Extensions and Proofs

EC.1. General Waiting Time Costs

In this section, we discuss a general setting that incorporates both server-based and appointment-based cases to model the waiting costs. We show that the models in Section 2 and solution methods in Section 3 can adapt for this general setting. For presentation brevity, we focus on DR expectation models and the adaptation of methods for DR CVaR models in EC.2 is similar.

To be general, we model the waiting time cost for each appointment i as $c_i^{\mathrm{s}} w_i + c_i^{\mathrm{a}} w_i q_i = (c_i^{\mathrm{s}} + c_i^{\mathrm{a}} q_i) w_i$, where c_i^{s} represents the server-based waiting time cost and c_i^{a} represents the appointment-based waiting time cost. Note that this setting applies for server-based and appointment-based cases, because we can set $c_i^{\mathrm{s}} := 0$ or $c_i^{\mathrm{a}} := 0$ if the corresponding cost does not apply. Accordingly, the general DR expectation model can be formulated as

$$\min_{x \in X} \sup_{\mathbb{P}_{q,s} \in \mathcal{F}(D,\mu,\nu)} \mathbb{E}_{\mathbb{P}_{q,s}} \left[Q^{\mathrm{G}}(x,q,s) \right], \tag{EC.1}$$

where $Q^{\mathrm{G}}(x,q,s)$ represents the cost function of the waiting, idleness, and overtime under the general setting. Replacing $c_i^{\mathrm{w}} w_i$ with $(c_i^{\mathrm{s}} + c_i^{\mathrm{a}} q_i) w_i$ in (11), we formulate the dual of $Q^{\mathrm{G}}(x,q,s)$ as

$$\begin{aligned}
Q^{\mathrm{G}}(x,q,s) \ = \ \max_{y} \ & \sum_{i=1}^{n} (q_i s_i - x_i) y_i && \text{(EC.2a)}\\
\text{s.t.} \ & y_{i-1} - y_i \le c_i^{\mathrm{s}} + c_i^{\mathrm{a}} q_i \quad \forall i = 2, \ldots, n && \text{(EC.2b)}\\
& -y_i \le c_i^{\mathrm{u}} \quad \forall i = 1, \ldots, n && \text{(EC.2c)}\\
& y_n \le c^{\mathrm{o}}, && \text{(EC.2d)}
\end{aligned}$$

and we let polyhedron $Y^{\mathrm{G}} := \{y : \text{(EC.2b)–(EC.2d)}\}$ represent the feasible region of variable y. As model (EC.2) is a linear program in variables y, there exists an optimal solution to (EC.2) that resides at an extreme point of Y^{G}. It can be observed (see, e.g., Zangwill (1966, 1969), Mak et al. (2015)) that any extreme point $\hat{y}$ of Y^{G} satisfy (i) either $\hat{y}_n = -c_n^{\mathrm{u}}$ or $\hat{y}_n = c^{\mathrm{o}}$, and (ii) for all $i = 1, \ldots, n-1$, dual constraint $\hat{y}_{i+1} + c_{i+1}^{\mathrm{s}} + c_{i+1}^{\mathrm{a}} q_{i+1} \ge \hat{y}_i \ge -c_i^{\mathrm{u}}$ is binding at either the lower bound or the upper bound. Similar to the analysis in Section 3.2, we define binary variables t_{kj} for all $1 \le k \le j \le n+1$ to represent extreme points $\hat{y}$, such that $t_{kj} = 1$ if $\hat{y}_j = -c_j^{\mathrm{u}}$ and $\hat{y}_i = \hat{y}_{i+1} + c_{i+1}^{\mathrm{s}} + c_{i+1}^{\mathrm{a}} q_{i+1}$, $\forall i = k, \ldots, j-1$. It follows that

$$\hat{y}_i = \pi_{ij}^{\mathrm{G}} := \begin{cases} -c_j^{\mathrm{u}} + \sum_{\ell=i+1}^{j} (c_\ell^{\mathrm{s}} + c_\ell^{\mathrm{a}} q_\ell) & 1 \le i \le j \le n, \\ c^{\mathrm{o}} + \sum_{\ell=i+1}^{n} (c_\ell^{\mathrm{s}} + c_\ell^{\mathrm{a}} q_\ell) & 1 \le i \le n, \ j = n+1, \end{cases} \tag{EC.3}$$

where $\hat{y}_{n+1} = \pi_{n+1,n+1}^{\mathrm{G}} := 0$. Based on this binary representation, we can rewrite $Q^{\mathrm{G}}(x,q,s)$ as

$$Q^{\mathrm{G}}(x,q,s) \ = \ \max_{t} \ \sum_{k=1}^{n+1} \sum_{j=k}^{n+1} \left(\sum_{i=k}^{j} (q_i s_i - x_i) \pi_{ij}^{\mathrm{G}} \right) t_{kj}$$

$$\begin{aligned}
&\equiv \sum_{k=1}^{n}\sum_{j=k}^{n}\sum_{i=k}^{j}\left[(q_is_i-x_i)\left(-c^{\mathrm{u}}_j+\sum_{\ell=i+1}^{j}(c^{\mathrm{s}}_\ell+c^{\mathrm{a}}_\ell q_\ell)\right)\right]t_{kj}+\\
&\quad \sum_{k=1}^{n}\sum_{i=k}^{n+1}\left[(q_is_i-x_i)\left(c^{\mathrm{o}}+\sum_{\ell=i+1}^{n}(c^{\mathrm{s}}_\ell+c^{\mathrm{a}}_\ell q_\ell)\right)\right]t_{k(n+1)} && \text{(EC.4a)}\\
\text{s.t.}\quad & \sum_{k=1}^{i}\sum_{j=i}^{n+1}t_{kj}=1 \quad \forall i=1,\dots,n+1 && \text{(EC.4b)}\\
& t_{kj}\in\{0,1\}, \quad \forall 1\le k\le j\le n+1. && \text{(EC.4c)}
\end{aligned}$$

Note that the objective function (EC.4a) contains multilinear terms $q_is_it_{kj}$ and $q_iq_\ell s_it_{kj}$ with binary variables q_i, q_ℓ, and t_{kj}, and continuous variables s_i. To linearize formulation (EC.4), we define $p_{ikj}\equiv q_it_{kj}$, $o_{ikj}\equiv q_is_it_{kj}$, and $r_{i\ell kj}=q_iq_\ell s_it_{kj}$ for all $1\le k\le i\le j\le n+1$ and $i+1\le \ell\le j$. We then linearize the multilinear terms by applying McCormick inequalities (18a)–(18b) for variables p_{ikj}, (18c)–(18d) for variables o_{ikj}, and (EC.5a)–(EC.5b) as follows for variables $r_{i\ell kj}$.

$$r_{i\ell kj}\ge 0,\ r_{i\ell kj}-q_\ell s_i\le 0, \tag{EC.5a}$$

$$r_{i\ell kj}-o_{ikj}\le 0,\ r_{i\ell kj}-o_{ikj}+s^{\mathrm{U}}_i(1-q_\ell)\ge 0. \tag{EC.5b}$$

If follows that $Q^{\mathrm{G}}(x,q,s)$ equals to the optimal objective value of the following MILP:

$$\begin{aligned}
\max_{t,p,o,r}\ & \sum_{k=1}^{n}\sum_{j=k}^{n}\sum_{i=k}^{j}\left[\left(-c^{\mathrm{u}}_j+\sum_{\ell=i+1}^{j}c^{\mathrm{s}}_\ell\right)o_{ikj}+\left(c^{\mathrm{u}}_j-\sum_{\ell=i+1}^{j}c^{\mathrm{s}}_\ell\right)x_it_{kj}+\sum_{\ell=i+1}^{j}c^{\mathrm{a}}_\ell r_{i\ell kj}-\sum_{\ell=i+1}^{j}c^{\mathrm{a}}_\ell x_ip_{\ell kj}\right]+\\
& \sum_{k=1}^{n}\sum_{i=k}^{n+1}\left[\left(c^{\mathrm{o}}+\sum_{\ell=i+1}^{n}c^{\mathrm{s}}_\ell\right)o_{ik(n+1)}-\left(c^{\mathrm{o}}+\sum_{\ell=i+1}^{n}c^{\mathrm{s}}_\ell\right)x_it_{k(n+1)}+\sum_{\ell=i+1}^{n}c^{\mathrm{a}}_\ell r_{i\ell k(n+1)}-\sum_{\ell=i+1}^{n}c^{\mathrm{a}}_\ell x_ip_{\ell k(n+1)}\right]\\
\text{s.t.}\ & \text{(18a)–(18d)}, \text{(EC.4b)}, \text{(EC.5a)–(EC.5b)}, && \text{(EC.6a)}\\
& t_{kj},p_{ikj}\in\{0,1\}, \quad \forall 1\le k\le i\le j\le n+1. && \text{(EC.6b)}
\end{aligned}$$

To finish reformulating the general DR expectation model (EC.1), we follow a similar dualization process described in Section 3 and rewrite formulation (EC.1) as

$$\min_{x\in X,\rho\in\mathbb{R}^n,\gamma\in\mathbb{R}^n,\theta\in\mathbb{R}}\ \sum_{i=1}^{n}\mu_i\rho_i+\sum_{i=1}^{n}\nu_i\gamma_i+\theta \tag{EC.7a}$$

$$\text{s.t.}\quad \theta\ge H^{\mathrm{G}}(x,\rho,\gamma)\equiv\max_{(q,s)\in D_q\times D_s}\left\{Q^{\mathrm{G}}(x,q,s)-\sum_{i=1}^{n}(\rho_is_i+\gamma_iq_i)\right\}. \tag{EC.7b}$$

Similar to Lemma 1, we observe that for any fixed variables x, ρ, and γ, $H^{\mathrm{G}}(x,\rho,\gamma)<+\infty$. Furthermore, function $H^{\mathrm{G}}(x,\rho,\gamma)$ is convex and piecewise linear in x, ρ, and γ with a finite number of pieces. Hence, we can adapt Algorithm 1 to solve model (EC.1) in a decomposition framework. We present this adaptation in Algorithm 2. Similar to Algorithm 1, we observe that Algorithm 2 is finite. Finally, for the separation problem in Step 3, we remark that (i) feasible region D_q can be set as $D_q^{(K)}$ for $K = 2, \dots, n+1$ based on the scheduler's targeted conservativeness, (ii) the separation problem is an MILP and can be solved by off-the-shelf software, and (iii) we can incorporate the same valid inequalities identified in Proposition 1 to accelerate solving the separation problem and hence the decomposition algorithm.

