Electronic Companion to "Closed-Form Approximations for Optimal (r,q) and (S,T) Policies in a Parallel Processing Environment"

Proof : **[of Lemma 3]** First we prove (41). By Proposition 3.1 in Yamazaki et al. (1992),

$$\mathsf{E}\Big(N^\lambda - \frac{\rho^\lambda}{q^\lambda}\Big)^2 \le \frac{\rho^\lambda}{q^\lambda} + 2\frac{\lambda}{q^\lambda}\cdot\frac{\theta}{\sqrt{q^\lambda}}\mathsf{E}L_{(2)}.$$

By the definition of $Y^\lambda(q^\lambda)$ given by (19) and $\left|\lim_{\lambda\to\infty} z^\lambda(i^\lambda, q^\lambda)\right| < \infty$, there exists a constant M such that

$$\mathsf{E}\left(z^\lambda(i^\lambda, q^\lambda) - Y^\lambda(q^\lambda)\right)^2 \le M.$$

Thus, the function $\hat{G}(\cdot)$ is uniformly integrable relative to the sequence of distribution functions given by $(z^\lambda(i^\lambda, q^\lambda) - Y^\lambda(q^\lambda))$. Hence, (41) directly follows from Theorem 2 on p.276 in Chow and Teicher (2003) and Lemma 2 (as Condition 1 holds and $\lim_{\lambda\to\infty} z^\lambda(i^\lambda, q^\lambda)$ exists).

Now we prove (42). This approximation result is mentioned by Zheng (1992) (see p.89, Zheng 1992). It may be hidden in some textbooks. For the completeness, here we give a proof. It suffices to show that for any $\varepsilon > 0$, there exists an Λ such that for $\lambda > \Lambda$,

$$\Big|\sum_{i=r^\lambda+1}^{r^\lambda+q^\lambda} C\left(z^\lambda(i,q^\lambda)\right) - \int_{r^\lambda}^{r^\lambda+q^\lambda} C\left(z^\lambda(x,q^\lambda)\right)\mathsf{d}x\Big| \le \varepsilon \times \int_{r^\lambda}^{r^\lambda+q^\lambda} C\left(z^\lambda(x,q^\lambda)\right)\mathsf{d}x. \tag{A-1}$$

Note that

$$\begin{aligned}
&\sum_{i=r^\lambda+1}^{r^\lambda+q^\lambda} C\left(z^\lambda(i,q^\lambda)\right) - \int_{r^\lambda}^{r^\lambda+q^\lambda} C\left(z^\lambda(x,q^\lambda)\right)\mathsf{d}x \\
&= \sum_{i=r^\lambda+1}^{r^\lambda+q^\lambda}\Big[C\left(z^\lambda(i,q^\lambda)\right) - \int_{i-1}^{i} C\left(z^\lambda(x,q^\lambda)\right)\mathsf{d}x\Big] \\
&= \sum_{i=r^\lambda+1}^{r^\lambda+q^\lambda}\int_{i-1}^{i}\Big[C\left(z^\lambda(i,q^\lambda)\right) - C\left(z^\lambda(x,q^\lambda)\right)\Big]\mathsf{d}x.
\end{aligned} \tag{A-2}$$

Using (35) and the convexity of $C(\cdot)$, for $i = r^\lambda + 1, \cdots, r^\lambda + q^\lambda$ and $x \in [i-1,\ i]$, if $z_* \notin (z^\lambda(i,q^\lambda),\ z^\lambda(i+1,q^\lambda))$, then

$$\begin{aligned}
\Big|C\left(z^\lambda(i,q^\lambda)\right) - C\left(z^\lambda(x,q^\lambda)\right)\Big| &\le \Big|C\left(z^\lambda(i+1,q^\lambda)\right) - C\left(z^\lambda(i,q^\lambda)\right)\Big| \\
&= h \times \Big|\Phi^1\left(-z^\lambda(i+1,q^\lambda)\right) - \Phi^1\left(-z^\lambda(i,q^\lambda)\right)\Big|
\end{aligned}$$

$$+p\times\left|\Phi^1\left(z^\lambda(i+1,q^\lambda)\right)-\Phi^1\left(z^\lambda(i,q^\lambda)\right)\right|$$
$$\le \frac{p}{\sigma^\lambda(q^\lambda)\sqrt{\lambda q^\lambda}}+\frac{h}{\sigma^\lambda(q^\lambda)\sqrt{\lambda q^\lambda}}, \tag{A-3}$$

and if $z_*\in(z^\lambda(i,q^\lambda),\ z^\lambda(i+1,q^\lambda))$, then

$$\left|C\left(z^\lambda(i,q^\lambda)\right)-C\left(z^\lambda(x,q^\lambda)\right)\right|$$
$$\le\max\left\{C\left(z^\lambda(i+1,q^\lambda)\right)-C(z_*),C\left(z^\lambda(i,q^\lambda)\right)-C(z_*)\right\}. \tag{A-4}$$

By Conditions 1–2, there exists an Λ_0 such that for $\lambda>\Lambda_0$,

$$\left|z^\lambda(i,q^\lambda)\right|\le M+1,\ i=r^\lambda+1,\cdots,r^\lambda+q^\lambda. \tag{A-5}$$

Thus, for $i=r^\lambda+1,\cdots,r^\lambda+q^\lambda$,

$$\int_{i-1}^{i}C\left(z^\lambda(x,q^\lambda)\right)\mathsf{d}x\ge h\times\Phi^1(M+1)+p\times\Phi^1(M+1). \tag{A-6}$$

From the definition of $\sigma^\lambda(q^\lambda)$ given by (17) and (A-6), for any $\varepsilon>0$, there exists an Λ_1 such that for $\lambda>\Lambda_1$,

$$\frac{p}{\sigma^\lambda(q^\lambda)\sqrt{\lambda q^\lambda}}+\frac{h}{\sigma^\lambda(q^\lambda)\sqrt{\lambda q^\lambda}}\le\varepsilon\times\int_{i-1}^{i}C\left(z^\lambda(x,q^\lambda)\right)\mathsf{d}x. \tag{A-7}$$

Combining (A-3) and (A-7) yields that for i with $z_*\notin(z^\lambda(i,q^\lambda),\ z^\lambda(i+1,q^\lambda))$,

$$\int_{i-1}^{i}\left|C\left(z^\lambda(i,q^\lambda)\right)-C\left(z^\lambda(x,q^\lambda)\right)\right|\mathsf{d}x\le\varepsilon\times\int_{i-1}^{i}C\left(z^\lambda(x,q^\lambda)\right)\mathsf{d}x. \tag{A-8}$$

With the help of (35), for i with $z_*\in(z^\lambda(i,q^\lambda),\ z^\lambda(i+1,q^\lambda))$, similarly, we can show that there exists an Λ_2 such that for $\lambda>\Lambda_2$,

$$\max\left\{C\left(z^\lambda(i+1,q^\lambda)\right)-C(z_*),C\left(z^\lambda(i,q^\lambda)\right)-C(z_*)\right\}\le\varepsilon\times\int_{i-1}^{i}C\left(z^\lambda(x,q^\lambda)\right)\mathsf{d}x. \tag{A-9}$$

Combining (A-4) and (A-9) yields that for i with $z_*\in(z^\lambda(i,q^\lambda),\ z^\lambda(i+1,q^\lambda))$,

$$\int_{i-1}^{i}\left|C\left(z^\lambda(i,q^\lambda)\right)-C\left(z^\lambda(x,q^\lambda)\right)\right|\mathsf{d}x\le\varepsilon\times\int_{i-1}^{i}C\left(z^\lambda(x,q^\lambda)\right)\mathsf{d}x. \tag{A-10}$$

Therefore, (A-1) holds for $\lambda>\max\{\Lambda_0,\Lambda_1,\Lambda_2\}$ directly from (A-2), (A-8) and (A-10). Thus the validity of the approximation given by (42) is proved. ■

Proof : **[of Lemma 4]** In view of (14) and (38), it suffices to show that

$$\lim_{\lambda\to\infty}\Big(q^\lambda\cdot\mathsf{E}\Big[\hat{G}\Big(IN^\lambda\Big)\Big]\Big)\Big/\Big(\gamma^\lambda(q^\lambda)\int_{r^\lambda}^{r^\lambda+q^\lambda}C\left(z^\lambda(x,q^\lambda)\right)\mathsf{d}x\Big)=1.$$

It follows from Lemma 3 that this is equivalent to show that

$$\lim_{\lambda\to\infty}\Big(q^\lambda\cdot\mathsf{E}\Big[\hat{G}\Big(IN^\lambda\Big)\Big]\Big)\Big/\Big(\gamma^\lambda(q^\lambda)\sum_{i=r^\lambda+1}^{r^\lambda+q^\lambda}C\left(z^\lambda(i,q^\lambda)\right)\Big)=1. \tag{A-11}$$

To prove (A-11), in view of (A-39), we only need to show that

$$\lim_{\lambda\to\infty}\Big(q^\lambda\sum_{i=r^\lambda+1}^{r^\lambda+q^\lambda}\mathsf{E}\left[I\Big\{J^\lambda=r^\lambda+q^\lambda-i\Big\}\times\hat{G}\left(z^\lambda(i,q^\lambda)-Y^\lambda(q^\lambda)\right)\right]\Big)\Big/\Big(\sum_{i=r^\lambda+1}^{r^\lambda+q^\lambda}C\left(z^\lambda(i,q^\lambda)\right)\Big)$$
$$=1. \tag{A-12}$$

