Online Appendices for
Supply and Demand Functions in Inventory Models

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A Modeling Choice

In this section, we discuss two variations of our model that have appeared in the literature. We show that the method developed in this paper applies to these variations and we also point out the subtle differences in the analysis under different modeling choices.

A.1 Instantaneous Replenishment

Our model assumes a one-period delivery lead time, while most of the inventory studies assume zero lead time. The sequence of events for the latter model is given in Figure 2. Because delivery happens immediately after the order, it is natural to restrict at most one order in a period. This is not essential for the case of deterministic and unlimited supply. With uncertain or limited supply, however, the restriction of one order per period becomes important. Without this restriction, one can potentially bring the stock to any desired level by placing multiple orders, one immediately after another.

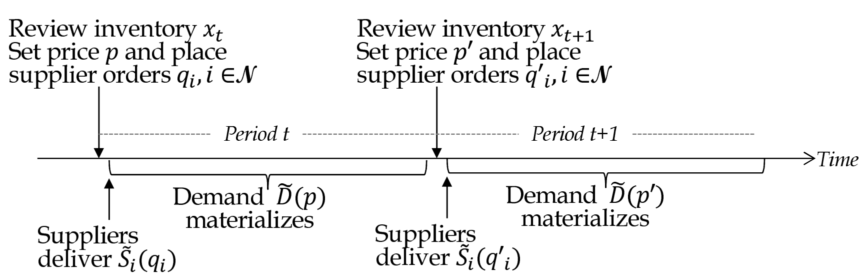

Figure 2: The model with instantaneous delivery.

The dynamic programming equation for the problem depicted in Figure 2 can be written as

$$V_t(x_t) = \max_{\substack{\underline{p} \le p \le \overline{p}, \\ q_i \ge 0, i \in \mathcal{N}}} J_t(x_t, p, q_1, \cdots, q_n), \tag{28}$$

$$\begin{aligned} J_t(x_t, p, q_1, \cdots, q_n) &= p\mathbb{E}[\tilde{D}(p)] - \sum_{i \in \mathcal{N}} c_i \mathbb{E}\big[\tilde{S}_i(q_i)\big] - \mathbb{E}\left[H\left(x_t - \tilde{D}(p) + \sum_{i \in \mathcal{N}} \tilde{S}_i(q_i)\right)\right] \\ &\quad + \rho\mathbb{E}\left[V_{t+1}\left(x_t - \tilde{D}(p) + \sum_{i \in \mathcal{N}} \tilde{S}_i(q_i)\right)\right]. \end{aligned} \tag{29}$$

Analytically, this model is not fundamentally different from our model and it can be shown that *all* the formal results derived in this paper continue to hold.

In the special case when the supply functions become deterministic, the different assumptions on delivery lead time can lead to subtle differences in the structure of the optimal policy. For the model with instantaneous delivery defined in (28)–(29), a multi-level modified base stock list price policy is optimal. (This can be easily established by following the proof of Theorem 2 in Feng and Shi (2012). All one needs is to treat $\mu_{Si} \le \bar{\mu}_{Si}$ as the capacity constraint.) Under this policy, there is a base stock level and a list price, both independent of the inventory level, for each supplier. The optimal order quantity to a supplier should bring the cumulative stock level up to that supplier's base stock level. If the final post-order inventory level equals a supplier's base stock level, the optimal product price equals the list price corresponding to that supplier. Such a policy, however, is generally suboptimal for the model with one-period delivery lead time defined in (11)–(12) based on our discussion on Corollary 1.

A.2 Responsive Pricing Decision

When procurement orders are delivered immediately, it is natural that one postpones the pricing decision *after* observing the delivery quantities (Li et al. 2013). This model is presented in Figure 3. The associated dynamic program can be written as

$$V_t(x_t) = \max_{q_i \ge 0, i \in \mathcal{N}} J_t^Q(x_t, q_1, \cdots, q_n), \tag{30}$$

$$J_t^Q(x_t, q_1, \cdots, q_n) = \mathbb{E}\left[\max_{\underline{p} \le p \le \overline{p}} J_t^P(x_t, p, \tilde{S}_1(q_1), \cdots, \tilde{S}_n(q_n))\right], \tag{31}$$

$$\begin{aligned} J_t^P(x_t, p, s_1, \cdots, s_n) &= p\mathbb{E}[\tilde{D}(p)] - \sum_{i \in \mathcal{N}} c_i s_i - \mathbb{E}\left[H\left(x_t - \tilde{D}(p) + \sum_{i \in \mathcal{N}} s_i\right)\right] \\ &\quad + \rho\mathbb{E}\left[V_{t+1}\left(x_t - \tilde{D}(p) + \sum_{i \in \mathcal{N}} s_i\right)\right]. \end{aligned} \tag{32}$$

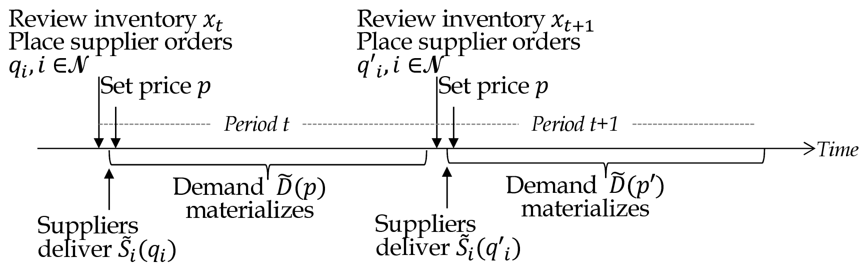

Figure 3: The model with instantaneous delivery with postponed pricing decision.

With the same conditions on the demand and supply functions as those in Theorem 2, we can prove that J_t^Q, J_t^P and V_t are concave. Also, Theorem 4 continues to hold. When the supply functions are deterministic, it is easy to check that the model defined in (30)–(32) is equivalent to that in (28)–(29).

B Proofs of Formal Results

Proof of Lemma 1. For ease of exposition, we proof the result by assuming that ϕ is differentiable. Note that ϕ is concave and thus can have at most countable number of nondifferentiable points (Rockafellar 1970, Theorem 25.3). In this case, we can approximate the concave function using a sequence of differentiable functions and apply the argument below to obtain the result.

