Appendix

In all sections in this Appendix, with the exception of §EC.5, the argument Q is dropped, since Q is assumed given in those parts.

EC.1. Proof of Theorem 1

Here we derive the solution to the quadratic hedging problem in (28), and thereby provide a proof for Theorem 1. We follow closely the numeraire-based approach of Gourieroux *et al.* (1998), but derive explicit results exploiting our specific problem context, as opposed to the more abstract (and general) setting in Gourieroux *et al.* (1998). We divide the proof into three parts: the first part sets things up and presents all required preliminary results, including a key Lemma 4 (without proof); the second part derives the main result, the optimal (minimal) value of $A(\lambda)$ in (30); the third part provides a proof to Lemma 4.

EC.1.1. Technical Preparations

First, we define the set of equivalent martingale measures with respect to X_t with square integrable densities (Radon-Nikodym derivatives):

$$\mathcal{M}_X := \{\mathsf{P}^{\bar{M}} \sim \mathsf{P} : \frac{d\mathsf{P}^{\bar{M}}}{d\mathsf{P}} \in L_2(\mathsf{P}),\, X_t \text{ is a } \mathsf{P}^{\bar{M}}\text{-martingale}\}. \tag{EC.1}$$

Recall the measure P^M defined in §3 is clearly in $\mathcal{M}_X$, hence $\mathcal{M}_X \neq \emptyset$. Indeed, P^M is known as the "variance-optimal martingale measure" in the literature (Gourieroux *et al.* (1998) and Schweizer (1992)), being the unique solution to $\min_{\mathsf{P}^{\bar{M}} \in \mathcal{M}_X} \mathsf{E}[(\frac{d\mathsf{P}^{\bar{M}}}{d\mathsf{P}})^2]$.

We remark that in general, the martingale requirement in $\mathcal{M}_X$ above should be replaced by local martingale, which is weaker (hence, more general). Refer to Gourieroux *et al.* (1998). But in our context, it is easy to verify that if X_t is a $\mathsf{P}^{\bar{M}}$-local martingale, and $\frac{d\mathsf{P}^{\bar{M}}}{d\mathsf{P}} \in L_2(\mathsf{P})$, then it is a $\mathsf{P}^{\bar{M}}$-martingale. Thus, there is no loss of generality in imposing the martingale requirement in $\mathcal{M}_X$.

Next, recall in §3, we briefly touched upon the basic requirements for a strategy to belong to the admissible class $\mathcal{A}_X$, i.e., it should be $\mathcal{F}_t$ predictable and X_t-integrable. Furthermore, for each admissible strategy $\vartheta \in \mathcal{A}_X$, the corresponding wealth process $\chi_t(\vartheta)$ should satisfy the following property: there exists a sequence of simple strategies $\{\vartheta^{(n)}\}$ (specified below), such that as $n \to \infty$,

$$\chi_t(\vartheta^{(n)}) \xrightarrow{\mathsf{P}} \chi_t(\vartheta) \quad \forall t \in [0,T]; \qquad \text{and} \qquad \chi_T(\vartheta^{(n)}) \xrightarrow{L_2} \chi_T(\vartheta). \tag{EC.2}$$

A *simple strategy* is a linear combination of strategies $h\mathbf{1}_{(\tau_1,\tau_2]}(t)$, where $\tau_1 \le \tau_2$ are stopping times such that $X_{t\wedge\tau_2}$ is bounded, and h is a bounded $\mathcal{F}_{\tau_1}$-measurable random variable. Refer to Cerny and Kallsen (2008), and Wang and Wissel (2013).

The above characterization then guarantees that $\{\chi_T(\vartheta) | \vartheta \in \mathcal{A}_X\}$, the set of terminal wealth attainable by admissible hedging strategies, is *closed*; specifically, it is the closure of the set of attainable terminal wealth achieved by simple strategies. (Refer to Lemma 2.6 of Cerny and Kallsen (2008).) This is necessary because the optimization problem in (13) is a mean-square-error (MSE) prediction problem in the Hilbert space; and the closedness of the space guarantees the existence of an optimal solution.

Following Cerny and Kallsen (2008), we can express $\mathcal{A}_X$ in the equivalent form below:

$$\mathcal{A}_X := \{\vartheta : \vartheta \text{ is } X_t\text{-integrable; } \chi_T(\vartheta) \in L_2(\mathsf{P});\, \forall \mathsf{P}^{\bar{M}} \in \mathcal{M}_X,\, \chi(\vartheta) \text{ is a } \mathsf{P}^{\bar{M}}\text{-martingale}\}. \tag{EC.3}$$

(It is also known that the above characterization of admissible strategies guarantees no arbitrage; refer to Delbaen and Schachermayer (1996).)

The key idea in solving the minimization problem in (28) is to turn it into the form of a mean-square-error (MSE) projection. To this end, working with the Radon-Nikodym derivative process Z_t in (16) is not good enough, for the simple reason it is not a P^M-martingale (it is a P-martingale). So, here we define

$$Z_t^M := \mathsf{E}^M(Z_T | \mathcal{F}_t). \tag{EC.4}$$

Then, Z_t^M is a P^M-martingale; and we can derive it explicitly, via the standard change of measure and making use of the expressions in (16):

$$Z_t^M = \mathsf{E}^M(Z_T|\mathcal{F}_t) = Z_t^{-1}\mathsf{E}(Z_T^2|\mathcal{F}_t) = e^{-\eta B_t - \frac{3}{2}\eta^2 t + \eta^2 T} = Z_t e^{\eta^2(T-t)}. \tag{EC.5}$$

Note, we have

$$Z_T^M = Z_T \quad \text{and} \quad Z_0^M = e^{\eta^2 T} = \mathsf{E}(Z_T^2). \tag{EC.6}$$

Hence, we can write

$$Z_t^M = Z_0^M e^{-\eta B_t^M - \eta^2 t/2},$$

where, as introduced earlier, $B_t^M = B_t + \eta t$ is a P^M-Brownian motion.

