \mu_1\psi(x+1,1) + \mu_1\tilde{c}_1 + \mu_2\psi(x+1,0).
\end{aligned} \tag{EC.24}$$

By Properties (P1)-(P3) for $\psi(x,y)$, we have

$$\begin{aligned}
&\lambda\triangle_{xy}\psi(x-1,0) + \mu_1\Big[\triangle_{xy}\psi(x,0) + \triangle_{xy}\psi(x,0)\Big] \\
&\quad \le \lambda\triangle_{xx}\psi(x-1,0) + \mu_1\Big[\triangle_{yy}\psi(x,0) + \triangle_{xx}\psi(x,0)\Big].
\end{aligned}$$

This implies

$$\begin{aligned}
&\lambda\psi(x,1) + \mu_1\psi(x+1,1) + \mu_2\psi(x+2,0) \\
&\quad + \lambda\psi(x,0) + \mu_1\psi(x+1,1) + \mu_1\tilde{c}_1 + \mu_2\psi(x+1,0) \\
&\qquad \le \lambda\psi(x-1,1) + \mu_1\psi(x,2) + \mu_1\tilde{c}_1 + \mu_2\psi(x+1,0) \\
&\qquad\quad + \lambda\psi(x+1,0) + \mu_1\psi(x+2,0) + \mu_2\psi(x+2,0).
\end{aligned} \tag{EC.25}$$

From (EC.23),

$$\begin{aligned}
\tilde{\mathcal{T}}\psi(x,1) + \tilde{\mathcal{T}}\psi(x+2,0) =& \lambda\psi(x-1,1) + \mu_1\psi(x,2) + \mu_1\tilde{c}_1 + \mu_2\psi(x+1,0) \\
&+ \lambda\psi(x+1,0) + \mu_1\psi(x+2,0) + \mu_2\psi(x+2,0).
\end{aligned} \tag{EC.26}$$

Hence, we obtain (EC.19) by (EC.24)-(EC.26).

Finally we prove inequality (EC.20). Let

$$\tilde{\mathcal{T}}\psi(x,2) = H(x,2,w_2) \quad \text{and} \quad \tilde{\mathcal{T}}\psi(x+1,0) = H(x+1,0,w_1).$$

Consider two cases:

Case c.i: $w_1 \ge w_2$. By (EC.21),

$$\begin{aligned}
\tilde{\mathcal{T}}\psi(x+1,1) + \tilde{\mathcal{T}}\psi(x,1) &\le H(x+1,1,w_1) + H(x,1,w_2) \\
&\le H(x+1,1,w_1) + H(x,1,w_1) + H(x,2,w_2) - H(x,2,w_1) \\
&\le H(x+1,0,w_1) + H(x,2,w_2) \\
&= \tilde{\mathcal{T}}\psi(x+1,0) + \tilde{\mathcal{T}}\psi(x,2).
\end{aligned}$$

Here, the second inequality comes from the submodularity of $H(x,y,w)$ in (y,w), and the last inequality is due to the fact that $H(x,y,w)$ satisfies Property (P3) for fixed w. Hence, we have (EC.20).

Case c.ii: $w_2 > w_1$. Similar to Case a.ii, we consider five subcases, $(w_1, w_2) = (0,3)$, $(w_1, w_2) = (1,3)$, $(w_1, w_2) = (2,3)$, $(w_1, w_2) = (0,2)$, and $(w_1, w_2) = (1,2)$.