To that end, we first consider each summand. Similar to (A-40), we have that

$$\begin{aligned}&\mathsf{E}\left[I\Big\{J^\lambda=r^\lambda+q^\lambda-i\Big\}\times\hat{G}\left(z^\lambda(i,q^\lambda)-Y^\lambda(q^\lambda)\right)\right]\\&\quad=\int_{-\infty}^{\infty}\hat{G}\left(z^\lambda(i,q^\lambda)-y\right)\mathsf{dPr}\Big(Y^\lambda(q^\lambda)\le y,J^\lambda=r^\lambda+q^\lambda-i\Big).\end{aligned} \tag{A-13}$$

By the first part of Theorem 1 and Theorem 1.A.3 on p.6, Shaked and Shanthikumar (2007), we know that

$$\int_{-\infty}^{\infty}\hat{G}\left(z^\lambda(i,q^\lambda)-y\right)\mathsf{dPr}\Big(Y^\lambda(q^\lambda)\le y,J^\lambda=r^\lambda+q^\lambda-i\Big) \tag{A-14}$$
$$\le\frac{1}{q^\lambda}\mathsf{E}\Big[h\cdot\Big(z^\lambda(i,q^\lambda)+\beta^\lambda(q^\lambda)-Y^\lambda(q^\lambda)\Big)^+ +p\cdot\Big(Y^\lambda(q^\lambda)+\beta^\lambda(q^\lambda)-z^\lambda(i,q^\lambda)\Big)^+\Big];$$

$$\int_{-\infty}^{\infty}\hat{G}\left(z^\lambda(i,q^\lambda)-y\right)\mathsf{dPr}\Big(Y^\lambda(q^\lambda)\le y,J^\lambda=r^\lambda+q^\lambda-i\Big) \tag{A-15}$$
$$\ge\frac{1}{q^\lambda}\mathsf{E}\Big[h\cdot\Big(z^\lambda(i,q^\lambda)-\beta^\lambda(q^\lambda)-Y^\lambda(q^\lambda)\Big)^+ +p\cdot\Big(Y^\lambda(q^\lambda)-\beta^\lambda(q^\lambda)-z^\lambda(i,q^\lambda)\Big)^+\Big].$$

Similar to the proof of (41) in Lemma 3, we can, by Conditions 1–2, show that

$$\begin{aligned}&\lim_{\lambda\to\infty}\mathsf{E}\Big[h\cdot\Big(z^\lambda(i,q^\lambda)+\beta^\lambda(q^\lambda)-Y^\lambda(q^\lambda)\Big)^+ +p\cdot\Big(Y^\lambda(q^\lambda)+\beta^\lambda(q^\lambda)-z^\lambda(i,q^\lambda)\Big)^+\Big]\\&\quad=\lim_{\lambda\to\infty}\mathsf{E}\Big[h\cdot\Big(z^\lambda(i,q^\lambda)-\beta^\lambda(q^\lambda)-Y^\lambda(q^\lambda)\Big)^+ +p\cdot\Big(Y^\lambda(q^\lambda)-\beta^\lambda(q^\lambda)-z^\lambda(i,q^\lambda)\Big)^+\Big]\\&\quad=\lim_{\lambda\to\infty}C\left(z^\lambda(i,q^\lambda)\right).\end{aligned} \tag{A-16}$$

Combining (A-13)-(A-16) yields that

$$\lim_{\lambda\to\infty}q^\lambda\cdot\mathsf{E}\left[I\Big\{J^\lambda=r^\lambda+q^\lambda-i\Big\}\times\hat{G}\left(z^\lambda(i,q^\lambda)-Y^\lambda(q^\lambda)\right)\right]=\lim_{\lambda\to\infty}C\left(z^\lambda(i,q^\lambda)\right),$$

which implies that (A-12) holds. Therefore, the lemma is proved. ■

Proof : **[of Lemma 6]** If $K>0$, from (58), we know that $\tau \neq 0$. Thus for positive τ and η, by the strict convexity of $C(\cdot)$, we know that there exists a unique $\alpha \in (0,1)$ (write as $g(\tau)$) such that

$$C\left(z_* - g(\tau)\tau\eta\right) = C\left(z_* + (1-g(\tau))\tau\eta\right).$$

Furthermore, this, by the strict convexity of $C(\cdot)$, implies that

$$\frac{\mathsf{d}\left(g(\tau)\tau\right)}{\mathsf{d}\tau} = \frac{C'(z_* + (1-g(\tau))\tau\eta)}{C'(z_* + (1-g(\tau))\tau\eta) - C'(z_* - g(\tau)\tau\eta)} \neq 0.$$

Plugging $g(\tau)$ into (58), we have

$$-\eta K + \tau \times C\left(z_* - g(\tau)\tau\eta\right) - \frac{1}{\eta}\int_{z_* - g(\tau)\tau\eta}^{z_* + (1-g(\tau))\tau\eta} C(y)\mathsf{d}y = 0.$$

Taking derivative on the left-hand side with respect to τ, we have

$$-\tau C'\left(z_* - g(\tau)\tau\eta\right) \cdot \frac{\mathsf{d}\left(g(\tau)\tau\right)}{\mathsf{d}\tau} \neq 0.$$

The existence of τ directly follows from the implicit function theorem (see Theorem 9.28 on p.224 of Rudin 1976).

Finally we show that $\tau \in (0,\infty)$. $\tau \neq 0$ directly follows from $K>0$, $\eta>0$, and (58). Suppose contrariwise that $\tau<0$. From (58), we have

$$\eta^2 K = \tau\eta \times C\left(z_* - \alpha\tau\eta\right) - \int_{z_* - \alpha\tau\eta}^{z_* + (1-\alpha)\tau\eta} C(y)\mathsf{d}y.$$

This is equivalent to

$$\eta^2 K = \int_{z_* - \alpha\tau\eta}^{z_* + (1-\alpha)\tau\eta} \Big(C\left(z_* - \alpha\tau\eta\right) - C(y)\Big)\mathsf{d}y. \tag{A-17}$$

By (59) and the convexity of $C(\cdot)$, and noticing that $z_* + (1-\alpha)\eta\tau < z_* - \alpha\eta\tau$ if $\tau<0$, we have that

$$C\left(z_* - \alpha\eta\tau\right) - C(y) \geq 0 \quad \text{for } y \in [z_* + (1-\alpha)\eta\tau, z_* - \alpha\eta\tau].$$

This implies that

$$\int_{z_* - \alpha\eta\tau}^{z_* + (1-\alpha)\eta\tau} \Big(C\left(z_* - \alpha\eta\tau\right) - C(y)\Big)\mathsf{d}y \leq 0.$$

Thus we get a contradiction from (A-17) as $\eta^2 K > 0$. Hence, $\tau \in (0,\infty)$. Thus, the proof of the lemma is completed. ■

Proof : **[of Lemma 7]** By the definitions of $\sigma^\lambda(q^\lambda)$ and $\beta^\lambda(q^\lambda)$ given by (17)-(18), in view of (46), it is sufficient to consider the differentiability of $\kappa(q^\lambda)$. By the strictly convexity of $C(\cdot)$ (see (36)), we know the continuity of $\kappa(\cdot)$. Using (45), for any $\delta > 0$,

$$\begin{aligned} &C\left(z_* - \kappa(q^\lambda+\delta)\right) - C\left(z_* - \kappa(q^\lambda)\right) \\ &\quad = C\left(z_* + \beta^\lambda(q^\lambda+\delta) - \kappa(q^\lambda+\delta)\right) - C\left(z_* + \beta^\lambda(q^\lambda) - \kappa(q^\lambda)\right). \end{aligned}$$

This implies that

$$\begin{aligned} &\frac{\kappa(q^\lambda+\delta)-\kappa(q^\lambda)}{\delta} \times \Bigg[\frac{C\left(z_*+\beta^\lambda(q^\lambda+\delta)-\kappa(q^\lambda+\delta)\right)-C\left(z_*+\beta^\lambda(q^\lambda)-\kappa(q^\lambda)\right)}{\left(\beta^\lambda(q^\lambda+\delta)-\kappa(q^\lambda+\delta)\right)-\left(\beta^\lambda(q^\lambda)-\kappa(q^\lambda)\right)} \\ &\qquad -\frac{C\left(z_*-\kappa(q^\lambda+\delta)\right)-C\left(z_*-\kappa(q^\lambda)\right)}{\kappa(q^\lambda)-\kappa(q^\lambda+\delta)}\Bigg] \\ &= \frac{C\left(z_*+\beta^\lambda(q^\lambda+\delta)-\kappa(q^\lambda+\delta)\right)-C\left(z_*+\beta^\lambda(q^\lambda)-\kappa(q^\lambda)\right)}{\left(\beta^\lambda(q^\lambda+\delta)-\kappa(q^\lambda+\delta)\right)-\left(\beta^\lambda(q^\lambda)-\kappa(q^\lambda)\right)} \times \frac{\beta^\lambda(q^\lambda+\delta)-\beta^\lambda(q^\lambda)}{\delta}. \end{aligned} \tag{A-18}$$

Letting δ go to zero, by the continuity of $\beta^\lambda(\cdot)$ and $\kappa(\cdot)$, we know the right-hand side of (A-18) does converge to

$$C'\left(z_*+\beta^\lambda(q^\lambda)-\kappa(q^\lambda)\right) \times \frac{\mathsf{d}\beta^\lambda(q^\lambda)}{\mathsf{d}q^\lambda}.$$

Similarly, the second factor of the left-hand side of (A-18) does converge to

$$C'\left(z_*+\beta^\lambda(q^\lambda)-\kappa(q^\lambda)\right) - C'\left(z_*-\kappa(q^\lambda)\right).$$

By (46), the strictly convexity of $C(\cdot)$ and the definition of z_* given by (36), we have that for $q^\lambda > 0$,

$$C'\left(z_*+\beta^\lambda(q^\lambda)-\kappa(q^\lambda)\right) > 0 \text{ and } C'\left(z_*-\kappa(q^\lambda)\right) < 0.$$