Algorithm 2 A decomposition algorithm for solving general DR expectation model (EC.1).

```
1: Input: feasible regions $X$ and $D_q \times D_s$; set of cuts $\{L(x,\rho,\gamma,\theta) \geq 0\} = \emptyset$.
2: Solve the master problem
       $\min_{x \in X, \rho, \gamma, \theta} \sum_{i=1}^{n} \mu_i \rho_i + \sum_{i=1}^{n} \nu_i \gamma_i + \theta$
       $\text{s.t.} \quad L(x,\rho,\gamma,\theta) \geq 0$
   and record an optimal solution $(x^*, \rho^*, \gamma^*, \theta^*)$.
3: With $(x,\rho,\gamma)$ fixed to be $(x^*, \rho^*, \gamma^*)$, solve the separation problem
       $\max_{t,p,q,s,o,r} \sum_{k=1}^{n} \sum_{j=k}^{n} \sum_{i=k}^{j} \left[ \left( -c_j^{\mathrm{u}} + \sum_{\ell=i+1}^{j} c_\ell^{\mathrm{s}} \right) o_{ikj} + \left( c_j^{\mathrm{u}} - \sum_{\ell=i+1}^{j} c_\ell^{\mathrm{s}} \right) x_i t_{kj} + \sum_{\ell=i+1}^{j} c_\ell^{\mathrm{a}} r_{i\ell kj} - \sum_{\ell=i+1}^{j} c_\ell^{\mathrm{a}} x_i p_{\ell kj} \right] +$
       $\sum_{k=1}^{n} \sum_{i=k}^{n+1} \left[ \left( c^{\mathrm{o}} + \sum_{\ell=i+1}^{n} c_\ell^{\mathrm{s}} \right) o_{ik(n+1)} - \left( c^{\mathrm{o}} + \sum_{\ell=i+1}^{n} c_\ell^{\mathrm{s}} \right) x_i t_{k(n+1)} + \sum_{\ell=i+1}^{n} c_\ell^{\mathrm{a}} r_{i\ell k(n+1)} - \sum_{\ell=i+1}^{n} c_\ell^{\mathrm{a}} x_i p_{\ell k(n+1)} \right] -$
       $\sum_{i=1}^{n} (\rho_i s_i + \gamma_i q_i)$
       $\text{s.t.} \quad \text{(18a)–(18d)}, \text{(EC.4b)}, \text{(EC.5a)–(EC.5b)},$
       $t_{kj}, p_{ikj} \in \{0,1\}, \ \forall 1 \leq k \leq i \leq j \leq n+1, \quad (q,s) \in D_q \times D_s$
   and record an optimal solution $(t^*, p^*, q^*, s^*, o^*, r^*)$.
4: if $\theta^*$ is greater than or equal to the optimal objective value of the separation problem, then
5:     stop and return $x^*$ as an optimal solution to formulation (EC.1).
6: else
7:     add the cut
       $\theta \geq \sum_{k=1}^{n} \sum_{j=k}^{n} \sum_{i=k}^{j} \left[ \left( -c_j^{\mathrm{u}} + \sum_{\ell=i+1}^{j} c_\ell^{\mathrm{s}} \right) o_{ikj}^* + \left( c_j^{\mathrm{u}} - \sum_{\ell=i+1}^{j} c_\ell^{\mathrm{s}} \right) t_{kj}^* x_i + \sum_{\ell=i+1}^{j} c_\ell^{\mathrm{a}} r_{i\ell kj}^* - \sum_{\ell=i+1}^{j} c_\ell^{\mathrm{a}} p_{\ell kj}^* x_i \right] +$
       $\sum_{k=1}^{n} \sum_{i=k}^{n+1} \left[ \left( c^{\mathrm{o}} + \sum_{\ell=i+1}^{n} c_\ell^{\mathrm{s}} \right) o_{ik(n+1)}^* - \left( c^{\mathrm{o}} + \sum_{\ell=i+1}^{n} c_\ell^{\mathrm{s}} \right) t_{k(n+1)}^* x_i + \sum_{\ell=i+1}^{n} c_\ell^{\mathrm{a}} r_{i\ell k(n+1)}^* - \sum_{\ell=i+1}^{n} c_\ell^{\mathrm{a}} p_{\ell k(n+1)}^* x_i \right] -$
       $\sum_{i=1}^{n} (s_i^* \rho_i + q_i^* \gamma_i)$
   to the set of cuts $\{L(x,\rho,\gamma,\theta) \geq 0\}$ and go to Step 2.
8: end if
```

EC.2. Solution Approaches for DR CVaR Models

In this section, we reformulate DR CVaR constraints in (6) with cost parameters c^{o}, c^{u}_i, and c^{w}_i. We first represent CVaR by an alternative definition (Rockafellar and Uryasev (2000, 2002)):

$$\mathrm{CVaR}_{1-\epsilon}\left(Q(x,q,s)\right) \;=\; \inf_{z\in\mathbb{R}}\left\{z+\frac{1}{\epsilon}\mathbb{E}_{\mathbb{P}_{q,s}}\left[Q(x,q,s)-z\right]^{+}\right\},$$

where $[a]^{+} := \max\{a,0\}$ for $a\in\mathbb{R}$. It follows that

$$\begin{aligned}
\sup_{\mathbb{P}_{q,s}\in\mathcal{F}(D,\mu,\nu)} \mathrm{CVaR}_{1-\epsilon}\left(Q(x,q,s)\right) &= \sup_{\mathbb{P}_{q,s}\in\mathcal{F}(D,\mu,\nu)}\inf_{z\in\mathbb{R}}\left\{z+\frac{1}{\epsilon}\mathbb{E}_{\mathbb{P}_{q,s}}\left[Q(x,q,s)-z\right]^{+}\right\}\\
&= \inf_{z\in\mathbb{R}}\sup_{\mathbb{P}_{q,s}\in\mathcal{F}(D,\mu,\nu)}\left\{z+\frac{1}{\epsilon}\mathbb{E}_{\mathbb{P}_{q,s}}\left[Q(x,q,s)-z\right]^{+}\right\} && \text{(EC.8a)}\\
&= \inf_{z\in\mathbb{R}}\left\{z+\frac{1}{\epsilon}\sup_{\mathbb{P}_{q,s}\in\mathcal{F}(D,\mu,\nu)}\mathbb{E}_{\mathbb{P}_{q,s}}\left[Q(x,q,s)-z\right]^{+}\right\}, && \text{(EC.8b)}
\end{aligned}$$

where (EC.8a) follows the Sion's minimax theorem (Sion 1958) because $z+\frac{1}{\epsilon}\mathbb{E}_{\mathbb{P}_{q,s}}\left[Q(x,q,s)-z\right]^{+}$ is convex in z, concave (in particular, linear) in variables $\mathbb{P}_{q,s}$, and $\mathcal{F}(D,\mu,\nu)$ is weakly compact.

EC.2.1. MILP Reformulation and Decomposition Algorithm

Based on a similar dualization process in Section 3 (see the primal and dual formulations (8) and (9)), we reformulate the inner maximization problem in (EC.8b) as a minimization problem, and combine it with the outer minimization problem to obtain

$$\begin{aligned}
&\inf_{z,\rho,\gamma,\theta}\quad z+\frac{1}{\epsilon}\left(\sum_{i=1}^{n}\mu_i\rho_i+\sum_{i=1}^{n}\nu_i\gamma_i+\theta\right)\\
&\text{s.t.}\quad \sum_{i=1}^{n}s_i\rho_i+\sum_{i=1}^{n}\gamma_i q_i+\theta \ge \left[Q(x,q,s)-z\right]^{+}\quad \forall (q,s)\in D_q\times D_s\\
=\;&\inf_{z,\rho,\gamma,\theta}\quad z+\frac{1}{\epsilon}\left(\sum_{i=1}^{n}\mu_i\rho_i+\sum_{i=1}^{n}\nu_i\gamma_i+\theta\right)\\
&\text{s.t.}\quad \sum_{i=1}^{n}s_i\rho_i+\sum_{i=1}^{n}\gamma_i q_i+\theta \ge 0\quad \forall (q,s)\in D_q\times D_s && \text{(EC.9a)}\\
&\qquad\ \sum_{i=1}^{n}s_i\rho_i+\sum_{i=1}^{n}\gamma_i q_i+\theta \ge Q(x,q,s)-z\quad \forall (q,s)\in D_q\times D_s, && \text{(EC.9b)}
\end{aligned}$$

where constraints (EC.9a) and (EC.9b) are derived based on the definition of $[\cdot]^{+}$. Thus, the DR CVaR constraint (6) is equivalent to

$$\overline{Q} \;\ge\; z+\frac{1}{\epsilon}\left(\sum_{i=1}^{n}\mu_i\rho_i+\sum_{i=1}^{n}\nu_i\gamma_i+\theta\right) \tag{EC.10a}$$

$$\min_{(q,s)\in D_q\times D_s}\left\{\sum_{i=1}^{n}\rho_i s_i+\sum_{i=1}^{n}\gamma_i q_i\right\}+\theta \;\ge\; 0 \tag{EC.10b}$$

$$\theta + z \ \geq \max_{(q,s)\in D_q\times D_s} \left\{ Q(x,q,s) - \sum_{i=1}^{n}(\rho_i s_i + \gamma_i q_i) \right\}, \tag{EC.10c}$$

where constraint (EC.10a) is linear, but (EC.10b) and (EC.10c) need further analysis. First, we replace constraint (EC.10b) by equivalent linear constraints in the following proposition, whose proof is relegated to EC.4.1.

PROPOSITION EC.1. *For fixed ρ and γ, and $D_q = D_q^{(K)}$ with $K \in \{2,\dots,n+1\}$,* (EC.10b) *is equivalent to linear constraints:*

$$\theta + \sum_{i=1}^{n-K+1} \beta_i + \sum_{i=1}^{n} (s_i^L \chi_i^L - s_i^U \chi_i^U - \eta_i) \ \geq 0, \tag{EC.11a}$$

$$-\eta_i + \sum_{j=\max\{i-K+1,1\}}^{\min\{i,n-K+1\}} \beta_j \ \leq \gamma_i \quad \forall 1\leq i\leq n, \tag{EC.11b}$$

$$\chi_i^L - \chi_i^U \ \leq \rho_i \quad \forall 1\leq i\leq n, \tag{EC.11c}$$

$$\beta_i,\ \chi_i^L,\ \chi_i^U, \eta_i \geq 0 \quad \forall 1\leq i\leq n. \tag{EC.11d}$$

Second, note that the right-hand side of constraint (EC.10c) is equivalent to that of constraint (13b) in the reformulated DR expectation model, and so the reformulated separation problem (19) and Algorithm 1 described in Section 3 can be easily adapted to handle constraint (EC.10c). Furthermore, the valid inequalities (20a)–(20f) can be incorporated to accelerate solving the adapted separation problem and implementing the decomposition algorithm.

