Electronic Companion to "A Non-cooperative Approach to Cost Allocation in Joint Replenishment"

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A. Additional Proof

Proof of Proposition 1. For any given replenishment policy $T \in \Gamma$, let π be a permutation of the indices $\{1,...,n\}$ such that $T_{\pi_1} \le T_{\pi_2} \le \cdots \le T_{\pi_n}$, and we use $\pi^{-1}(k) = r$ to denote that retailer k is the r^{th} smallest in the above ranking list (w.l.o.g., ties are broken such that the retailer with a smaller index is ranked first), i.e., $\pi_r = k$. Consider any arbitrary retailer $i \in N$, and let $Q = \{j \in N : T_j > T_i\}$ denote the subset of retailers whose replenishment interval is strictly larger than that of retailer i. Suppose $|Q| = m$, or equivalently, $\pi_{n-m} = i$. Then by (2), the share of the average major setup cost paid by retailer i is given by

$$\begin{aligned} x_i(T_i;T_{-i}) &= \sum_{k=n-m}^{n} \left(\frac{1}{T_{\pi^{-1}(k)}} - \frac{1}{T_{\pi^{-1}(k+1)}} \right) \frac{K_0}{k} \\ &= \frac{1}{T_{\pi^{-1}(n-m)}} \frac{K_0}{n-m} + \sum_{k=n-m}^{n-1} \frac{1}{T_{\pi^{-1}(k+1)}} \left(\frac{K_0}{k+1} - \frac{K_0}{k} \right). \end{aligned} \tag{EC.1}$$

Consider the coefficient of $\frac{1}{T_{\pi^{-1}(n-m)}}$, or equivalently, $\frac{1}{T_i}$. In what follows, we compare the coefficient of $\frac{1}{T_i}$ in (4) with the coefficient of $\frac{1}{T_{\pi^{-1}(n-m)}}$ in (EC.1). We will show that the former is equal to the latter, which is $\frac{K_0}{n-m}$. The comparison between the coefficients of the remaining terms, i.e., $\frac{1}{T_{\pi^{-1}(k+1)}}$ for all $n-m \le k \le n-1$ can be done in a similar fashion and hence the details are omitted. It is also clear that the coefficient of $\frac{1}{T_r}$ on the right-hand side of (4) for all $r \in N \setminus Q$ is equal to zero. Therefore, the right-hand side of (4) is identical to that of (EC.1), which would complete the proof of Proposition 1.

Now we will show the coefficient of $\frac{K_0}{T_i}$ in (4) is equal to $\frac{1}{n-m}$. Notice that for each $S \subseteq N \setminus \{i\}$, the corresponding term in (4) is equal to zero if there exists some $j \in S$ with $T_j \le T_i$. Therefore, it suffices to only consider $S \subseteq Q \cup \emptyset$. Consider an arbitrary $S \subseteq Q \cup \emptyset$, and let $|S| = q$ where $0 \le q \le m$. The coefficient of $\frac{K_0}{T_i}$ that corresponds to this particular S is equal to $\frac{q!(n-q-1)!}{n!}$. Since the number of subsets $S \subseteq Q \cup \emptyset$ with $|S| = q$ is $\binom{m}{q} = \frac{m!}{q!(m-q)!}$, it follows that the coefficient of $\frac{K_0}{T_i}$ in (4) is given by

$$\sum_{q=0}^{m} \frac{q!(n-q-1)!}{n!} \frac{m!}{q!(m-q)!} = \frac{m!}{n!} \sum_{q=0}^{m} \frac{(n-q-1)!}{(m-q)!} = \frac{m!}{n!} \cdot \frac{A}{m!} = \frac{A}{n!}, \tag{EC.2}$$

where

$$\begin{aligned} A &= (n-1)! + (n-2)! \cdot m + (n-3)! \cdot m(m-1) + \cdots + (n-m-1)! \cdot m! \\ &= \sum_{t=1}^{m+1} (n-t)! \frac{m!}{(m-t+1)!} = (n-m-1)! \sum_{t=1}^{m+1} \frac{(n-t)!}{(n-m-1)!} \frac{m!}{(m-t+1)!}. \end{aligned} \tag{EC.3}$$

By simple algebraic manipulation, it is straightforward to verify that for each $2 \le p \le m$,

$$\sum_{t=m+2-p}^{m+1} \frac{(n-t)!}{(n-m-1)!} \frac{m!}{(m-t+1)!} = \left(\prod_{x=p}^{m} x \right) \left(\prod_{y=1}^{p-1} (n-m+y) \right).$$

Therefore, we have

$$\begin{aligned} \sum_{t=1}^{m+1} \frac{(n-t)!}{(n-m-1)!} \frac{m!}{(m-t+1)!} &= \frac{(n-1)!}{(n-m-1)!} + \sum_{t=2}^{m+1} \frac{(n-t)!}{(n-m-1)!} \frac{m!}{(m-t+1)!} \\ &= \frac{(n-1)!}{(n-m-1)!} + m \cdot \prod_{y=1}^{m-1} (n-m+y) = \frac{n!}{(n-m)!}. \end{aligned} \tag{EC.4}$$

It then follows from (EC.2), (EC.3) and (EC.4) that the coefficient of $\frac{K_0}{T_i}$ in (4) is equal to

$$\frac{(n-m-1)!}{n!} \frac{n!}{(n-m)!} = \frac{1}{n-m},$$

which completes the proof. □

Proof of Lemma 1. (a) A sufficient condition for T^* being an N.E. is that for each $i \in N$, we have

$$f_i(T_i; T_{-i}^*) \ge f_i(T_i^*; T_{-i}^*), \quad \forall T_i \in \Gamma_i. \tag{EC.5}$$

Since all the retailers have the same replenishment interval under T^*, each of them shares $1/n$ of the major setup cost. Therefore, retailer i's cost under T^* is given by

$$f_i(T_i^*; T_{-i}^*) = HT_i^* + \frac{K}{T_i^*} + \frac{K_0}{nT_i^*}. \tag{EC.6}$$

Suppose retailer i reduces his replenishment interval to some $T_i \le \frac{1}{2}T_i^*$. In this case, retailer i has the smallest replenishment interval among all retailers, and by (2) we have

$$f_i(T_i;T_{-i}^*) = HT_i + \frac{K}{T_i} + x_i(T_i;T_{-i}^*) = HT_i + \frac{K}{T_i} + \left(\frac{K_0}{T_i} - \frac{K_0}{T_i^*}\right) + \frac{K_0}{nT_i^*}.$$

Then condition (EC.5) reduces to

$$HT_i + \frac{K+K_0}{T_i} \ge HT_i^* + \frac{K+K_0}{T_i^*}, \quad \forall T_i \in \Gamma_i \text{ such that } T_i \le \frac{1}{2}T_i^*. \tag{EC.7}$$

Consider the function $g(x) = Hx + \frac{K+K_0}{x}$ defined on $x \in \Gamma_i$. The optimal solution that minimizes $g(x)$ is give by $x^* \overset{POT}{=} \sqrt{\frac{K+K_0}{H}}$. Notice that $g(x)$ is nonincreasing on $(0,x^*] \cap \Gamma_i$. It then follows that $g(T_i) \ge g(T_i^*)$ for all $T_i \le \frac{1}{2}T_i^*$ with $T_i^c \le T_i^* \le \tilde{T}_i$. Therefore, (EC.7) holds.