Let $\mathrm{d}\phi(y)/\mathrm{d}y = \zeta(y) + \beta$ such that $\zeta(y) \geq 0$ and β is a constant. We consider the case where $\mathrm{d}\phi(y)/\mathrm{d}y$ is bounded from the below by $\beta > -\infty$. Usual limiting argument can be used to treat the general case. With (6), the expression in (3) can be written as

$$\psi[\bar{F}] = -\int_0^\infty \phi(y)\mathrm{d}\bar{F}(y) = \int_0^\infty \bar{F}(y)\mathrm{d}\phi(y) + \phi(0) = \int_0^\infty \zeta(y)\bar{F}(y)\mathrm{d}y + \beta\mu + \phi(0). \tag{33}$$

Therefore, maximizing $\psi[\bar{F}]$ boils down to maximizing the first term on the right-hand side. If $\mu = \mathbb{E}[Z]$, then it is clear that $\bar{F} = \bar{F}_Z$ maximizes $\psi[\bar{F}]$ under constraint (5). If, however, $\mu < \mathbb{E}[Z]$, this solution violates constraint (6). In this case, we take $\bar{F}^0$ that corresponds to the survival function in the original problem, i.e, $\bar{F}^0(y) = \Pr\{\min\{x, Z\} \geq y\}$ for some x such that $\mathbb{E}[\min\{x, Z\}] = \mu$. Consider the Gâteaux derivative of ψ at $\bar{F}^0$

$$\mathrm{d}_h\psi[\bar{F}]|_{\bar{F}=\bar{F}^0} = \lim_{\tau\to 0} \frac{\psi[\bar{F}^0 + \tau h] - \psi[\bar{F}^0]}{\tau} = \int_0^\infty \zeta(y)h(y)\mathrm{d}y$$

along the feasible direction satisfying

$$h(y) < (>) 0, y < (>) y^0, \text{ for some } y^0 \in (0, \infty),$$
$$\int_0^\infty h(y)\mathrm{d}y = 0.$$

The first condition suggests that along the feasible direction $\int_0^y \bar{F}^0(u)\mathrm{d}u$ and $\int_0^y (\bar{F}^0(u) + \tau h(u))\mathrm{d}u$ are ordered (see Theorem 3.A.44 in Shaked and Shanthikumar 2006), which takes care of constraint (5). The second condition ensures that the perturbed survival function $\bar{F}^0(\cdot) + \tau h(\cdot)$ satisfies (6). Note that given that $\bar{F}^0$ corresponds to the survival function of the original problem, h must satisfy these two conditions. Because $\zeta(\cdot)$ is decreasing and nonnegative, the above two conditions imply

$$\begin{aligned} \mathrm{d}_h\psi[\bar{F}]|_{\bar{F}=\bar{F}^0} &= \int_0^{y^0} \zeta(y)h(y)\mathrm{d}y + \int_{y^0}^\infty \zeta(y)h(y)\mathrm{d}y \le \int_0^{y^0} \zeta(y^0)h(y)\mathrm{d}y + \int_{y^0}^\infty \zeta(y)h(y)\mathrm{d}y \\ &= -\int_{y^0}^\infty \zeta(y^0)h(y)\mathrm{d}y + \int_{y^0}^\infty \zeta(y)h(y)\mathrm{d}y = \int_{y_0}^\infty (\zeta(y) - \zeta(y_0))h(y)\mathrm{d}y \le 0. \end{aligned}$$

In other words, the Gâteaux derivative of $\psi[\bar{F}]$ at $\bar{F} = \bar{F}^0$ is negative along the feasible direction. Therefore, the objective is maximized at $\bar{F}^0$. Note that the optimality of $\bar{F}^0$ is independent of the choice of $\phi(\cdot)$, as long as $\phi(\cdot)$ is concave.

Next we show that $\Psi(\mu)$ is concave in μ. Let $\hat{x}(\mu)$ be the smallest x such that $\int_0^{\hat{x}(\mu)} \bar{F}_Z(u)\mathrm{d}u = \mu$ for $\mu \in [0, \mathbb{E}[Z]]$. Take $\mu_1 < \mu_2$, $i = 1, 2$, then we have from (33)

$$\begin{aligned} & 2\Psi\left(\frac{\mu_1+\mu_2}{2}\right) - \Psi(\mu_1) - \Psi(\mu_2) \\ = \ & 2\int_0^{\hat{x}(\frac{\mu_1+\mu_2}{2})} \zeta(y)\bar{F}_Z(y)\mathrm{d}y - \int_0^{\hat{x}(\mu_1)} \zeta(y)\bar{F}_Z(y)\mathrm{d}y - \int_0^{\hat{x}(\mu_2)} \zeta(y)\bar{F}_Z(y)\mathrm{d}y \\ = \ & \int_{\hat{x}(\mu_1)}^{\hat{x}(\frac{\mu_1+\mu_2}{2})} \zeta(y)\bar{F}_Z(y)\mathrm{d}y - \int_{\hat{x}(\frac{\mu_1+\mu_2}{2})}^{\hat{x}(\mu_2)} \zeta(y)\bar{F}_Z(y)\mathrm{d}y \\ \ge \ & \zeta\left(\hat{x}(\frac{\mu_1+\mu_2}{2})\right)\left(\frac{\mu_1+\mu_2}{2} - \mu_1\right) - \zeta\left(\hat{x}(\frac{\mu_1+\mu_2}{2})\right)\left(\mu_2 - \frac{\mu_1+\mu_2}{2}\right) \\ = \ & 0. \end{aligned}$$

The inequality follows from the facts that $\zeta(\cdot)$ is decreasing and $\int_0^{\hat{x}(\mu)} \bar{F}_Z(y)\mathrm{d}y = \mu$. We conclude that $\Psi(\cdot)$ is concave. □

Proof of Lemma 2. Pick $x_1 \in \mathcal{X}$ and $x_2 \in \mathcal{X}$. Because $\{Y(x), x \in \mathcal{X}\} \in SL(mp)$, there exist two random variables $\hat{Y}(x_1)$ and $\hat{Y}(x_2)$ with the same distribution as $Y(x_1)$ and $Y(x_2)$, respectively, such that $\frac{\hat{Y}(x_1)+\hat{Y}(x_2)}{2} \leq_{cv} Y(\frac{x_1+x_2}{2})$. For any concave $\phi(\cdot)$, we have

$$\frac{\mathbb{E}[\phi(Y(x_1))] + \mathbb{E}[\phi(Y(x_2))]}{2} = \frac{\mathbb{E}[\phi(\hat{Y}(x_1))] + \mathbb{E}[\phi(\hat{Y}(x_2))]}{2}$$
$$\leq \mathbb{E}\left[\phi\left(\frac{(\hat{Y}(x_1) + \hat{Y}(x_2)}{2}\right)\right] \leq \mathbb{E}\left[\phi\left(Y\left(\frac{x_1 + x_2}{2}\right)\right)\right].$$

The first inequality follows from the concavity of ϕ and the second inequality follows from the relation $\frac{\hat{Y}(x_1)+\hat{Y}(x_2)}{2} \leq_{cv} Y(\frac{x_1+x_2}{2})$. Since x_1 and x_2 are arbitrarily chosen, we conclude that $\mathbb{E}[\phi(Y(x))]$ is concave in x and thus $\{Y(x), x \in \mathcal{X}\} \in SL$. □

Proof of Lemma 3. Let $\phi_A(\cdot)$ and $\phi_B(\cdot)$ be two increasing functions such that $\phi_A(z)$ crosses $\phi_B(z)$ from the above at most once over $z \in \mathcal{Z}$. Then, there exists a $\bar{\theta}$ such that $\{z \in \mathcal{Z} : \phi_A(z) < \theta\} \subseteq (\supseteq)\{z \in \mathcal{Z} : \phi_B(z) < \theta\}$ or equivalently $\Pr\{\phi_A(Z) < \theta\} \leq (\geq)\Pr\{\phi_B(Z) < \theta\}$ for $\theta < (>)\bar{\theta}$. Take $\phi_A(z) = \hat{\varphi}((\mu_1 + \mu_2)/2, z)$ and $\phi_B(z) = (\hat{\varphi}(\mu_1, z) + \hat{\varphi}(\mu_2, z))/2$. Because $\varphi(x, z)$ is increasing in (x, z), $\hat{\varphi}(\mu, z)$ is increasing in (μ, z) and thus $\phi_A(z)$ and $\phi_B(z)$ are increasing in z. Because of the single-crossing property of $\hat{\varphi}$, we deduce that the distribution function of $\hat{\varphi}((\mu_1 + \mu_2)/2, Z)$ crosses that of $(\hat{\varphi}(\mu_1, Z) + \hat{\varphi}(\mu_2, Z))/2$ from the below at most once. We conclude from Theorem 3.A.44 of Shaked and Shanthikumar (2006) that $Y(\hat{x}(\mu))$ is stochastically linear in mid-point.