EC.2.2. LP Reformulations of the DR CVaR Model

We derive LP reformulations for the DR CVaR constraint (6) when $D_q = D_q^{(2)}$ (i.e., no consecutive no-shows) and when $D_q = D_q^{(n+1)}$ (i.e., arbitrary no-shows).

Case 1. (No Consecutive No-Shows) Recall that DR CVaR constraint (6) is equivalent to constraints (EC.10a), (EC.11a)–(EC.11d) with $K = 2$, and (EC.10c). When $D_q = D_q^{(2)}$, we apply Theorem 1 to further reformulate (EC.10c) as linear constraints $\theta + z \geq \sum_{i=1}^{n+1} (\alpha_i + s_i^{\mathrm{U}}\tau_i^{\mathrm{U}} - s_i^{\mathrm{L}}\tau_i^{\mathrm{L}})$ and (22b)–(22g), resulting in the following proposition.

PROPOSITION EC.2. *When $D_q = D_q^{(2)}$, the DR CVaR constraint* (6) *is equivalent to linear constraints* (EC.10a), (EC.11a)–(EC.11d) *with $K = 2$, $\sum_{i=1}^{n+1} (\alpha_i + s_i^U\tau_i^U - s_i^L\tau_i^L) \leq \theta + z$, and* (22b)–(22g)*.*

We remark that the LP reformulation in Proposition EC.2 is of the size $\mathcal{O}(n^3)$ because constraints (22b)–(22g) incorporate $\mathcal{O}(n^3)$ decision variables and linear constraints. In this section, we focus on a specific DR CVaR constraint (6) that restricts overtime only. That is, $c_i^{\mathrm{u}} = c_i^{\mathrm{w}} = 0$ for all $1 \leq i \leq n$ and $c^{\mathrm{o}} = 1$, and $Q(x,q,s) = Q^W(x,q,s) := \min_{w,u,W} W$ subject to constraints (3b)–(3d). Next, we derive a more compact LP reformulation of this DR CVaR constraint with $\mathcal{O}(n^2)$ variables and constraints. To that end, we derive an $\mathcal{O}(n^2)$ LP reformulation for constraint (EC.10c). We begin by specializing the extreme point representation of polyhedron Y for $Q(x,q,s)=Q^W(x,q,s)$.

LEMMA EC.1. *When $c_i^u = c_i^w = 0$ for all $1 \le i \le n$ and $c^o = 1$, the set of extreme points of polyhedron Y defined in* (15) *is $\{\sum_{\ell=k}^n e_\ell : k = 1,\dots,n\} \bigcup \{\mathbf{0}_n\}$, where e_ℓ represents an n-dimensional unit vector with component ℓ equaling to one and any other component equaling to zero; $\mathbf{0}_n$ is an n-dimensional zero vector.*

Recall the observation in Section 3.2 that each extreme point $(y_1,\dots,y_{n+1})$ of Y is associated with a partition of set $\{1,\dots,n+1\}$ into intervals. The result in Lemma EC.1 follows from (16) when the cost parameters take the above specified values. Define binary variables t_k for all $1 \le k \le n$ to represent the set of extreme points of Y, such that $t_k = 1$ if the extreme point is $\sum_{\ell=k}^n e_\ell$ and $t_k = 0$ otherwise. Note that extreme point $\mathbf{0}_n$ is represented by $t_k = 0$ for all $1 \le k \le n$. For a valid representation, we require $\sum_{k=1}^n t_k \le 1$. It follows that the right-hand side of (EC.10c) (with $Q(x,q,s)=Q^W(x,q,s)$) is equivalent to

$$\begin{aligned} \max_{t,q,s} \quad & \sum_{k=1}^n \left(\sum_{i=k}^n (q_i s_i - x_i) \right) t_k - \sum_{i=1}^n (\rho_i s_i + \gamma_i q_i) \\ \text{s.t.} \quad & \sum_{k=1}^n t_k \le 1, \quad q \in D_q, \quad s \in D_s, \quad t \in \{0,1\}^n \end{aligned}$$

as a mixed-integer bilinear program with binary vectors q and t, and continuous vector s. We linearize the bilinear terms by defining $p_{ki} \equiv t_k q_i$ and $o_{ki} \equiv t_k q_i s_i$ for all $1 \le k \le i \le n$. Also, we introduce McCormick inequalities (EC.12b)–(EC.12c) and (EC.12d)–(EC.12e) for variables p_{ki} and o_{ki}, respectively to further reformulate the separation problem as a mixed-integer linear program:

$$\max_{t,q,s,p,o} \quad \sum_{k=1}^n \sum_{i=k}^n (o_{ki} - x_i t_k) - \sum_{i=1}^n (\rho_i s_i + \gamma_i q_i) \tag{EC.12a}$$

$$\text{s.t.} \quad p_{ki} - t_k \le 0 \quad \forall 1 \le k \le i \le n, \tag{EC.12b}$$

$$p_{ki} - q_i \le 0,\ p_{ki} - q_i - t_k \ge -1,\ p_{ki} \ge 0 \quad \forall 1 \le k \le i \le n, \tag{EC.12c}$$

$$o_{ki} - s_i^{\mathrm{L}} p_{ki} \ge 0,\ o_{ki} - s_i^{\mathrm{U}} p_{ki} \le 0 \quad \forall 1 \le k \le i \le n, \tag{EC.12d}$$

$$o_{ki} - s_i + s_i^{\mathrm{L}}(1 - p_{ki}) \le 0,\ o_{ki} - s_i + s_i^{\mathrm{U}}(1 - p_{ki}) \ge 0 \quad \forall 1 \le k \le i \le n, \tag{EC.12e}$$

$$\sum_{k=1}^n t_k \le 1, \tag{EC.12f}$$

$$q \in D_q,\ s \in D_s,\ t \in \{0,1\}^n. \tag{EC.12g}$$

Similar as before, we aim to derive the convex hull of the feasible region of problem (EC.12), i.e., the mixed-integer feasible region described by constraints (EC.12b)–(EC.12g). We denote the feasible region as set G and derive $\mathrm{conv}(G)$ in the following theorem, whose proof is in EC.4.2.

THEOREM EC.1. *When $D_q = D_q^{(2)}$, the following inequalities are valid for set $G = \{(t,q,s,p,o) :$ (EC.12b)–(EC.12g)$\}$:*

$$\sum_{k=1}^{n} p_{kn} \le q_n, \tag{EC.13a}$$

$$p_{ki} + p_{k(i+1)} \ge t_k \quad \forall 1 \le k \le i \le n-1, \tag{EC.13b}$$

$$\sum_{k=1}^{i} (p_{ki} - t_k) \ge q_i - 1 \quad \forall 1 \le i \le n, \tag{EC.13c}$$

$$\sum_{k=1}^{i} \big(p_{ki} + p_{k(i+1)}\big) \le \sum_{k=1}^{i} t_k + q_i + q_{i+1} - 1 \quad \forall 1 \le i \le n-1, \tag{EC.13d}$$

$$s_i - \sum_{k=1}^{i} (o_{ki} - s_i^L p_{ki}) \ge s_i^L \quad \forall 1 \le i \le n, \tag{EC.13e}$$

$$s_i - \sum_{k=1}^{i} (o_{ki} - s_i^U p_{ki}) \le s_i^U \quad \forall 1 \le i \le n. \tag{EC.13f}$$

Furthermore, polyhedron $CG := \{(t,q,s,p,o) :$ (EC.12b), (EC.12d), (EC.13a)–(EC.13f)$\}$ is the convex hull of set G, i.e., $CG = conv(G)$.

Theorem EC.1 provides us a compact LP reformulation of the right-hand side of constraint (EC.10c) with $\mathcal{O}(n^2)$ variables and constraints:

$$\max_{t,q,s,p,o} \quad \sum_{k=1}^{n} \sum_{i=k}^{n} (o_{ki} - x_i t_k) - \sum_{i=1}^{n} (\rho_i s_i + \gamma_i q_i) \tag{EC.14a}$$

$$\text{s.t.} \quad (t,q,s,p,o) \in CG. \tag{EC.14b}$$

Finally, by resorting to the dual formulation of (EC.14), we represent constraint (EC.10c) as

$$\sum_{i=1}^{n} (\alpha_i - s_i^{\mathrm{L}} \tau_i^{\mathrm{L}} + s_i^{\mathrm{U}} \tau_i^{\mathrm{U}}) - \sum_{i=1}^{n-1} \phi_i \le \theta + z \tag{EC.15a}$$

$$\sum_{i=k}^{n} (\alpha_i - \sigma_{ki}) + \sum_{i=k}^{n-1} (\lambda_{ki} - \phi_i) \ge -\sum_{i=k}^{n} x_i \quad \forall 1 \le k \le n, \tag{EC.15b}$$

$$\alpha_i - \sum_{\ell=\max\{i-1,1\}}^{\min\{i,n-1\}} \phi_\ell - \sum_{\ell=n}^{\max\{i,n-1\}} \zeta \ge -\gamma_i \quad \forall 1 \le i \le n, \tag{EC.15c}$$

$$\tau_i^{\mathrm{U}} - \tau_i^{\mathrm{L}} \ge -\rho_i \quad \forall 1 \le i \le n, \tag{EC.15d}$$

$$\begin{aligned} &\sigma_{ki} + s_i^{\mathrm{L}} \varphi_{ki}^{\mathrm{L}} - s_i^{\mathrm{U}} \varphi_{ki}^{\mathrm{U}} - \alpha_i - s_i^{\mathrm{L}} \tau_i^{\mathrm{L}} + s_i^{\mathrm{U}} \tau_i^{\mathrm{U}} \\ &+ \sum_{\ell=n}^{\max\{i,n-1\}} \zeta + \sum_{\ell=\max\{i-1,k\}}^{\min\{i,n-1\}} (\phi_\ell - \lambda_{k\ell}) \ge 0 \quad \forall 1 \le k \le i \le n, \end{aligned} \tag{EC.15e}$$

$$-\varphi_{ki}^{\mathrm{L}} + \varphi_{ki}^{\mathrm{U}} + \tau_i^{\mathrm{L}} - \tau_i^{\mathrm{U}} \ge 1 \quad \forall 1 \le k \le i \le n, \tag{EC.15f}$$

$$\sigma_{ki},\ \varphi_{ki}^{\mathrm{L}},\ \varphi_{ki}^{\mathrm{U}},\ \zeta,\ \lambda_{ki},\ \alpha_i,\ \phi_i,\ \tau_i^{\mathrm{L}},\ \tau_i^{\mathrm{U}} \ge 0 \quad \forall 1 \le k \le i \le n, \tag{EC.15g}$$

where dual variables σ_{ki}, $\varphi_{ki}^{\mathrm{L/U}}$, ζ, λ_{ki}, α_i, ϕ_i, and $\tau_i^{\mathrm{L/U}}$ are associated with constraints (EC.12b), (EC.12d), (EC.13a), (EC.13b), (EC.13c), (EC.13d), and (EC.13e)–(EC.13f), respectively (after transforming all "$\geq$" inequalities into the "$\leq$" form), and dual constraints (EC.15b)–(EC.15f) are associated with primal variables t_k, q_i, s_i, p_{ki}, and o_{ki} respectively. This results in an $\mathcal{O}(n^2)$ LP reformulation of the DR CVaR constraint on overtime.