In view of the above, and analogous to using $Z_T = d\mathsf{P}^M/d\mathsf{P}$ to change measure from P to P^M, we can use Z_T^M/Z_0^M to further change measure from P^M to yet another measure denoted P^R, such that $B_t^R := B_t + 2\eta t$ is a P^R-martingale. Here's a summary of the associated R-N derivatives:

$$\frac{d\mathsf{P}^M}{d\mathsf{P}} = Z_T, \qquad \frac{d\mathsf{P}^R}{d\mathsf{P}^M} = \frac{Z_T^M}{Z_0^M} = \frac{Z_T}{Z_0^M}, \qquad \frac{d\mathsf{P}^R}{d\mathsf{P}} = \frac{(Z_T^M)^2}{Z_0^M} = \frac{Z_T^2}{\mathsf{E}(Z_T^2)}. \tag{EC.7}$$

To derive $A(\lambda)$, denote $\hat{H}_T(\lambda) := \lambda - H_T$, and change measure from P to P^R to obtain:

$$\begin{aligned} A(\lambda) &= \inf_{\vartheta \in \mathcal{A}_X} \mathsf{E}\Big[\Big(\hat{H}_T(\lambda) - \chi_T(\vartheta)\Big)^2\Big] \\ &= \inf_{\vartheta \in \mathcal{A}_X} Z_0^M \mathsf{E}\Big[\frac{(Z_T^M)^2}{Z_0^M}\Big(\frac{\hat{H}_T(\lambda)}{Z_T^M} - \frac{\chi_T(\vartheta)}{Z_T^M}\Big)^2\Big] \\ &= Z_0^M \inf_{\vartheta \in \mathcal{A}_X} \mathsf{E}^R\Big[\Big(\frac{\hat{H}_T(\lambda)}{Z_T^M} - \frac{\chi_T(\vartheta)}{Z_T^M}\Big)^2\Big]. \end{aligned} \tag{EC.8}$$

Next, instead of using X_t as tradable assets, we use another set of tradable assets, $N_t = (N_t^0, N_t^1)$ defined as follows:

$$N_t^0 := \frac{1}{Z_t^M},\ t \in [0,T], \qquad N_t^1 := \frac{X_t}{Z_t^M},\ t \in [0,T]. \tag{EC.9}$$

It is straightforward to verify the following:

$$N_t^0 = e^{\eta B_t^R - \frac{1}{2}\eta^2 t - \eta^2 T} \quad \text{and} \quad N_t^1 = X_0 e^{(\eta+\sigma)B_t^R - \frac{1}{2}(\eta+\sigma)^2 t - \eta^2 T}, \tag{EC.10}$$

where as introduced above, $B_t^R = B_t + 2\eta t$ is a P^R-martingale; hence, both N_t^0 and N_t^1 are P^R-martingale.

Now, in parallel with $\mathcal{M}_X$ and $\mathcal{A}_X$, we define

$$\mathcal{M}_N := \{\mathsf{P}^{\bar{R}} \sim P : \frac{1}{Z_T^M}\frac{d\mathsf{P}^{\bar{R}}}{d\mathsf{P}} \in L_2(\mathsf{P}),\ N_t^0 \text{ and } N_t^1 \text{ are } \mathsf{P}^{\bar{R}}\text{-martingales}\}, \tag{EC.11}$$

and

$$\mathcal{A}_N := \{\varphi : \varphi \text{ is } N_t\text{-integrable};\ Z_T^M \pi_T(\varphi) \in L_2(\mathsf{P});\ \forall \mathsf{P}^{\bar{R}} \in \mathcal{M}_N,\ \pi(\varphi) \text{ is a } \mathsf{P}^{\bar{R}}\text{-martingale }\} \tag{EC.12}$$

where the notation parallels $\chi(\varphi)$: $\pi(\varphi) := \{\pi_t(\varphi),\, t \in [0,T]\}$, with $\varphi := \{\phi_t = (\phi_t^0, \phi_t^1),\, t \in [0,T]\}$ being a $\mathcal{F}$-predictable two-dimensional process, and

$$\pi_t(\varphi) := \int_0^t \phi_s \cdot dN_s = \int_0^t \phi_s^0 dN_s^0 + \int_0^t \phi_s^1 dN_s^1. \tag{EC.13}$$

Likewise, we define $\pi_T(\mathcal{A}_N) := \{\pi_T(\varphi) | \varphi \in \mathcal{A}_N\}$ to be the attainable terminal wealth by admissible strategies in $\mathcal{A}_N$. There is a one-to-one relationship between the two admissible classes $\mathcal{A}_X$ and $\mathcal{A}_N$ as specified bellow.