Thus we know that the limit of the second factor of the left-hand side of (A-18) is positive. Hence we know the limit of the first factor of the left-hand side of (A-18) does exist, which gives the differentiability of $\kappa(\cdot)$. ■

Proof : **[of Lemma 8]** According to the definitions of $\kappa(\tilde{q}_*^\lambda)$ and $\overline{\kappa}(\tilde{q}_*^\lambda)$ given by (46), it suffices to show the first part of the lemma, namely,

$$\lim_{\lambda\to\infty} \tilde{q}_*^\lambda = \infty \quad \text{and} \quad \lim_{\lambda\to\infty} \frac{\tilde{q}_*^\lambda}{\lambda} = 0. \tag{A-19}$$

If the first equation does not hold, then, by (46), the right-hand side of (65) will go to zero while the left-hand side is fixed at $K > 0$. And if the second equation does not hold, the right-hand side of (65) will go to infinite while the left-hand side is fixed at K. And thus $\tilde{q}_*^\lambda$ cannot be a solution of (65) when (A-19) does not hold. Thus we have (A-19). This in turn implies the lemma. ■

Proof : **[of Proposition 1]** First, assume $K > 0$. By Lemma 8 and the Taylor expansion, and recalling (36)-(37), we have

$$\begin{aligned}
&\int_{z_*-\kappa(\tilde{q}_*^\lambda)}^{z_*+\overline{\kappa}(\tilde{q}_*^\lambda)} C(y)\mathsf{d}y \\
&= \int_{z_*-\kappa(\tilde{q}_*^\lambda)}^{0} C(y)\mathsf{d}y + \int_0^{z_*+\overline{\kappa}(\tilde{q}_*^\lambda)} C(y)\mathsf{d}y \\
&= \int_{z_*}^{0} C(y)\mathsf{d}y + C_*\kappa(\tilde{q}_*^\lambda) - \frac{1}{2!}C'(z_*)\Big(\kappa(\tilde{q}_*^\lambda)\Big)^2 + \frac{1}{3!}C''(z_*)\left(\kappa(\tilde{q}_*^\lambda)\right)^3 + O\Big(\Big(\beta^\lambda(\tilde{q}_*^\lambda)\Big)^4\Big) \\
&\quad + \int_0^{z_*} C(y)\mathsf{d}y + C_*\overline{\kappa}(\tilde{q}_*^\lambda) + \frac{1}{2!}C'(z_*)\left(\overline{\kappa}(\tilde{q}_*^\lambda)\right)^2 + \frac{1}{3!}C''(z_*)\left(\overline{\kappa}(\tilde{q}_*^\lambda)\right)^3 + O\Big(\Big(\beta^\lambda(\tilde{q}_*^\lambda)\Big)^4\Big) \\
&= C_*\beta^\lambda(\tilde{q}_*^\lambda) + \frac{C_*}{3!}\left(\beta^\lambda(\tilde{q}_*^\lambda)\right)^3\left(1 - 3\alpha(\tilde{q}_*^\lambda) + 3\alpha^2(\tilde{q}_*^\lambda)\right) + O\Big(\Big(\beta^\lambda(\tilde{q}_*^\lambda)\Big)^4\Big). \qquad \text{(A-20)}
\end{aligned}$$

By again the Taylor expansion,

$$\begin{aligned}
C(z_* - \kappa(\tilde{q}_*^\lambda)) &= C_* - C'(z_*)\kappa(\tilde{q}_*^\lambda) + \frac{1}{2!}C''(z_*)\left(\kappa(\tilde{q}_*^\lambda)\right)^2 + O\Big(\Big(\beta^\lambda(\tilde{q}_*^\lambda)\Big)^3\Big) \\
&= C_* + \frac{1}{2!}C_*\cdot\left(\kappa(\tilde{q}_*^\lambda)\right)^2 + O\Big(\Big(\beta^\lambda(\tilde{q}_*^\lambda)\Big)^3\Big). \qquad \text{(A-21)}
\end{aligned}$$

Note that

$$\frac{\nu q + 2\theta^2\mathsf{E}[L_{(2)}]}{\sqrt{\nu q + \theta^2\mathsf{E}[L_{(2)}]}} = \sqrt{\nu q + \theta^2\mathsf{E}[L_{(2)}]} + \frac{\theta^2\mathsf{E}[L_{(2)}]}{\sqrt{\nu q + \theta^2\mathsf{E}[L_{(2)}]}}. \qquad \text{(A-22)}$$

It follows from Lemma 8 and (65)-(A-22) that

$$\tilde{q}_*^\lambda = \left(\frac{2K}{C_*\sqrt{\nu}}\right)^{2/3}\cdot\lambda^{1/3} + o\left(\lambda^{1/3}\right), \qquad \text{(A-23)}$$

which is (A.i) for $K > 0$.

Now we examine $\tilde{r}_*^\lambda$. By the Taylor expansion of both sides of (45) (expanding to the second moment), we have

$$\begin{aligned}
&C_* - C'(z_*)\kappa(\tilde{q}_*^\lambda) + \frac{1}{2}C''(z_*)\left(\kappa(\tilde{q}_*^\lambda)\right)^2 \\
&\quad = C_* + C'(z_*)\overline{\kappa}(\tilde{q}_*^\lambda) + \frac{1}{2}C''(z_*)\left(\overline{\kappa}(\tilde{q}_*^\lambda)\right)^2 + O\left(\left(\beta^\lambda(\tilde{q}_*^\lambda)\right)^3\right).
\end{aligned}$$

Applying (36)-(37) yields $(1/2)(1-2\alpha(\tilde{q}_*^\lambda))\cdot C_* = O(\beta^\lambda(\tilde{q}_*^\lambda))$. Because C_* is a positive constant, we know that

$$\alpha(\tilde{q}_*^\lambda) = \frac{1}{2} + O\left(\beta^\lambda(\tilde{q}_*^\lambda)\right). \tag{A-24}$$

Thus, by (19), (44) and (46),

$$\tilde{r}_*^\lambda = \rho^\lambda + z_* \cdot \left(\frac{2K\nu}{C_*}\right)^{1/3} \cdot \lambda^{2/3} + o\left(\lambda^{2/3}\right). \tag{A-25}$$

This is (A.ii) for $K > 0$.

For the optimal cost of System-$\tilde{\mathcal{S}}^\lambda$, following (47), (A-20), and (A-23)-(A-25),

$$\begin{aligned}
\widetilde{\mathcal{A}C}(\tilde{r}_*^\lambda, \tilde{q}_*^\lambda) &= \frac{\lambda K}{\tilde{q}_*^\lambda} + \left(\sigma^\lambda(\tilde{q}_*^\lambda)\right)^2 \lambda \int_{z_*-\kappa(\tilde{q}_*^\lambda)}^{z_*+\overline{\kappa}(\tilde{q}_*^\lambda)} C(y)\mathsf{d}y \\
&= \lambda K \Big/ \left[\left(\frac{2K}{C_*\sqrt{\nu}}\right)^{2/3} \cdot \lambda^{1/3} + o\left(\lambda^{1/3}\right)\right] \\
&\quad + \left(\sigma^\lambda(\tilde{q}_*^\lambda)\right)^2 \lambda \left[C_*\beta^\lambda(\tilde{q}_*^\lambda) + \frac{C_*}{4!}\left(\beta^\lambda(\tilde{q}_*^\lambda)\right)^3 + O\left(\left(\beta^\lambda(\tilde{q}_*^\lambda)\right)^4\right)\right] \\
&= 3\left(\frac{K\nu C_*^2}{4}\right)^{1/3} \lambda^{2/3} + o\left(\lambda^{2/3}\right).
\end{aligned} \tag{A-26}$$

This is (A.iii) for $K > 0$.

When $K = 0$, $\tilde{q}_*^\lambda = 1$ is our assumption; (A.ii) and (A.iii) are given by (38). ■

Proof : **[of Proposition 2]** First by (47),

$$\min_{r^\lambda} \widetilde{\mathcal{A}C}(r^\lambda, \tilde{q}^\lambda) = \frac{\lambda K}{\tilde{q}^\lambda} + \left(\sigma^\lambda(\tilde{q}^\lambda)\right)^2 \lambda \int_{z_*-\kappa(\tilde{q}^\lambda)}^{z_*+\overline{\kappa}(\tilde{q}^\lambda)} C(x)\mathsf{d}x. \tag{A-27}$$

If $\overline{\lim}_{\lambda\to\infty} \tilde{q}^\lambda/\lambda > 0$, then, by (46), we have

$$\overline{\lim_{\lambda\to\infty}} \int_{z_*-\kappa(\tilde{q}^\lambda)}^{z_*+\overline{\kappa}(\tilde{q}^\lambda)} C(x)\mathsf{d}x > 0.$$

This together with (A-27) gives that

$$\overline{\lim_{\lambda\to\infty}} \frac{1}{\lambda}\left\{\min_{r^\lambda} \widetilde{\mathcal{A}C}(r^\lambda, \tilde{q}^\lambda)\right\} > 0.$$

This, by Proposition 1, implies that

$$\overline{\lim_{\lambda\to\infty}} \frac{\min_{r^\lambda} \widetilde{\mathcal{A}C}(r^\lambda, \tilde{q}^\lambda)}{\widetilde{\mathcal{A}C}(\tilde{r}_*^\lambda, \tilde{q}_*^\lambda)} = \infty.$$