A sufficient condition for the single-crossing property to hold is that

$$(\hat{\varphi}(\mu_2, z) - \hat{\varphi}((\mu_1 + \mu_2)/2, z)) - (\hat{\varphi}((\mu_1 + \mu_2)/2, z) - \hat{\varphi}(\mu_1, z))$$

is increasing in z, which leads to (9). □

Proof of Theorem 1. Following the proof of Lemma 3, take $\phi_A(z) = \varphi(\check{x}, z)$ and $\varphi_B(z) = (\varphi(x_1, z)+\varphi(x_2, z))/2$ with $x_1 < x_2$ and $\check{x}$ satisfying $\mu(\check{x}) = (\mu(x_1)+\mu(x_2))/2$. It is easy to see that $x_1 < \check{x} < x_2$. Therefore, the distribution function of $\varphi(\check{x}, Z)$ crosses that of $(\varphi(x_1, Z)+\varphi(x_2, Z))/2$ from the above at most once. We deduce from Theorem 3.A.44 of Shaked and Shanthikumar (2006) that $Y(\hat{x}(\mu))$ is stochastically linear in mid-point.

A sufficient condition for the single-crossing property to hold is that

$$\Gamma = (\varphi(x_2, z) - \varphi(\check{x}, z)) - (\varphi(\check{x}, z) - \varphi(x_1, z))$$

is increasing in z. Let $\bar{x} = (x_1 + x_2)/2$. We have

$$\Gamma = (\varphi(x_2, z) - \varphi(\bar{x}, z)) - (\varphi(\bar{x}, z) - \varphi(x_1, z)) + 2(\varphi(\bar{x}, z) - \varphi(\check{x}, z)).$$

The sum of the first two terms is increasing in z when $\frac{\partial^3\varphi(x,z)}{\partial z \partial x^2} \geq 0$, and the last term is increasing in z when $\frac{\partial^2\varphi(x,z)}{\partial z \partial x} \geq 0$ and $\bar{x} > \check{x}$. When $\bar{x} \leq \check{x}$, the sum of the first two terms is negative because

$\varphi(x,z)$ is concave in x and the last term is negative because $\varphi(x,z)$ is increasing in x. Hence, $\varphi(x,z)$ satisfies the single-crossing property. $\square$

Proof of Theorem 2. It follows immediately from Definition 1 that $\hat{J}_T$ is concave. Because concavity is preserved under maximization and expectation, the concavity of V_t and $\hat{J}_t$ can be easily established using induction. $\square$

Proof of Theorem 3. Take $n = 1$, $D(\mu_D) = \mu_D$, and $H(x) = hx^+ + bx^-$ for some $h > 0$ and $b > 0$, where $x^+ = \max\{0, x\}$ and $x^- = \max\{0, -x\}$. Let $\bar{F}_{Y(\mu)}$ be the survival function of the supply function $Y(\mu), 0 \le \mu \le \overline{\mu}$. Then J_T is unimodal in q if and only if

$$\pi(\mu) = c\mu + h\mathbb{E}[S(\mu) - \mu_D]^+ + b\mathbb{E}[\mu_D - S(\mu)]^+$$

is quasiconvex in μ. We note that

$$\begin{aligned}\pi(\mu) &= c\mu + b(\mu_D - \mu) + (h+b)\int_{\mu_D}^{\infty} \bar{F}_{Y(\mu)}(y)\mathrm{d}y \\ &= b\mu_D + (h+b)\Big(\int_{\mu_D}^{\infty} \bar{F}_{Y(\mu)}(y)\mathrm{d}y - \frac{b-c}{h+b}\mu\Big).\end{aligned}$$

Note that $(b-c)/(h+b)$ can be any positive number for $c > 0$, $b > -h$ and $b > 0$. Therefore, $\pi(\mu)$ is quasiconvex for any h, b, and c if and only if $\int_d^\infty \bar{F}_{Y(\mu)}(y)\mathrm{d}y$ is convex in μ for any $\mu_D \ge 0$, which is equivalent to $\{Y(\mu), 0 \le \mu \le \overline{\mu}\} \in SL$. $\square$

Proof of Lemma 4. By definition, $\{Y(x), x \in \mathcal{X}\} \in SI(disp)$ is equivalent to that $F^{inv}_{Y(x_2)}(\alpha) - F^{inv}_{Y(x_1)}(\alpha)$ is increasing in $\alpha \in (0,1)$ for any $x_1 < x_2$ with $x_1, x_2 \in \mathcal{X}$, where $F^{inv}_{Y(x)}(\cdot)$ is the inverse of the distribution function of $Y(x)$. The latter statement is, in turn, equivalent to that $F^{inv}_{Y(x)}(\alpha_2) - F^{inv}_{Y(x)}(\alpha_1)$ is increasing in $x \in \mathcal{X}$ for any $\alpha_1 < \alpha_2$ with $\alpha_1, \alpha_2 \in (0,1)$. Let $F^{inv}_Z(\cdot)$ denote the inverse of the distribution of Z. By definition, $F^{inv}_{Y(x)}(\alpha) = \varphi(x, F^{inv}_Z(\alpha))$ and $F^{inv}_Z(\cdot)$ is increasing. Therefore,

$$F^{inv}_{Y(x)}(\alpha_2) - F^{inv}_{Y(x)}(\alpha_1) = \varphi(x, F^{inv}_Z(\alpha_2)) - \varphi(x, F^{inv}_Z(\alpha_1))$$

is increasing in x if and only if $\varphi(x,z)$ is supermodular. $\square$

The proof of Theorem 4 needs Lemmas 6 and 7.

Lemma 6 *If $\{Y(x), x \in \mathcal{X} = [\underline{x}, \overline{x}] \subseteq [0,\infty)\} \in SI(disp)$ and $\mathbb{E}[Y(x)] = x$, then*

$$\lim_{\Delta \downarrow 0} \frac{\mathbb{E}[\phi(Y(x+\Delta))] - \mathbb{E}[\phi(Y(x))]}{\Delta} \le \frac{\partial \mathbb{E}[\phi(Y(x)+a)]}{\partial a}\Big|_{a\downarrow 0} \tag{34}$$

for any concave $\phi(\cdot)$.