Lemma 4 (i) For any given X_t-admissible trading strategy $\vartheta = \{\theta_t,\, t\in[0,T]\} \in \mathcal{A}_X$, there exists an N_t-admissible strategy $\varphi = \{\phi_t,\, t\in[0,T]\}\in\mathcal{A}_N$, such that $\forall t\in[0,T]$,

$$\frac{\chi_t(\vartheta)}{Z_t^M} = \pi_t(\varphi), \qquad \text{and} \qquad \phi_t = (\chi_t(\vartheta) - \theta_t X_t,\, \theta_t).$$

(ii) Conversely, given any N_t-admissible strategy $\varphi = \{\phi_t = (\phi_t^0, \phi_t^1),\, t\in[0,T]\}\in\mathcal{A}_N$, there exists an X_t-admissible trading strategy $\vartheta = \{\theta_t,\, t\in[0,T]\}\in\mathcal{A}_X$, such that $\forall t\in[0,T]$,

$$\frac{\chi_t(\vartheta)}{Z_t^M} = \pi_t(\varphi), \qquad \text{and} \qquad \theta_t = \zeta_t(\pi_t(\varphi) - \phi_t\cdot N_t) + \phi_t^1 \quad \text{with} \quad \zeta_t := -\frac{\eta}{\sigma}\frac{Z_t^M}{X_t}.$$

(iii) Combining (i) and (ii), we have:

$$\pi_T(\mathcal{A}_N) = \frac{\chi_T(\mathcal{A}_X)}{Z_T^M} := \Big\{\frac{\chi_T(\vartheta)}{Z_T^M} : \vartheta\in\mathcal{A}_X\Big\}.$$

Consequently, $\pi_T(\mathcal{A}_N)$ is closed under $L_2(\mathsf{P}^R)$.

We defer the proof of the lemma to §EC.1.3 below, but want to first note here that the trading strategy as spelled out in the above lemma splits the wealth χ_t at time t into two components: a position of θ_t in asset X_t (corresponding to η as market price of risk), and the rest in a risk-free asset (with a nominal zero market price of risk), but with both assets denominated in Z_t^M.

EC.1.2. Deriving (30)

Applying Lemma 4, part (iii) in particular, to the $A(\lambda)$ expression in (EC.8), we have

$$A(\lambda) = Z_0^M \inf_{\vartheta\in\mathcal{A}_X} \mathsf{E}^R\Big[\Big(\frac{\hat{H}_T(\lambda)}{Z_T^M} - \frac{\chi_T(\vartheta)}{Z_T^M}\Big)^2\Big] = Z_0^M \inf_{\varphi\in\mathcal{A}_N} \mathsf{E}^R\Big[\Big(\frac{\hat{H}_T(\lambda)}{Z_T^M} - \pi_T(\varphi)\Big)^2\Big]. \tag{EC.14}$$

Since $\mathsf{E}^R\big[\frac{\hat{H}_T(\lambda)}{Z_T^M}\big|\mathcal{F}_t\big]$ is a P^R-martingale, it admits the following decomposition (similar to the decomposition in (19)):

$$\mathsf{E}^R\Big[\frac{\hat{H}_T(\lambda)}{Z_T^M}\Big|\mathcal{F}_t\Big] = \mathsf{E}^R\Big[\frac{\hat{H}_T(\lambda)}{Z_T^M}\Big] + \int_0^t \phi_s^H\cdot dN_s + \int_0^t \gamma_s d\tilde{B}_s, \qquad t\in[0,T]. \tag{EC.15}$$

where the ϕ_t^H process may not be unique. This point will be addressed below in §EC.2. In addition, we need to verify that $\phi_t^H\in\mathcal{A}_N$, which is also addressed in §EC.2.

To continue, from Lemma 4, part (i) or part (ii) and along with (EC.13), we can write

$$\frac{\chi_T(\vartheta)}{Z_T^M} = \pi_T(\varphi) = \int_0^T \phi_t\cdot dN_t.$$

Combining the above with (EC.15) (letting $t=T$), and taking into account (EC.15), we can re-express the expectation in (EC.8) as follows:

$$\begin{aligned} &\mathsf{E}^R\Big[\Big(\frac{\hat{H}_T(\lambda)}{Z_T^M} - \frac{\chi_T(\vartheta)}{Z_T^M}\Big)^2\Big] = \mathsf{E}^R\Big[\Big(\frac{\hat{H}_T(\lambda)}{Z_T^M} - \pi_T(\varphi)\Big)^2\Big] \\ &= \mathsf{E}^R\Big[\Big(\mathsf{E}^R\Big[\frac{\hat{H}_T(\lambda)}{Z_T^M}\Big] + \int_0^T \gamma_t d\tilde{B}_t + \int_0^T \phi_t^H\cdot dN_t - \int_0^T \phi_t\cdot dN_t\Big)^2\Big] \\ &= \mathsf{E}^R\Big[\Big(\mathsf{E}^R\Big[\frac{\hat{H}_T(\lambda)}{Z_T^M}\Big] + \int_0^T \gamma_t d\tilde{B}_t\Big)^2\Big] + \mathsf{E}^R\Big[\Big(\int_0^T \phi_t^H\cdot dN_t - \int_0^T \phi_t\cdot dN_t\Big)^2\Big], \end{aligned} \tag{EC.16}$$