Subcase c.ii.1: $(w_1, w_2) = (0,3)$. By (EC.21),

$$\begin{aligned}
&\tilde{\mathcal{T}}\psi(x+1,1) + \tilde{\mathcal{T}}\psi(x,1) \\
&\quad \le \lambda\psi(x,1) + \mu_1\psi(x+1,2) + \mu_1\tilde{c}_1 + \mu_2\psi(x+2,0) \\
&\qquad + h_2 + c_2 + \lambda\psi(x-1,2) + \mu_1\psi(x,2) + \mu_2\psi(x+1,1) \\
&\quad = \lambda\psi(x-1,2) + \mu_1\psi(x,2) + \mu_2\psi(x+1,1) \\
&\qquad + h_2 + c_2 + \lambda\psi(x,1) + \mu_1\psi(x+1,2) + \mu_1\tilde{c}_1 + \mu_2\psi(x+2,0) \\
&\quad = \tilde{\mathcal{T}}\psi(x,2) + \tilde{\mathcal{T}}\psi(x+1,0).
\end{aligned}$$

This implies (EC.20).

Subcase c.ii.2: $(w_1, w_2) = (1,3)$. (EC.20) directly follows from the following inequality:

$$\begin{aligned}
&\tilde{\mathcal{T}}\psi(x+1,1) + \tilde{\mathcal{T}}\psi(x,1) \\
&\quad \le \lambda\psi(x,1) + \mu_1\psi(x+1,1) + \mu_2\psi(x+2,0) \\
&\qquad + h_2 + c_2 + \lambda\psi(x-1,2) + \mu_1\psi(x,2) + \mu_2\psi(x+1,1) \\
&\quad = \lambda\psi(x-1,2) + \mu_1\psi(x,2) + \mu_2\psi(x+1,1) \\
&\qquad + h_2 + c_2 + \lambda\psi(x,1) + \mu_1\psi(x+1,1) + \mu_2\psi(x+2,0) \\
&\quad = \tilde{\mathcal{T}}\psi(x,2) + \tilde{\mathcal{T}}\psi(x+1,0).
\end{aligned}$$

Subcase c.ii.3: $(w_1, w_2) = (2,3)$. By (16),

$$\lambda\triangle_{xy}\psi(x-1,0) + \mu_2\triangle_x\psi(x+1,0) \le \lambda\triangle_{yy}\psi(x-1,0) + \mu_2\triangle_y\psi(x+1,0).$$

Hence,

$$\begin{aligned}
&\lambda\psi(x,1) + \mu_1\psi(x+1,1) + \mu_2\psi(x+2,0) \\
&\quad + \lambda\psi(x-1,1) + \mu_1\psi(x,2) + \mu_1\tilde{c}_1 + \mu_2\psi(x+1,0) \\
&\qquad \le \lambda\psi(x-1,2) + \mu_1\psi(x,2) + \mu_2\psi(x+1,1) \\
&\qquad\quad + \lambda\psi(x,0) + \mu_1\psi(x+1,1) + \mu_1\tilde{c}_1 + \mu_2\psi(x+1,0).
\end{aligned}$$

This implies

$$\begin{aligned}
&\tilde{\mathcal{T}}\psi(x+1,1) + \tilde{\mathcal{T}}\psi(x,1) \\
&\quad \le \lambda\psi(x,1) + \mu_1\psi(x+1,1) + \mu_2\psi(x+2,0)
\end{aligned}$$

$$
\begin{aligned}
&\quad + \lambda\psi(x-1,1) + \mu_1\psi(x,2) + \mu_1\tilde{c}_1 + \mu_2\psi(x+1,0)\\
&\le \lambda\psi(x-1,2) + \mu_1\psi(x,2) + \mu_2\psi(x+1,1)\\
&\quad + \lambda\psi(x,0) + \mu_1\psi(x+1,1) + \mu_1\tilde{c}_1 + \mu_2\psi(x+1,0)\\
&= \tilde{\mathcal{T}}\psi(x,2) + \tilde{\mathcal{T}}\psi(x+1,0),
\end{aligned}
$$

which yields (EC.20).