Proof of Lemma 6. Take $x_1, x_2 \in \mathcal{X}$. Then, $Y(x_1) \leq_{disp} Y(x_2)$. By condition (3.B.14) on page 149 in Shaked and Shanthikumar (2006), we have $Y(x_2) =^{st} Y(x_1) + \psi(Y(x_1))$ for some increasing function ψ. Let $\hat{Y}_1 =^{st} Y(x_1)$ with $\hat{Y}_1$ independent of $Y(x_1)$. Let ψ^{inv} denote the inverse of ψ. Then,

$$\begin{aligned}
& \Pr\{\psi(Y(x_1)) \leq \psi_0, Y(x_1) \leq y_1\} \\
= \;& \Pr\{Y(x_1) \leq \psi^{inv}(\psi_0), Y(x_1) \leq y_1\} \\
= \;& F_{Y(x_1)}(\min\{\psi^{inv}(\psi_0), y_1\}) \\
= \;& \min\{F_{Y(x_1)}(\psi^{inv}(\psi_0)), F_{Y(x_1)}(y_1)\} \\
\geq \;& \min\{F_{Y(x_1)}(\psi^{inv}(\psi_0)), F_{Y(x_1)}(y_1)\} \cdot \max\{F_{Y(x_1)}(\psi^{inv}(\psi_0)), F_{Y(x_1)}(y_1)\} \\
= \;& F_{Y(x_1)}(\psi^{inv}(\psi_0)) \cdot F_{Y(x_1)}(y_1) \\
= \;& \Pr\{\psi(\hat{Y}_1) \leq \psi_0\} \Pr\{Y(x_1) \leq y_1\}.
\end{aligned}$$

Therefore, $(\psi(Y_1(x_1)), Y_1(x_1))$ is stochastically larger than $(\psi(\hat{Y}_1), Y(x_1))$ in the supermodular order (see §§9.A Shaked and Shanthikumar 2006). We have

$$\begin{aligned}
\mathbb{E}[\phi(Y(x_2))] &= \mathbb{E}[\phi(\psi(Y(x_1)) + Y(x_1))] \\
&\leq \mathbb{E}[\phi(\psi(\hat{Y}_1) + Y(x_1))] \\
&\leq \mathbb{E}[\phi(\mu(x_2) - \mu(x_1) + Y(x_1))] \\
&= \mathbb{E}[\phi(x_2 - x_1 + Y(x_1))].
\end{aligned}$$

The first inequality follows from the definition of supermodular order and the fact that $\phi(a + b)$ is submodular in (a, b) because $\phi(\cdot)$ is concave. The second inequality follows from the Jensen's inequality. We have also used the relation $\mathbb{E}[\psi(\hat{Y}_1)] = \mathbb{E}[Y(x_2)] - \mathbb{E}[Y(x_1)] = \mu(x_2) - \mu(x_1)$. Then,

$$\frac{\mathbb{E}[\phi(Y(x_2))] - \mathbb{E}[\phi(Y(x_1))]}{x_2 - x_1} \leq \frac{\mathbb{E}[\phi(x_2 - x_1 + Y(x_1))] - \mathbb{E}[\phi(Y(x_1))]}{x_2 - x_1}.$$

Taking the limit as $x_2 \to x_1$, we obtain the result. □

Lemma 7 *For a differentiable $\phi(\cdot)$ and any $\epsilon > 0$, if $Y(x), x \in \mathcal{X} = [\underline{x}, \overline{x}] \subseteq [0, \infty)$, with $\mathbb{E}[Y(x)] = x$ and $Y(0) = 0$ satisfies*

(i) $\frac{1}{\Delta}\mathbb{E}[Y(\Delta)\mathbb{I}_{\{Y(\Delta) > \epsilon\}}] \to 0$ *as* $\Delta \downarrow 0$, *and*

(ii) $\frac{1}{\Delta}\mathbb{E}[(\phi(Y(\Delta)) - \phi(0))\mathbb{I}_{\{Y(\Delta) > \epsilon\}}] \to 0$ *as* $\Delta \downarrow 0$,

then

$$\lim_{\Delta\downarrow 0}\frac{\mathbb{E}[\phi(Y(\Delta))]-\phi(0)}{\Delta}=\frac{\mathrm{d}\phi(a)}{\mathrm{d}a}\bigg|_{a\downarrow 0}. \tag{35}$$

Moreover, if $\mathrm{d}\phi(y)/\mathrm{d}y$ *is bounded, then condition (i) implies condition (ii).*

Proof of Lemma 7. We show the result for the case where $\mathrm{d}\phi(y)/\mathrm{d}y|_{y=0}$ is finite. Usual limiting argument can be used to treat the general case. We have

$$\begin{aligned}\frac{\mathbb{E}[\phi(Y(\Delta))]-\mathbb{E}[\phi(Y(0))]}{\Delta} &= \frac{1}{\Delta}\int_0^{\epsilon}(\phi(y)-\phi(0))\mathrm{d}F_{Y(\Delta)}(y)+\frac{1}{\Delta}\int_{\epsilon}^{\infty}(\phi(y)-\phi(0))\mathrm{d}F_{Y(\Delta)}(y)\\ &= \int_0^{\epsilon}\frac{\phi(y)-\phi(0)}{y-0}\frac{y}{\Delta}\mathrm{d}F_{Y(\Delta)}(y)+\frac{1}{\Delta}\mathbb{E}[(\phi(Y(\Delta))-\phi(0))\mathbb{I}_{\{Y(\Delta)>\epsilon\}}]. \end{aligned}\tag{36}$$

Note that $\int_0^{\epsilon}\frac{y}{\Delta}\mathrm{d}F_{Y(\Delta)}(y)=\frac{\mathbb{E}[Y(\Delta)]}{\Delta}-\frac{1}{\Delta}\mathbb{E}[Y(\Delta)\mathbb{I}_{Y(\Delta)>\epsilon}]$ goes to 1 as $\Delta\downarrow 0$ in view of condition (i). Therefore, taking the limit on both sides of (36) as $\Delta\downarrow 0$ gives (35).