where the last equality is due to the vanishing cross term, since $\tilde{B}_t$ and N_t are independent, and the two integrals involving dN_t and $d\tilde{B}_t$ are square integrable martingales under P^R. From the last line above, we can immediately conclude that the optimal trading strategy that solves the minimization problem in (EC.16) is to choose the $\mathcal{A}_N$-strategy to be $\phi_t = \phi_t^H$ for all $t \in [0,T]$, which makes the second expectation in (EC.16) vanish. Hence, the resulting minimal A is:

$$\begin{aligned} A(\lambda) &= \mathsf{E}^R\Big[\Big(\mathsf{E}^R\Big[\frac{\hat{H}_T(\lambda)}{Z_T^M}\Big] + \int_0^T \gamma_t d\tilde{B}_t\Big)^2\Big] \\ &= Z_0^M\Big(\mathsf{E}^R\Big[\frac{\hat{H}_T(\lambda)}{Z_T^M}\Big]\Big)^2 + Z_0^M \int_0^T \mathsf{E}^R(\gamma_t^2)dt. \end{aligned} \tag{EC.17}$$

Furthermore, comparing (EC.15) with (19), we have

$$\gamma_t = -\frac{\delta_t}{Z_t^M}. \tag{EC.18}$$

What remains is to translate back from the P^R measure to the P^M and the P measures, which is straightforward from (EC.7):

$$\mathsf{E}^R\left[\frac{\hat{H}_T(\lambda)}{Z_T^M}\right] = \mathsf{E}^M\left[\frac{\hat{H}_T(\lambda)}{Z_T^M}\frac{Z_T^M}{Z_0^M}\right] = \frac{1}{Z_0^M}\mathsf{E}^M\left[\hat{H}_T(\lambda)\right]; \tag{EC.19}$$

and

$$\mathsf{E}^R(\gamma_t^2) = \mathsf{E}^M\Big[\Big(\frac{\delta_t}{Z_t^M}\Big)^2\frac{Z_t^M}{Z_0^M}\Big] = \frac{1}{Z_0^M}\mathsf{E}\Big[\frac{\delta_t^2}{Z_t^M}Z_t\Big] = \frac{e^{-\eta^2(T-t)}}{Z_0^M}\mathsf{E}(\delta_t^2), \tag{EC.20}$$

where the last equality makes use of (EC.5). Substituting the above back into (EC.17), we have, first under P^M and then under P:

$$\begin{aligned} A(\lambda) &= \frac{\big(\mathsf{E}^M[\hat{H}_T(\lambda)]\big)^2}{Z_0^M} + \int_0^T \mathsf{E}^M\Big(\frac{\delta_t^2}{Z_t^M}\Big)dt \\ &= e^{-\eta^2 T}(\lambda - V_0)^2 + \int_0^T e^{-\eta^2(T-t)}\mathsf{E}(\delta_t^2)dt. \end{aligned} \tag{EC.21}$$

The last line above is the expression in (30). (Recall $V_0 = \mathsf{E}^M(H_T)$.)

EC.1.3. Proof of Lemma 4

First, recall that for any given $\vartheta \in \mathcal{A}_X$, we have $\chi_t(\vartheta) = \mathsf{E}^M[\chi_T(\vartheta)|\mathcal{F}_t]$. Now define:

$$M_t := \mathsf{E}^R\Big[\frac{\chi_T(\vartheta)}{Z_T^M}\,\Big|\,\mathcal{F}_t\Big] = \mathsf{E}^M\Big[\frac{\chi_T(\vartheta)}{Z_T^M}\frac{Z_T^M}{Z_t^M}\,\Big|\,\mathcal{F}_t\Big] = \frac{\chi_t(\vartheta)}{Z_t^M}. \tag{EC.22}$$

Hence, M_t is a P^R-square-integrable martingale. Next we want to show that M_t can be expressed as a stochastic integral with respect to N_t, the π_t expression in (EC.13), as specified in the lemma.

From (EC.10) we have,

$$dN_t^0 = \eta N_t^0 dB_t^R \qquad dN_t^1 = (\eta+\sigma)N_t^1 dB_t^R = \Big(1+\frac{\sigma}{\eta}\Big)X_t dN_t^0, \tag{EC.23}$$

where the last equality takes into account $N_t^1 = X_t N_t^0$. Applying Itô's product rule, we have,

$$dN_t^1 = d(X_t N_t^0) = X_t dN_t^0 + N_t^0 dX_t + dN_t^0 dX_t,$$

which, in comparison with dN_t^1 in (EC.23), yields

$$N_t^0 dX_t + dN_t^0 dX_t = \frac{\sigma}{\eta}X_t dN_t^0.$$

Applying Itô's product rule again, this time to $M_t = \frac{\chi_t}{Z_t^M} = \chi_t N_t^0$, and taking into account $d\chi_t = \theta_t dX_t$, we have,

$$dM_t = d(\chi_t N_t^0) = \chi_t dN_t^0 + \theta_t (N_t^0 dX_t + dN_t^0 dX_t).$$

Combining the last two equations, we have

$$dM_t = \Big(\chi_t + \frac{\sigma}{\eta}\theta_t X_t\Big) dN_t^0; \tag{EC.24}$$

which, along with the dN_t^1 and dN_t^0 relation in (EC.23), leads to

$$dM_t = \Big(\chi_t + \frac{\sigma}{\eta}\theta_t X_t - (1+\frac{\sigma}{\eta})\theta_t X_t\Big) dN_t^0 + \theta_t dN_t^1 = \big(\chi_t - \theta_t X_t\big) dN_t^0 + \theta_t dN_t^1. \tag{EC.25}$$

The last one is the desired result in the lemma. But it is also clear that we can choose to represent dM_t in terms of dN_t^0 alone as in (EC.24) (or in terms of dN_t^1 alone). Since we have established $\frac{\chi_t(\vartheta)}{Z_t^M} = \pi_t(\varphi)$, we will use the two quantities interchangeably below.