If retailer i increases his policy to some $T_i \ge 2T_i^*$, it can be similarly shown that condition (EC.5) reduces to

$$HT_i + \frac{K_0+nK}{nT_i} \ge HT_i^* + \frac{K_0+nK}{nT_i^*}, \quad \forall T_i \in \Gamma_i \text{ such that } T_i \ge 2T_i^*. \tag{EC.8}$$

To show (EC.8) holds, consider the function $\phi(y) = Hy + \frac{K_0+nK}{ny}$ defined on Γ_i. The optimal solution that minimizes $\phi(y)$ is given by $y^* \overset{POT}{=} \sqrt{\frac{K_0+nK}{nH}}$, which is exactly T_i^c. It follows that $\phi(\cdot)$ is increasing on $[y^*,\infty) \cap \Gamma_i$, and therefore (EC.8) holds.

(b) We prove by contradiction. Let T^* be an N.E., and let $\pi = (\pi_1,...,\pi_n)$ be a permutation of the indices such that $T_{\pi_1}^* \le \cdots \le T_{\pi_n}^*$. Suppose that not all the retailers have the same replenishment interval, i.e., $T_i^* \ne T_j^*$ for some $i,j \in N$. Let $m = |\{i \in N : T_i^* = T_{\pi_1}^*\}|$. Then by assumption, we must have $m < n$ and $T_{\pi_{m+1}}^* > T_{\pi_1}^*$. Since T^* is an N.E., retailer π_1 cannot have a profitable unilateral deviation from $T_{\pi_1}^*$ to $T_{\pi_{m+1}}^*$, i.e.,

$$f_{\pi_1}(T_{\pi_1}^*;T_{-\pi_1}^*) \le f_{\pi_1}(T_{\pi_{m+1}}^*;T_{-\pi_1}^*),$$

which implies

$$HT_{\pi_1}^* + \left(\frac{1}{T_{\pi_1}^*} - \frac{1}{T_{\pi_{m+1}}^*}\right)\left(\frac{K_0}{m} + K\right) \le HT_{\pi_{m+1}}^*. \tag{EC.9}$$

Similarly, retailer π_{m+1} cannot benefit from unilaterally deviating from $T_{\pi_{m+1}}^*$ to $T_{\pi_1}^*$, i.e.,

$$f_{\pi_{m+1}}(T_{\pi_{m+1}}^*;T_{-\pi_{m+1}}^*) \le f_{\pi_{m+1}}(T_{\pi_1}^*;T_{-\pi_{m+1}}^*),$$

which results in

$$HT^*_{\pi_{m+1}} \le HT^*_{\pi_1} + \left(\frac{1}{T^*_{\pi_1}} - \frac{1}{T^*_{\pi_{m+1}}}\right)\left(\frac{K_0}{m+1} + K\right). \tag{EC.10}$$

Notice that (EC.9) and (EC.10) can simultaneously hold only when $T^*_{\pi_{m+1}} = T^*_{\pi_1}$, which contradicts the assumption that $T^*_{\pi_{m+1}} > T^*_{\pi_1}$. And this completes the proof. □

Proof of Lemma 2. In view of (5), it suffices to show

$$x_i(T_i^1; T_{-i}^1) \le x_i(T_i^2; T_{-i}^2), \quad \forall T^1, T^2 \in \Gamma \text{ such that } T^1 \le T^2 \text{ with } T_i^1 = T_i^2. \tag{EC.11}$$

To see (EC.11), recall that $x_i(T_i; T_{-i})$ is given by (4). For each $i \in N$ and $S \subseteq N \setminus \{i\}$, define

$$g_i^S(T_i; T_{-i}) = \frac{K_0}{\min_{j \in S \cup \{i\}} T_j} - \frac{K_0}{\min_{j \in S} T_j}.$$

Notice that $g_i^S(\cdot)$ is nonnegative by definition. If $\min_{j\in S} T_j^1 \le T_i^1$, then $g_i^S(T_i^1; T_{-i}^1) = 0 \le g_i^S(T_i^2; T_{-i}^2)$. If $\min_{j\in S} T_j^1 > T_i^1$, then we have

$$g_i^S(T_i^1; T_{-i}^1) = \frac{K_0}{T_i^1} - \frac{K_0}{\min_{j\in S} T_j^1} \le \frac{K_0}{T_i^2} - \frac{K_0}{\min_{j\in S} T_j^2} = g_i^S(T_i^2; T_{-i}^2),$$

where the inequality follows from $T^1 \le T^2$ and $T_i^1 = T_i^2$. Therefore, it holds that

$$g_i^S(T_i^1; T_{-i}^1) \le g_i^S(T_i^2; T_{-i}^2), \quad \forall T^1, T^2 \in \Gamma \text{ such that } T^1 \le T^2 \text{ with } T_i^1 = T_i^2$$

for each $i \in N$ and $S \subseteq N \setminus \{i\}$, which implies (EC.11) by (4). □

Proof of Lemma 3. Since T^{least} is the least element of the set $\{T \in \Gamma : R(T) \le T\}$, it remains to show that if (16) holds, then $R(\hat{T}) \le \hat{T}$. It is clear that retailer i's share of the average major setup cost is bounded from above by

$$x_i(\hat{T}_i; \hat{T}_{-i}) \le \frac{K_0}{m_i \hat{T}_i}. \tag{EC.12}$$

Notice that $R(\hat{T}) \le \hat{T}$ holds if retailer i is no better off when he increases his replenishment interval $\hat{T}_i$ to $2^z \hat{T}_i$ for some integer $z \ge 1$ while $\hat{T}_{-i}$ remains fixed. More precisely, a sufficient condition for $R(\hat{T}) \le \hat{T}$ is $f_i(2^z \hat{T}_i; \hat{T}_{-i}) \ge f_i(\hat{T}_i; \hat{T}_{-i})$ for all integers $z \ge 1$, i.e.,

$$2^z H_i \hat{T}_i + \frac{K_i}{2^z \hat{T}_i} + x_i(2^z \hat{T}_i; \hat{T}_{-i}) \ge H_i \hat{T}_i + \frac{K_i}{\hat{T}_i} + x_i(\hat{T}_i; \hat{T}_{-i}), \quad \forall z \in \mathbb{Z} \text{ s.t. } z \ge 1$$

or equivalently

$$H_i\hat{T}_i \geq \frac{K_i}{2^z\hat{T}_i} + \frac{x_i(\hat{T}_i;\hat{T}_{-i}) - x_i(2^z\hat{T}_i;\hat{T}_{-i})}{2^z-1}, \quad \forall z\in\mathbb{Z} \text{ s.t. } z\geq 1 \tag{EC.13}$$

To conclude the proof, we notice that by (EC.12), condition (16) implies

$$H_i\hat{T}_i \geq \frac{K_i}{2^z\hat{T}_i} + \frac{x_i(\hat{T}_i;\hat{T}_{-i})}{2^z-1}, \quad \forall z\in\mathbb{Z} \text{ s.t. } z\geq 1,$$

which in turn is a sufficient condition for (EC.13). □

Proof of Lemma 4. In light of Lemma 3, it suffices to show $\hat{T}$ satisfies the sufficient condition (16), or equivalently,

$$2H_i\hat{T}_i^2 \geq K_i + \frac{2K_0}{m_i}, \quad \forall i\in N.$$

We proceed in two steps. Recall that $m_i = |\{j\in N : \hat{T}_j \leq \hat{T}_i\}|$. We first show

$$2H_{u_i}(\hat{T}_{u_i})^2 \geq K_{u_i} + \frac{2K_0}{m_{u_i}}, \quad \forall u_i\in U. \tag{EC.14}$$