Hence, to prove the proposition, it suffices to consider $\overline{\lim}_{\lambda\to\infty}\tilde{q}^\lambda/\lambda=0$. Under this condition, by the Taylor expansion given by (A-20),

$$\int_{z_*-\kappa(\tilde{q}^\lambda)}^{z_*+\overline{\kappa}(\tilde{q}^\lambda)} C(x)\mathsf{d}x = C_*\beta^\lambda(\tilde{q}^\lambda)+\frac{C_*}{3!}\left(\beta^\lambda(\tilde{q}^\lambda)\right)^3\left(1-3\alpha(\tilde{q}^\lambda)+3\alpha^2(\tilde{q}^\lambda)\right)+O\left(\left(\beta^\lambda(\tilde{q}^\lambda)\right)^4\right).$$

Hence,

$$\begin{aligned}
&\frac{\lambda K}{\tilde{q}^\lambda}+\left(\sigma^\lambda(\tilde{q}^\lambda)\right)^2\lambda\int_{z_*-\kappa(\tilde{q}^\lambda)}^{z_*+\overline{\kappa}(\tilde{q}^\lambda)} C(x)\mathsf{d}x \\
&=\frac{\lambda K}{\Delta(\lambda)\tilde{q}^\lambda_*}+\left(\sigma^\lambda(\tilde{q}^\lambda)\right)^2\lambda\left[C_*\beta^\lambda(\tilde{q}^\lambda)+\frac{C_*}{3!}\left(\beta^\lambda(\tilde{q}^\lambda)\right)^3\left(1-3\alpha(\tilde{q}^\lambda)+3\alpha^2(\tilde{q}^\lambda)\right)\right. \\
&\qquad\qquad\left.+O\left(\left(\beta^\lambda(\tilde{q}^\lambda)\right)^4\right)\right] \\
&=\left[\left(\frac{K\nu C_*^2}{4}\right)^{1/3}\cdot\frac{1}{\Delta(\lambda)}+\left(2K\nu C_*^2\right)^{1/3}\sqrt{\Delta(\lambda)}\right]\cdot\lambda^{2/3}+o\left(\lambda^{2/3}\right). \qquad \text{(A-28)}
\end{aligned}$$

Let

$$U(x)=\left[\left(\frac{K\nu C_*^2}{4}\right)^{1/3}\cdot\frac{1}{x}+\left(2K\nu C_*^2\right)^{1/3}\sqrt{x}\right]$$

It is direct to verify that $-U(x)$ is unimodal, and $\arg\min_x U(x)=1$. Therefore, (A-28) implies part (i) of the proposition.

Next, consider part (ii). Note, by (38), that

$$\widetilde{\mathcal{A}C}(\tilde{r}^\lambda,\tilde{q}^\lambda)=\frac{\lambda K}{\tilde{q}^\lambda}+\left(\sigma^\lambda(\tilde{q}^\lambda)\right)^2\lambda\int_{z^\lambda(\tilde{r}^\lambda,\tilde{q}^\lambda)}^{z^\lambda(\tilde{r}^\lambda,\tilde{q}^\lambda)+\beta^\lambda(\tilde{q}^\lambda)} C(y)\mathsf{d}y. \qquad \text{(A-29)}$$

If $\overline{\lim}_{\lambda\to\infty} z^\lambda(\tilde{r}^\lambda,\tilde{q}^\lambda)\neq z_*$ or $\underline{\lim}_{\lambda\to\infty} z^\lambda(\tilde{r}^\lambda,\tilde{q}^\lambda)\neq z_*$, then there exists a subsequence, again writing as λ, such that

$$\lim_{\lambda\to\infty} z^\lambda(\tilde{r}^\lambda,\tilde{q}^\lambda)=b\neq z_*. \qquad \text{(A-30)}$$

Now making the Taylor expansion (expanding to the second moment) for the last term in (A-29), we obtain

$$\left(\sigma^\lambda(\tilde{q}^\lambda)\right)^2\lambda\int_{z^\lambda(\tilde{r}^\lambda,\tilde{q}^\lambda)}^{z^\lambda(\tilde{r}^\lambda,\tilde{q}^\lambda)+\beta^\lambda(\tilde{q}^\lambda)} C(y)\mathsf{d}y=\sigma^\lambda(\tilde{q}^\lambda)\cdot C(z^\lambda(\tilde{r}^\lambda,\tilde{q}^\lambda))\sqrt{\lambda\tilde{q}^\lambda}+o\left(\lambda^{2/3}\right). \qquad \text{(A-31)}$$

By the definition of z_* and (A-30), we know that $\lim_{\lambda\to\infty}C(z^\lambda(\tilde{r}^\lambda,\tilde{q}^\lambda))=C(b)>C(z_*)$. This together with (A-29) and (A-31) yields part (ii). ■

Proof : [**of Proposition 3**] Suppose contrariwise that the proposition is not true. Then there exists a subsequence $\{\lambda_k : k \geq 1\}$ such that

$$\lim_{k\to\infty} q_*^{\lambda_k} < \infty \quad \text{or} \quad \lim_{k\to\infty} \frac{q_*^{\lambda_k}}{\lambda_k} > 0. \tag{A-32}$$

To simplify notation, we write the sequence as λ (In the remainder of the paper, for the same reason, the subsequences will be always written as λ). By Lemma 1, we know that given q^λ, $q^\lambda \| (J^\lambda + q^\lambda \cdot N^\lambda)$ is uniformly distributed on $\mathcal{Q}$. Here, again, "$\|$" is the modulo operator. Now let

$$\Delta_1^\lambda = \Big\{0, \cdots, \lfloor \frac{q_*^\lambda}{4} \rfloor - 1\Big\}, \quad \Delta_2^\lambda = \Big\{\lfloor \frac{q_*^\lambda}{4} \rfloor, \cdots, 2\lfloor \frac{q_*^\lambda}{4} \rfloor - 1\Big\},$$
$$\Delta_3^\lambda = \Big\{2\lfloor \frac{q_*^\lambda}{4} \rfloor, \cdots, 3\lfloor \frac{q_*^\lambda}{4} \rfloor - 1\Big\}, \quad \Delta_4^\lambda = \Big\{3\lfloor \frac{q_*^\lambda}{4} \rfloor, \cdots, q_*^\lambda - 1\Big\}.$$

When $(q_*^\lambda \| r_*^\lambda) \in \Delta_1^\lambda$, we have

$$\Big| r_*^\lambda + q_*^\lambda - J^\lambda - q_*^\lambda \cdot N^\lambda \Big| \times I\{J^\lambda \in \Delta_3^\lambda\} \geq \lfloor \frac{q_*^\lambda}{4} \rfloor - 1.$$

Hence, if $(q_*^\lambda \| r_*^\lambda) \in \Delta_1^\lambda$ and the second inequality in (A-32) holds, then

$$\begin{aligned}
\overline{\lim_{\lambda\to\infty}} \frac{\mathcal{A}C(r_*^\lambda, q_*^\lambda)}{\lambda} &\geq \overline{\lim_{\lambda\to\infty}} \frac{1}{\lambda} \mathsf{E}\left[\hat{G}\left(r_*^\lambda + q_*^\lambda - J^\lambda - q_*^\lambda \cdot N^\lambda\right)\right] \quad \text{(by (14))} \\
&\geq \overline{\lim_{\lambda\to\infty}} \frac{1}{\lambda} \mathsf{E}\left[\hat{G}\left(r_*^\lambda + q_*^\lambda - J^\lambda - q_*^\lambda \cdot N^\lambda\right) \times I\{J^\lambda \in \Delta_3^\lambda\}\right] \\
&\geq \overline{\lim_{\lambda\to\infty}} \frac{1}{\lambda} \mathsf{E}\Big[\min\{p, h\} \times \Big(\lfloor \frac{q_*^\lambda}{4} \rfloor - 1\Big) \times I\{J^\lambda \in \Delta_3^\lambda\}\Big] \\
&\geq \lim_{\lambda\to\infty} \frac{\min\{p, h\}}{\lambda} \times \frac{1}{4} \times \Big(\lfloor \frac{q_*^\lambda}{4} \rfloor - 1\Big) \\
&> 0.
\end{aligned} \tag{A-33}$$

Similarly, we can show that for $(q_*^\lambda \| r_*^\lambda) \in \Delta_i^\lambda$ $(i = 2, 3, 4)$, (A-33) still holds if the second inequality in (A-32) holds.

If the first inequality in (A-32) holds, then by $\mathsf{E}[\hat{G}(IN)] \geq 0$ and (14),

$$\overline{\lim_{\lambda\to\infty}} \frac{\mathcal{A}C(r_*^\lambda, q_*^\lambda)}{\lambda} \geq \overline{\lim_{\lambda\to\infty}} \frac{K}{q_*^\lambda} > 0. \tag{A-34}$$

By the definition of $z^\lambda(i, q^\lambda)$ (see (19)), for $i = \lfloor \rho^\lambda \rfloor + 1, \cdots, \lfloor \rho^\lambda \rfloor + \lfloor \sqrt{\lambda} \rfloor$, $\lim_{\lambda\to\infty} z^\lambda(i, \lfloor \sqrt{\lambda} \rfloor) = 0$. So when policy $(r^\lambda, q^\lambda) = (\lfloor \rho^\lambda \rfloor, \lfloor \sqrt{\lambda} \rfloor)$ is implemented, by (14) and (39)-(41), we have

$$\begin{aligned}
\lim_{\lambda\to\infty} \frac{\mathcal{A}C(\lfloor \rho^\lambda \rfloor, \lfloor \sqrt{\lambda} \rfloor)}{\lambda} &= \lim_{\lambda\to\infty} \frac{1}{\lambda} \left\{ \frac{\lambda K}{\lfloor \sqrt{\lambda} \rfloor} + \frac{\sigma^\lambda(\lfloor \sqrt{\lambda} \rfloor)\sqrt{\lambda \lfloor \sqrt{\lambda} \rfloor}}{\lfloor \sqrt{\lambda} \rfloor} \sum_{i=\lfloor \rho^\lambda \rfloor + 1}^{\lfloor \rho^\lambda \rfloor + \lfloor \sqrt{\lambda} \rfloor} C\left(z^\lambda(i, \lfloor \sqrt{\lambda} \rfloor)\right) \right\} \\
&\leq \lim_{\lambda\to\infty} \frac{\sigma^\lambda(\lfloor \sqrt{\lambda} \rfloor)\lfloor \sqrt{\lambda} \rfloor}{\sqrt{\lambda \lfloor \sqrt{\lambda} \rfloor}} \cdot \max_{\lfloor \rho^\lambda \rfloor + 1 \leq i \leq \lfloor \rho^\lambda \rfloor + \lfloor \sqrt{\lambda} \rfloor} C\left(z^\lambda(i, \lfloor \sqrt{\lambda} \rfloor)\right) \\
&= 0.
\end{aligned}$$