What remains is to verify that $\varphi = (\chi_t - \theta_t X_t, \theta_t) \in \mathcal{A}_N$. The first two defining conditions in $\mathcal{A}_N$ clearly hold. To verify the last (martingale) condition, we make use of the following one-to-one correspondence between $\mathcal{M}_X$ and $\mathcal{M}_N$: $\mathsf{P}^{\bar{R}} \in \mathcal{M}_N$ if any only if there exists $\mathsf{P}^{\bar{M}} \in \mathcal{M}_X$ such that

$$\frac{d\mathsf{P}^{\bar{R}}}{d\mathsf{P}} = \frac{d\mathsf{P}^{\bar{M}}}{d\mathsf{P}} \cdot \frac{Z_T^M}{Z_0^M} \Big(= \frac{d\mathsf{P}^{\bar{M}}}{d\mathsf{P}} \cdot \frac{d\mathsf{P}^R}{d\mathsf{P}^M} \Big). \tag{EC.26}$$

Refer to Proposition 3.1 in Gourieroux *et al.* (1998). Hence, starting from $\mathsf{P}^{\bar{R}} \in \mathcal{A}_N$ that defines φ, we pick the $\mathsf{P}^{\bar{M}}$ that corresponds to $\mathsf{P}^{\bar{R}}$ in (EC.26); thus, $\mathsf{P}^{\bar{M}} \in \mathcal{M}_X$. From (EC.26), we have $d\mathsf{P}^{\bar{R}}/d\mathsf{P}^{\bar{M}} = Z_T^M/Z_0^M$; and it is easy to verify that Z_t^M is a $\mathsf{P}^{\bar{M}}$-martingale, taking into account X_t a $\mathsf{P}^{\bar{M}}$-martingale as part of the specification of $\mathcal{M}_X$. Consequently, we can use Z_t^M to change measure from $\mathsf{P}^{\bar{R}}$ to $\mathsf{P}^{\bar{M}}$:

$$\mathsf{E}^{\bar{R}}\Big[\frac{\chi_T(\vartheta)}{Z_T^M} \Big| \mathcal{F}_t\Big] = \mathsf{E}^{\bar{M}}\Big[\frac{\chi_T(\vartheta)}{Z_T^M}\frac{Z_T^M}{Z_t^M} \Big| \mathcal{F}_t\Big] = \frac{\chi_t(\vartheta)}{Z_t^M}, \tag{EC.27}$$

where the second equality follows from the definition of $\mathcal{A}_X$, i.e., χ is is a $\mathsf{P}^{\bar{M}}$-martingale. Thus, the above confirms that $\pi_t = \frac{\chi_t}{Z_t^M}$ is a $\mathsf{P}^{\bar{M}}$-martingale.

For the converse part, consider any given $\varphi = \{\phi_t\}_{t\in[0,T]} \in \mathcal{A}_N$, and let's start with the one-dimensional case. Specifically, assuming N_t^1 is 0 and let N_t be N_t^0, then drop the superscript 0 for simplicity. We want to have:

$$\chi_t = \int_0^t \theta_s dX_s = Z_t^M \cdot \int_0^t \phi_s dN_s, \tag{EC.28}$$

for some $\vartheta = \{\theta_t,\, t \in [0,T]\}$. Equivalently,

$$\theta_t dX_t = dZ_t^M \int_0^t \phi_s dN_s + \phi_t Z_t^M dN_t + \phi_t dZ_t^M dN_t. \tag{EC.29}$$

Denote

$$\zeta_t := \frac{dZ_t^M}{dX_t} = -\frac{\eta}{\sigma}\frac{Z_t^M}{X_t}. \tag{EC.30}$$

To derive θ_t from (EC.29), it helps to write out key terms involved in both sides explicitly in B_t^R, the P^R-martingale (and recall $B_t^R = B_t^M + \eta t = B_t + 2\eta t$):

$$dX_t = X_t(-\mu dt + \sigma dB_t^R), \quad dZ_t^M = Z_t^M(-\eta dB_t^R + \eta^2 dt), \quad dN_t = N_t \eta dB_t^R.$$

Hence, there are two kinds of terms on both sides of (EC.29), involving dB_t^R and dt. (Note, the third term on the right hand side is a dt term.) Matching the dB_t^R terms on the two sides, we have

$$\theta_t = \zeta_t\Big(\int_0^t \phi_s dN_s - \phi_t N_t\Big). \tag{EC.31}$$

(Matching the dt terms on the two sides gets the same result.)