To see (EC.14), suppose Step 2 in Algorithm 1 Part I results in $\hat{T}_{u_i} = T_l^a$ for some $l\geq i$. Then by the definition of $\hat{T}_{u_i}$, we have $\hat{T}_{u_j} = T_l^a$ for all $i\leq j\leq l$ and $\hat{T}_{u_j}\leq T_l^a$ for all $1\leq j\leq i-1$. Therefore, we must have $m_{u_i}^a \geq l$, where $m_{u_i}^a$ is the number of retailers in U whose replenishment interval is at most $\hat{T}_{u_i}$, i.e., $m_{u_i}^a = |\{u_j\in U : \hat{T}_{u_j}\leq \hat{T}_{u_i}\}|$. This together with $H_{u_i}\geq H_{u_l}$ implies

$$2(\hat{T}_{u_i})^2 = 2(T_l^a)^2 \geq s + \frac{2K_0}{lH_{u_l}} \geq s + \frac{2K_0}{m_{u_i}^a H_{u_i}}. \tag{EC.15}$$

By the definition of set U, we have $sH_{u_i}\geq K_{u_i}$ for each $u_i\in U$. Moreover, it is clear that $m_{u_i}\geq m_{u_i}^a$. Therefore, (EC.15) implies (EC.14).

It remains to show

$$2H_{v_i}(\hat{T}_{v_i})^2 \geq K_{v_i} + \frac{2K_0}{m_{v_i}}, \quad \forall v_i\in V. \tag{EC.16}$$

But this is obviously true because from the definition of S_l in Algorithm 1, we have $2H_{v_i}(\hat{T}_{v_i})^2 \geq K_{v_i} + \frac{2K_0}{q_l}$ for each $v_i\in S_l$, which together with $m_{v_i}\geq q_l$ completes the proof. □

Proof of Theorem 5. Recall that by Corollary 1 and (15), it suffices to focus on bounding the system-wide holding cost under $\hat{T}$. We first consider retailers in U. By the definition of $\hat{T}_{u_i}$ and T_i^a, we have

$$(\hat{T}_{u_i})^2 \le (T_i^a)^2 \le 2(s + \frac{2K_0}{iH_{u_i}}), \quad \forall\, u_i \in U,$$

which implies

$$H_{u_i}\hat{T}_{u_i} \le H_{u_i}\sqrt{2s + \frac{4K_0}{iH_{u_i}}} \le \sqrt{2s}H_{u_i} + 2\sqrt{\frac{K_0 H_{u_i}}{i}}, \quad \forall\, u_i \in U. \tag{EC.17}$$

By the definition of s, we have

$$s \le \frac{K(U) + K_0}{H(U)}. \tag{EC.18}$$

Moreover, by Cauchy-Schwarz inequality, we have $\sum_{u_i \in U} \sqrt{H_{u_i}/i} \le \sqrt{\sum_{u_i \in U} H_{u_i}}\sqrt{\sum_{u_i \in U} 1/i}$. It then follows from (EC.17) and (EC.18) that

$$\begin{aligned}\sum_{u_i \in U} H_{u_i}\hat{T}_{u_i} &\le \sqrt{2s}H(U) + 2\sqrt{K_0}\sum_{u_i \in U}\sqrt{\frac{H_{u_i}}{i}} \\ &\le \sqrt{2H(U)(K(U)+K_0)} + 2\sqrt{H(U)K_0}\sqrt{\sum_{u_i \in U}\frac{1}{i}}. \end{aligned} \tag{EC.19}$$

By definition the system-wide cost under the optimal centralized solution is equal to

$$C(T^c) = H(U)T^c_{\min} + \frac{K(U)}{T^c_{\min}} + \sum_{i \in V}\left(H_i T_i^c + \frac{K_i}{T_i^c}\right) + \frac{K_0}{T^c_{\min}},$$

which implies

$$C(T^c) \ge 2\sqrt{H(U)(K(U)+K_0)}. \tag{EC.20}$$

It is known that

$$\sum_{i=1}^{n}\frac{1}{i} \le \ln n + 1. \tag{EC.21}$$

It then follows from (EC.19), (EC.20) and (EC.21) that

$$\sum_{u_i \in U} H_{u_i}\hat{T}_{u_i} \le (1 + \sqrt{\ln|U| + 1})\, C(T^c). \tag{EC.22}$$

Next we consider retailers in V. For each $v_i \in S_l$, define

$$a_{v_i} = \frac{4K_0}{(T^l)^2 H_{v_i} - 2K_{v_i}}. \tag{EC.23}$$

According to Step 2 of Algorithm 1 Part II, the number of retailers in S_l is equal to $q_l - q_{l-1}$. W.l.o.g, let $S_l = \{v_{q_{l-1}+1}, ..., v_{q_l}\}$ such that for all $i, j \in [q_{l-1}+1, q_l]$, we have $i < j$ if $0 < a_{v_i} \le a_{v_j}$ or $a_{v_i} \le a_{v_j} \le 0$, and $i < j$ whenever $a_{v_i} > 0$ and $a_{v_j} \le 0$. In words, S_l is ordered in such a way that retailers with positive a_{v_i} values are in front of those with nonpositive a_{v_i}, and within each subset of retailers whose a_{v_i} values have the same sign, we order them by nondecreasing a_{v_i}. To bound the holding cost incurred by retailers in V under policy $\hat{T}$, we first present an auxiliary result.

Lemma 5. *For each $v_i \in V$ with $a_{v_i} > 0$, we have $a_{v_i} \ge i$.*

Proof of Lemma 5. We prove by contradiction. Suppose there exists some $v_j \in S_l$ such that $0 < a_{v_j} < j$. Then by the ordering of S_l, we must have $0 < a_{v_i} < j$ for all $q_{l-1} + 1 \le i \le j$, or equivalently

$$\frac{2K_0}{j} + K_{v_i} < \frac{1}{2}(T^l)^2 H_{v_i} = 2(T^{l-1})^2 H_{v_i}, \quad \forall\, q_{l-1} + 1 \le i \le j.$$

The above inequality implies that $S'_{l-1} = S_{l-1} \cup \{v_{q_{l-1}+1}, ..., v_j\}$ is a feasible solution to the problem

$$\begin{aligned} &\max_{S \subseteq V^{l-2}} \ |S| \\ &\text{s.t.} \qquad 2(T^{l-1})^2 H_{v_i} \ge K_{v_i} + \frac{2K_0}{q_{l-2} + |S|}, \quad \forall\, v_i \in S. \end{aligned}$$

This contradicts the definition of S_{l-1}, which completes the proof. $\square$

Now we are ready to bound the holding cost incurred by retailers in V. By the definition of a_{v_i} in (EC.23), the following inequality holds for each $v_i \in S_l$ such that $a_{v_i} < 0$:

$$H_{v_i}(T^l)^2 < 2K_{v_i}.$$

For each $v_i \in S_l$ such that $a_{v_i} > 0$, it follows from (EC.23) and Lemma 5 that

$$H_{v_i}(T^l)^2 \le \frac{4K_0}{i} + 2K_{v_i}.$$

Combining the two cases results in

$$\frac{1}{2} H_{v_i}(T^l)^2 \le \frac{2K_0}{i} + K_{v_i}, \quad \forall\, v_i \in S_l. \qquad \text{(EC.24)}$$