Proposition EC.3. *When $D_q = D_q^{(2)}$, $c_i^u = c_i^w = 0$ for all $1 \leq i \leq n$ and $c^o = 1$, the DR CVaR constraint* (6) *on overtime is equivalent to linear constraints* (EC.10a), (EC.11a)–(EC.11d) *with $K = 2$, and* (EC.15a)–(EC.15g).

Case 2. (Arbitrary No-Shows) Recall that DR CVaR constraint (6) is equivalent to constraints (EC.10a), (EC.11a)–(EC.11d) with $K = n+1$, and (EC.10c). As $D_q = D_q^{(n+1)}$, we can apply the results in Section 4 (see Case 2) to further reformulate (EC.10c) as linear constraints $\theta + z \geq \sum_{i=1}^{n+1} \alpha_i$, (24b), (24d), and (25a)–(25d). This results in the following proposition.

Proposition EC.4. *When $D_q = D_q^{(n+1)}$, the DR CVaR constraint* (6) *is equivalent to constraints* (EC.10a), (EC.11a)–(EC.11d) *with $K = n+1$, $\sum_{i=1}^{n+1} \alpha_i \leq \theta + z$,* (24b), (24d), *and* (25a)–(25d).

EC.3. Proofs for the DR Expectation Model

EC.3.1. Proof of Lemma 1

Proof of Lemma 1 First, feasible regions Y and $D_q \times D_s$ are both independent of x, ρ, and γ, and bounded. Hence, $\max_{y \in Y} h(x, y, \rho, \gamma) \equiv \max_{y \in Y, (q,s) \in D_q \times D_s} \{\sum_{i=1}^n (q_i s_i - x_i) y_i - \sum_{i=1}^n (\rho_i s_i + \gamma_i q_i)\} < +\infty$. Second, for any fixed y, q, and s, $\sum_{i=1}^n (q_i s_i - x_i) y_i - \sum_{i=1}^n (\rho_i s_i + \gamma_i q_i)$ is a linear function of x, ρ, and γ. It follows that $\max_{y \in Y} h(x, y, \rho, \gamma)$ is the maximum of a set of linear functions of x, ρ, and γ, and hence convex and piecewise linear. Third, it is clear that each linear piece of function $\max_{y \in Y} h(x, y, \rho, \gamma)$ is associated with one distinct extreme point of polyhedra Y, D_q, and D_s respectively. Therefore, the number of pieces of function $\max_{y \in Y} h(x, y, \rho, \gamma)$ is finite because each of these polyhedra has a finite number of extreme points. This completes the proof. $\square$

EC.3.2. Proof of Lemma 2

Proof of Lemma 2 For fixed x, ρ, and γ, in view of the definition of function $h(x, y, \rho, \gamma)$ in (12c), we have $h(x, y, \rho, \gamma) = \max_{(q,s) \in D_q \times D_s} H(q, s, y)$, where $H(q, s, y)$ is a linear function of variable y. It follows that $h(x, y, \rho, \gamma)$ is the supremum of a set of convex functions of y, and hence itself convex in variable y. $\square$

EC.3.3. Proof of Proposition 1

Proof of Proposition 1 First, because $p_{ikj} \equiv q_i t_{kj}$, equality (20a) can be obtained via multiplying equalities $\sum_{k=1}^{i} \sum_{j=i}^{n+1} t_{kj} = 1$ by q_i on both sides.

Second, because $o_{ikj} \equiv q_i s_i t_{kj} \equiv s_i p_{ikj}$, and by equalities (20a) and $s_i \in [s_i^{\mathrm{L}}, s_i^{\mathrm{U}}]$, we have

$$\begin{aligned}
\sum_{k=1}^{i} \sum_{j=i}^{n+1} (o_{ikj} - s_i^{\mathrm{L}} p_{ikj}) \; &= \; (s_i - s_i^{\mathrm{L}}) \sum_{k=1}^{i} \sum_{j=i}^{n+1} p_{ikj} \; = \; (s_i - s_i^{\mathrm{L}}) q_i \; \le \; (s_i - s_i^{\mathrm{L}}), \\
\sum_{k=1}^{i} \sum_{j=i}^{n+1} (o_{ikj} - s_i^{\mathrm{U}} p_{ikj}) \; &= \; (s_i - s_i^{\mathrm{U}}) \sum_{k=1}^{i} \sum_{j=i}^{n+1} p_{ikj} \; = \; (s_i - s_i^{\mathrm{U}}) q_i \; \ge \; (s_i - s_i^{\mathrm{U}}),
\end{aligned}$$

which shows the validity of inequalities (20b) and (20c).

Third, for $1 \le k < j \le n+1$ and $k \le i \le j - K + 1$, because $\sum_{\ell=i}^{i+K-1} q_\ell \ge 1$ by the definition of $D_q^{(K)}$, we have

$$\sum_{\ell=i}^{i+K-1} p_{\ell kj} \; = \; \sum_{\ell=i}^{i+K-1} q_\ell t_{kj} \; = \; \left(\sum_{\ell=i}^{i+K-1} q_\ell \right) t_{kj} \; \ge \; t_{kj},$$

which shows the validity of inequalities (20d).

Fourth, for $i = 1, \ldots, n$, because $\sum_{k=1}^{i} \sum_{j=i}^{n+1} t_{kj} = 1$ and $\sum_{k=1}^{i+1} \sum_{j=i+1}^{n+1} t_{kj} = 1$ by constraints (17b), we have

$$0 \; = \; \sum_{k=1}^{i} \sum_{j=i}^{n+1} t_{kj} - \sum_{k=1}^{i+1} \sum_{j=i+1}^{n+1} t_{kj} \; = \; \sum_{k=1}^{i} t_{ki} - \sum_{j=i+1}^{n+1} t_{(i+1)j}. \tag{EC.16}$$

We show the validity of inequalities (20e) for all $i = K-1, \ldots, n$. If $\sum_{k=1}^{i-K+2} t_{ki} = 0$, then the conclusion holds because each $p_{ikj} \ge 0$. Now suppose that $\sum_{k=1}^{i-K+2} t_{ki} = 1$, then $\sum_{j=i+1}^{n+1} t_{(i+1)j} = 1$ in view of (EC.16). It follows that

$$\begin{aligned}
\sum_{k=1}^{i-K+2} \sum_{\ell=i-K+2}^{i} p_{\ell ki} + \sum_{j=i+1}^{n+1} p_{(i+1)(i+1)j} \; &= \; \left(\sum_{\ell=i-K+2}^{i} q_\ell \right) \left(\sum_{k=1}^{i-K+2} t_{ki} \right) + q_{i+1} \sum_{j=i+1}^{n+1} t_{(i+1)j} \\
&= \; \sum_{\ell=i-K+2}^{i+1} q_\ell \; \ge \; 1,
\end{aligned}$$

where the last inequality is due to the definition of $D_q^{(K)}$.

Finally, we show the validity of inequalities (20f) for all $i = 1, \ldots, n-K+2$. If $\sum_{j=i+K-1}^{n+1} t_{(i+1)j} = 0$, then the conclusion holds because each $p_{ikj} \ge 0$. Now suppose that $\sum_{j=i+K-1}^{n+1} t_{(i+1)j} = 1$, then $\sum_{k=1}^{i} t_{ki} = 1$ in view of (EC.16). It follows that

$$\begin{aligned}
\sum_{k=1}^{i} p_{iki} + \sum_{\ell=i+1}^{i+K-1} \sum_{j=i+K-1}^{n+1} p_{\ell(i+1)j} \; &= q_i \left(\sum_{k=1}^{i} t_{ki} \right) + \left(\sum_{\ell=i+1}^{i+K-1} q_\ell \right) \left(\sum_{j=i+K-1}^{n+1} t_{(i+1)j} \right) \\
&= \sum_{\ell=i}^{i+K-1} q_\ell \; \ge \; 1,
\end{aligned}$$

where the last inequality is due to the definition of $D_q^{(K)}$. $\square$

EC.3.4. Proof of Theorem 1

Recall that polyhedron $CF = \{(t,q,s,p,o) : \text{(17b), (18a), (18c), (20a)–(20d), (21)}\}$ in Theorem 1. We first study the extreme points of polyhedron CF and show their properties as follows.

Proposition EC.5. *Every extreme point (t,q,s,p,o) of CF satisfies the following:*

1. $t_{kj}, p_{ikj} \in \{0,1\}$ *for all* $1 \le k \le j \le n+1$ *and* $k \le i \le j$;
2. $q_i \in \{0,1\}$ *for all* $1 \le i \le n+1$;
3. $p_{ikj} = q_i t_{kj}$ *and* $o_{ikj} = q_i s_i t_{kj}$ *for all* $1 \le k \le j \le n+1$ *and* $k \le i \le j$.

Proof of Proposition EC.5 Consider arbitrary cost coefficients c^{q}_i and c^{s}_i for all $1 \le i \le n+1$, c^{t}_{kj} for all $1 \le k \le j \le n+1$, and c^{p}_{ikj} and c^{o}_{ikj} for all $1 \le k \le j \le n+1$ and $k \le i \le j$. We construct a related linear program

$$\begin{aligned}
\text{(LP-CF)} \quad \min_{t,q,s,p,o} \quad & \sum_{i=1}^{n+1} (c^{\mathrm{q}}_i q_i + c^{\mathrm{s}}_i s_i) + \sum_{k=1}^{n+1}\sum_{j=k}^{n+1} \left(c^{\mathrm{t}}_{kj} t_{kj} + \sum_{i=k}^{j} \left(c^{\mathrm{p}}_{ikj} p_{ikj} + c^{\mathrm{o}}_{ikj} o_{ikj} \right) \right) \\
\text{s.t.} \quad & (t,q,s,p,o) \in CF.
\end{aligned}$$

To prove that each extreme point of CF satisfies properties 1, 2, and 3, we show for any values of c^{t}_{kj}, c^{q}_i, c^{s}_i, c^{p}_{ikj}, and c^{o}_{ikj}, there exists an optimal solution $(t^*, q^*, s^*, p^*, o^*)$ to (LP-CF) that satisfies properties 1, 2, and 3 (cf. Wolsey 1998, Nemhauser and Wolsey 1999).