Finally, we note that when $\mathrm{d}\phi(y)/\mathrm{d}y$ is bounded, we must have $\beta_1 y\le\phi(y)-\phi(0)\le\beta_2 y$ for any a and some constants β_1 and β_2. Therefore,

$$\frac{1}{\Delta}\mathbb{E}[\beta_1 Y(\Delta)\mathbb{I}_{\{Y(\Delta)>\epsilon\}}]\le\frac{1}{\Delta}\mathbb{E}[(\phi(Y(\Delta))-\phi(0))\mathbb{I}_{\{Y(\Delta)>\epsilon\}}]\le\frac{1}{\Delta}\mathbb{E}[\beta_2 Y(\Delta)\mathbb{I}_{\{Y(\Delta)>\epsilon\}}]$$

When condition (i) hold, the limit of the above as $\Delta\downarrow 0$ gives rise to condition (ii). □

Proof of Theorem 4. Let $(\mu^*_{S1}(\cdot),\cdots,\mu^*_{Sn}(\cdot))$ be a maximizer of (14). Because the mean demand is finite and $\lim_{x\to\infty}H(x)=\infty$, it is easy to see that $\mu^*_{Sj}(x)=0$ for a large enough x. Define

$$\phi(y,x)=\rho\mathbb{E}\left[V_{t+1}\left(y+\sum_{k\in\mathcal{N}\setminus\{j\}}S_k(\mu^*_{Sk},x)-D\right)\right].$$

By Theorem 2, $\phi(y,x)$ is concave in y and $\mathbb{E}[\phi(x+S_j(\mu^*_{Sj}(x_0)),x_0)]|_{x0=x}$ is concave in x. For given $\mu^*_{Si}(x)$, $i\neq j$, if there can be multiple solutions of $\mu^*_{Sj}(x)$, we choose the smallest one. Suppose there exist $x_1<x_2<x_3$ such that $\mu^*_{Sj}(x)=0$ for $x\in[x_1,x_2]$ and $\mu^*_{Sj}(x_3)>0$. Because $\mu^*_{Sj}(x_3)>0$ is optimal, we must have

$$\mathbb{E}[\phi(x_3+S_j(\mu^*_{Sj}(x_3)+\Delta),x_3)]-c_j(\mu^*_{Sj}(x_3)+\Delta)<\mathbb{E}[\phi(x_3+S_j(\mu^*_{Sj}(x_3)),x_3)]-c_j\mu^*_{Sj}(x_3). \tag{37}$$

Suppose

$$\frac{\mathbb{E}[\phi(x_3+\Delta+S_j(\mu^*_{Sj}(x_3)),x_3)]-\mathbb{E}[\phi(x_3+S_j(\mu^*_{Sj}(x_3)),x_3)]}{\Delta}\le c_j, \tag{38}$$

then by Lemma 6, we must have

$$\frac{\mathbb{E}[\phi(x_3 + S_j(\mu^*_{Sj}(x_3) + \Delta), x_3)] - \mathbb{E}[\phi(x_3 + S_j(\mu^*_{Sj}(x_3)), x_3)]}{\Delta}$$

$$\leq \frac{\mathbb{E}[\phi(x_3 + \Delta + S_j(\mu^*_{Sj}(x_3)), x_3)] - \mathbb{E}[\phi(x_3 + S_j(\mu^*_{Sj}(x_3)), x_3)]}{\Delta} \leq c_j,$$

which contradicts (37). Therefore, (38) cannot hold and we must have

$$\frac{\mathbb{E}[\phi(x_3 + \Delta + S_j(\mu^*_{Sj}(x_3)), x_3)] - \mathbb{E}[\phi(x_3 + S_j(\mu^*_{Si}(x_3)), x_3)]}{\Delta} > c_j. \tag{39}$$

We also note that

$$\frac{V_t(x+\Delta) - V_t(x)}{\Delta} \geq \frac{\phi(x + \Delta + S_j(\mu^*_{Sj}(x)), x) - \mathbb{E}[\phi(x + S_j(\mu^*_{Si}(x)), x)]}{\Delta},$$

because the solution $(\mu^*_{S1}(x), \cdots, \mu^*_{S1}(x))$ can be suboptimal when the inventory level is $x + \Delta$. Thus, (39) suggests that the slope or, more specifically, the right derivative of $V_t(x)$ is strictly above c_j for $x < x_3$, as V_t is concave.

Now note that a concave function can have at most a countable number of nondifferentiable points (Rockafellar 1970, Theorem 25.3). Then we can pick an $\hat{x} \in [x_1, x_2]$ at which V_t is differentiable at $\hat{x}$. By the envelope theorem, we have

$$\begin{aligned}\left.\frac{\mathrm{d}V_t(x)}{\mathrm{d}x}\right|_{x=\hat{x}} &= \left.\frac{\partial \mathbb{E}[\phi(x + S_j(\mu^*_{Sj}(\hat{x})), \hat{x})]}{\partial x}\right|_{x=\hat{x}} \\ &= \left.\frac{\mathrm{d}\phi(x, \hat{x})}{\mathrm{d}x}\right|_{x=\hat{x}} = \lim_{\Delta \downarrow 0} \frac{\mathbb{E}[\phi(\hat{x} + S_j(\Delta), \hat{x})] - \mathbb{E}[\phi(\hat{x} + S_j(0), \hat{x})]}{\Delta} \leq c_j.\end{aligned}$$

The last inequality follows from the fact that $\mu^*_{Sj}(x) = 0$ for any $x \in [x_1, x_2]$. The above relation contradicts the facts that $V_t(x)$ is concave and that the right derivative of $V_t(x)$ is strictly larger than c_j for $x < x_3$. Hence, we cannot have $\mu^*_{Sj}(x) = 0$ for $x \in [x_1, x_2]$ and $\mu^*_{Sj}(x_3) > 0$ for $x_1 < x_2 < x_3$. In other words, there exists a threshold $\bar{x}_{t,j}$ such that $\mu^*_{Sj}(x) = 0$ for any $x \geq \bar{x}_{t,j}$, and $\mu^*_{Sj}(x) > 0$ for $x < \bar{x}_{t,j}$ except for a countable set of points.

We further note that when $V_t(\cdot)$ is differentiable, $\phi(\cdot)$ is differentiable. The above analysis suggests that $\mathrm{d}V_t(x)/\mathrm{d}x > c_j$ when $\mu^*_{Sj}(x) > 0$ and $\mathrm{d}V_t(x)/\mathrm{d}x \leq c_j$ when $\mu^*_{Sj}(x) = 0$. Therefore, a threshold policy must be optimal, i.e., $\Xi_{t,j} = \emptyset$ and $\bar{x}_{t,j}$ is higher when c_j is lower. □

Proof of Theorem 5. For part (i), $\varphi(q, z) = qZ$ is almost surely linear in q and supermodular in (q, z). Thus $\{S(\mu_S), 0 \leq \mu_S \leq \overline{\mu}_S\} \in SL(mp) \cup SI(disp)$. For part (ii), stochastic linearity in midpoint follows immediately from Lemma 1. Alternatively, it is easy to verify that $\varphi(q, z) = \min\{q, Z\}$ satisfies the conditions in Theorem 1. It is also straightforward to check that $\varphi(q, z)$ is supermodular and thus $S(\mu_S)$ is stochastically increasing in the dispersive order.