It then follows from (EC.24) that the following inequality holds for all $v_i \in V$:

$$2H_{v_i}\hat{T}_{v_i} \leq \frac{H_{v_i}\hat{T}_{v_i}^2}{2T_{v_i}^c\sqrt{\ln|V|+1}} + 2T_{v_i}^c\sqrt{\ln|V|+1}\cdot H_{v_i}$$
$$\leq \frac{\frac{2K_0}{i}+K_{v_i}}{T_{v_i}^c\sqrt{\ln|V|+1}} + 2T_{v_i}^c\sqrt{\ln|V|+1}\cdot H_{v_i}.$$

Therefore, the holding cost of retailers in V under $\hat{T}$ is bounded from above by

$$\sum_{v_i\in V} H_{v_i}\hat{T}_{v_i} \leq \frac{1}{2\sqrt{\ln|V|+1}}\sum_{v_i\in V}\frac{\frac{2K_0}{i}+K_{v_i}}{T_{v_i}^c} + \sqrt{\ln|V|+1}\sum_{v_i\in V}H_{v_i}T_{v_i}^c$$
$$\leq \frac{1}{\sqrt{\ln|V|+1}}\frac{K_0}{T_{\min}^c}\sum_{v_i\in V}\frac{1}{i} + \sum_{v_i\in V}\left(\frac{1}{2\sqrt{\ln|V|+1}}\frac{K_{v_i}}{T_{v_i}^c} + \sqrt{\ln|V|+1}H_{v_i}T_{v_i}^c\right)$$
$$\leq \sqrt{\ln|V|+1}\frac{K_0}{T_{\min}^c} + \sqrt{\ln|V|+1}\sum_{v_i\in V}\left(\frac{K_{v_i}}{T_{v_i}^c} + H_{v_i}T_{v_i}^c\right) \tag{EC.25}$$
$$\leq \sqrt{\ln|V|+1}\,C(T^c), \tag{EC.26}$$

where (EC.25) holds because $\sum_{v_i\in V} 1/i \leq \ln|V|+1$ and $\frac{1}{2\sqrt{\ln|V|+1}} \leq \sqrt{\ln|V|+1}$ for all $|V|\geq 1$.

By Corollary 1, T^{least} has the smallest total cost among all Nash equilibria. Then by (15), (EC.22) and (EC.26), we have

$$PoS = \inf_{T\in E}\frac{C(T)}{C(T^c)} = \frac{C(T^{least})}{C(T^c)} \leq 2+2\sqrt{\ln n+1},$$

and this completes the proof of Theorem 5. □

Proof of Proposition 2. Notice that by (17), it follows that

$$\sqrt{2}\cdot 2^{z-1}B \leq \sqrt{s} \leq 2^z B, \quad \text{for some integer } z\in\mathbb{Z}.$$

Since the nearest POT point to any real value x belongs to $[x/\sqrt{2}, \sqrt{2}x]$, the above condition ensures that $\sqrt{s}$ will be rounded up and we have $T_{\min}^c = 2^z B \geq \sqrt{s}$. Moreover, it is clear that if the following condition holds:

$$\sqrt{\frac{K_i}{H_i}} \leq \sqrt{2s}, \quad \forall i\in V, \tag{EC.27}$$

then $T_i^c = T_{\min}^c$ for all $i\in V$, and in this case all the retailers will order together in the centralized solution.

To see (EC.27), notice that by the definition of α and β, we have $K_i \le \alpha K_{\min}$, $K_i \ge \frac{1}{\alpha} K_{\max}$ and $H_i \le \beta H_{\min}$, $H_i \ge \frac{1}{\beta} H_{\max}$ for all $i \in N$. It then follows that for any $i \in V$, we have

$$\frac{K_i}{H_i} \le \alpha\beta \cdot \frac{K_{\min}}{H_{\max}} \le 2 \cdot \frac{K_{\max}}{\alpha\beta H_{\min}} \le 2 \cdot \frac{\sum_{j=1}^{i^*} K_j}{\sum_{j=1}^{i^*} H_j} \le 2 \cdot \frac{K_0 + \sum_{j=1}^{i^*} K_j}{\sum_{j=1}^{i^*} H_j},$$

where the second inequality follows from the assumption that $\alpha\beta \le 2$. Therefore, (EC.27) holds and the optimal centralized solution is given by

$$T_i^c = T_{\min}^c \overset{POT}{=} \sqrt{\frac{K_0 + \sum_{i=1}^n K_i}{\sum_{i=1}^n H_i}}, \quad \forall i \in N. \tag{EC.28}$$

We next show T^c given by (EC.28) is an N.E., which implies that PoS is equal to 1. A sufficient condition for T^c being an N.E. is that for each $i \in N$, we have

$$f_i(T_i; T_{-i}^c) \ge f_i(T_i^c; T_{-i}^c), \quad \forall T_i \in \Gamma_i. \tag{EC.29}$$

Since all the retailers have the same replenishment interval under T^c, it is clear that

$$f_i(T_i^c; T_{-i}^c) = H_i T_{\min}^c + \frac{K_i}{T_{\min}^c} + \frac{K_0}{n T_{\min}^c}.$$

Suppose retailer i reduces his replenishment interval to some $T_i \le \frac{1}{2} T_i^c$. In this case, condition (EC.29) reduces to

$$H_i T_i + \frac{K_i}{T_i} + \left(\frac{1}{T_i} - \frac{1}{T_{\min}^c}\right) K_0 + \frac{K_0}{n T_{\min}^c} \ge H_i T_{\min}^c + \frac{K_i}{T_{\min}^c} + \frac{K_0}{n T_{\min}^c}, \quad \forall T_i \in \Gamma_i \text{ such that } T_i \le \frac{1}{2} T_{\min}^c,$$

or equivalently,

$$H_i T_i + \frac{K_i + K_0}{T_i} \ge H_i T_{\min}^c + \frac{K_i + K_0}{T_{\min}^c}, \quad \forall T_i \in \Gamma_i \text{ such that } T_i \le \frac{1}{2} T_{\min}^c. \tag{EC.30}$$

Consider the function $g(x) = H_i x + \frac{K_i + K_0}{x}$ defined on $x \in \Gamma_i$. The optimal solution that minimizes $g(x)$ is give by $x^* \overset{POT}{=} \sqrt{\frac{K_i + K_0}{H_i}}$. Notice that if the following inequality holds:

$$\sqrt{\frac{K_0 + \sum_{i=1}^n K_i}{\sum_{i=1}^n H_i}} \le \sqrt{2} \sqrt{\frac{K_i + K_0}{H_i}}, \tag{EC.31}$$

then we must have $T^c_{\min} \le x^*$ because both $T^c_{\min}$ and x^* are POT policies. It then follows that $g(x)$ is nonincreasing on $(0, T^c_{\min}] \cap \Gamma_i$, which implies (EC.30). To see (EC.31), note that since $K_j \le \alpha K_i$ and $H_j \ge \frac{1}{\beta} H_i$ for all j, the following inequality is a sufficient condition for (EC.31)

$$\frac{\beta K_0 + \alpha\beta n K_i}{n H_i} \le \frac{2(K_i + K_0)}{H_i},$$

or equivalently

$$(\beta - 2n)K_0 \le (2 - \alpha\beta) n K_i,$$

which is true under the assumption that $\alpha\beta \le 2$. Therefore, (EC.30) holds.