First, in view of equalities (20a), we can assume that $c^{\mathrm{q}}_i = 0$ for all $1 \le i \le n+1$ w.l.o.g., because we can always replace each c^{p}_{ikj} with $c^{\mathrm{p}}_{ikj} + c^{\mathrm{q}}_i$ so that variables p_{ikj} will carry the cost of decisions q_i. It follows that we can ignore variables q_i in (LP-CF) because they do not contribute to the objective function and their values entirely depend on p_{ikj} by constraints (20a). Also, we note that (i) $s^{\mathrm{L}}_i p_{ikj} \le o_{ikj} \le s^{\mathrm{U}}_i p_{ikj}$ by (18c), and so $p_{ikj} \ge 0$ for all $1 \le k \le i \le j \le n+1$, and (ii) for all $1 \le i \le n+1$, $\sum_{k=1}^{i} \sum_{j=i}^{n+1} p_{ikj} \le \sum_{k=1}^{i} \sum_{j=i}^{n+1} t_{kj} = 1$ by (18a) and (17b).

Second, we rewrite (LP-CF) as a two-stage formulation as follows:

$$\begin{aligned}
\min_{t,p} \quad & \sum_{k=1}^{n+1}\sum_{j=k}^{n+1} \Big(c^{\mathrm{t}}_{kj} t_{kj} + \sum_{i=k}^{j} c^{\mathrm{p}}_{ikj} p_{ikj} \Big) + V(p) \\
\text{s.t.} \quad & (t,p) \in CF_{t,p},
\end{aligned}$$

where polyhedron $CF_{t,p} := \{(t,p) : \text{(17b), (18a), (20d), (21)}\}$ and $V(p)$ represents a value function of p defined as

$$\begin{aligned}
\text{(LP-CF}(p)) \quad V(p) := \min_{s,o} \quad & \sum_{i=1}^{n+1} c^{\mathrm{s}}_i s_i + \sum_{k=1}^{n+1}\sum_{j=k}^{n+1}\sum_{i=k}^{j} c^{\mathrm{o}}_{ikj} o_{ikj} \\
\text{s.t.} \quad & (s,o) \in CF_{s,o}(p),
\end{aligned}$$

where

$$CF_{s,o}(p) = \Bigg\{ (s,o): \ o_{ikj} \geq s_i^{\mathrm{L}} p_{ikj} \quad \forall 1 \leq k \leq j \leq n+1, \ \forall k \leq i \leq j, \tag{EC.17a}$$

$$o_{ikj} \leq s_i^{\mathrm{U}} p_{ikj} \quad \forall 1 \leq k \leq j \leq n+1, \ \forall k \leq i \leq j, \tag{EC.17b}$$

$$s_i - \sum_{k=1}^{i} \sum_{j=i}^{n+1} (o_{ikj} - s_i^{\mathrm{L}} p_{ikj}) \geq s_i^{\mathrm{L}} \quad \forall 1 \leq i \leq n+1, \tag{EC.17c}$$

$$s_i - \sum_{k=1}^{i} \sum_{j=i}^{n+1} (o_{ikj} - s_i^{\mathrm{U}} p_{ikj}) \leq s_i^{\mathrm{U}} \quad \forall 1 \leq i \leq n+1 \Bigg\} \tag{EC.17d}$$

represents a parametric polyhedron depending on the values of p_{ikj}. We solve (LP-CF(p)) by considering its dual formulation

$$V(p) = \max_{\psi,\omega} \ \sum_{i=1}^{n+1} \sum_{k=1}^{i} \sum_{j=i}^{n+1} \left(s_i^{\mathrm{L}} p_{ikj} \psi_{ikj}^{\mathrm{L}} - s_i^{\mathrm{U}} p_{ikj} \psi_{ikj}^{\mathrm{U}} \right) + \sum_{i=1}^{n+1} \left[s_i^{\mathrm{L}} \left(1 - \sum_{k=1}^{i} \sum_{j=i}^{n+1} p_{ikj} \right) \omega_i^{\mathrm{L}} - s_i^{\mathrm{U}} \left(1 - \sum_{k=1}^{i} \sum_{j=i}^{n+1} p_{ikj} \right) \omega_i^{\mathrm{U}} \right]$$

$$\text{s.t.} \ \ \psi_{ikj}^{\mathrm{L}} - \psi_{ikj}^{\mathrm{U}} - \omega_i^{\mathrm{L}} + \omega_i^{\mathrm{U}} = c_{ikj}^{\mathrm{o}} \quad \forall 1 \leq k \leq j \leq n+1, \ \forall k \leq i \leq j, \tag{EC.18a}$$

$$\omega_i^{\mathrm{L}} - \omega_i^{\mathrm{U}} = c_i^{\mathrm{s}} \quad \forall 1 \leq i \leq n+1, \tag{EC.18b}$$

where dual variables $\psi_{ikj}^{\mathrm{L/U}}$ and $\omega_i^{\mathrm{L/U}}$ are associated with primal constraints (EC.17a)–(EC.17b) and (EC.17c)–(EC.17d), respectively (after transforming all "$\leq$" inequalities into the "$\geq$" form), and dual constraints (EC.18a) and (EC.18b) are associated with primal variables o_{ikj} and s_i, respectively. Because $p_{ikj} \geq 0$ for all $1 \leq k \leq i \leq j \leq n+1$ and $1 - \sum_{k=1}^{i} \sum_{j=i}^{n+1} p_{ikj} \geq 0$ for all $1 \leq i \leq n+1$, a dual optimal solution to problem (LP-CF(p)) is $\psi_{ikj}^{\mathrm{L}*} = (c_{ikj}^{\mathrm{o}} + c_i^{\mathrm{s}})^+$, $\psi_{ikj}^{\mathrm{U}*} = (-c_{ikj}^{\mathrm{o}} - c_i^{\mathrm{s}})^+$, $\omega_i^{\mathrm{L}*} = (c_i^{\mathrm{s}})^+$, and $\omega_i^{\mathrm{U}*} = (-c_i^{\mathrm{s}})^+$. It follows that

$$V(p) = \sum_{i=1}^{n+1} \left[s_i^{\mathrm{L}} (c_i^{\mathrm{s}})^+ - s_i^{\mathrm{U}} (-c_i^{\mathrm{s}})^+ \right] + \sum_{i=1}^{n+1} \sum_{k=1}^{i} \sum_{j=i}^{n+1} \left[s_i^{\mathrm{L}} (c_{ikj}^{\mathrm{o}} + c_i^{\mathrm{s}})^+ - s_i^{\mathrm{U}} (-c_{ikj}^{\mathrm{o}} - c_i^{\mathrm{s}})^+ - s_i^{\mathrm{L}} (c_i^{\mathrm{s}})^+ + s_i^{\mathrm{U}} (-c_i^{\mathrm{s}})^+ \right] p_{ikj}$$

is a linear function of p. Therefore, (LP-CF) is equivalent to optimizing a linear function of (t,p) on polyhedron $CF_{t,p}$. It follows that there exists an optimal solution (t^*, p^*, s^*, o^*) to (LP-CF) where (t^*, p^*) is an extreme point of polyhedron $CF_{t,p}$.

Third, we show that all extreme points of $CF_{t,p}$ are integral. To this end, we show that the constraint matrix describing $CF_{t,p}$ is totally unimodular (TU). For presentation convenience, we rewrite the constraints defining $CF_{t,p}$ in inequalities as follows:

$$\sum_{k=1}^{i} \sum_{j=i}^{n+1} t_{kj} \ \geq 1 \quad \forall i = 1, \ldots, n+1, \tag{EC.19a}$$

$$-\sum_{k=1}^{i} \sum_{j=i}^{n+1} t_{kj} \ \geq -1 \quad \forall i = 1, \ldots, n+1, \tag{EC.19b}$$

$$-t_{kj} + p_{ikj} + p_{(i+1)kj} \ \geq 0 \quad \forall 1 \leq k < j \leq n+1, \ \forall k \leq i \leq j-1, \tag{EC.19c}$$

$$-\sum_{k=1}^{i} t_{ki} + \sum_{k=1}^{i} p_{iki} + \sum_{j=i+1}^{n+1} p_{(i+1)(i+1)j} \ \geq 0 \quad \forall i = 1, \ldots, n, \tag{EC.19d}$$

$$t_{kj} - p_{ikj} \ \geq 0 \quad \forall 1 \leq k \leq j \leq n+1, \ \forall k \leq i \leq j, \tag{EC.19e}$$

and we denote the constraint matrix as

$$\mathcal{CF}^0_{t,p} \ := \ \begin{bmatrix} (\sum_{k=1}^{i}\sum_{j=i}^{n+1} t_{kj}), \quad \forall 1 \leq i \leq n+1 \\ (-\sum_{k=1}^{i}\sum_{j=i}^{n+1} t_{kj}), \quad \forall 1 \leq i \leq n+1 \\ (-t_{kj} + p_{ikj} + p_{(i+1)kj}), \quad \forall 1 \leq k < j \leq n+1, \forall k \leq i \leq j-1 \\ (-\sum_{k=1}^{i} t_{ki} + \sum_{k=1}^{i} p_{iki} + \sum_{j=i+1}^{n+1} p_{(i+1)(i+1)j}), \quad \forall 1 \leq i \leq n \\ (t_{kj} - p_{ikj}), \quad \forall 1 \leq k \leq i \leq j \leq n+1 \end{bmatrix},$$

where the five row sub-matrices in matrix $\mathcal{CF}^0_{t,p}$ are associated with the left-hand side of constraints (EC.19a)–(EC.19e), respectively. To show that matrix $\mathcal{CF}^0_{t,p}$ is TU, we conduct pivot operations on the matrix with variables p_{ikj} and t_{kj}. Note that a matrix is TU if and only if it remains TU after pivot operations (Nemhauser and Wolsey 1999). We conduct the following pivot operations in order.