Subcase c.ii.4: $(w_1,w_2)=(0,2)$. Similar to Subcase c.ii.2, (EC.20) directly follows from

$$
\begin{aligned}
&\tilde{\mathcal{T}}\psi(x+1,1) + \tilde{\mathcal{T}}\psi(x,1)\\
&\quad \le \lambda\psi(x,1) + \mu_1\psi(x+1,2) + \mu_1\tilde{c}_1 + \mu_2\psi(x+2,0)\\
&\qquad + h_2 + c_2 + \lambda\psi(x-1,2) + \mu_1\psi(x,3) + \mu_1\tilde{c}_1 + \mu_2\psi(x+1,1)\\
&\quad = \lambda\psi(x-1,2) + \mu_1\psi(x,3) + \mu_1\tilde{c}_1 + \mu_2\psi(x+1,1)\\
&\qquad + h_2 + c_2 + \lambda\psi(x,1) + \mu_1\psi(x+1,2) + \mu_1\tilde{c}_1 + \mu_2\psi(x+2,0)\\
&\quad = \tilde{\mathcal{T}}\psi(x,2) + \tilde{\mathcal{T}}\psi(x+1,0).
\end{aligned}
$$

Subcase c.ii.5: $(w_1,w_2)=(1,2)$. Similar to Subcase c.ii.2, (EC.20) directly follows from

$$
\begin{aligned}
&\tilde{\mathcal{T}}\psi(x+1,1) + \tilde{\mathcal{T}}\psi(x,1)\\
&\quad \le \lambda\psi(x,1) + \mu_1\psi(x+1,1) + \mu_2\psi(x+2,0)\\
&\qquad + h_2 + c_2 + \lambda\psi(x-1,2) + \mu_1\psi(x,3) + \mu_1\tilde{c}_1 + \mu_2(x+1,1)\\
&\quad = \lambda\psi(x-1,2) + \mu_1\psi(x,3) + \mu_1\tilde{c}_1 + \mu_2\psi(x+1,1)\\
&\qquad + h_2 + c_2 + \lambda\psi(x,1) + \mu_1\psi(x+1,1) + \mu_2\psi(x+2,0)\\
&\quad = \tilde{\mathcal{T}}\psi(x,2) + \tilde{\mathcal{T}}\psi(x+1,0).
\end{aligned}
$$

The proof of the lemma is now complete. □

Proof of Lemma 10. It is sufficient to show that

$$
\tilde{\mathcal{T}}\psi(x+1,y) \le \tilde{\mathcal{T}}\psi(x,y+1). \tag{EC.27}
$$

Note that

$$
\begin{aligned}
&\lambda\psi(x-1,y+1) + \mu_1\Big(\psi(x,y+2) + \tilde{c}_1\Big) + \mu_2\psi(x+1,y)\\
&\quad \ge \lambda\psi(x,y) + \mu_1\Big(\psi(x+1,y+1) + \tilde{c}_1\Big)\\
&\qquad + \mu_2[\mathbf{I}_{\{y>0\}}\psi(x+2,y-1) + \mathbf{I}_{\{y=0\}}\psi(x+1,y)]\\
&\quad \ge \tilde{\mathcal{T}}\psi(x+1,y), \qquad\qquad (\text{EC.28})\\
&\lambda\psi(x-1,y+1) + \mu_1\psi(x,y+1) + \mu_2\psi(x+1,y)\\
&\quad \ge \lambda\psi(x,y) + \mu_1\psi(x+1,y) + \mu_2[\mathbf{I}_{\{y>0\}}\psi(x+2,y-1) + \mathbf{I}_{\{y=0\}}\psi(x+1,y)]\\
&\quad \ge \tilde{\mathcal{T}}\psi(x+1,y), \qquad\qquad (\text{EC.29})
\end{aligned}
$$

and by Lemma 7,

$$\begin{aligned}\tilde{\mathcal{T}}\psi(x,y+1) = \min\Big\{&\lambda\psi(x-1,y+1)+\mu_1\Big(\psi(x,y+2)+\tilde{c}_1\Big)+\mu_2\psi(x+1,y),\\ &\lambda\psi(x-1,y+1)+\mu_1\psi(x,y+1)+\mu_2\psi(x+1,y)\Big\}.\end{aligned} \tag{EC.30}$$