So in view of (A-33)-(A-34), when (A-32) holds,

$$\lim_{\lambda\to\infty} \frac{\mathcal{A}C(r_*^\lambda, q_*^\lambda)}{\mathcal{A}C(\lfloor \rho^\lambda \rfloor, \lfloor \sqrt{\lambda} \rfloor)} = \infty,$$

which implies that $(r_*^\lambda, q_*^\lambda)$ cannot be optimal, a contradiction. Thus, the proposition holds. ■

Proof : **[of Proposition 4]** According to the definition of Condition 2, it is sufficient to show

$$\left| \lim_{\lambda\to\infty} z^\lambda(r_*^\lambda, q_*^\lambda) \right| < \infty. \tag{A-35}$$

To that end, we first show that

$$\left| \overline{\lim_{\lambda\to\infty}} z^\lambda(r_*^\lambda, q_*^\lambda) \right| < \infty. \tag{A-36}$$

Suppose contrariwise that this does not hold. Then we have two possible cases:

$$\text{Case A:} \quad \underline{\lim}_{\lambda\to\infty} z^\lambda(r_*^\lambda, q_*^\lambda) = -\infty; \qquad \text{Case B:} \quad \overline{\lim}_{\lambda\to\infty} z^\lambda(r_*^\lambda, q_*^\lambda) = +\infty. \tag{A-37}$$

First, consider Case A. In view of Proposition 3, we have that if $K > 0$,

$$\underline{\lim}_{\lambda\to\infty} z^\lambda(i, q_*^\lambda) = -\infty \text{ for } i = r_*^\lambda + 1, \cdots, r_*^\lambda + q_*^\lambda. \tag{A-38}$$

If $K = 0$, by $q_*^\lambda = 1$, (A-38) also holds under Case A. Then there exists a subsequence $\{\lambda_k : k \geq 1\}$ such that $\lim_{k\to\infty} z^{\lambda_k}(i, q_*^\lambda) = -\infty$. We still write this subsequence as λ. By (39)-(40), for any policy (r^λ, q^λ),

$$\begin{aligned}
\mathsf{E}\left[\hat{G}\left(IN^\lambda\right)\right] &= \sum_{i=r^\lambda+1}^{r^\lambda+q^\lambda} \mathsf{E}\left[I\left\{J^\lambda = r^\lambda + q^\lambda - i\right\} \times \gamma^\lambda(q^\lambda) \times \hat{G}\left(z^\lambda(i, q^\lambda) - Y^\lambda(q^\lambda)\right)\right] \\
&= \gamma^\lambda(q^\lambda) \sum_{i=r^\lambda+1}^{r^\lambda+q^\lambda} \mathsf{E}\left[I\left\{J^\lambda = r^\lambda + q^\lambda - i\right\} \hat{G}\left(z^\lambda(i, q^\lambda) - Y^\lambda(q^\lambda)\right)\right].
\end{aligned} \tag{A-39}$$

We first consider each summand. Note that

$$\begin{aligned}
&\mathsf{E}\left[I\left\{J^\lambda = r^\lambda + q^\lambda - i\right\} \times \hat{G}\left(z^\lambda(i, q^\lambda) - Y^\lambda(q^\lambda)\right)\right] \\
&= \int_{-\infty}^{\infty} \hat{G}\left(z^\lambda(i, q^\lambda) - y\right) \mathsf{d}\mathsf{Pr}\Big(Y^\lambda(q^\lambda) \leq y, J^\lambda = r^\lambda + q^\lambda - i\Big).
\end{aligned} \tag{A-40}$$

By the first part of Theorem 1 and Theorem 1.A.3 (a) in Shaked and Shanthikumar (2007), we know that

$$\begin{aligned}&\int_{-\infty}^{\infty} p\cdot\left(z^\lambda(i,q^\lambda)-y\right)^- \mathsf{dPr}\Big(Y^\lambda(q^\lambda)\le y, J^\lambda=r^\lambda+q^\lambda-i\Big)\\ &\quad\ge \frac{1}{q^\lambda}\int_{-\infty}^{\infty} p\cdot\left(z^\lambda(i,q^\lambda)-y\right)^- \mathsf{dPr}\Big(Y^\lambda(q^\lambda)-\beta^\lambda(q^\lambda)\le y\Big), \qquad \text{(A-41)}\\ &\int_{-\infty}^{\infty} h\cdot\left(z^\lambda(i,q^\lambda)-y\right)^+ \mathsf{dPr}\Big(Y^\lambda(q^\lambda)\le y, J^\lambda=r^\lambda+q^\lambda-i\Big)\\ &\quad\ge \frac{1}{q^\lambda}\int_{-\infty}^{\infty} h\cdot\left(z^\lambda(i,q^\lambda)-y\right)^+ \mathsf{dPr}\Big(Y^\lambda(q^\lambda)+\beta^\lambda(q^\lambda)\le y\Big). \qquad \text{(A-42)}\end{aligned}$$

Combining (A-41)-(A-42) yields

$$\begin{aligned}&\int_{-\infty}^{\infty} \hat{G}\left(z^\lambda(i,q^\lambda)-y\right) \mathsf{dPr}\Big(Y^\lambda(q^\lambda)\le y, J^\lambda=r^\lambda+q^\lambda-i\Big) \qquad \text{(A-43)}\\ &\ge \frac{1}{q^\lambda}\mathsf{E}\Big[h\cdot\Big(z^\lambda(i,q^\lambda)-\beta^\lambda(q^\lambda)-Y^\lambda(q^\lambda)\Big)^+ + p\cdot\Big(Y^\lambda(q^\lambda)-\beta^\lambda(q^\lambda)-z^\lambda(i,q^\lambda)\Big)^+\Big].\end{aligned}$$

Considering policy $(r_*^\lambda,q_*^\lambda)$, we have, by Proposition 3, that for $i=r_*^\lambda+1,\cdots,r_*^\lambda+q_*^\lambda$,

$$\begin{aligned}&\lim_{\lambda\to\infty}\mathsf{E}\Big[h\cdot\Big(z^\lambda(i,q_*^\lambda)-\beta^\lambda(q_*^\lambda)-Y^\lambda(q_*^\lambda)\Big)^+ + p\cdot\Big(Y^\lambda(q_*^\lambda)-\beta^\lambda(q_*^\lambda)-z^\lambda(i,q_*^\lambda)\Big)^+\Big]\\ &\ge \lim_{\lambda\to\infty}\mathsf{E}\Big[p\cdot\Big(Y^\lambda(q_*^\lambda)-\beta^\lambda(q_*^\lambda)-z^\lambda(i,q_*^\lambda)\Big)^+\Big]\\ &\ge \lim_{\lambda\to\infty}\mathsf{E}\Big[p\cdot\Big(-\beta^\lambda(q_*^\lambda)-z^\lambda(i,q_*^\lambda)\Big)^+\times I\Big\{Y^\lambda(q_*^\lambda)\le 0\Big\}\Big]\\ &=\frac{p}{2}\times\lim_{\lambda\to\infty}\Big(-\beta^\lambda(q_*^\lambda)-z^\lambda(i,q_*^\lambda)\Big)^+ \quad \text{(by Lemma 2)}\\ &=\infty. \quad \text{(by (A-38))}\end{aligned}$$

It follows from (A-43) that for policy $(r_*^\lambda,q_*^\lambda)$,

$$\sum_{i=r_*^\lambda+1}^{r_*^\lambda+q_*^\lambda}\mathsf{E}\left[I\Big\{J^\lambda=r_+^\lambda+q_*^\lambda-i\Big\}\hat{G}\left(z^\lambda(i,q_*^\lambda)-Y^\lambda(q_*^\lambda)\right)\right]\to\infty \ \text{ as } \lambda\to\infty. \qquad \text{(A-44)}$$

On the other hand, consider another policy $(r_0^\lambda,q_*^\lambda)$ with $r_0^\lambda=\Big\lfloor \rho^\lambda+\gamma^\lambda(q_*^\lambda)\Big\rfloor$. It is direct to verify that the sequence of $(r_0^\lambda,q_*^\lambda)$-policies satisfies Condition 2. Furthermore, by Proposition 3, the sequence of ordering quantities $\{q_*^\lambda\}$ satisfies Condition 1. Similar to the proof of (41) in Lemma 3, we can, by Conditions 1–2, show that for $i=r_0^\lambda+1,\cdots,r_0^\lambda+q_*^\lambda$,

$$\begin{aligned}&\lim_{\lambda\to\infty}\mathsf{E}\Big[h\cdot\Big(z^\lambda(i,q_*^\lambda)+\beta^\lambda(q_*^\lambda)-Y^\lambda(q_*^\lambda)\Big)^+ + p\cdot\Big(Y^\lambda(q_*^\lambda)+\beta^\lambda(q_*^\lambda)-z^\lambda(i,q_*^\lambda)\Big)^+\Big]\\ &=\lim_{\lambda\to\infty}\mathsf{E}\Big[h\cdot\Big(z^\lambda(i,q_*^\lambda)-\beta^\lambda(q_*^\lambda)-Y^\lambda(q_*^\lambda)\Big)^+ + p\cdot\Big(Y^\lambda(q_*^\lambda)-\beta^\lambda(q_*^\lambda)-z^\lambda(i,q_*^\lambda)\Big)^+\Big]\\ &=\lim_{\lambda\to\infty}C\left(z^\lambda(i,q_*^\lambda)\right). \qquad \text{(A-45)}\end{aligned}$$