(i) For all $1 \leq k \leq j \leq n+1$ and $k \leq i \leq j$, pivot with variable p_{ikj} based on the component -1 in sub-matrix $(t_{kj} - p_{ikj})$ (corresponding to constraints (EC.19e)). This pivot operation is equivalent to (a) adding $t_{kj} - p_{ikj}$, for all $1 \leq k \leq j \leq n+1$ and $k \leq i \leq j$, to the left-hand side of every constraint (EC.19c)–(EC.19d) in which variable p_{ikj} has coefficient 1, and (b) multiplying the left-hand side of each constraint (EC.19e) by -1. As a result, the matrix after pivoting becomes

$$\mathcal{CF}^1_{t,p} \ := \ \begin{bmatrix} (\sum_{k=1}^{i}\sum_{j=i}^{n+1} t_{kj}), \quad \forall 1 \leq i \leq n+1 \\ (-\sum_{k=1}^{i}\sum_{j=i}^{n+1} t_{kj}), \quad \forall 1 \leq i \leq n+1 \\ (t_{kj}), \quad \forall 1 \leq k < j \leq n+1, \forall k \leq i \leq j-1 \\ (\sum_{j=i+1}^{n+1} t_{(i+1)j}), \quad \forall 1 \leq i \leq n \\ (-t_{kj} + p_{ikj}), \quad \forall 1 \leq k \leq i \leq j \leq n+1 \end{bmatrix}.$$

Note that sub-matrix $(-t_{kj} + p_{ikj} + p_{(i+1)kj})$ becomes (t_{kj}) because each $-t_{kj} + p_{ikj} + p_{(i+1)kj}$ on the left-hand side of (EC.19c) is summed with $t_{kj} - p_{ikj}$ and $t_{kj} - p_{(i+1)kj}$, and so the coefficient of each t_{kj} changes from -1 to 1 after pivoting.

(ii) For all $1 \leq k < j \leq n+1$, pivot with variable t_{kj} based on any component 1 in sub-matrix (t_{kj}) (note that there are multiple components 1 corresponding to each variable t_{kj} in sub-matrix (t_{kj}) and we can pick any one of them). Since all components in each row of sub-matrix (t_{kj}) are zeros except one equaling 1, these pivot operations (a) make all coefficients of all variables t_{kj} zeros in matrix $\mathcal{CF}^1_{t,p}$ as long as $1 \le k < j \le n+1$, and (b) keep all coefficients of all variables p_{ikj} unchanged. As a result, the matrix after pivoting becomes

$$\mathcal{CF}^2_{t,p} := \begin{bmatrix} (t_{ii}), \quad \forall 1 \le i \le n+1 \\ (-t_{ii}), \quad \forall 1 \le i \le n+1 \\ (t_{kj}), \quad \forall 1 \le k < j \le n+1, \forall k \le i \le j-1 \\ (t_{(i+1)(i+1)}), \quad \forall 1 \le i \le n \\ \begin{cases} (-t_{ii} + p_{iii}), \quad \forall 1 \le i \le n+1 \\ (p_{ikj}), \quad 1 \le k < j \le n+1, \forall k \le i \le j-1 \end{cases} \end{bmatrix}.$$

It follows that matrix $\mathcal{CF}^2_{t,p}$ contains only $\{-1, 0, 1\}$ entries, has no more than two nonzero entries in each row, and the sum of the entries is zero for each row containing two nonzero entries. Hence, matrix $\mathcal{CF}^2_{t,p}$ is TU, and so is matrix $\mathcal{CF}^0_{t,p}$.

Therefore, the extreme points of polyhedron $CF_{t,p}$ are integral and so property 1 is proved.

Fourth, to show property 2, we consider any extreme point (t, q, s, p, o) of polyhedron CF. By constraints (18a) and (20a), we have $q_i = \sum_{k=1}^{i} \sum_{j=i}^{n+1} p_{ikj} \le \sum_{k=1}^{i} \sum_{j=i}^{n+1} t_{kj} = 1$, and so $q_i \in \{0, 1\}$ because each $p_{ikj} \in \{0, 1\}$ by property 1. This shows property 2.

Finally, to show property 3, we consider any extreme point (t, q, s, p, o) of polyhedron CF. We show $p_{ikj} = q_i t_{kj}$ by discussing the following cases.

(i) If $q_i = 0$, then $p_{ikj} = 0$ for all $1 \le k \le i$ and $i \le j \le n+1$ because $q_i = \sum_{k=1}^{i} \sum_{j=i}^{n+1} p_{ikj}$. It follows that $p_{ikj} = q_i t_{kj}$.

(ii) If $q_i = 1$, then there exist $1 \le k^* \le i$ and $i \le j^* \le n+1$ such that $p_{ik^*j^*} = 1$ and any other $p_{ikj} = 0$. It follows that $t_{k^*j^*} = 1$ because $p_{ikj} - t_{kj} \le 0$ by constraint (18a), and any other $t_{kj} = 0$ because $\sum_{k=1}^{i} \sum_{j=i}^{n+1} t_{kj} = 1$ given by (17b). Therefore, we have $p_{ik^*j^*} = q_i t_{k^*j^*} = 1$ and $p_{ikj} = q_i t_{kj} = 0$ for all other $1 \le k \le i$ and $i \le j \le n+1$.

For all $1 \le i \le n+1$, since $\sum_{k=1}^{i} \sum_{j=i}^{n+1} t_{kj} = 1$, there exist $1 \le k^* \le i$ and $i \le j^* \le n+1$ such that $t_{k^*j^*} = 1$ and any other $t_{kj} = 0$. Since (t, q, s, p, o) is an extreme point of polyhedron CF, each o_{ikj} satisfies either inequality (EC.17a) or (EC.17b) at equality, and each s_i satisfies either inequality (EC.17c) or (EC.17d) at equality. We discuss the following two cases to show $o_{ikj} = q_i s_i t_{kj}$.

(i) If $q_i = 0$, then $p_{ikj} = q_i t_{kj} = 0$ for all $1 \le k \le i$ and $i \le j \le n+1$. It follows from inequalities (EC.17a)–(EC.17b) that each corresponding $o_{ikj} = 0$. Therefore, we have $o_{ikj} = s_i p_{ikj} = 0$, or equivalently $o_{ikj} = q_i s_i t_{kj} = 0$, for all $1 \le k \le j \le n+1$ and $k \le i \le j$.

(ii) If $q_i = 1$, then $p_{ik^*j^*} = q_i t_{k^*j^*} = 1$ and $p_{ikj} = 0$ for all other $1 \le k \le i$ and $i \le j \le n+1$. Then, inequalities (EC.17a)–(EC.17b) yield $o_{ikj} = s_i p_{ikj} = 0$ for all $1 \le k \le i$ and $i \le j \le n+1$ such that $(k, j) \ne (k^*, j^*)$. Furthermore, inequalities (EC.17c)–(EC.17d) yield

$$s_i - \sum_{k=1}^{i} \sum_{j=i}^{n+1} \left(o_{ikj} - s_i^{\mathrm{L}} p_{ikj}\right) \; = \; s_i - o_{ik^*j^*} + s_i^{\mathrm{L}} p_{ik^*j^*} \; \ge \; s_i^{\mathrm{L}},$$

$$s_i - \sum_{k=1}^{i}\sum_{j=i}^{n+1}\left(o_{ikj} - s_i^{\mathrm{U}} p_{ikj}\right) \;=\; s_i - o_{ik^*j^*} + s_i^{\mathrm{U}} p_{ik^*j^*} \;\leq\; s_i^{\mathrm{U}}.$$

It follows that $s_i = o_{ik^*j^*}$. Therefore, we have $o_{ik^*j^*} = s_i p_{ik^*j^*}$.

□

We are now ready to show Theorem 1.

Proof of Theorem 1 ($CF \supseteq \mathrm{conv}(F)$) By Proposition 1, since polyhedron CF consists of either trivial equalities/inequalities or valid inequalities of set F, we have $(t,q,s,p,o) \in CF$ if $(t,q,s,p,o) \in F$. It follows that $CF \supseteq \mathrm{conv}(F)$.

($CF \subseteq \mathrm{conv}(F)$) By Proposition EC.5, since each extreme point (t,q,s,p,o) of CF satisfies properties 1, 2, and 3, $(t,q,s,p,o) \in F$. By the Minkowski's Theorem on polyhedron, we have $x \in \mathrm{conv}(F)$ if $x \in CF$. It follows that $CF \subseteq \mathrm{conv}(F)$. This completes the proof. □

EC.4. Proofs for the DR CVaR Model

EC.4.1. Proof of Proposition EC.1

Proof of Proposition EC.1 We analyze the following two cases based on the value of K.

When $K \in \{2,\dots,n\}$: For the embedded minimization problem in constraint (EC.10b), we observe that the constraint matrix of D_q, described by constraints $\sum_{j=i}^{i+K-1} q_j \geq 1$ for all $1 \leq i \leq n-K+1$, is an interval matrix and thus TU. It follows that $\mathrm{conv}(D_q) = \{q \in [0,1]^n : \sum_{j=i}^{i+K-1} q_j \geq 1,\ \forall 1 \leq i \leq n-K+1\}$. Because the feasible regions of variables q and s (i.e., D_q and D_s) are disjoint in (EC.10b), we can replace D_q with $\mathrm{conv}(D_q)$ and obtain

$$\theta + \min_{q,s} \left\{\sum_{i=1}^{n} \rho_i s_i + \sum_{i=1}^{n} \gamma_i q_i\right\} \;\geq\; 0 \tag{EC.20a}$$

$$\text{s.t. } s_i^{\mathrm{L}} \leq s_i \leq s_i^{\mathrm{U}} \quad \forall 1 \leq i \leq n, \tag{EC.20b}$$

$$\sum_{j=i}^{i+K-1} q_j \geq 1 \quad \forall 1 \leq i \leq n-K+1, \tag{EC.20c}$$

$$q_i \leq 1 \quad \forall 1 \leq i \leq n, \tag{EC.20d}$$

$$q_i, s_i \geq 0 \quad \forall 1 \leq i \leq n. \tag{EC.20e}$$

Presenting linear program (EC.20) in its dual form yields (EC.11a)–(EC.11d), where dual variables $\chi_i^{\mathrm{L/U}}$, β_i, and η_i are associated with constraints (EC.20b), (EC.20c), and (EC.20d) respectively, and dual constraints (EC.11b) and (EC.11c) are associated with primal variables q_i and s_i, respectively.

When $K = n+1$: In this case, $D_q = \{0,1\}^n$ and so $\mathrm{conv}(D_q) = [0,1]^n$. It follows that constraint (EC.10b) is equivalent to

$$\theta + \min_{q,s} \left\{\sum_{i=1}^{n} \rho_i s_i + \sum_{i=1}^{n} \gamma_i q_i\right\} \;\geq\; 0$$

$$
\begin{aligned}
\text{s.t. } & s_i^{\mathrm{L}} \le s_i \le s_i^{\mathrm{U}} \quad \forall 1 \le i \le n, \\
& q_i \le 1 \quad \forall 1 \le i \le n, \\
& q_i, s_i \ge 0 \quad \forall 1 \le i \le n,
\end{aligned}
$$

Similar to the case when $K \in \{2, \dots, n\}$, we can present the embedded LP in its dual form to obtain the following linear constraints:

$$
\begin{aligned}
\theta + \sum_{i=1}^{n} (s_i^{\mathrm{L}}\chi_i^{\mathrm{L}} - s_i^{\mathrm{U}}\chi_i^{\mathrm{U}} - \eta_i) \ & \ge 0, \\
-\eta_i \ & \le \gamma_i \quad \forall 1 \le i \le n, \\
\chi_i^{\mathrm{L}} - \chi_i^{\mathrm{U}} \ & \le \rho_i \quad \forall 1 \le i \le n, \\
\chi_i^{\mathrm{L}},\ \chi_i^{\mathrm{U}}, \eta_i \ge 0 \ & \quad \forall 1 \le i \le n.
\end{aligned}
$$

We note that these linear constraints are equivalent to (EC.11a)–(EC.11d) because $\sum_{i=1}^{n-K+1} \beta_i = \sum_{i=1}^{0} \beta_i = 0$ and $\sum_{j=\max\{i-K+1,1\}}^{\min\{i,n-K+1\}} \beta_j = \sum_{j=1}^{0} \beta_j = 0$. The proof is completed. ☐

EC.4.2. Proof of Theorem EC.1

We take the following three steps to prove Theorem EC.1.