If retailer i increases his policy to some $T_i \ge 2T^c_i$, it can be similarly shown that condition (EC.29) reduces to

$$H_i T_i + \frac{K_0 + nK_i}{nT_i} \ge H_i T^c_{\min} + \frac{K_0 + nK_i}{nT^c_{\min}}, \quad \forall T_i \in \Gamma_i \text{ such that } T_i \ge 2T^c_{\min}. \tag{EC.32}$$

To show (EC.32) holds, consider the function $\phi(y) = H_i y + \frac{K_0 + nK_i}{ny}$ defined on Γ_i. The optimal solution that minimizes $\phi(y)$ is given by $y^* \overset{POT}{=} \sqrt{\frac{K_0 + nK_i}{nH_i}}$. Since both $T^c_{\min}$ and y^* are POT policies, $T^c_{\min} \ge y^*$ holds if the following inequality holds:

$$\sqrt{\frac{K_0 + \sum_{i=1}^n K_i}{\sum_{i=1}^n H_i}} \ge \frac{1}{\sqrt{2}} \sqrt{\frac{K_0 + nK_i}{nH_i}}. \tag{EC.33}$$

It can be similarly shown that a sufficient condition for (EC.33) is given by

$$\alpha(\beta - 2)K_0 \le (2 - \alpha\beta) n K_i,$$

which is true by $\alpha\beta \le 2$. It then follows that $\phi(\cdot)$ is increasing on $[T^c_{\min}, \infty) \cap \Gamma_i$, and therefore (EC.32) holds. □

B. Additional Numerical Results on the PoA and the PoS

Table 3 Price of Stability and Price of Anarchy in 10,000 instances: $n = 30$

K_i	H_i	K_0	PoA			PoS		
			min	average	max	min	average	max
$\mathcal{U}(0.5,3)$	$\mathcal{U}(0.2,2)$	5	1.0083	1.1074	1.2147	1.0000	1.0026	1.0339
$\mathcal{U}(0.5,3)$	$\mathcal{U}(0.2,2)$	20	1.3137	1.4969	1.7289	1.0000	1.0198	1.0910
$\mathcal{U}(0.5,3)$	$\mathcal{U}(0.2,2)$	50	1.1790	1.3450	2.3032	1.0000	1.0083	1.0924
$\mathcal{U}(0.5,3)$	$\mathcal{U}(2,5)$	5	1.0000	1.0356	1.1040	1.0000	1.0110	1.0479
$\mathcal{U}(0.5,3)$	$\mathcal{U}(2,5)$	20	1.2846	1.3871	1.5395	1.0000	1.0000	1.0213
$\mathcal{U}(0.5,3)$	$\mathcal{U}(2,5)$	50	1.9002	2.1471	2.3510	1.0000	1.0017	1.0235
$\mathcal{U}(0.5,3)$	$\mathcal{U}(5,10)$	5	1.1500	1.2432	1.3621	1.0000	1.0002	1.0067
$\mathcal{U}(0.5,3)$	$\mathcal{U}(5,10)$	20	1.6996	1.8899	2.1346	1.0000	1.0056	1.0812
$\mathcal{U}(0.5,3)$	$\mathcal{U}(5,10)$	50	1.4695	1.5799	1.7500	1.0000	1.0322	1.1000
$\mathcal{U}(4,10)$	$\mathcal{U}(0.2,2)$	5	1.0000	1.0001	1.0255	1.0000	1.0001	1.0023
$\mathcal{U}(4,10)$	$\mathcal{U}(0.2,2)$	20	1.0100	1.0978	1.1968	1.0000	1.0026	1.0400
$\mathcal{U}(4,10)$	$\mathcal{U}(0.2,2)$	50	1.0000	1.0598	1.6632	1.0000	1.0268	1.0758
$\mathcal{U}(4,10)$	$\mathcal{U}(2,5)$	5	1.0000	1.0251	1.0831	1.0000	1.0006	1.0115
$\mathcal{U}(4,10)$	$\mathcal{U}(2,5)$	20	1.0000	1.0162	1.0709	1.0000	1.0112	1.0376
$\mathcal{U}(4,10)$	$\mathcal{U}(2,5)$	50	1.3472	1.4262	1.5354	1.0000	1.0000	1.0211
$\mathcal{U}(4,10)$	$\mathcal{U}(5,10)$	5	1.0000	1.0000	1.0002	1.0000	1.0000	1.0002
$\mathcal{U}(4,10)$	$\mathcal{U}(5,10)$	20	1.1583	1.2417	1.3192	1.0000	1.0001	1.0017
$\mathcal{U}(4,10)$	$\mathcal{U}(5,10)$	50	1.1215	1.1963	1.2698	1.0000	1.0008	1.0094
$\mathcal{U}(11,20)$	$\mathcal{U}(0.2,2)$	5	1.0000	1.0013	1.0159	1.0000	1.0004	1.0065
$\mathcal{U}(11,20)$	$\mathcal{U}(0.2,2)$	20	1.0000	1.0001	1.0027	1.0000	1.0001	1.0027
$\mathcal{U}(11,20)$	$\mathcal{U}(0.2,2)$	50	1.0000	1.2059	1.3543	1.0000	1.0006	1.0057
$\mathcal{U}(11,20)$	$\mathcal{U}(2,5)$	5	1.0000	1.0000	1.0002	1.0000	1.0000	1.0002
$\mathcal{U}(11,20)$	$\mathcal{U}(2,5)$	20	1.0000	1.0005	1.1811	1.0000	1.0001	1.0013
$\mathcal{U}(11,20)$	$\mathcal{U}(2,5)$	50	1.0852	1.1745	1.2478	1.0000	1.0006	1.0052
$\mathcal{U}(11,20)$	$\mathcal{U}(5,10)$	5	1.0000	1.0011	1.0350	1.0000	1.0002	1.0049
$\mathcal{U}(11,20)$	$\mathcal{U}(5,10)$	20	1.0000	1.0151	1.0514	1.0000	1.0052	1.0201
$\mathcal{U}(11,20)$	$\mathcal{U}(5,10)$	50	1.0000	1.0010	1.0325	1.0000	1.0010	1.0229