Combining (A-40)-(A-45) yields that for $i = r_0^\lambda + 1, \cdots, r_0^\lambda + q_*^\lambda$,

$$\lim_{\lambda\to\infty} q_*^\lambda \cdot \mathsf{E}\left[I\left\{ J^\lambda = r^\lambda + q_*^\lambda - i \right\} \times \hat{G}\left(z^\lambda(i, q_*^\lambda) - Y^\lambda(q_*^\lambda) \right) \right] = \lim_{\lambda\to\infty} C\left(z^\lambda(i, q_*^\lambda) \right) < \infty. \quad \text{(A-46)}$$

Thus, from (A-44) and (A-46),

$$\frac{\sum_{i=r_*^\lambda+1}^{r_*^\lambda+q_*^\lambda} \mathsf{E}\left[I\left\{ J^\lambda = r_*^\lambda + q_*^\lambda - i \right\} \hat{G}\left(z^\lambda(i, q_*^\lambda) - Y^\lambda(q_*^\lambda) \right) \right]}{\sum_{i=r_0^\lambda+1}^{r_0^\lambda+q_*^\lambda} \mathsf{E}\left[I\left\{ J^\lambda = r_0^\lambda + q_*^\lambda - i \right\} \hat{G}\left(z^\lambda(i, q_*^\lambda) - Y^\lambda(q_*^\lambda) \right) \right]} \to \infty \quad \text{as } \lambda \to \infty.$$

which, by (14) and (39), contradicts the optimality of $(r_*^\lambda, q_*^\lambda)$. Therefore, Case A does not hold. Similarly, we can show Case B does not hold also. Hence (A-36) is proved.

To prove (A-35), with the help of (A-36), it is sufficient to show that for any convergent subsequence of $z^\lambda(r_*^\lambda, q_*^\lambda)$ (for the sake of notation simplicity, we still write it as $z^\lambda(r_*^\lambda, q_*^\lambda)$), its limit is always z_*. That is, we only need to prove

$$\lim_{\lambda\to\infty} z^\lambda(r_*^\lambda, q_*^\lambda) = z_*. \quad \text{(A-47)}$$

The convergence of the subsequence of $z^\lambda(r_*^\lambda, q_*^\lambda)$ implies that its corresponding subsequence of $(r_*^\lambda, q_*^\lambda)$ satisfies Condition 2. In view of Proposition 3, we know that $(r_*^\lambda, q_*^\lambda)$ satisfies Conditions 1–2 in Lemma 4. Thus, by Lemma 4,

$$\lim_{\lambda\to\infty} \frac{\mathcal{AC}(r_*^\lambda, q_*^\lambda)}{\widetilde{\mathcal{AC}}(r_*^\lambda, q_*^\lambda)} = 1. \quad \text{(A-48)}$$

Using Proposition 1, we know that the sequence of $(\tilde{r}_*^\lambda, \tilde{q}_*^\lambda)$-policies satisfies Conditions 1–2 with $\lim_{\lambda\to\infty} \tilde{q}_*^\lambda = \infty$. It follows from Lemma 4 that

$$\lim_{\lambda\to\infty} \frac{\mathcal{AC}(\tilde{r}_*^\lambda, \tilde{q}_*^\lambda)}{\widetilde{\mathcal{AC}}(\tilde{r}_*^\lambda, \tilde{q}_*^\lambda)} = 1. \quad \text{(A-49)}$$

On the other hand, by the optimality of $(r_*^\lambda, q_*^\lambda)$ for System-$\mathcal{S}^\lambda$ and the optimality $(\tilde{r}_*^\lambda, \tilde{q}_*^\lambda)$ for System-$\tilde{\mathcal{S}}^\lambda$, we have $\mathcal{AC}(r_*^\lambda, q_*^\lambda) \le \mathcal{AC}(\tilde{r}_*^\lambda, \tilde{q}_*^\lambda)$ and $\widetilde{\mathcal{AC}}(r_*^\lambda, q_*^\lambda) \ge \widetilde{\mathcal{AC}}(\tilde{r}_*^\lambda, \tilde{q}_*^\lambda)$. Hence from (A-48) and (A-49),

$$\lim_{\lambda\to\infty} \frac{\widetilde{\mathcal{AC}}(r_*^\lambda, q_*^\lambda)}{\widetilde{\mathcal{AC}}(\tilde{r}_*^\lambda, \tilde{q}_*^\lambda)} = 1. \quad \text{(A-50)}$$

With the help of Proposition 2, we, by (A-50), know that $\lim_{\lambda\to\infty} z^\lambda(r_*^\lambda, q_*^\lambda) = z_*$, which proves (A-47). This implies (A-35). The second part of the proposition ((62) and (63)) directly follows from (A-48) and (A-50). ■

Proof : [**of Proposition 5**] First we consider $K>0$ case. Similar to the proof of Proposition 1, we need to establish the result similar to Lemma 8. Namely,

$$\lim_{\lambda\to\infty}\tilde{q}^{\lambda}_{*c}=\infty \quad\text{and}\quad \frac{\tilde{q}^{\lambda}_{*c}}{\sqrt{\lambda}} \text{ is bounded.} \tag{A-51}$$

If the first equation is not true, then there exists a subsequence $\{\lambda_k : k\geq 1\}$ such that

$$\lim_{k\to\infty}\lambda_k=\infty \quad\text{and}\quad \lim_{k\to\infty}\tilde{q}^{\lambda_k}_{*c}=a<\infty. \tag{A-52}$$

We still label the subsequence of (A-52) by λ. Under (A-52), by the definition of γ_c^{λ} given by (48), we have

$$\kappa_c(\tilde{q}^{\lambda}_{*c})\to 0 \quad\text{and}\quad \overline{\kappa}_c(\tilde{q}^{\lambda}_{*c})\to 0, \tag{A-53}$$

which implies

$$\lim_{\lambda\to\infty}C(z_*-\kappa_c(\tilde{q}^{\lambda}_{*c}))=C_*. \tag{A-54}$$

This plus the mean-value theorem for integration yields

$$\frac{\gamma_c^{\lambda}}{\tilde{q}^{\lambda}_{*c}}\int_{z_*-\kappa_c(\tilde{q}^{\lambda}_{*c})}^{z_*+\overline{\kappa}_c(\tilde{q}^{\lambda}_{*c})}C(x)\mathsf{d}x\to C_*. \tag{A-55}$$

Combining (A-54)–(A-55) yields that the right-hand-side of (57) converges to zero. However, (A-52) implies that $\frac{\lambda K}{\gamma_c^{\lambda}\tilde{q}^{\lambda}_{*c}}\to\infty$. This produces a contradiction to (57). Therefore, we have the first equation of (A-51).

Next we show the second equation of (A-51). Suppose contrariwise that there exists a sequence of $\{\lambda_k, k\geq 1\}$ such that

$$\lim_{k\to\infty}\frac{\tilde{q}^{\lambda_k}_{*c}}{\sqrt{\lambda_k}}=\infty. \tag{A-56}$$

Again, for simpler notation, we label the sequence by λ. From (55) and the strict convexity of $C(\cdot)$, in view of (A-56), we know that

$$\lim_{\lambda\to\infty}\frac{\alpha_c(\tilde{q}^{\lambda}_{*c})\cdot\tilde{q}^{\lambda}_{*c}}{\sqrt{\lambda}}=\infty,\quad \lim_{\lambda\to\infty}\frac{[1-\alpha_c(\tilde{q}^{\lambda}_{*c})]\cdot\tilde{q}^{\lambda}_{*c}}{\sqrt{\lambda}}=\infty. \tag{A-57}$$

It follows from (A-57) and the strict convexity of $C(\cdot)$ that

$$\begin{aligned}
&C(z_*-\kappa_c(\tilde{q}^{\lambda}_{*c}))-\frac{\gamma_c^{\lambda}}{\tilde{q}^{\lambda}_{*c}}\int_{z_*-\kappa_c(\tilde{q}^{\lambda}_{*c})}^{z_*+\overline{\kappa}_c(\tilde{q}^{\lambda}_{*c})}C(x)\mathsf{d}x\\
&=\lim_{\lambda\to\infty}\frac{\gamma_c^{\lambda}}{\tilde{q}^{\lambda}_{*c}}\int_{z_*-\kappa_c(\tilde{q}^{\lambda}_{*c})}^{z_*+\overline{\kappa}_c(\tilde{q}^{\lambda}_{*c})}\Big(C(z_*-\kappa_c(\tilde{q}^{\lambda}_{*c}))-C(x)\Big)\mathsf{d}x\\
&>0.
\end{aligned}$$

But, from (A-56), $\lim_{\lambda\to\infty}\lambda K/(\gamma_c^\lambda \tilde{q}_{*c}^\lambda)=0$. Thus, we reach a contradiction to (57). In other words, (A-56) cannot hold, and we must have the second equation of (A-51).