Next we examine part (iii). Let $\varphi(q,z) = \frac{qz}{q+\alpha z^\kappa}$ and $\tilde{\varphi}(a,z) = \varphi(1/a,z)$. It is easy to check that $\varphi(q,z)$ is concave in q and $\tilde{\varphi}(a,z)$ is concave in a. We have

$$\frac{\partial^3 \tilde{\varphi}(a,z)}{\partial z \partial a^2} = \frac{2\alpha^2 z^{2\kappa}(1+2\kappa+\alpha(1-\kappa)az^\kappa)}{(1+\alpha a z^\kappa)^4} > 0$$

for any $\kappa \le 1$. Therefore, for any $q_1 < q_2$ with $\bar{q} = 2q_1q_2/(q_1+q_2) \le (q_1+q_2)/2$,

$$2\varphi(\bar{q},z) - (\varphi(q_1,z)+\varphi(q_2,z)) = 2\tilde{\varphi}(1/\bar{q},z) - (\tilde{\varphi}(1/q_1,z)+\tilde{\varphi}(1/q_2,z))$$

is decreasing in z. Furthermore, for $\kappa \le 1$,

$$\frac{\partial^2 \varphi}{\partial q \partial z} = \frac{az^\kappa(q(1+\kappa)+\alpha(1-\kappa)z^\kappa)}{(q+\alpha z^\kappa)^2} > 0. \tag{40}$$

For any q satisfying $q_1 < q < q_2$, we have

$$2\varphi(q,z) - (\varphi(q_1,z)+\varphi(q_2,z)) = (2\varphi(\bar{q},z) - (\varphi(q_1,z)+\varphi(q_2,z))) + 2(\varphi(q,z) - \varphi(\bar{q},z)).$$

When $q \le \bar{q}$, the right-hand side is decreasing in z. When $q > \bar{q}$, the right-hand side is positive. To see that, we note that the first term is positive because $\tilde{\varphi}(a,z)$ is concave in a and the second term is positive because $\varphi(q,z)$ is increasing in q. Hence, $\varphi(q,z)$ satisfies the single-crossing property stated in Theorem 1. Together with (40), we conclude $\{S(\mu_S), 0 \le \mu_S \le \overline{\mu}_S\} \in SL(mp) \cup SI(disp)$.

To see part (iv), let $\varphi(q,z) = \frac{q}{q+z}k, q \ge 0, z \ge 0$. For any fixed $q_1 < q < q_2$, we have

$$\begin{aligned}
\delta(q,z) &= 2\varphi(q,z) - (\varphi(q_1,z)+\varphi(q_2,z)) \\
&= \frac{2q}{q+z}k - \frac{q_1}{q_1+z}k - \frac{q_2}{q_2+z}k \\
&= \frac{(q(q_1+q_2)-2q_1q_2)z + (2q-q_1-q_2)z^2}{(q+z)(q_1+z)(q_2+z)}k \\
&= \frac{(q_1+q_2)(q-\check{q})z - 2(\bar{q}-q)z^2}{(q+z)(q_1+z)(q_2+z)},
\end{aligned}$$

where $\check{q} = 2q_1q_2/(q_1+q_2)$ and $\bar{q} = (q_1+q_2)/2$. Note that if $q \le \check{q} \le \bar{q}$, $\delta(y,z) \le 0$ for any $z \ge 0$. Otherwise, we have $\delta(q,z) \ge (\le) 0$ for $z \le (\ge) \check{z} = \bar{q}\frac{q-\check{q}}{\bar{q}-q} > 0$. Hence, $\varphi(q,z)$ satisfies the single-crossing property stated in Theorem 1. □

Proof of Theorem 6. Take μ_1, μ_2, and $\bar{\mu} = (\mu_1+\mu_2)/2$. Define $s(\mu) = \sigma_D(p(\mu))$. Let $D_1 =^d \mu_1 A + s(\mu_1)B$ and $D_2 =^d \mu_2 A + s(\mu_2)B$ so that $(D_1+D_2)/2 = \bar{\mu}A + B(s(\mu_1)+s(\mu_2))/2$. Because $s(\cdot)$ is convex, we must have $(s(\mu_1)+s(\mu_2))/2 \ge s(\bar{\mu})$. We note that

$$[\frac{D_1+D_2}{2}|A=a] =^d \bar{\mu}a + \frac{s(\mu_1)+s(\mu_2)}{2}B \le_{cv} \bar{\mu}a + s(\bar{\mu})B =^d [D(\bar{\mu})|A=a].$$

This inequality follows directly from the fact that $\alpha B \leq_{cv} B$ for any $\alpha \geq 1$. Applying Theorem 3.A.12(b) of Shaked and Shanthikumar (2006), we conclude that

$$\frac{D_1 + D_2}{2} \leq_{cv} D(\bar{\mu}).$$

Thus, we conclude the proof. □

Proof of Theorem 7. We first show that $\hat{q}_1(q, \mu)$ is concave. Take (q_a, μ_a) and (q_b, μ_b) such that $\mu_a \geq q_a \geq 0$ and $\mu_b \geq q_b \geq 0$, and define $\bar{q} = (q_a + q_b)/2$ and $\bar{\mu} = (\mu_a + \mu_b)/2$. Let $q_{1,a} = \hat{q}_1(q_a, \mu_a)$, $q_{1,b} = \hat{q}_1(q_b, \mu_b)$, and $\bar{q}_1 = (q_{1,a} + q_{1,b})/2$. Because $\hat{\mu}(q_1, q)$ is jointly convex in (q_1, q), we have $\check{\mu} \equiv \hat{\mu}(\bar{q}_1, \bar{q}) \leq (\mu_a + \mu_b)/2 = \bar{\mu}$ because $\mu_a = \hat{\mu}(q_{1,a}, q_a)$ and $\mu_b = \hat{\mu}(q_{1,b}, q_b)$ by definition of $q_{1,a}$ and $q_{1,b}$. Further note that $\hat{\mu}(q_1, q)$ is increasing in q_2 and thus $\hat{q}_1(q, \mu)$ is increasing in μ. Therefore,

$$\hat{q}_1(q_a, \mu_a) + \hat{q}_1(q_b, \mu_b) - 2\hat{q}_1(\bar{q}, \bar{\mu}) \leq q_{1,a} + q_{1,b} - 2\hat{q}_1(\bar{q}, \check{\mu}) = 2\bar{q}_1 - 2\bar{q}_1 = 0.$$

We deduce that $\hat{q}_1(q, \mu)$ is jointly concave in (q, μ).

Now let $\phi(q, \mu, z) = \max\{q, \hat{q}_1(q, \mu)z\}$. We show that the sign of

$$\phi(q_a, \mu_a, z) + \phi(q_b, \mu_b, z) - 2\phi(\bar{q}, \bar{\mu}, z) = \max\{q_a, q_{1,a}z\} + \max\{q_b, q_{1,b}z\} - 2\max\{\bar{q}, \hat{q}_1(\bar{q}, \bar{\mu})z\} \quad (41)$$

changes at most once and the change is from negative to positive. Without loss of generality, we assume that $q_a < q_b$. When $q_{1,a} > 0$ and $q_{1,b} > 0$, let $z_a = q_a/q_{1,a}$ and $z_b = q_b/q_{1,b}$. We have two cases to consider:

Case 1: If $z_b < z_a$, we have

$$\phi(q_a, \mu_a, z) + \phi(q_b, \mu_b, z) = \begin{cases} 2\bar{q}, & z < z_b, \\ q_a + q_{1,b}z, & z_b \leq z \leq z_a, \\ 2\bar{q}_1 z, & z > z_a. \end{cases}$$

From our previous argument, $\bar{q}_1 \leq \hat{q}_1(\bar{q}, \bar{\mu})$. Thus, as z increases, $2\max\{\bar{q}, \hat{q}_1(\bar{q}, \bar{\mu})z\}$ can across $\phi(q_a, \mu_a, z) + \phi(q_b, \mu_b, z)$ at most once and the crossing is from the above.