Table 4 Price of Stability and Price of Anarchy in 10,000 instances: $n = 50$

K_i	H_i	K_0	PoA			PoS		
			min	average	max	min	average	max
$\mathcal{U}(0.5,3)$	$\mathcal{U}(0.2,2)$	5	1.0419	1.1215	1.2110	1.0000	1.0008	1.0080
$\mathcal{U}(0.5,3)$	$\mathcal{U}(0.2,2)$	20	1.4064	1.6000	1.8249	1.0000	1.0334	1.0825
$\mathcal{U}(0.5,3)$	$\mathcal{U}(0.2,2)$	50	1.2954	1.4241	1.5446	1.0000	1.0027	1.0292
$\mathcal{U}(0.5,3)$	$\mathcal{U}(2,5)$	5	1.0007	1.0494	1.1059	1.0000	1.0028	1.0286
$\mathcal{U}(0.5,3)$	$\mathcal{U}(2,5)$	20	1.3443	1.4322	1.5674	1.0000	1.0018	1.0401
$\mathcal{U}(0.5,3)$	$\mathcal{U}(2,5)$	50	2.1690	2.3492	2.5241	1.0000	1.0000	1.0000
$\mathcal{U}(0.5,3)$	$\mathcal{U}(5,10)$	5	1.1725	1.2557	1.3434	1.0000	1.0001	1.0054
$\mathcal{U}(0.5,3)$	$\mathcal{U}(5,10)$	20	1.8091	1.9985	2.1925	1.0000	1.0015	1.0093
$\mathcal{U}(0.5,3)$	$\mathcal{U}(5,10)$	50	1.6057	1.7713	1.9593	1.0000	1.0313	1.1189
$\mathcal{U}(4,10)$	$\mathcal{U}(0.2,2)$	5	1.0000	1.0000	1.0007	1.0000	1.0000	1.0007
$\mathcal{U}(4,10)$	$\mathcal{U}(0.2,2)$	20	1.0442	1.1120	1.1856	1.0000	1.0008	1.0066
$\mathcal{U}(4,10)$	$\mathcal{U}(0.2,2)$	50	1.0170	1.0818	1.6455	1.0000	1.0074	1.0595
$\mathcal{U}(4,10)$	$\mathcal{U}(2,5)$	5	1.0000	1.0209	1.0846	1.0000	1.0002	1.0042
$\mathcal{U}(4,10)$	$\mathcal{U}(2,5)$	20	1.0000	1.0287	1.0712	1.0000	1.0065	1.0276
$\mathcal{U}(4,10)$	$\mathcal{U}(2,5)$	50	1.3926	1.4595	1.5656	1.0000	1.0036	1.0359
$\mathcal{U}(4,10)$	$\mathcal{U}(5,10)$	5	1.0000	1.0000	1.0001	1.0000	1.0000	1.0001
$\mathcal{U}(4,10)$	$\mathcal{U}(5,10)$	20	1.1850	1.2551	1.3178	1.0000	1.0000	1.0005
$\mathcal{U}(4,10)$	$\mathcal{U}(5,10)$	50	1.1743	1.2258	1.2849	1.0000	1.0002	1.0027
$\mathcal{U}(11,20)$	$\mathcal{U}(0.2,2)$	5	1.0000	1.0022	1.0132	1.0000	1.0005	1.0038
$\mathcal{U}(11,20)$	$\mathcal{U}(0.2,2)$	20	1.0000	1.0000	1.0089	1.0000	1.0000	1.0089
$\mathcal{U}(11,20)$	$\mathcal{U}(0.2,2)$	50	1.0000	1.1823	1.3540	1.0000	1.0002	1.0016
$\mathcal{U}(11,20)$	$\mathcal{U}(2,5)$	5	1.0000	1.0000	1.0001	1.0000	1.0000	1.0001
$\mathcal{U}(11,20)$	$\mathcal{U}(2,5)$	20	1.0000	1.0000	1.0004	1.0000	1.0000	1.0004
$\mathcal{U}(11,20)$	$\mathcal{U}(2,5)$	50	1.1280	1.1895	1.2591	1.0000	1.0002	1.0018
$\mathcal{U}(11,20)$	$\mathcal{U}(5,10)$	5	1.0000	1.0002	1.0288	1.0000	1.0001	1.0011
$\mathcal{U}(11,20)$	$\mathcal{U}(5,10)$	20	1.0000	1.0211	1.0488	1.0000	1.0016	1.0142
$\mathcal{U}(11,20)$	$\mathcal{U}(5,10)$	50	1.0000	1.0079	1.0344	1.0000	1.0071	1.0249

C. Discussion on the equal-division rule and the Shapley Value

For the multi-retailer joint replenishment problem with any given joint replenishment strategy T, define a cooperative game on N with the following characteristic function:

$$g(\emptyset) = 0, \quad \text{and} \quad g(S) = \sum_{j \in S} \left(H_j T_j + \frac{K_j}{T_j} \right) + \max_{j \in S} \frac{K_0}{T_j}, \quad \forall \emptyset \subset S \subseteq N, \tag{EC.34}$$

where $g(S)$ can be interpreted as the long-run average cost for coalition S under joint replenishment strategy T. Recall that the Shapley value for the game (N, g) is defined by

$$\phi_i(g) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(n - |S| - 1)!}{n!} \left[g(S \cup \{i\}) - g(S) \right], \quad \forall i \in N.$$

In view of Proposition 1 and the additivity property of the Shapley value (cf. Shapley 1953), the equal-division rule turns out to generate precisely the Shapley value of the long-run average total cost in the system. The Shapley value is the unique distribution (allocation) of surplus/cost among the players that satisfies a certain collection of desirable properties, which arguably makes it a fair distribution. For more discussions on the subject, we refer to a survey by Hart (2008).

Here we would like to remark that the above game with characteristic function (EC.34) has a nice connection with a special class of cooperative games, called *airport games*. This class of games has the property that the value of the characteristic function is determined by the "largest" player in the coalition. The name of this game originates from the problem of allocating capital costs for runways to different types of planes. The capital cost of building an airport runway essentially depends on the largest type of plane to land there. Therefore, the cost of any subset is determined by the largest player in that subset.

To see the connection between game (EC.34) and airport games, we define a cooperative game on N with the following characteristic function:

$$g^m(\emptyset) = 0, \quad \text{and} \quad g^m(S) = \max_{j \in S} \frac{K_0}{T_j}, \quad \forall \emptyset \subset S \subseteq N,$$

where $g^m(S)$ can be interpreted as the long-run average major setup cost for coalition S. Note that $c_j \equiv K_0/T_j$ is the long-run average major setup cost of retailer j if he places orders by himself.

Therefore, the characteristic function g^m has the property that the cost of any subset of retailers is equal to the cost of the "largest" retailer in that subset, i.e., $g^m(S) = \max_{j\in S} c_j$. Recall that $\pi = (\pi_1, ..., \pi_n)$ is a permutation of the indices $\{1, ..., n\}$ such that $T_{\pi_1} \leq \cdots \leq T_{\pi_n}$, i.e., we have $c_{\pi_1} \geq \cdots \geq c_{\pi_n} \geq c_{\pi_{n+1}} \equiv 0$. By a well-known result in Littlechild and Owen (1973), the Shapley value of (airport) game g^m has a simple closed-form expression:

$$\phi_{\pi_i}(g^m) = \sum_{m=i}^{n} \frac{c_{\pi_m} - c_{\pi_{m+1}}}{m} = \sum_{m=i}^{n} \left(\frac{1}{T_{\pi_m}} - \frac{1}{T_{\pi_{m+1}}} \right) \frac{K_0}{m} = x_{\pi_i}(T_{\pi_i}; T_{-\pi_i}),$$

which provides an alternative proof for Proposition 1.

Finally, it is appropriate to point out that the equal-division rule coincides with the Shapley value of the cooperative game (EC.34) for a *fixed* joint strategy profile T, *not* the game with characteristic function g' defined as the optimal average total cost of coalition S when they restrict themselves to POT policies:

$$g'(\emptyset) = 0, \quad \text{and} \quad g'(S) = \min_{T_j \in \Gamma_j, \forall j \in S} \left[\sum_{j\in S} \left(H_j T_j + \frac{K_j}{T_j} \right) + \max_{j\in S} \frac{K_0}{T_j} \right], \quad \forall \emptyset \subset S \subseteq N.$$

D. Example 1 and Example 2 in Section 4.

Example 1. Let $K_0 = 1$. Let $K_i = 0$ and $H_i = 2$ for each $i \in N$.

By (6), the total cost of the system under joint policy $T \in \Gamma$ is equal to

$$C(T) = 2\sum_{i=1}^{n} T_i + \frac{1}{\min_{i\in N} T_i},$$

and the optimal centralized solution is given by

$$T_i^c \overset{POT}{=} \sqrt{\frac{1}{2n}}, \quad \forall i \in N.$$

Therefore, the optimal centralized system-wide cost $C(T^c)$ is at most 1.06 times $2\sqrt{2n}$.