As $\{\alpha_c(\tilde{q}_{*c}^\lambda)\ :\ \lambda>0\}$ is also bounded, in view of (A-51), we pick up any two convergence sequences, say $\{\frac{\tilde{q}_{*c}^{\lambda_k}}{\sqrt{\lambda_k}}\ :\ k\geq 1\}$ and $\{\alpha_c(\tilde{q}_{*c}^{\lambda_k})\ :\ k\geq 1\}$, from $\{\frac{\tilde{q}_{*c}^\lambda}{\sqrt{\lambda}}\ :\ \lambda>0\}$ and $\{\alpha_c(\tilde{q}_{*c}^\lambda)\ :\ \lambda>0\}$ (again, write them as λ sequences). Let

$$\lim_{k\to\infty}\frac{\tilde{q}_{*c}^\lambda}{\sqrt{\lambda}}=\overline{\tau}\quad\text{and}\quad\lim_{k\to\infty}\alpha_c(\tilde{q}_{*c}^\lambda)=\overline{\alpha}. \tag{A-58}$$

These imply

$$\lim_{\lambda\to\infty}\big(\kappa_c(\tilde{q}_{*c}^\lambda)-\alpha\tau\eta\big)=\lim_{\lambda\to\infty}\big(\overline{\kappa}_c(\tilde{q}_{*c}^\lambda)-(1-\alpha)\tau\eta\big)=0.$$

We have, by (55), that

$$\lim_{\lambda\to\infty}\frac{\gamma_c^\lambda}{\tilde{q}_{*c}^\lambda}\int_{z_*-\kappa_c(\tilde{q}_{*c}^\lambda)}^{z_*+\overline{\kappa}_c(\tilde{q}_{*c}^\lambda)}C(x)\mathsf{d}x=\frac{1}{\overline{\tau}\eta}\int_{z_*-\overline{\alpha}\overline{\tau}\eta}^{z_*+(1-\overline{\alpha})\overline{\tau}\eta}C(y)\mathsf{d}y, \tag{A-59}$$

$$\lim_{\lambda\to\infty}\frac{\lambda K}{\gamma_c^\lambda\tilde{q}_{*c}^\lambda}=\frac{K\eta}{\overline{\tau}}, \tag{A-60}$$

$$C\left(z_*-\overline{\alpha}\overline{\tau}\eta\right)=C\left(z_*+(1-\overline{\alpha})\overline{\tau}\eta\right). \tag{A-61}$$

It follows from (57), (A-59)-(A-60) that

$$\frac{K\eta}{\overline{\tau}}=C\left(z_*-\overline{\alpha}\overline{\tau}\eta\right)-\frac{1}{\overline{\tau}\eta}\int_{z_*-\overline{\alpha}\overline{\tau}\eta}^{z_*+(1-\overline{\alpha})\overline{\tau}\eta}C(y)\mathsf{d}y. \tag{A-62}$$

Thus the limits of any convergence sequences of $\{\frac{\tilde{q}_{*c}^\lambda}{\sqrt{\lambda}}\}$ and $\{\alpha(\tilde{q}_{*c}^\lambda)\}$ satisfy (A-61)-(A-62). By Lemma 6, we proved (B.i) and (B.ii) for $(\tilde{r}_{*c}^\lambda,\tilde{q}_{*c}^\lambda)$ of system-$\tilde{\mathcal{S}}_c^\lambda$.

Now consider (B.iii) for $\widetilde{\mathcal{A}C}_c(\tilde{r}_{*c}^\lambda,\tilde{q}_{*c}^\lambda)$. Similar to (A-20), using the Taylor expansion, we get

$$\begin{aligned}\int_{z_*-\kappa_c(\tilde{q}_{*c}^\lambda)}^{z_*+\overline{\kappa}_c(\tilde{q}_{*c}^\lambda)}C(y)\mathsf{d}y&=\int_{z_*-\kappa_c(\tilde{q}_{*c}^\lambda)}^{0}C(y)\mathsf{d}y+\int_0^{z_*+\overline{\kappa}_c(\tilde{q}_{*c}^\lambda)}C(y)\mathsf{d}y\\&=\int_{z_*-\alpha\tau\eta}^{0}C(y)\mathsf{d}y+C\left(z_*-\alpha\tau\eta\right)\times\big(\kappa_c(\tilde{q}_{*c}^\lambda)-\alpha\tau\eta\big)+\int_0^{z_*+(1-\alpha)\tau\eta}C(y)\mathsf{d}y\\&\quad+C\left(z_*+(1-\alpha)\tau\eta\right)\times\big(\overline{\kappa}_c(\tilde{q}_{*c}^\lambda)-(1-\alpha)\tau\eta\big)+o(1).\end{aligned} \tag{A-63}$$

This, by the first part of the proposition and (A-63), implies that

$$\begin{aligned}\widetilde{\mathcal{A}C}_c(\tilde{r}_{*c}^\lambda,\tilde{q}_{*c}^\lambda)&=\frac{\lambda K}{\tilde{q}_{*c}^\lambda}+\gamma_c^\lambda\cdot\frac{\gamma_c^\lambda}{\tilde{q}_{*c}^\lambda}\int_{z_*-\kappa_c(\tilde{q}_{*c}^\lambda)}^{z_*+\overline{\kappa}_c(\tilde{q}_{*c}^\lambda)}C(y)\mathsf{d}y\\&=\left(\frac{K}{\tau}+\frac{1}{\tau\eta^2}\int_{z_*-\alpha\tau\eta}^{z_*+(1-\alpha)\tau\eta}C(y)\mathsf{d}y\right)\sqrt{\lambda}+o(\sqrt{\lambda}).\end{aligned} \tag{A-64}$$

Therefore, (B.iii) for $\widetilde{\mathcal{A}C}_c(\tilde r^\lambda_{*c}, \tilde q^\lambda_{*c})$ is proved.

Now consider $K=0$. $\tilde q^\lambda_{*c}=1$ directly follows from (51) and convexity of $C(\cdot)$. (B.ii) and (B.iii) are given by (48) and (51). ■

Proof : **[of Proposition 6]** We first prove (i). Suppose that

$$\overline{\lim_{\lambda\to\infty}}\, \Delta_c(\lambda)=\infty. \tag{A-65}$$

Then, by (51),

$$\begin{aligned}\lim_{\lambda\to\infty}\frac{\min_{r^\lambda}\widetilde{\mathcal{A}C}_c(r^\lambda,\tilde q^\lambda)}{\sqrt{\lambda}} &= \lim_{\lambda\to\infty}\frac{\gamma_c^\lambda}{\tilde q^\lambda\sqrt{\lambda}}\int_{\tilde r^\lambda_{*c}(\tilde q^\lambda)}^{\tilde r^\lambda_{*c}(\tilde q^\lambda)+\tilde q^\lambda} C\left(z_c^\lambda(y)\right)\mathsf{d}y\\ &= \frac{(\gamma_c^\lambda)^2}{\sqrt{\lambda}\tilde q^\lambda}\int_{z_c^\lambda(\tilde r^\lambda_{*c}(\tilde q^\lambda))}^{z_c^\lambda(\tilde r^\lambda_{*c}(\tilde q^\lambda))+\beta_c^\lambda(\tilde q^\lambda)} C(y)\ \mathsf{d}y.\end{aligned} \tag{A-66}$$

Applying L'Hôpital's rule, we have

$$\lim_{\lambda\to\infty}\int_{z_c^\lambda(\tilde r^\lambda_{*c}(\tilde q^\lambda))}^{z_c^\lambda(\tilde r^\lambda_{*c}(\tilde q^\lambda))+\beta_c^\lambda(\tilde q^\lambda)} C(y)\ \mathsf{d}y\Big/\frac{\tilde q^\lambda}{\sqrt{\lambda}}=\infty. \tag{A-67}$$

Hence, by (A-66)-(A-67) and Proposition 5, we have that

$$\overline{\lim_{\lambda\to\infty}}\,\frac{\min_{r^\lambda}\widetilde{\mathcal{A}C}_c(r^\lambda,\tilde q^\lambda)}{\widetilde{\mathcal{A}C}_c(\tilde r^\lambda_{*c},\tilde q^\lambda_{*c})}=\infty,$$

which implies (i). Now suppose that $\overline{\lim}_{\lambda\to\infty}\Delta_c(\lambda)<\infty$ but $\overline{\lim}_{\lambda\to\infty}\Delta_c(\lambda)\neq\underline{\lim}_{\lambda\to\infty}\Delta_c(\lambda)$. Then there exist two convergence sequences, say $\{\frac{\tilde q^{\lambda_k}}{\sqrt{\lambda_k}}\ :\ k\geq 1\}$ and $\{\alpha_c(\tilde q^{\lambda_k})\ :\ k\geq 1\}$, from $\{\frac{\tilde q^\lambda}{\sqrt{\lambda}}\ :\ \lambda>0\}$ and $\{\alpha_c(\tilde q^\lambda)\ :\ \lambda>0\}$ (again, write them as λ sequences) such that

$$\lim_{k\to\infty}\frac{\tilde q^\lambda}{\sqrt{\lambda}}=\tilde\tau\neq\tau \quad\text{and}\quad \lim_{k\to\infty}\alpha_c(\tilde q^\lambda)=\tilde\alpha\neq\alpha. \tag{A-68}$$

Exactly going along the line (A-63)-(A-64), we have

$$\widetilde{\mathcal{A}C}_c(\tilde r^\lambda_{*c}(\tilde q^\lambda),\tilde q^\lambda)=\left(\frac{K}{\tilde\tau}+\frac{1}{\tilde\tau\eta^2}\int_{z_*-\tilde\alpha\tilde\tau\eta}^{z_*+(1-\tilde\alpha)\tilde\tau\eta}C(y)\mathsf{d}y\right)\sqrt{\lambda}+o(\sqrt{\lambda}), \tag{A-69}$$

where $\tilde\alpha$, using (55), satisfies

$$C\left(z_*-\tilde\alpha\tilde\tau\eta\right)=C\left(z_*+(1-\tilde\alpha)\tilde\tau\eta\right). \tag{A-70}$$