(EC.27) directly follows from (EC.28)-(EC.30). ☐

Proof of Lemma 11. Properties (P1)-(P3) follow from the convexity of $c(x,y)=hx^+ + bx^- + h_2 y$ with respect to x and y separately. Note that

$$\triangle_y c(x,y) = h_2 \quad \text{and} \quad \triangle_x c(x,y) = \begin{cases} h, & \text{if } x \ge 0,\\ -b, & \text{if } x < 0.\end{cases}$$

Thus, $h_2 \ge h \ge 0$ implies (14) and (16). ☐

Proof of Proposition 1. We use sample-path analysis and the coupling method to prove this result. First, note that any policy for System **M** can be implemented by System **A**. Thus,

$$\tilde{f}(x,y) \le \hat{f}(x,y,z). \tag{EC.31}$$

Next, we show that the optimal policy in System **A** can also be implemented to System **M**. Comparing the two systems, the only scenario where a policy of System **A** might not be applicable to System **M** is when System **A** turns off server 1 while it's busy. However, for state $(x,0)$, when $R < S(1) < S(0)$, from Theorem 1, if server 1 is busy and some other event happens, System **A** will keep server 1 busy. Similarly, for state (x,y) with $y>0$, from Lemma 5, we know $S(y) < S(y-1)$, and again System **A** will keep server 1 busy. Thus, the above scenario will never occur. Hence, the optimal policy for System **A** can indeed be implemented in System **M**. Thus,

$$\tilde{f}(x,y) \ge \hat{f}(x,y,z). \tag{EC.32}$$

Combining (EC.31)-(EC.32) yields the result. ☐

Proof of Proposition 2. It is straightforward to verify that, if $c_2(1-\lambda)-\mu_1\tilde{c}_1 \ge \mu_2 b/\alpha$, then

$$h_2 + (1-\lambda)c_2 - \mu_1\tilde{c}_1 + \mu_2\triangle_x c(x,y) \ge 0.$$

Hence, to prove the proposition, it is sufficient to show that if $g(x,y)$ satisfies (18) and Properties (P1)-(P3), then

$$h_2 + (1-\lambda)c_2 - \mu_1\tilde{c}_1 + \mu_2\triangle_x[hx^+ + bx^- + h_2 y + \tilde{\mathcal{T}}g(x,y)] \ge 0. \tag{EC.33}$$

Define

$$\begin{aligned} S^g(y) &= \min\{x \,:\, \triangle_y g(x,y) + \tilde{c}_1 \ge 0\},\\ R^g &= \min\Big\{x : h_2 + c_2 - \mu_1\tilde{c}_1 + \lambda\triangle_y g(x-1,0) + \mu_2\triangle_x g(x,0)\\ &\qquad \ge -\mu_1(\triangle_y g(x,1)+\tilde{c}_1)\mathbf{I}_{\{x<S^g(1)\}} - \mu_1(\triangle_y g(x,0)+\tilde{c}_1)\mathbf{I}_{\{x\ge S^g(0)\}}\Big\}.\end{aligned}$$

Based on the discussion above, we have

$$R^g < S^g(1) < S^g(0).$$

To prove (EC.33), first consider $y = 0$. If $x + 1 < R^g$, then (EC.33) holds, because