By Lemma 1 (a), the joint replenishment policy $\tilde{T}$ with $\tilde{T}_i = 1$ for all $i \in N$ is an N.E.. Since $C(\tilde{T}) = 2n + 1$ and $C(T^c) \leq 1.06 \cdot 2\sqrt{2n}$, we have

$$PoA \geq \frac{C(\tilde{T})}{C(T^c)} \geq \frac{\sqrt{n}}{1.06\sqrt{2}}.$$

Example 2. Let $K_0 = 1$. Let $K_i = 0$ and $H_i = 1/i$ for each $i \in N$.

Similar to the analysis of Example 1, the optimal centralized solution is $T_i^c \overset{POT}{=} \sqrt{1/(\sum_{i=1}^n \frac{1}{i})}$ for each $i \in N$, and therefore $C(T^c)$ is at most 1.06 times $2\sqrt{\sum_{i=1}^n \frac{1}{i}}$. For the decentralized system, following the proof of Lemma 1, it can be similarly shown that the strategy profile $(\tilde{T}_i : i \in N)$ with $\tilde{T}_i = 1$ for all $i \in N$ is an N.E. with a total cost of $\sum_{i=1}^n \frac{1}{i} + 1$. Since the n^{th} harmonic number $\sum_{i=1}^n 1/i$ is $O(\ln n)$, it follows that the ratio $\frac{C(\tilde{T})}{C(T^c)}$ is $O(\sqrt{\ln n})$. In fact, as shown below in Lemma 6, $\tilde{T}$ is the least N.E. of this game. Then by Corollary 1, the price of stability of this game is $O(\sqrt{\ln n})$.

LEMMA 6. *The joint replenishment policy $\tilde{T}$ with $\tilde{T}_i = 1$ for all $i \in N$ is the least N.E. in Example 2.*

Proof. Suppose T with $T_i \leq 1$ for all $i \in N$ is an N.E.. We will prove by induction that $T_i = 1$ for all $i \in N$. We first consider retailer n. Let $\pi = (\pi_1, ..., \pi_n)$ be a permutation of the indices $\{1, ..., n\}$ such that $T_{\pi_1} \leq \cdots \leq T_{\pi_n}$, and w.l.o.g. we assume $T_n = T_{\pi_j}$ for some $1 \leq j \leq n$. The cost of retailer n under T is given by

$$f_n(T_n; T_{-n}) = \frac{T_n}{n} + \sum_{m=j}^{n} \left(\frac{1}{T_{\pi_m}} - \frac{1}{T_{\pi_{m+1}}} \right) \frac{1}{m},$$

where $1/T_{\pi_{n+1}} = 0$. Suppose retailer n increases his replenishment interval to 1, and his cost becomes $f_n(1; T_{-n}) = \frac{2}{n}$. Since T is an N.E., retailer n cannot have a profitable unilateral deviation, i.e., $f_n(1; T_{-n}) \geq f_n(T_n; T_{-n})$, or

$$\begin{aligned} \frac{2}{n} &\geq \frac{T_n}{n} + \sum_{m=j}^{n} \left(\frac{1}{T_{\pi_m}} - \frac{1}{T_{\pi_{m+1}}} \right) \frac{1}{m} \\ &\geq \frac{T_n}{n} + \sum_{m=j}^{n} \left(\frac{1}{T_{\pi_m}} - \frac{1}{T_{\pi_{m+1}}} \right) \frac{1}{n} \\ &= \frac{T_n}{n} + \frac{1}{nT_n}, \end{aligned}$$

which implies

$$\frac{1 - T_n}{n} \geq \frac{1 - T_n}{nT_n}.$$

If $T_n < 1$, then the above inequality implies $T_n \geq 1$, which is a contradiction. Therefore, we must have $T_n = 1$.

Assume $T_j = 1$ for all $k \leq j \leq n$, which implies $T_{\pi_k} = T_{\pi_{k+1}} = \cdots = T_{\pi_n} = 1$. We next show $T_{k-1} = 1$. Assume w.l.o.g. that $T_{k-1} = T_{\pi_r}$ for some $r \leq k-1$. The cost of retailer $k-1$ under T is given by

$$f_{k-1}(T_{k-1}; T_{-(k-1)}) = \frac{T_{k-1}}{k-1} + \sum_{m=r}^{k-1} \left(\frac{1}{T_{\pi_m}} - \frac{1}{T_{\pi_{m+1}}} \right) \frac{1}{m} + \frac{1}{n}.$$

Suppose retailer $k-1$ increases his replenishment interval to 1, and his new cost is equal to $f_{k-1}(1; T_{-(k-1)}) = \frac{1}{k-1} + \frac{1}{n}$. Since T is an N.E., retailer $k-1$ cannot have a profitable unilateral deviation, i.e., $f_{k-1}(1; T_{-(k-1)}) \geq f_{k-1}(T_{k-1}; T_{-(k-1)})$, or

$$\begin{aligned}
\frac{1}{k-1} + \frac{1}{n} &\geq \frac{T_{k-1}}{k-1} + \sum_{m=r}^{k-1} \left(\frac{1}{T_{\pi_m}} - \frac{1}{T_{\pi_{m+1}}} \right) \frac{1}{m} + \frac{1}{n} \\
&\geq \frac{T_{k-1}}{k-1} + \sum_{m=r}^{k-1} \left(\frac{1}{T_{\pi_m}} - \frac{1}{T_{\pi_{m+1}}} \right) \frac{1}{k-1} + \frac{1}{n} \\
&= \frac{T_{k-1}}{k-1} + \frac{1}{(k-1)T_{k-1}} - \frac{1}{k-1} + \frac{1}{n},
\end{aligned}$$

which implies

$$\frac{1 - T_{k-1}}{k-1} \geq \frac{1 - T_{k-1}}{(k-1)T_{k-1}}.$$

It is clear that the above inequality holds only when $T_{k-1} = 1$, and this completes the proof. □

E. An Example with Proportional Sharing Rule for K_0

Consider a system with two retailers, and let T_{-i} be the replenishment interval of the retailer other than retailer i for $i \in \{1, 2\}$. Under the proportional sharing rule where K_0 is allocated to retailer i proportionally to his order quantity $d_i T_i$, the cost functions are given by (the superscript P stands for *proportional*):

$$f_i^P(T_i, T_{-i}) = H_i T_i + \frac{K_i}{T_i} + x_i^P(T_i, T_{-i}),$$

where

$$x_i^P(T_i, T_{-i}) = \begin{cases} K_0 \left(\frac{1}{T_i} - \frac{1}{T_{-i}} \right) + \frac{K_0}{T_{-i}} \cdot \frac{T_i d_i}{T_i d_i + T_{-i} d_{-i}}, & \text{if } T_i \leq T_{-i}, \\ \frac{K_0}{T_i} \cdot \frac{T_i d_i}{T_i d_i + T_{-i} d_{-i}}, & \text{if } T_i > T_{-i}. \end{cases}$$

Consider the following parameters for the costs and demand:

- $K_0 = 11$
- $K_1 = 6,\ K_2 = 24$
- $H_1 = 2,\ H_2 = 1$
- $d_1 = 2,\ d_2 = 5$

Let $\Gamma_i = \{T_i : \sqrt{\frac{K_i}{2H_i}} \le T_i \le \sqrt{\frac{2(K_i+K_0)}{H_i}}$ such that $T_i = 2^{z_i}, z_i \in \mathbb{Z}\}$ be the set of feasible strategies of retailer i (the base planning period is assumed w.l.o.g. to be 1). Then we have $\Gamma_1 = \{2,4,8\}$ and $\Gamma_2 = \{4,8,16\}$, and the retailers' costs for all feasible replenishment strategies $\Gamma = \Gamma_1 \times \Gamma_2$ are computed as follows:

Table 5 Retailers' costs under the proportional sharing rule

(T_1,T_2)	(2,4)	(4,4)	(8,4)	(2,8)	(4,8)	(8,8)	(2,16)	(4,16)	(8,16)
$f_1^P(T_1,T_2)$	10.21	10.29	17.36	11.25	11.10	17.14	11.85	11.63	17.55
$f_2^P(T_1,T_2)$	12.29	11.96	12.14	12.25	12.15	11.98	18.15	18.13	18.07

It is straightforward to verify that Nash equilibrium does not exist in this example.