Back to the two-dimensional case, all that's needed is to write dN_t as (dN_t^0, dN_t^1), and $\phi_t dN_t$ as $\phi_t^0 dN_t^0 + \phi_t^1 dN_t^1$. The new term dN_t^1 completely parallels the above derivation, replacing the coefficient η by $\eta+\sigma$ (refer to (EC.23)). So, the final result is:

$$\begin{aligned}\theta_t &= \zeta_t\Big(\int_0^t (\phi_s^0 dN_s^0 + \phi_s^1 dN_s^1) - \phi_t^0 N_t^0 - (1+\frac{\sigma}{\eta})\phi_t^1 N_t^1\Big)\\ &= \zeta_t\Big(\int_0^t (\phi_s^0 dN_s^0 + \phi_s^1 dN_s^1) - \phi_t^0 N_t^0 - \phi_t^1 N_t^1\Big) + \phi_t^1,\end{aligned} \tag{EC.32}$$

which is the desired conclusion in part (ii) of the lemma.

Finally, verifying $\vartheta = \{\theta_t, t\in[0,T]\} \in \mathcal{A}_X$ is completely analogous to verifying $\varphi \in \mathcal{A}_N$. This completes the proof of (ii). □

EC.2. Proof of Theorem 3

The P^R-martingale, $\hat{M}_t := \mathsf{E}^R\big[\frac{\hat{H}_T(\lambda)}{Z_T^M}\,\big|\,\mathcal{F}_t\big]$ admits the decomposition in (EC.15), and for this section, we drop the superscript H in ϕ_t^H for simplicity and write $\phi_t^H = \phi_t := (\phi_t^0, \phi_t^1)$. On the other hand, $\hat{M}_t$ admits the following martingale representation:

$$\hat{M}_t = \mathsf{E}^R\Big[\frac{\hat{H}_T(\lambda)}{Z_T^M}\Big] + \int_0^t y_s dB_s^R + \int_0^t \gamma_s d\tilde{B}_s, \tag{EC.33}$$

which is unique. Comparing (EC.33) with the decomposition in (EC.15), taking into account (EC.23), we have

$$y_t = \eta\phi_t^0 N_t^0 + (\eta+\sigma)\phi_t^1 N_t^1, \qquad t\in[0,T]. \tag{EC.34}$$

That is, while the process ϕ_t need not be unique, as long as it satisfies (EC.34), it will satisfy (EC.33), and all such ϕ_t will give the same wealth process since

$$\int_0^t \phi_s \cdot dN_s = \int_0^t y_s dB_s^R.$$

Using the fact that $\hat{M}_t$ is a square integrable martingale under P^R, it is easily verified that $\phi_t \in \mathcal{A}_N$; ϕ_t is then an optimal solution to (EC.16), as shown in §EC.1.2.

Next, substituting (EC.34) into (EC.32), we have

$$\theta_t = \zeta_t\Big(\int_0^t y_s dB_s^R - \frac{1}{\eta} y_t\Big). \tag{EC.35}$$

Thus, even there might be multiple ϕ_t satisfying (EC.34), θ_t above is unique, as it is represented by the unique y_t from the martingale representation of $\hat{M}_t$ in (EC.33). (In other words, whereas there are different ways to choose the underlying assets to carry out the trading strategy with respect to N_t, they will not alter the value dynamics of θ_t.)

Now,

$$\frac{\chi_t(\vartheta)}{Z_t^M} = \pi_t(\varphi) = \int_0^t \phi_s \cdot dN_s = \int_0^t y_s dB_s^R,$$

where the first equality follows from Lemma 4, the second is (EC.13), and the last has just been argued above. Substituting the above into (EC.35) yields

$$\theta_t = \zeta_t\Big(\frac{\chi_t(\vartheta)}{Z_t^M} - \frac{1}{\eta}y_t\Big). \tag{EC.36}$$

Changing measure from P^R to P^M, we have

$$\hat{M}_t = \mathsf{E}^R\Big[\frac{\hat{H}_T(\lambda)}{Z_T^M}\,\Big|\,\mathcal{F}_t\Big] = \mathsf{E}^M\Big[\frac{\hat{H}_T(\lambda)}{Z_T^M}\frac{Z_T^M}{Z_t^M}\,\Big|\,\mathcal{F}_t\Big] = \frac{\lambda - V_t}{Z_t^M}.$$

Thus, writing out the martingale representations of both $\hat{M}_t$ and V_t, and noting $\mathsf{E}(\hat{M}_t) = \frac{\lambda - V_0}{Z_0^M}$, we have

$$Z_t^M\Big(\frac{\lambda - V_0}{Z_0^M} + \int_0^t y_s dB_s^R + \int_0^t \gamma_s d\tilde{B}_s\Big) = \lambda - V_0 - \int_0^t \xi_s dX_s - \int_0^t \delta_s d\tilde{B}_s. \tag{EC.37}$$

Applying Itô's product rule to the both sides, and equating the terms that involve dB_t^R, we have

$$-\eta(\lambda - V_t) + Z_t^M y_t = -\sigma \xi_t X_t;$$

and solving for y_t yields

$$y_t = \frac{\eta(\lambda - V_t) - \sigma X_t \xi_t}{Z_t^M}. \tag{EC.38}$$

Substituting the above into (EC.36) leads to the optimal trading strategy (writing θ_t as θ_t^* and χ_t as χ_t^*):

$$\theta_t^* = -\xi_t + \frac{\eta}{\sigma X_t}(\lambda - V_t - \chi_t^*),$$

By Lemma 4 (ii), θ_t^* is the optimal solution to (13), which is the desired result in (33). □

EC.3. Proof of Proposition 2

From (19), we have $dV_t = \xi_t dX_t + \delta_t d\tilde{B}_t$, which, combined with $d\chi_t^*$ in (38), yields

$$dW_t = dV_t + d\chi_t^* = \eta(\lambda - W_t)(\eta dt + dB_t) + \delta_t d\tilde{B}_t;$$

and direct verification shows that W_t in (34) uniquely solves the above stochastic differential equation.