$$\begin{aligned}
& h_2 + (1-\lambda)c_2 - \mu_1\tilde{c}_1 + \mu_2\triangle_x(hx^+ + bx^- + \tilde{\mathcal{T}}g(x,0)) \\
&\quad = h_2 + (1-\lambda)c_2 - \mu_1\tilde{c}_1 + \mu_2(h\mathbf{I}_{\{x\geq 0\}} - b\mathbf{I}_{\{x<0\}}) + \mu_2[\lambda\triangle_x g(x-1,1) \\
&\qquad + \mu_1\triangle_x g(x,2) + \mu_2\triangle_x g(x+1,0)] \\
&\quad \geq [h_2 + (1-\lambda)c_2 - \mu_1\tilde{c}_1]\alpha - \mu_2 b + [h_2 + (1-\lambda)c_2 - \mu_1\tilde{c}_1](\lambda + \mu_1 + \mu_2) \\
&\qquad + \mu_2[\lambda\triangle_x g(x-1,1) + \mu_1\triangle_x g(x,2) + \mu_2\triangle_x g(x+1,0)] \\
&\quad \geq \lambda[h_2 + (1-\lambda)c_2 - \mu_1\tilde{c}_1 + \mu_2\triangle_x g(x-1,1)] \\
&\qquad + \mu_1[h_2 + (1-\lambda)c_2 - \mu_1\tilde{c}_1 + \mu_2\triangle_x g(x,2)] \\
&\qquad + \mu_2[h_2 + (1-\lambda)c_2 - \mu_1\tilde{c}_1 + \mu_2\triangle_x g(x+1,0)] \\
&\quad \geq 0.
\end{aligned} \tag{EC.34}$$

Here, the second inequality comes from the assumption $h_2 + c_2(1-\lambda) - \mu_1\tilde{c}_1 \geq \mu_2 b/\alpha$, and the last inequality holds because $g(x,y)$ satisfies (18).

If $x + 1 = R^g$, (EC.33) holds, provided that

$$\begin{aligned}
& h_2 + (1-\lambda)c_2 - \mu_1\tilde{c}_1 + \mu_2\triangle_x(hx^+ + bx^- + \tilde{\mathcal{T}}g(x,0)) \\
&\quad = h_2 + (1-\lambda)c_2 - \mu_1\tilde{c}_1 + \mu_2(h\mathbf{I}_{\{x\geq 0\}} - b\mathbf{I}_{\{x<0\}}) \\
&\qquad + \mu_2\Big(-h_2 - c_2 + \lambda\triangle_x g(x-1,0) + \mu_1\triangle_x g(x,1)\Big) \\
&\qquad - \mu_2\Big(\lambda\triangle_y g(x-1,0) + \mu_1\triangle_y g(x,1)\Big) \\
&\quad \geq 0.
\end{aligned} \tag{EC.35}$$

But (EC.35) is equivalent to

$$\begin{aligned}
& h_2 + (1-\lambda)c_2 - \mu_1\tilde{c}_1 + \mu_2(h\mathbf{I}_{\{x\geq 0\}} - b\mathbf{I}_{\{x<0\}}) \\
&\quad + \mu_2\Big(-h_2 - c_2 + \lambda\triangle_x g(x-1,0) + \mu_1\triangle_x g(x,1)\Big) \\
&\quad \geq \mu_2\Big(\lambda\triangle_y g(x-1,0) + \mu_1\triangle_y g(x,1)\Big)
\end{aligned}$$

Adding $\mu_2^2\triangle_x g(x,0) + \mu_2(h_2 + c_2)$ to both sides of the inequality above, similar to (EC.34), the left-hand side is nonnegative, and the right-hand side is negative, since $x < x + 1 = R^g$ implies

$$\mu_2\Big(h_2 + c_2 + \lambda\triangle_y g(x-1,0) + \mu_2\triangle_x g(x,0) + \mu_1\triangle_y g(x,1)\Big) < 0.$$

If $R^g \le x < x+1 < S^g(0)$, then (EC.33) holds, because

$$\begin{aligned}
&h_2 + (1-\lambda)c_2 - \mu_1\tilde{c}_1 + \mu_2\triangle_x(hx^+ + bx^- + \tilde{\mathcal{T}}g(x,0)) \\
&\quad = h_2 + (1-\lambda)c_2 - \mu_1\tilde{c}_1 + \mu_2(h\mathbf{I}_{\{x\ge 0\}} - b\mathbf{I}_{\{x<0\}}) \\
&\qquad + \mu_2\Big(\lambda\triangle_x g(x-1,0) + \mu_1\triangle_x g(x,1) + \mu_2\triangle_x g(x,0)\Big) \\
&\quad \ge 0.
\end{aligned} \tag{EC.36}$$