F. Proportional Sharing Rule vs. Equal-division Rule for a Two-Retailer Game

Consider a system with two retailers who have identical minor setup cost and holding cost parameters. In what follows, we compare the cost allocation under the equal-division rule and the proportional sharing rule, and demonstrate that retailers with a smaller demand rate would prefer the proportional sharing rule over the equal-division rule.

PROPOSITION 3. *Assume $H_i = H$ and $K_i = K$ for both $i \in \{1,2\}$. If T is a Nash equilibrium under the proportional sharing rule, then we must have $T_1 = T_2$.*

Proof. We prove by contradiction. Without loss of generality, let $T_1 \le T_2$. Assume $T_1 \ne T_2$ and hence we have $T_1 < T_2$. Since T is an N.E., retailer 1 cannot have a profitable unilateral deviation from T_1 to T_2, i.e.,

$$f_1^P(T_1,T_2) \le f_1^P(T_2,T_2), \tag{EC.35}$$

where

$$f_1^P(T_1,T_2) = HT_1 + \frac{K}{T_1} + K_0\left(\frac{1}{T_1} - \frac{1}{T_2}\right) + \frac{K_0}{T_2}\frac{T_1 d_1}{T_1 d_1 + T_2 d_2},$$
$$f_1^P(T_2,T_2) = HT_2 + \frac{K}{T_2} + \frac{K_0}{T_2}\frac{T_2 d_1}{T_2 d_1 + T_2 d_2}.$$

After some algebraic manipulation, (EC.35) implies

$$H - \frac{K}{T_1 T_2} \geq \frac{K_0}{T_1 T_2} - \frac{K_0 d_1 d_2}{(T_1 d_1 + T_2 d_2)(T_2 d_1 + T_2 d_2)}. \quad \text{(EC.36)}$$

Similarly, retailer 2 cannot benefit from unilaterally deviating from T_2 to T_1, i.e.,

$$f_2^P(T_1,T_2) \leq f_2^P(T_1,T_1), \quad \text{(EC.37)}$$

where

$$f_2^P(T_1,T_2) = HT_2 + \frac{K}{T_2} + \frac{K_0}{T_2}\frac{T_2 d_2}{T_1 d_1 + T_2 d_2},$$
$$f_2^P(T_1,T_1) = HT_1 + \frac{K}{T_1} + \frac{K_0}{T_1}\frac{T_1 d_2}{T_1 d_1 + T_1 d_2}.$$

It then follows that (EC.37) results in

$$H - \frac{K}{T_1 T_2} \leq \frac{K_0 d_2^2}{(T_1 d_1 + T_2 d_2)(T_1 d_1 + T_1 d_2)}. \quad \text{(EC.38)}$$

Comparing the right-hand side of (EC.36) and (EC.38), we obtain

$$\begin{aligned}
&\frac{K_0 d_2^2}{(T_1 d_1 + T_2 d_2)(T_1 d_1 + T_1 d_2)} - \left(\frac{K_0}{T_1 T_2} - \frac{K_0 d_1 d_2}{(T_1 d_1 + T_2 d_2)(T_2 d_1 + T_2 d_2)}\right) \\
&= \frac{K_0 d_2}{T_1 T_2 (d_1 + d_2)} - \frac{K_0}{T_1 T_2} \\
&< 0.
\end{aligned}$$

Therefore, (EC.36) and (EC.38) do not hold simultaneously, which implies that the original assumption $T_1 \neq T_2$ cannot be true. □

It immediately follows from Proposition 3 that under any equilibrium outcome (if there exists any), the retailer with a smaller demand rate d_i (and hence with a smaller order volume) bears a smaller share of the total cost under the proportional sharing rule. Recall that for the equal-division rule, retailers with identical K_i and H_i also have the same replenishment interval under any Nash equilibrium (cf. Lemma 1). Therefore, the retailers always share an equal amount of the total cost under the equal-division rule regardless of their order volume. In case the retailers can observe each other's order quantities, the above comparison illustrates that retailers with a smaller demand rate would prefer the proportional sharing rule over the equal-division rule.

Here we would like to point out that Proposition 3 does not extend to the general case $n > 2$. To see a counterexample, consider the following parameters for the costs and demand:

- $n = 4$
- $d = [6, 1, 1, 1]$
- $H_i = K_i = 1$ for all $i \in \{1, 2, 3, 4\}$
- $K_0 = 20$

The set of feasible strategies for each retailer $i \in \{1, 2, 3, 4\}$ is given by $\Gamma_i = \{1, 2, 4, 8\}$, and the set of Nash equilibria can be obtained by computing the retailers' costs for all possible joint replenishment strategies $\Gamma = \times_{i=1}^{4} \Gamma_i$. It can be shown (see Table 6 for details) that the joint strategy profile $T^* = [4, 2, 2, 2]$ is an N.E., which implies that under the proportional sharing rule, it is not necessarily true that $T_i = T_j$ for all $1 \leq i, j \leq n$ even when all the retailers have identical K_i and H_i for $n > 2$.

Table 6 Retailers' costs under different strategy profiles show that $T^* = [4, 2, 2, 2]$ is an N.E.

$T = (T_1, T_2, T_3, T_4)$	$f_1^P(T)$	$f_2^P(T)$	$f_3^P(T)$	$f_4^P(T)$
(1, 2, 2, 2)	17.00	4.17	4.17	4.17
(2, 2, 2, 2)	9.17	3.61	3.61	3.61
(4, 2, 2, 2)	**8.25**	4.50	4.50	4.50
(8, 2, 2, 2)	10.35	5.09	5.09	5.09
(4, 1, 2, 2)	8.39	13.17	4.84	4.84
(4, 2, 2, 2)	8.25	**4.50**	4.50	4.50
(4, 4, 2, 2)	8.00	4.88	5.31	5.31
(4, 8, 2, 2)	8.06	8.68	5.32	5.32
(4, 2, 1, 2)	8.39	4.84	13.17	4.84
(4, 2, 2, 2)	8.25	4.50	**4.50**	4.50
(4, 2, 4, 2)	8.00	5.31	4.88	5.31
(4, 2, 8, 2)	8.06	5.32	8.68	5.32
(4, 2, 2, 1)	8.39	4.84	4.84	13.17
(4, 2, 2, 2)	8.25	4.50	4.50	**4.50**
(4, 2, 2, 4)	8.00	5.31	5.31	4.88
(4, 2, 2, 8)	8.06	5.32	5.32	8.68