Step 1: We prove the validity of inequalities (EC.13a)–(EC.13f) in the following proposition.

PROPOSITION EC.6. *When* $D_q = D_q^{(2)}$*, inequalities* (EC.13a)–(EC.13f) *are valid for set* $G = \{(t, q, s, p, o) : \text{(EC.12b)–(EC.12g)}\}$.

Proof of Proposition EC.6 First, since $\sum_{k=1}^{n} t_k \le 1$ by constraint (EC.12f) and $q_n \ge 0$, we have

$$\sum_{k=1}^{n} p_{kn} \ = \ q_n \sum_{k=1}^{n} t_k \ \le \ q_n,$$

which shows inequality (EC.13a).

Second, for all $1 \le k \le i \le n-1$, since $q_i + q_{i+1} \ge 1$ by the definition of D_q and $t_k \ge 0$, we have

$$p_{ki} + p_{k(i+1)} \ = \ t_k(q_i + q_{i+1}) \ \ge \ t_k,$$

which shows inequalities (EC.13b).

Third, for all $1 \le i \le n$, since $t_i \ge 0$, $\forall i$ and thus $\sum_{k=1}^{i} t_k \le \sum_{k=1}^{n} t_k \le 1$ by constraint (EC.12f) and since $q_i \le 1 \ \Rightarrow \ q_i - 1 \le 0$, we have

$$\sum_{k=1}^{i} (p_{ki} - t_k) \ = \ (q_i - 1) \sum_{k=1}^{i} t_k \ \ge \ q_i - 1,$$

which shows inequality (EC.13c).

Fourth, for all $1 \le i \le n-1$ because (a) $\sum_{k=1}^{i} t_k \le \sum_{k=1}^{n} t_k \le 1$ by constraint (EC.12f) and $t_k \ge 0$, $\forall k$, and (b) $q_i + q_{i+1} \ge 1$ by the definition of D_q, we have

$$\begin{aligned}\sum_{k=1}^{i}(p_{ki} + p_{k(i+1)}) - (q_i + q_{i+1}) \;&=\; (q_i + q_{i+1})\sum_{k=1}^{i} t_k - (q_i + q_{i+1})\\ =\; (q_i + q_{i+1})\left(\sum_{k=1}^{i} t_k - 1\right) \;&\le\; \sum_{k=1}^{i} t_k - 1,\end{aligned}$$

which shows inequality (EC.13d).

Finally, for each $1 \le i \le n$, since $\sum_{k=1}^{n} t_k \le 1$ by constraint (EC.12f), and $t_k, q_i \in [0,1]$, we have $\sum_{k=1}^{i} p_{ki} = q_i(\sum_{k=1}^{i} t_k) \le q_i(\sum_{k=1}^{n} t_k) \le 1$. Also, because $s_i \in [s_i^{\mathrm{L}}, s_i^{\mathrm{U}}]$, it follows that

$$\begin{aligned}\sum_{k=1}^{i}(o_{ki} - s_i^{\mathrm{L}} p_{ki}) \;&=\; (s_i - s_i^{\mathrm{L}})\sum_{k=1}^{i} p_{ki} \;\le\; s_i - s_i^{\mathrm{L}},\\ \sum_{k=1}^{i}(o_{ki} - s_i^{\mathrm{U}} p_{ki}) \;&=\; (s_i - s_i^{\mathrm{U}})\sum_{k=1}^{i} p_{ki} \;\ge\; s_i - s_i^{\mathrm{U}},\end{aligned}$$

which shows inequalities (EC.13e)–(EC.13f). □

Step 2: We show the properties of the extreme points of polyhedron CG in the following proposition. Recall that $CG = \{(t,q,s,p,o) : \text{(EC.12b), (EC.12d), (EC.13a)–(EC.13f)}\}$.

PROPOSITION EC.7. *Each extreme point (t,q,s,p,o) of CG has the following properties:*

1. $t_k, q_i, p_{ki} \in \{0,1\}$ *for all* $1 \le k \le i \le n$*;*
2. $p_{ki} = t_k q_i$ *and* $o_{ki} = t_k q_i s_i$ *for all* $1 \le k \le i \le n$*.*

Proof of Proposition EC.7 For any c_k^{t}, c_i^{q}, c_i^{s}, c_{ki}^{p}, and c_{ki}^{o} for all $1 \le k \le i \le n$, we consider linear program

$$\begin{aligned}\text{(LP-CG)} \quad \min_{t,q,s,p,o} \quad & \sum_{i=1}^{n}(c_i^{\mathrm{q}} q_i + c_i^{\mathrm{s}} s_i) + \sum_{k=1}^{n}\left(c_k^{\mathrm{t}} t_k + \sum_{i=k}^{n}(c_{ki}^{\mathrm{p}} p_{ki} + c_{ki}^{\mathrm{o}} o_{ki})\right)\\ \text{s.t.} \quad & (t,q,s,p,o) \in CG.\end{aligned}$$

To prove that each extreme point of CG satisfies properties 1 and 2, we show that for any c_k^{t}, c_i^{q}, c_i^{s}, c_{ki}^{p}, and c_{ki}^{o}, there exists an optimal solution $(t^*, q^*, s^*, p^*, o^*)$ to (LP-CG) that satisfies properties 1 and 2. First, we rewrite (LP-CG) as a two-stage formulation as follows:

$$\begin{aligned}\min_{t,q,p} \quad & \sum_{i=1}^{n} c_i^{\mathrm{q}} q_i + \sum_{k=1}^{n}\left(c_k^{\mathrm{t}} t_k + \sum_{i=k}^{n} c_{ki}^{\mathrm{p}} p_{ki}\right) + V(p)\\ \text{s.t.} \quad & (t,q,p) \in CG_{t,q,p},\end{aligned}$$

where polyhedron $CG_{t,q,p} := \{(t,q,p) : \text{(EC.12b), (EC.13a)–(EC.13d)}\}$ and $V(p)$ represents a value function of p defined as

$$\text{(LP-CG}(p)\text{)} \quad V(p) := \min_{s,o}\left\{\sum_{i=1}^{n} c_i^{\mathrm{s}} s_i + \sum_{k=1}^{n}\sum_{i=k}^{n} c_{ki}^{\mathrm{o}} o_{ki} : (s,o) \in CG_{s,o}(p)\right\},$$

$$\text{where} \quad CG_{s,o}(p) = \left\{ (s,o): \ \text{(EC.12d), (EC.13e), (EC.13f)} \right\}$$

$$= \left\{ (s,o): \ o_{ki} \geq s_i^{\mathrm{L}} p_{ki} \quad \forall 1 \leq k \leq i \leq n, \right. \tag{EC.21a}$$

$$o_{ki} \leq s_i^{\mathrm{U}} p_{ki} \quad \forall 1 \leq k \leq i \leq n, \tag{EC.21b}$$

$$s_i - \sum_{k=1}^{i} (o_{ki} - s_i^{\mathrm{L}} p_{ki}) \geq s_i^{\mathrm{L}} \quad \forall 1 \leq i \leq n, \tag{EC.21c}$$

$$\left. s_i - \sum_{k=1}^{i} (o_{ki} - s_i^{\mathrm{U}} p_{ki}) \leq s_i^{\mathrm{U}} \quad \forall 1 \leq i \leq n \right\} \tag{EC.21d}$$

represents a parametric polyhedron depending on the values of p_{ki}. We solve (LP-CG(p)) by considering its dual formulation

$$V(p) = \max_{\psi,\omega} \quad \sum_{i=1}^{n}\sum_{k=1}^{i} (s_i^{\mathrm{L}} p_{ki} \psi_{ki}^{\mathrm{L}} - s_i^{\mathrm{U}} p_{ki} \psi_{ki}^{\mathrm{U}}) + \sum_{i=1}^{n} \left[s_i^{\mathrm{L}} \left(1 - \sum_{k=1}^{i} p_{ki}\right) \omega_i^{\mathrm{L}} - s_i^{\mathrm{U}} \left(1 - \sum_{k=1}^{i} p_{ki}\right) \omega_i^{\mathrm{U}} \right]$$

$$\text{s.t.} \quad \psi_{ki}^{\mathrm{L}} - \psi_{ki}^{\mathrm{U}} - \omega_i^{\mathrm{L}} + \omega_i^{\mathrm{U}} = c_{ki}^{\mathrm{o}} \quad \forall 1 \leq k \leq i \leq n, \tag{EC.22a}$$

$$\omega_i^{\mathrm{L}} - \omega_i^{\mathrm{U}} = c_i^{\mathrm{s}} \quad \forall 1 \leq i \leq n, \tag{EC.22b}$$

where dual variables $\psi_{ki}^{\mathrm{L/U}}$ and $\omega_i^{\mathrm{L/U}}$ are associated with primal constraints (EC.21a)–(EC.21b) and (EC.21c)–(EC.21d), respectively (after transforming all "$\leq$" inequalities into the "$\geq$" form), and dual constraints (EC.22a) and (EC.22b) are associated with primal variables o_{ki} and s_i, respectively. Because (i) $s_i^{\mathrm{L}} p_{ki} \leq o_{ki} \leq s_i^{\mathrm{U}} p_{ki}$ by (EC.12d), and so $p_{ki} \geq 0$ for all $1 \leq k \leq i \leq n$, and (ii) $s_i^{\mathrm{L}}(1 - \sum_{k=1}^{i} p_{ki}) \leq s_i - \sum_{k=1}^{i} o_{ki} \leq s_i^{\mathrm{U}}(1 - \sum_{k=1}^{i} p_{ki})$ by (EC.13e)–(EC.13f), and so $1 - \sum_{k=1}^{i} p_{ki} \geq 0$ for all $1 \leq i \leq n$, a dual optimal solution to problem (LP-CG(p)) is $\psi_{ki}^{\mathrm{L}*} = (c_{ki}^{\mathrm{o}} + c_i^{\mathrm{s}})^+$, $\psi_{ki}^{\mathrm{U}*} = (-c_{ki}^{\mathrm{o}} - c_i^{\mathrm{s}})^+$, $\omega_i^{\mathrm{L}*} = (c_i^{\mathrm{s}})^+$, and $\omega_i^{\mathrm{U}*} = (-c_i^{\mathrm{s}})^+$. It follows that

$$V(p) = \sum_{i=1}^{n} \left[s_i^{\mathrm{L}} (c_i^{\mathrm{s}})^+ - s_i^{\mathrm{U}} (-c_i^{\mathrm{s}})^+ \right] + \sum_{i=1}^{n}\sum_{k=1}^{i} \left[s_i^{\mathrm{L}} (c_{ki}^{\mathrm{o}} + c_i^{\mathrm{s}})^+ - s_i^{\mathrm{U}} (-c_{ki}^{\mathrm{o}} - c_i^{\mathrm{s}})^+ - s_i^{\mathrm{L}} (c_i^{\mathrm{s}})^+ + s_i^{\mathrm{U}} (-c_i^{\mathrm{s}})^+ \right] p_{ki}$$

is a linear function of p. Therefore, (LP-CG) is equivalent to optimizing a linear function of (t,q,p) on polyhedron $CG_{t,q,p}$. It follows that there exists an optimal solution $(t^*, q^*, s^*, p^*, o^*)$ to (LP-CG) where (t^*, q^*, p^*) is an extreme point of polyhedron $CG_{t,q,p}$.