If $S^g(0) = x+1$, then (EC.33) holds, because

$$\begin{aligned}
&h_2 + (1-\lambda)c_2 - \mu_1\tilde{c}_1 + \mu_2\triangle_x(hx^+ + bx^- + \tilde{\mathcal{T}}g(x,0)) \\
&\quad = h_2 + (1-\lambda)c_2 - \mu_1\tilde{c}_1 + \mu_2(h\mathbf{I}_{\{x\ge 0\}} - b\mathbf{I}_{\{x<0\}}) \\
&\qquad + \mu_2\Big(\lambda\triangle_x g(x-1,0) + \mu_1\triangle_x g(x,0) + \mu_2\triangle_x g(x,0)\Big) - \mu_2\mu_1[\triangle_y g(x,0) + \tilde{c}_1] \\
&\quad \ge 0,
\end{aligned}$$

where the last inequality holds, since $\triangle_y g(x,0) + \tilde{c}_1 < 0$, by $x < x+1 = S^g(0)$.

If $x \ge S^g(0)$, (EC.33) holds, because

$$\begin{aligned}
&h_2 + (1-\lambda)c_2 - \mu_1\tilde{c}_1 + \mu_2\triangle_x(hx^+ + bx^- + \tilde{\mathcal{T}}g(x,0)) \\
&\quad = h_2 + (1-\lambda)c_2 - \mu_1\tilde{c}_1 + \mu_2(h\mathbf{I}_{\{x\ge 0\}} - b\mathbf{I}_{\{x<0\}}) \\
&\qquad + \mu_2\Big(\lambda\triangle_x g(x-1,0) + \mu_1\triangle_x g(x,0) + \mu_2\triangle_x g(x,0)\Big) \\
&\quad \ge 0.
\end{aligned} \tag{EC.37}$$

Finally, consider $y > 0$. If $x+1 < S^g(y)$, then (EC.33) can be proven similar to (EC.36). If $x \ge S^g(y)$, one can prove (EC.33) along the lines of the proof of (EC.37). If $x+1 = S^g(y)$, (EC.33) holds, by

$$\begin{aligned}
&h_2 + (1-\lambda)c_2 - \mu_1\tilde{c}_1 + \mu_2\triangle_x(hx^+ + bx^- + h_2 y + \tilde{\mathcal{T}}g(x,y)) \\
&\quad = h_2 + (1-\lambda)c_2 - \mu_1\tilde{c}_1 + \mu_2(h\mathbf{I}_{\{x\ge 0\}} - b\mathbf{I}_{\{x<0\}}) \\
&\qquad + \mu_2\Big(\lambda\triangle_x g(x-1,y) + \mu_1\triangle_x g(x,y) + \mu_2\triangle_x g(x+1,y-1)\Big) \\
&\qquad - \mu_2\mu_1[\triangle_y g(x,y) + \tilde{c}_1],
\end{aligned}$$

where $\triangle_y g(x,y) + \tilde{c}_1 \le 0$ by $x < S^g(y)$. ☐

Proof of Lemma 12. It is direct to verify that $\mathcal{T}^{(1)}\psi(x,y)$ satisfies (21). Here we omit this verification. We now prove the supermodularity (Property (P1)) of $\mathcal{T}^{(1)}\psi(x,y)$:

$$\mathcal{T}^{(1)}\psi(x+1,y+1) + \mathcal{T}^{(1)}\psi(x,y) \ge \mathcal{T}^{(1)}\psi(x,y+1) + \mathcal{T}^{(1)}\psi(x+1,y), \tag{EC.38}$$

As $\psi(x,y)$ satisfies (21), we have that for any x and $y \geq 1$,

$$\mathcal{T}^{(1)}\psi(x,y) = \lambda\psi(x-1,y) + \mu_2\psi(x+1,y-1). \tag{EC.39}$$