For $\mathsf{E}(W_t)$, observe $\mathsf{E}(Z_t) = 1$; and $\mathsf{E}\big(Z_t \int_0^t \frac{\delta_s}{e^{-\eta^2 s} Z_s} d\tilde{B}_s\big) = 0$, since Z_t and $\int_0^t \frac{\delta_s}{(e^{-\eta^2 s} Z_s)} d\tilde{B}_s$ are two square-integrable martingales under $L_2(\mathsf{P})$ that have zero covariation. Then, the stated expression for $\mathsf{E}(W_t)$ follows immediately.

To obtain $\mathsf{Var}(W_t)$, it remains to compute $\mathsf{E}(W_t^2)$ as follows. First, by direct algebra we have:

$$\begin{aligned} W_t^2 \;=&\; \lambda^2 + k^2 e^{2\eta^2(T-t)} Z_t^2 + e^{2\eta^2(T-t)} Z_t^2 \Big(\int_0^t e^{-\eta^2(T-s)} \frac{\delta_s}{Z_s} d\tilde{B}_s\Big)^2 + (2\lambda k) e^{\eta^2(T-t)} Z_t \\ &+ (2\lambda) e^{\eta^2(T-t)} Z_t \int_0^t e^{-\eta^2(T-s)} \frac{\delta_s}{Z_s} d\tilde{B}_s + (2k) e^{2\eta^2(T-t)} Z_t^2 \int_0^t e^{-\eta^2(T-s)} \frac{\delta_s}{Z_s} d\tilde{B}_s, \end{aligned}$$

with the constant $k := e^{-\eta^2 T}(V_0 - \lambda)$. Note, $\mathsf{E}(Z_t) = 1$ and $\mathsf{E}\big(Z_t \int_0^t e^{-\eta^2(T-s)} \frac{\delta_s}{Z_s} d\tilde{B}_s\big) = 0$ from the derivation of $\mathsf{E}(W_t)$, and $\mathsf{E}(Z_t^2) = e^{\eta^2 t}$. We derive the remaining two terms as follows. Recognizing $Z_t^2 = Z_0^M e^{-\eta^2(T-t)} \mathsf{E}(\frac{d\mathsf{P}^R}{dP} \,|\, \mathcal{F}_t)$, it follows that:

$$\begin{aligned} \mathsf{E}\Big[Z_t^2\Big(\int_0^t e^{-\eta^2(T-s)} \frac{\delta_s}{Z_s} d\tilde{B}_s\Big)^2\Big] &= Z_0^M e^{-\eta^2(T-t)} \mathsf{E}^R\Big[\Big(\int_0^t e^{-\eta^2(T-s)} \frac{\delta_s}{Z_s} d\tilde{B}_s\Big)^2\Big] \\ &= Z_0^M e^{-\eta^2(T-t)} \int_0^t e^{-2\eta^2(T-s)} \mathsf{E}^R\Big(\frac{\delta_s^2}{Z_s^2}\Big) ds \\ &= Z_0^M e^{-\eta^2(T-t)} \int_0^t e^{-2\eta^2(T-s)} \mathsf{E}\Big[\Big(\frac{\delta_s^2}{Z_s^2}\Big) \frac{e^{\eta^2(T-s)}}{Z_0^M} Z_s^2\Big] ds \\ &= e^{-\eta^2(T-t)} \int_0^t e^{-\eta^2(T-s)} \mathsf{E}(\delta_s^2) ds. \end{aligned}$$

Similarly, we can derive

$$\mathsf{E}\Big[Z_t^2\int_0^t e^{-\eta^2(T-s)}\frac{\delta_s}{Z_s}d\tilde{B}_s\Big]=0.$$

Putting all the terms together yield the desired formula for $\mathsf{Var}(W_t)$. □

EC.4. Proof of Proposition 5

For any given λ, we can write

$$\mathsf{Var}(H_T)=\mathsf{Var}(\lambda-H_T)=\mathsf{E}[(\lambda-H_T)^2]-(\lambda-m)^2.$$

As before, write $\hat{H}_T:=\lambda-H_T$. Changing measure to P^R, we have

$$\mathsf{E}\big(\hat{H}_T^2\big)=Z_0^M\mathsf{E}^R\Big[\Big(\frac{\hat{H}_T}{Z_T^M}\Big)^2\Big]=Z_0^M\mathsf{E}^R(\hat{M}_T^2),$$

where $\hat{M}_t$ is the P^R-martingale defined at the beginning of §EC.2. Making use of (EC.33), we have

$$\mathsf{E}^R(\hat{M}_T^2)=\Big[\mathsf{E}^R\Big(\frac{\hat{H}_T}{Z_T^M}\Big)\Big]^2+\int_0^T\mathsf{E}^R(y_t^2)dt+\int_0^T\mathsf{E}^R(\gamma_t^2)dt.$$