Case 2: If $z_a \leq z_b$, we have

$$\phi(q_a, \mu_a, z) + \phi(q_b, \mu_b, z) = \begin{cases} 2\bar{q}, & z < z_a, \\ q_{1,a}z + q_b, & z_a \leq z \leq z_b, \\ 2\bar{q}_1 z, & z > z_b. \end{cases}$$

From our previous argument, $\bar{q}_1 \leq \hat{q}_1(\bar{q}, \bar{\mu})$. Thus, as z increases, $2\max\{\bar{q}, \hat{q}_1(\bar{q}, \bar{\mu})z\}$ can across $\phi(q_a, \mu_a, z) + \phi(q_b, \mu_b, z)$ at most once and the crossing is from the above.

If $q_{1,a} = 0$ or $q_{1,b} = 0$ then we set $z_a = \infty$ or $z_b = \infty$, respectively, in the above analysis. For the case where $q_{1,a} = q_{1,b} = 0$, one can apply the above argument by setting $z_a = z_b = \infty$.

Taking all the cases together, we deduce that the sign of (41) changes at most once and the change is from negative to positive. Hence, from the variation diminishing property under integration, we obtain

$$\int_0^{z_0} (\phi(q_a, \mu_a, z) + \phi(q_b, \mu_b, z))\mathrm{d}F_Z(z) - 2\int_0^{z_0} \phi(\bar{q}, \bar{\mu}, z)\mathrm{d}F_Z(z) \geq 0, \forall z_0 \geq 0.$$

Also note that $\mathbb{E}[\phi(q_a, \mu_a, Z) + \phi(q_b, \mu_b, Z)] = \mu_a + \mu_b = \bar{\mu} = 2\mathbb{E}[\phi(\bar{q}, \bar{\mu}, Z)]$. Thus, by Theorem 3.A.5. of Shaked and Shanthikumar (2006), we deduce

$$\phi(q_a, \mu_a, Z) + \phi(q_b, \mu_b, Z) \leq_{cv} 2\phi(\bar{q}, \bar{\mu}, Z).$$

We conclude that $\{S(q, \mu), q \geq 0, \mu \geq 0\} \in SL(mp)$. □

Proof of Lemma 5. It is easy to see that (19) hold by the definitions of g^i and h^i because $\mathbb{E}[\phi^i(q^i(\mu), z)] = \mu$.

We first consider $\{Y^{min}(q), q \geq 0\}$. Let $\hat{q}^{min} = q^{min}(\bar{\mu}^{min})$. Note that $q_a < \hat{q}^{min} < \bar{q} = (q_a + q_b)/2 < q_b$. Then, it can be easily verified that

$$g^{min}(z) \leq h^{min}(z) \text{ for } z < 2\hat{q}^{min} - q_a,$$
$$g^{min}(z) \geq h^{min}(z) \text{ for } z > 2\hat{q}^{min} - q_a.$$

In other words, the sign of $g^{min}(z) - h^{min}(z)$ changes from negative to positive at most once. From the variation diminishing property under integration, we obtain (18).

We now consider $\{Y^{max}(q), q \geq 0\}$. Let $\hat{q}^{max} = q^{max}(\bar{\mu}^{max})$. Note that $q_a < \bar{q} < \hat{q}^{max} < q_b$. It is easy to that

$$g^{max}(z) \leq h^{max}(z) \text{ for } z < q_a + 2(\hat{q}^{max} - \bar{q}),$$
$$g^{max}(z) \geq h^{max}(z) \text{ for } z > q_a + 2(\hat{q}^{max} - \bar{q}).$$

In other words, the sign of $g^{max}(z) - h^{max}(z)$ changes from negative to positive at most once. From the variation diminishing property under integration, we obtain (18).

Finally, we note that in both cases, we have shown that $\phi^i(q, z), i \in \{min, max\}$ satisfies the single-crossing property defined in Lemma 3. Thus, $\{Y^{min}(q^{min}(\mu)), 0 \leq \mu \leq \mu_Z\} \in SL(mp)$ and $\{Y^{max}(\hat{q}^{max}(\mu)), \mu \geq \mu_Z\} \in SL(mp)$. □

Proof of Theorem 8. Take $(q_a^{min}, q_a^{max}) \in \mathcal{Q}$, $(q_b^{min}, q_b^{max}) \in \mathcal{Q}$ with $q_a^{min} < q_b^{min}$ and $q_a^{max} < q_b^{max}$. Let $\mu_a^{min} = \mu^{min}(q_a^{min})$, $\mu_b^{min} = \mu^{min}(q_b^{min})$, $\mu_a^{max} = \mu^{max}(q_a^{max})$, $\mu_b^{max} = \mu^{max}(q_b^{max})$, $\bar{\mu}^{min} = (\mu^{min}(q_a^{min}) + \mu^{min}(q_b^{min}))/2$ and $\bar{\mu}^{max} = (\mu^{max}(q_a^{max}) + \mu^{max}(q_b^{max}))/2$. Also, define

$$\begin{aligned} g^{min}(z) &= \frac{1}{2}(\phi^{min}(q_a^{min}, z) + \phi^{min}(q_b^{min}, z)), \\ h^{min}(z) &= \phi^{min}(q^{min}(\bar{\mu}^{min})), \\ g^{max}(z) &= \frac{1}{2}(\phi^{max}(q_a^{max}, z) + \phi^{max}(q_b^{max}, z), \\ h^{max}(z) &= \phi^{max}(q^{max}(\bar{\mu}^{max})). \end{aligned}$$

Now let

$$\begin{aligned} g(z) &= g^{min}(z) + g^{max}(z) - z, \\ h(z) &= h^{min}(z) + h^{max}(z) - z. \end{aligned}$$

Then from (18) and (19) in Lemma 5, we deduce that

$$\begin{aligned} &\int_{z_0}^{\infty} (g(z) - h(z)) \mathrm{d}F_Z(z) \geq 0, \ \forall z_0 \geq 0, \\ &\int_{0}^{\infty} (g(z) - h(z)) \mathrm{d}F_Z(z) = 0. \end{aligned}$$

By definition, both $g(z)$ and $h(z)$ are increasing in z. By Theorem 3.A.5. of Shaked and Shanthikumar (2006), the above relations imply that $g(Z) \geq_{cv} h(Z)$. Now note that

$$S(\mu_1, \mu_2) = \max\{\hat{q}_2(\mu_2 + \mu_Z), Z\} + \min\{\hat{q}_1(\mu_1), Z\} - Z, \ (\mu_1, \mu_2) \in \mathcal{M}.$$