Consider (EC.38) for $y \geq 1$. In view of (EC.39), we have

$$\begin{aligned} &\mathcal{T}^{(1)}\psi(x+1,y+1) + \mathcal{T}^{(1)}\psi(x,y) \\ &\quad = \lambda\psi(x,y+1) + \mu_2\psi(x+2,y) + \lambda\psi(x-1,y) + \mu_2\psi(x+1,y-1), \end{aligned} \tag{EC.40}$$

$$\begin{aligned} &\mathcal{T}^{(1)}\psi(x,y+1) + \mathcal{T}^{(1)}\psi(x+1,y) \\ &\quad = \lambda\psi(x-1,y+1) + \mu_2\psi(x+1,y) + \lambda\psi(x,y) + \mu_2\psi(x+2,y-1). \end{aligned} \tag{EC.41}$$

By $\triangle_{xy}\psi(x-1,y) \geq 0$ and $\triangle_{xy}\psi(x+1,y-1) \geq 0$, (EC.38) follows directly from (EC.40)-(EC.41).

Next, consider (EC.38) for $y = 0$. Let

$$\begin{aligned} u_1(x) = \arg\min_{u \geq 0} \Big\{ &(c_2+h_2)u + \lambda\psi(x-1,u) \\ &+ \mu_2\Big[\mathbf{I}_{\{u>0\}}\psi(x+1,u-1) + \mathbf{I}_{\{u=0\}}\psi(x,u)\Big]\Big\}. \end{aligned}$$

To prove (EC.38) for $y = 0$, consider two cases:

$$Case\ \ a.i\ \ u_1(x) > 0; \quad Case\ \ a.ii\ \ u_1(x) = 0.$$

In Case a.i, by the fact that $\psi(x,y)$ satisfies (21), we must have $u_1(x) = 1$. Hence, again by (EC.39),

$$\begin{aligned} &\mathcal{T}^{(1)}\psi(x+1,1) + \mathcal{T}^{(1)}\psi(x,0) \\ &\quad = \lambda\psi(x,1) + \mu_2\psi(x+2,0) + (c_2+h_2) + \lambda\psi(x-1,1) + \mu_2\psi(x+1,0), \end{aligned} \tag{EC.42}$$

$$\begin{aligned} &\mathcal{T}^{(1)}\psi(x,1) + \mathcal{T}^{(1)}\psi(x+1,0) \\ &\quad \leq \lambda\psi(x-1,1) + \mu_2\psi(x+1,0) + (c_2+h_2) + \lambda\psi(x,1) + \mu_2\psi(x+2,0). \end{aligned} \tag{EC.43}$$

This implies (EC.38).

In Case a.ii, using (EC.39),

$$\begin{aligned} &\mathcal{T}^{(1)}\psi(x+1,1) + \mathcal{T}^{(1)}\psi(x,0) \\ &\quad = \lambda\psi(x,1) + \mu_2\psi(x+2,0) + \lambda\psi(x-1,0) + \mu_2\psi(x,0), \end{aligned} \tag{EC.44}$$

$$\begin{aligned} &\mathcal{T}^{(1)}\psi(x,1) + \mathcal{T}^{(1)}\psi(x+1,0) \\ &\quad \leq \lambda\psi(x-1,1) + \mu_2\psi(x+1,0) + \lambda\psi(x,0) + \mu_2\psi(x+1,0). \end{aligned} \tag{EC.45}$$

By $\triangle_{xy}\psi(x-1,0) \geq 0$ and $\triangle_{xx}\psi(x,0) \geqq 0$, (EC.44)-(EC.45) imply (EC.38).