Translating back into the P measure (refer to the end of the proof of Proposition 1 in §EC.1), and recognizing $\mathsf{E}^R(y_t^2)=e^{-\eta^2t}\mathsf{E}(y_t^2Z_t^2)$, we have

$$\begin{aligned}\mathsf{Var}(H_T)&=e^{-\eta^2T}(\lambda-V_0)^2+\int_0^T e^{-\eta^2(T-t)}\mathsf{E}(\delta_t^2)dt+\int_0^T e^{\eta^2(T-t)}\mathsf{E}(y_t^2Z_t^2)dt-(\lambda-m)^2\\&=A(\lambda)-(\lambda-m)^2+\int_0^T e^{\eta^2(T-t)}\mathsf{E}(y_t^2Z_t^2)dt.\end{aligned}$$

Setting λ as in (32) yields $A(\lambda)-(\lambda-m)^2=B(m,Q)$, and hence the desired result, taking into account $Z_t^M=Z_te^{\eta^2(T-t)}$ from (EC.5). □

EC.5. Efficient Frontier for One-Shot Hedging

Recall, in the one-shot hedging case (§4.3), we want to solve the following problem:

$$\min_{Q\ge 0,\bar{\theta}}\bar{B}(m,Q)=\mathsf{Var}[H_T(Q)]+\bar{\theta}^2\mathsf{Var}(X_T)+2\bar{\theta}\mathsf{Cov}[H_T(Q),X_T],\quad \text{s.t. }\mathsf{E}[H_T(Q)]+\bar{\theta}\mathsf{E}(X_T-X_0)=m. \tag{EC.39}$$

It turns out that we need to relax the "$=m$" constraint above to "$\ge m$"; and the reason will become evident below. To simplify notation, re-express the above problem as follows:

$$\min_{x\ge 0,y}f(x,y),\qquad \text{s.t.}\quad h(x)+ay\ge m; \tag{EC.40}$$

where $(x,y):=(Q,\bar{\theta})$, $f(x,y)$ represents the right side of $\bar{B}(m,Q)$ in (EC.39), $h(x):=\mathsf{E}[H_T(x)]$, and $a:=\mathsf{E}(X_T-X_0)>0$. With f_x,f_y,h' denoting the derivatives, and $\ell\ge 0$, the Lagrangian multiplier, the Karush-Kuhn-Tucker (KKT) conditions (necessary for optimality) are (in addition to the constraints $h(x)+ay\ge m$ and $x\ge 0$):

$$f_x=\ell h',\quad f_y=\ell a,\quad \ell[h(x)+ay-m]=0. \tag{EC.41}$$

Let (x_m,y_m) denote a (any) solution to the above equations. Then,

$$\frac{d}{dm}f(x_m,y_m)=f_x\frac{dx_m}{dm}+f_y\frac{dy_m}{dm}.$$

We want to show the above derivative is non-negative. Substituting (EC.41) into the above, we have, separating the two cases $\ell = 0$ and $\ell > 0$,

$$\ell = 0 \Rightarrow \frac{d}{dm} f(x_m, y_m) = 0; \qquad \ell > 0 \Rightarrow \frac{d}{dm} f(x_m, y_m) = \ell\Big(h' \frac{dx_m}{dm} + a\frac{dy_m}{dm}\Big) = \ell; \tag{EC.42}$$

where the last equality follows from taking derivatives with respect to m on both sides of $h(x) + ay = m$, an equation implied by the last KKT condition in (EC.41) with $\ell > 0$. Therefore, we have established that $\bar{B}(m, \bar{Q}_m)$, with $\bar{Q}_m$ being the optimal order quantity associated with the one-shot hedging $\bar{\theta}$ (also optimized), is increasing in m, and hence forming an efficient frontier.

From (EC.42), it is clear that on the efficient-frontier, there will in general be flat segments corresponding to $\ell = 0$. Let $[m_0, m_1]$ denote such a segment. Then, for a given $m \in [m_0, m_1)$, the above algorithm returns a solution (x_m, y_m) satisfying $f_x = f_y = 0$, the first two equations in (EC.41) with $\ell = 0$. (The third equation is automatically satisfied when $\ell = 0$.) Furthermore, the solution will achieve the target return m_1, i.e., $h(x_m) + ay_m = m_1$. In other words, the algorithm will skip the flat segment, getting directly to its right end-point.

Now return to the original "$= m$" constraint in (EC.39). Then, (EC.41) becomes

$$f_x = \ell h', \quad f_y = \ell a, \quad h(x) + ay - m = 0;$$

where ℓ is not necessarily non-negative. Indeed, the above (three equations with three unknowns) can return a solution (x_m, y_m) along with a negative ℓ, which will lead to (similar to the second part in (EC.42))

$$\frac{d}{dm} f(x_m, y_m) = \ell < 0,$$

implying the solution cannot be efficient: an increase (sufficiently small) in m will result in a decrease in the variance. And this is exactly what happens in the case of m falling into a flat segment of the efficient frontier, which is skipped over in the relaxed formulation as explained above.

Proposition 9 For any given m, there exists a production quantity $\bar{Q}_m$ that minimizes the variance, along with a one-shot hedging position $\bar{\theta}$. Furthermore, the pair $(\bar{Q}_m, \bar{\theta})$, expressed as (x, y), is a solution to the KKT conditions in (EC.41) and achieves a target mean that is at least m. Consequently, $\bar{B}(m, \bar{Q}_m)$ is increasing in m, forming an efficient frontier. □

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