It is easy to see that $g(Z) = S(\mu_a^{min}, \mu_a^{max}) + S(\mu_b^{min}, \mu_b^{max})$ and $h(Z) = S(\bar{\mu}^{min}, \bar{\mu}^{max})$. Therefore, by the definition of stochastic mid-point linearity, we conclude that $\{S(\mu_1, \mu_2), (\mu_1, \mu_2) \in \mathcal{M}\} \in SL(mp)$. □

Proof of Theorem 9. We first show $\hat{q}_2(q, \mu)$ is convex. Take (q_a, μ_a) and (q_b, μ_b), and define $\bar{q} = (q_a + q_b)/2$ and $\bar{\mu} = (\mu_a + \mu_b)/2$. Let $q_{2,a} = \hat{q}_2(q_a, \mu_a)$, $q_{2,b} = \hat{q}_2(q_b, \mu_b)$, and $\bar{q}_2 = (q_{2,a} + q_{2,b})/2$. Because $\hat{\mu}(q, q_2)$ is jointly concave in (q, q_2), we have $\check{\mu} \equiv \hat{\mu}(\bar{q}, \bar{q}_2) \geq (\mu_a + \mu_b)/2 = \bar{\mu}$. Further note that $\hat{\mu}(q, q_2)$ is increasing in q_2 and thus $\hat{q}_2(q, \mu)$ is increasing in μ. Therefore,

$$\hat{q}_2(q_a, \mu_a) + \hat{q}_2(q_b, \mu_b) - 2\hat{q}_2(\bar{q}, \bar{\mu}) \geq q_{2,a} + q_{2,b} - 2\hat{q}_2(\bar{q}, \check{\mu}) = 2\bar{q}_2 - 2\bar{q}_2 = 0.$$

We deduce that $\hat{q}_2(q, \mu)$ is jointly convex in (q, μ).

Now let $\phi(q,\mu,z)=\min\{\hat{q}_2(q,\mu),qz\}$. We show the sign of

$$\phi(q_a,\mu_a,z)+\phi(q_b,\mu_b,z)-2\phi(\bar{q},\bar{\mu},z)=\min\{q_{2,a},q_az\}+\min\{q_{2,b},q_bz\}-2\min\{\hat{q}_2(\bar{q},\bar{\mu}),\bar{q}z\} \quad (42)$$

changes at most once and the change is from negative to positive at once. Without loss of generality, we assume that $q_a<q_b$. Let $z_a=q_{2,a}/q_a$ and $z_b=q_{2,b}/q_b$. We have two cases to consider:

Case 1: If $z_b<z_a$, we have

$$\phi(q_a,\mu_a,z)+\phi(q_b,\mu_b,z)=\begin{cases}2\bar{q}z, & z<z_b,\\ q_az+q_{2,b}, & z_b\le z\le z_a,\\ 2\bar{q}_2, & z>z_a.\end{cases}$$

From our previous argument, $\bar{q}_2\ge\hat{q}_2(\bar{q},\bar{\mu})$. Thus, as z increases, $2\min\{\hat{q}_2(\bar{q},\bar{\mu}),\bar{q}z\}$ can across $\phi(q_a,\mu_a,z)+\phi(q_b,\mu_b,z)$ at most once and the crossing is from the above.

Case 2: If $z_a\le z_b$, we have

$$\phi(q_a,\mu_a,z)+\phi(q_b,\mu_b,z)=\begin{cases}2\bar{q}z, & z<z_a,\\ q_{2,a}+q_bz, & z_a\le z\le z_b,\\ 2\bar{q}_2, & z>z_b.\end{cases}$$

From our previous argument, $\bar{q}_2\ge\hat{q}_2(\bar{q},\bar{\mu})$. Thus, as z increases, $2\min\{\hat{q}_2(\bar{q},\bar{\mu}),\bar{q}z\}$ can across $\phi(q_a,\mu_a,z)+\phi(q_b,\mu_b,z)$ at most once and the crossing is from the above.

Taking the above two cases together, we deduce that the sign of (42) changes at most once and the change is from negative to positive. Hence, from the variation diminishing property under integration, we obtain

$$\int_0^{z_0}(\phi(q_a,\mu_a,z)+\phi(q_b,\mu_b,z))\mathrm{d}F_Z(z)-2\int_0^{z_0}\phi(\bar{q},\bar{\mu},z)\mathrm{d}F_Z(z)\ge 0,\forall z_0\ge 0.$$

Also note that $\mathbb{E}[\phi(q_a,\mu_a,Z)+\phi(q_b,\mu_b,Z)]=\mu_a+\mu_b=\bar{\mu}=2\mathbb{E}[\phi(\bar{q},\bar{\mu},Z)]$. Thus, by Theorem 3.A.5. of Shaked and Shanthikumar (2006), we deduce

$$\phi(q_a,\mu_a,Z)+\phi(q_b,\mu_b,Z)\le_{cv}2\phi(\bar{q},\bar{\mu},Z).$$

We conclude that $\{S(q,\mu),q\ge 0,\mu\ge 0\}\in SL(mp)$. □

C Results Used from Shaked and Shanthikumar (2006)

Theorem 3.A.44. *Let Z_1 and Z_2 be two random variables with equal finite means, density functions f_{Z_1} and f_{Z_2}, distribution functions F_{Z_1} and F_{Z_2}, and survival functions $\bar{F}_{Z_1}$ and $\bar{F}_{Z_2}$, respec-*

tively. Then $Z_1 \leq_{cx} Z_2$ if any of the following conditions hold:

$$S^-(f_{Z_2} - f_{Z_1}) = 2 \quad \textit{and the sign sequence is } +, -, +,$$
$$S^-(\bar{F}_{Z_1} - \bar{F}_{Z_2}) = 1 \quad \textit{and the sign sequence is } +, -,$$
$$S^-(F_{Z_2} - F_{Z_1}) = 1 \quad \textit{and the sign sequence is } +, -,$$

where $S^-(g)$ denotes the number of sign changes of a function g.

Condition (3.B.14). *For continuous random variables Z_1 and Z_2, we have that $Z_1 \leq_{disp} Z_2$ if, and only if*

$$Z_2 =_{st} Z_1 + \psi(Z_1)$$

for some increasing function ψ.

Theorem 3.A.5. *Let Z_1 and Z_2 be two random variables with distributions F_{Z_1} and F_{Z_2}, respectively, and with equal finite means. Then each of the following statements is a necessary and sufficient condition for $Z_1 \leq_{cx} Z_2$:*

$$\int_0^\alpha F_{Z_1}^{inv}(u)\mathrm{d}u \geq \int_0^\alpha F_{Z_2}^{inv}(u) \quad \textit{for all } \alpha \in [0, 1],$$

and

$$\int_\alpha^1 F_{Z_1}^{inv}(u)\mathrm{d}u \geq \int_\alpha^1 F_{Z_2}^{inv}(u) \quad \textit{for all } \alpha \in [0, 1].$$

Theorem 3.A.12(b). *Let Z_1, Z_2, and Θ be random variables such that $[Z_1|\Theta = \theta] \leq_{cx} [Z_2|\Theta = \theta]$ for all θ in the support of Θ. Then $Z_1 \leq_{cx} Z_2$. That is, the convex order is closed under mixture.*