Finally, we prove that $\mathcal{T}^{(1)}\psi(x,y)$ satisfies Property (P2). We need to show that

$$\mathcal{T}^{(1)}\psi(x+2,y) + \mathcal{T}^{(1)}\psi(x,y+1) \geq \mathcal{T}^{(1)}\psi(x+1,y+1) + \mathcal{T}^{(1)}\psi(x+1,y). \tag{EC.46}$$

Along the same lines as the proof of (EC.38), one can show (EC.46). We omit the details. □

Proof of Theorem 4. First, we show that $f^{(1)}(\cdot,\cdot)$ satisfies (21). It follows from (20) that

$$\begin{aligned}
&c_2+\triangle_y f^{(1)}(x,y)\\
&\quad=c_2+h_2+\min_{u\geq y+1}\Big\{(c_2+h_2)(u-y-1)+\lambda f^{(1)}(x-1,u)\\
&\qquad\qquad+\mu_2[\mathbf{I}_{\{u>0\}}f^{(1)}(x+1,u-1)+\mathbf{I}_{\{u=0\}}f^{(1)}(x,0)]\Big\}\\
&\qquad-\min_{u\geq y}\Big\{(c_2+h_2)(u-y)+\lambda f^{(1)}(x-1,u)\\
&\qquad\qquad+\mu_2[\mathbf{I}_{\{u>0\}}f^{(1)}(x+1,u-1)+\mathbf{I}_{\{u=0\}}f^{(1)}(x,0)]\Big\}\\
&\quad=\min_{u\geq y+1}\Big\{(c_2+h_2)(u-y)+\lambda f^{(1)}(x-1,u)\\
&\qquad\qquad+\mu_2[\mathbf{I}_{\{u>0\}}f^{(1)}(x+1,u-1)+\mathbf{I}_{\{u=0\}}f^{(1)}(x,0)]\Big\}\\
&\qquad-\min_{u\geq y}\Big\{(c_2+h_2)(u-y)+\lambda f^{(1)}(x-1,u)\\
&\qquad\qquad+\mu_2[\mathbf{I}_{\{u>0\}}f^{(1)}(x+1,u-1)+\mathbf{I}_{\{u=0\}}f^{(1)}(x,0)]\Big\}\\
&\quad\geq 0.
\end{aligned}$$

Hence, we have (21).

Now we show that $f^{(1)}(\cdot,\cdot)$ satisfies Properties (P1)-(P2). Let $\psi(x,y)=hx^+ +bx^-$. Then we know that $\psi(\cdot,\cdot)$ satisfies Properties (P1)-(P2). Then by Lemma 12, similar to the arguments in the proof of Theorem 2, we know that $f^{(1)}(\cdot,\cdot)$ satisfies Properties (P1)-(P2).

It follows from the definition of S given by (22) that S is unique. The optimality of not ordering when $y>0$ directly follows from (21). For $y=0$, we consider only two possible actions, placing an order and not ordering. Then, the optimality equation (20) simplifies to

$$\begin{aligned}
f^{(1)}(x,0)&=hx^+ +bx^- +\min_{u\in\{0,1\}}\Big\{(c_2+h_2)u+\lambda f^{(1)}(x-1,u)\\
&\qquad+\mu_2\Big[\mathbf{I}_{\{u>0\}}f^{(1)}(x+1,u-1)+\mathbf{I}_{\{u=0\}}f^{(1)}(x,0)\Big]\Big\}\\
&=hx^+ +bx^- +\min\Big\{c_2+h_2+\lambda f^{(1)}(x-1,1)+\mu_2 f^{(1)}(x+1,0),\\
&\qquad\lambda f^{(1)}(x-1,0)+\mu_2 f^{(1)}(x,0)\Big\}\\
&=hx^+ +bx^- +\lambda f^{(1)}(x-1,0)+\mu_2 f^{(1)}(x,0)\\
&\quad+\min\Big\{c_2+h_2+\lambda\triangle_y f^{(1)}(x-1,0)+\mu_2\triangle_x f^{(1)}(x,0),0\Big\}.
\end{aligned}$$

The theorem now follows. $\square$