Consider function

$$g_c(\tilde\tau)=\frac{K}{\tilde\tau}+\frac{1}{\tilde\tau\eta^2}\int_{z_*-\tilde\alpha\tilde\tau\eta}^{z_*+(1-\tilde\alpha)\tilde\tau\eta}C(y)\mathsf{d}y.$$

If $-g_c(\tilde{\tau})$ is strictly unimodal and its maximizer is given by τ, then we have (i). Thus to complete the proof of (i), it is sufficient to show that the strict unimodality of $-g_c(\tilde{\tau})$ and its maximizer is τ. Note, by (A-70), that

$$\frac{\mathsf{d}g_c(\tilde{\tau})}{\mathsf{d}\tilde{\tau}} = -\frac{K}{\tilde{\tau}^2} - \frac{1}{\tilde{\tau}^2\eta^2}\int_{z_*-\tilde{\alpha}\tilde{\tau}\eta}^{z_*+(1-\tilde{\alpha})\tilde{\tau}\eta} C(y)\mathsf{d}y + \frac{1}{\tilde{\tau}\eta}C(z_* - \tilde{\alpha}\tilde{\tau}\eta).$$

Letting $\mathsf{d}g_c(\tilde{\tau})/\mathsf{d}\tilde{\tau} = 0$, we have

$$\eta^2 K = \tilde{\tau}\eta \times C\left(z_* - \tilde{\alpha}\tilde{\tau}\eta\right) - \int_{z_*-\tilde{\alpha}\tilde{\tau}\eta}^{z_*+(1-\tilde{\alpha})\tilde{\tau}\eta} C(y)\mathsf{d}y. \tag{A-71}$$

Making a comparison with (58), we know τ is minimizer of $g(\tilde{\tau})$. Considering

$$\tilde{\tau}\eta \times C\left(z_* - \tilde{\alpha}\tilde{\tau}\eta\right) - \int_{z_*-\tilde{\alpha}\tilde{\tau}\eta}^{z_*+(1-\tilde{\alpha})\tilde{\tau}\eta} C(y)\mathsf{d}y$$

as a function of $\tilde{\tau}\eta$, by (A-70) and Lemma 6 in Zheng (1992), it is strict increasing. Hence, we know that

$$-\frac{K}{\tilde{\tau}^2} - \frac{1}{\tilde{\tau}^2\eta^2}\int_{z_*-\tilde{\alpha}\tilde{\tau}\eta}^{z_*+(1-\tilde{\alpha})\tilde{\tau}\eta} C(y)\mathsf{d}y + \frac{1}{\tilde{\tau}\eta}C(z_* - \tilde{\alpha}\tilde{\tau}\eta) < 0 \quad \text{for } \tilde{\tau} < \tau;$$
$$-\frac{K}{\tilde{\tau}^2} - \frac{1}{\tilde{\tau}^2\eta^2}\int_{z_*-\tilde{\alpha}\tilde{\tau}\eta}^{z_*+(1-\tilde{\alpha})\tilde{\tau}\eta} C(y)\mathsf{d}y + \frac{1}{\tilde{\tau}\eta}C(z_* - \tilde{\alpha}\tilde{\tau}\eta) > 0 \quad \text{for } \tilde{\tau} > \tau.$$

Thus the unimodality of $-g_c(\tilde{\tau})$ is proven.

Finally we prove (ii). Suppose that $\overline{\lim}_{\lambda\to\infty}|\varpi^\lambda| < \infty$ and one of $\overline{\lim}_{\lambda\to\infty}\varpi^\lambda \neq 1$ and $\underline{\lim}_{\lambda\to\infty}\varpi^\lambda \neq 1$ holds. Then there exists a convergence sequences, say $\{\varpi^{\lambda_k} : k \geq 1\}$ from $\{\varpi^\lambda : \lambda > 0\}$ (again, write them as λ sequences) such that

$$\lim_{\lambda\to\infty}\varpi^\lambda = b \neq 1. \tag{A-72}$$

Similar to (A-64),

$$\lim_{\lambda\to\infty}\frac{\widetilde{\mathcal{A}C}_c(\tilde{r}^\lambda, \tilde{q}^\lambda_{*c})}{\sqrt{\lambda}} = \frac{K}{\tau} + \frac{1}{\tau\eta^2}\int_{b(z_*-\alpha\tau\eta)}^{b(z_*-\alpha\tau\eta)+\tau\eta} C(y)\mathsf{d}y.$$

It is direct to verify the above function has a unique minimizer at $b = 1$. Hence we have (ii). ■

Proof : **[of Proposition 7]** By (10)-(11) and (49), going along the line of the proof of Propositions 3 and 4, we can show the proposition holds. Here the details are omitted. ■

Proof : **[of Equation (79)]** Suppose contrariwise that

$$\overline{\lim_{\lambda\to\infty}}\sqrt{\lambda}\hat{T}_*^\lambda=\infty.$$

Then there exists a subsequence $\{\lambda_k : k\geq 1\}$ (label it as λ sequence) such that

$$\lim_{k\to\infty}\sqrt{\lambda}\hat{T}_*^\lambda=\infty. \tag{A-73}$$

By (75) and (78),

$$\lim_{\lambda\to\infty}\Bigg[C\left(\frac{z_*\theta\sqrt{\rho^\lambda}+M^\lambda(\hat{T}_*^\lambda)-\lambda\hat{T}_*^\lambda}{\theta\sqrt{\lambda\hat{T}_*^\lambda+\rho^\lambda}}\right)\sqrt{\hat{T}_*^\lambda+\frac{1}{\mu}} -\frac{1}{\hat{T}_*^\lambda}\int_0^{\hat{T}_*^\lambda}C\left(\frac{z_*\theta\sqrt{\rho^\lambda}+M^\lambda(\hat{T}_*^\lambda)-\lambda t}{\theta\sqrt{\lambda t+\rho^\lambda}}\right)\sqrt{t+\frac{1}{\mu}}\mathsf{d}t\Bigg]=0. \tag{A-74}$$

The remainder of the proof is divided into three cases.

$$\textbf{Case A}\quad \lim_{\lambda\to\infty}\hat{T}_*^\lambda=0;\ \textbf{Case B}\quad \lim_{\lambda\to\infty}\hat{T}_*^\lambda=a\in(0,\infty);\ \textbf{Case C}\quad \lim_{\lambda\to\infty}\hat{T}_*^\lambda=\infty.$$

For each case, we will get a contradiction with (A-74) if (A-73) holds. First we look at Case A. This case will be further divided into subcases by (75) and (A-73):

Subcase A.1 $\lim_{\lambda\to\infty}\hat{T}_*^\lambda=0$ and $\lim_{\lambda\to\infty}\frac{M^\lambda(\hat{T}_*^\lambda)-\lambda\hat{T}_*^\lambda}{\theta\sqrt{\lambda\hat{T}_*^\lambda+\rho^\lambda}}=-\infty$.

Under Subcase A.1, we have

$$\lim_{\lambda\to\infty}\frac{z_*\theta\sqrt{\rho^\lambda}+M^\lambda(\hat{T}_*^\lambda)-\lambda\hat{T}_*^\lambda}{\theta\sqrt{\lambda\hat{T}_*^\lambda+\rho^\lambda}}=-\infty \quad\text{and}\quad \lim_{\lambda\to\infty}\frac{z_*\theta\sqrt{\rho^\lambda}+M^\lambda(\hat{T}_*^\lambda)}{\theta\sqrt{\rho^\lambda}}\geq z_*.$$

Then it follows from the strict convexity of $C(\cdot)$ that

$$\lim_{\lambda\to\infty}\left|C\left(\frac{z_*\theta\sqrt{\rho^\lambda}+M^\lambda(\hat{T}_*^\lambda)-\lambda\hat{T}_*^\lambda}{\theta\sqrt{\lambda\hat{T}_*^\lambda+\rho^\lambda}}\right)-\frac{1}{\hat{T}_*^\lambda}\int_0^{\hat{T}_*^\lambda}C\left(\frac{z_*\theta\sqrt{\rho^\lambda}+M^\lambda(\hat{T}_*^\lambda)-\lambda t}{\theta\sqrt{\lambda t+\rho^\lambda}}\right)\mathsf{d}t\right|=\infty,$$

this, in view of $\lim_{\lambda\to\infty}\hat{T}_*^\lambda=0$, contradicts with (A-74).

Subcase A.2 $\lim_{\lambda\to\infty}\hat{T}_*^\lambda=0$ and $\lim_{\lambda\to\infty}\frac{M^\lambda(\hat{T}_*^\lambda)-\lambda\hat{T}_*^\lambda}{\theta\sqrt{\lambda\hat{T}_*^\lambda+\rho^\lambda}}=b$ with $|b|<\infty$.

For this subcase, by (A-73), we have

$$\lim_{\lambda\to\infty}\frac{z_*\theta\sqrt{\rho^\lambda}+M^\lambda(\hat{T}_*^\lambda)-\lambda\hat{T}_*^\lambda}{\theta\sqrt{\lambda\hat{T}_*^\lambda+\rho^\lambda}}=z_*+b \quad\text{and}\quad \lim_{\lambda\to\infty}\frac{z_*\theta\sqrt{\rho^\lambda}+M^\lambda(\hat{T}_*^\lambda)}{\theta\sqrt{\rho^\lambda}}=\infty.$$

Similar to Subcase A.1, by the strict convexity of $C(\cdot)$, we get a contradiction with (A-74). Cases B and C can be analyzed along the same line. ■

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