Second, we show that all extreme points of $CG_{t,q,p}$ are integral. To this end, we show that the constraint matrix describing $CG_{t,q,p}$ is TU. For presentation convenience, we rewrite the constraints defining $CG_{t,q,p}$ as follows:

$$q_i + q_{i+1} + \sum_{k=1}^{i} t_k - \sum_{k=1}^{i} (p_{ki} + p_{k(i+1)}) \ \geq 1 \quad \forall 1 \leq i \leq n-1 \tag{EC.23a}$$

$$q_n - \sum_{k=1}^{n} p_{kn} \ \geq 0, \tag{EC.23b}$$

$$-q_i - \sum_{k=1}^{i} t_k + \sum_{k=1}^{i} p_{ki} \ \geq \ -1 \quad \forall 1 \leq i \leq n, \tag{EC.23c}$$

$$-t_k + p_{ki} + p_{k(i+1)} \ \geq 0 \quad \forall 1 \leq k \leq i \leq n-1, \tag{EC.23d}$$

$$t_k - p_{ki} \ \geq 0 \quad \forall 1 \leq k \leq i \leq n, \tag{EC.23e}$$

and we denote the constraint matrix as

$$\mathcal{CG}^0_{t,q,p} := \begin{bmatrix} (q_i + q_{i+1} + \sum_{k=1}^{i} t_k - \sum_{k=1}^{i}(p_{ki} + p_{k(i+1)})), \quad \forall 1 \leq i \leq n-1 \\ (q_n - \sum_{k=1}^{n} p_{kn}) \\ (-q_i - \sum_{k=1}^{i} t_k + \sum_{k=1}^{i} p_{ki}), \quad \forall 1 \leq i \leq n \\ (-t_k + p_{ki} + p_{k(i+1)}), \quad \forall 1 \leq k \leq i \leq n-1 \\ (t_k - p_{ki}), \quad \forall 1 \leq k \leq i \leq n \end{bmatrix},$$

where the five rows of sub-matrices are associated with constraints (EC.23a)–(EC.23e), respectively. To show that matrix $\mathcal{CG}^0_{t,q,p}$ is TU, we conduct pivot operations on the matrix with variables p_{ki}, t_k, and q_i. Note that a matrix is TU if and only if it remains TU after pivot operations (cf. Nemhauser and Wolsey 1999). We conduct the following pivot operations in order.

(i) For all $1 \leq k \leq i \leq n$, pivot with variable p_{ki} based on the component -1 in sub-matrix $(t_k - p_{ki})$ (corresponding to constraints (EC.23e)). This pivot operation is equivalent to (a) adding $t_k - p_{ki}$, for all $1 \leq k \leq i \leq n$, to the left-hand side of every constraint (EC.23c)–(EC.23d) in which variable p_{ki} has coefficient 1, (b) adding $p_{ki} - t_k$, for all $1 \leq k \leq i \leq n$, to the left-hand side of every constraint (EC.23a)–(EC.23b) in which variable p_{ki} has coefficient -1 and (c) multiplying each left-hand side of constraint (EC.23e) by -1. As a result, the matrix after pivoting becomes

$$\mathcal{CG}^1_{t,q,p} := \begin{bmatrix} (q_i + q_{i+1} - \sum_{k=1}^{i} t_k), \quad \forall 1 \leq i \leq n-1 \\ (q_n - \sum_{k=1}^{n} t_k) \\ (-q_i), \quad \forall 1 \leq i \leq n \\ (t_k), \quad \forall 1 \leq k \leq i \leq n-1 \\ (-t_k + p_{ki}), \quad \forall 1 \leq k \leq i \leq n \end{bmatrix},$$

Note that sub-matrix $(-t_k + p_{ki} + p_{k(i+1)})$ becomes (t_k) because the left-hand side of each constraint (EC.23d), $-t_k + p_{ki} + p_{k(i+1)}$, is summed with $t_k - p_{ki}$ and $t_k - p_{k(i+1)}$ and so the coefficient of each t_k changes from -1 to 1 after pivoting. Meanwhile, sub-matrix $(-q_i - \sum_{k=1}^{i} t_k + \sum_{k=1}^{i} p_{ki})$ becomes $(-q_i)$ after pivoting because, for each $1 \leq i \leq n$, $-q_i - \sum_{k=1}^{i} t_k + \sum_{k=1}^{i} p_{ki}$ is summed with $t_k - p_{ki}$ for all $1 \leq k \leq i$.

(ii) For all $1 \leq k \leq n-1$, pivot with variable t_k based on any component 1 in sub-matrix (t_k) (note that there are multiple components 1 associated with each variable t_k in (t_k) and we can pick any one of them). Since all components in each row of sub-matrix (t_k) are zeros except one equaling 1, these pivot operations (a) make all coefficients of all variables t_k zeros in matrix $\mathcal{CG}^1_{t,q,p}$ as long as $1 \le k \le n-1$, and (b) keep all coefficients of all variables q_i and p_{ki} unchanged. As a result, the matrix after pivoting becomes

$$\mathcal{CG}^2_{t,q,p} := \begin{bmatrix} (q_i + q_{i+1}), \quad \forall 1 \le i \le n-1 \\ (q_n - t_n) \\ (-q_i), \quad \forall 1 \le i \le n \\ (t_k), \quad \forall 1 \le k \le i \le n-1 \\ \begin{cases} (p_{ki}), \quad \forall 1 \le k \le i \le n-1 \\ (-t_n + p_{nn}), \end{cases} \end{bmatrix}.$$

(iii) For all $1 \le i \le n$, pivot with variable q_i based on any component -1 in sub-matrix $(-q_i)$ in $\mathcal{CG}^2_{t,q,p}$. Since $(-q_i)$ is an identity matrix, these pivot operations eliminate all coefficients of variables q_i in all other sub-matrices. As a result, the matrix after pivoting becomes

$$\mathcal{CG}^2_{t,q,p} := \begin{bmatrix} (0), \quad \forall 1 \le i \le n-1 \\ (-t_n) \\ (q_i), \quad \forall 1 \le i \le n \\ (t_k), \quad \forall 1 \le k \le i \le n-1 \\ \begin{cases} (p_{ki}), \quad \forall 1 \le k \le i \le n-1 \\ (-t_n + p_{nn}), \end{cases} \end{bmatrix}.$$

It follows that matrix $\mathcal{CG}^3_{t,q,p}$ contains only $\{-1, 0, 1\}$ entries, has no more than two nonzero entries in each row, and the sum of the entries is zero for each row containing two nonzero entries. Hence, matrix $\mathcal{CG}^3_{t,q,p}$ is TU, and so is matrix $\mathcal{CG}^0_{t,q,p}$.

Therefore, the extreme points of polyhedron $CG_{t,q,p}$ are integral and so property 1 is proved.

Third, to show property 2, we consider any extreme point (t, q, s, p, o) of polyhedron CG. Because $\sum_{k=1}^n t_k \le 1$ and each $t_k \in \{0, 1\}$ by property 1, we show $p_{ki} = t_k q_i$ by discussing the following two cases on values of t_k.

(i) If $t_k = 0$ for all $1 \le k \le n$, then $p_{ki} = 0$ for all $1 \le k \le i \le n$ because $p_{ki} \le t_k$. It follows that $p_{ki} = t_k q_i = 0$.

(ii) If there exists $1 \le k^* \le n$ such that $t_{k^*} = 1$, then any other $t_k = 0$. It follows that $p_{ki} = 0$, and so $p_{ki} = t_k q_i = 0$ for all $1 \le k \le i \le n$ and $k \ne k^*$. For all $k^* \le i \le n$, constraints (EC.13c) yield

$$-q_i - \sum_{k=1}^{i} t_k + \sum_{k=1}^{i} p_{ki} \;=\; -q_i - 1 + p_{k^*i} \;\ge\; -1 \quad \Rightarrow \quad p_{k^*i} \ge q_i.$$

Also, for all $k^* \le i \le n-1$, constraints (EC.13d) yield

$$q_i + q_{i+1} + \sum_{k=1}^{i} t_k - \sum_{k=1}^{i} (p_{ki} + p_{k(i+1)}) \;=\; q_i + q_{i+1} + 1 - p_{k^*i} - p_{k^*(i+1)} \;\ge\; 1$$

$$\Rightarrow \; p_{k^*i} + p_{k^*(i+1)} \le q_i + q_{i+1}.$$

It follows that $p_{k^*i} + p_{k^*(i+1)} = q_i + q_{i+1}$ for all $k^* \le i \le n-1$. Furthermore, constraint (EC.13a) implies $q_n - \sum_{k=1}^{n} p_{kn} = q_n - p_{k^*n} \ge 0$, and so $q_n = p_{k^*n}$. Therefore, $q_i = p_{k^*i}$, or equivalently $q_i = t_{k^*} p_{k^*i}$ since $t_{k^*} = 1$, for all $k^* \le i \le n$.

Since (t, q, s, p, o) is an extreme point of polyhedron CG, each o_{ki} satisfies either inequality (EC.21a) or (EC.21b) at equality, and each s_i satisfies either inequality (EC.21c) or (EC.21d) at equality. We discuss the following cases to show $o_{ki} = s_i p_{ki} = t_k q_i s_i$.

(i) If $t_k = 0$ for all $1 \le k \le n$, then $p_{ki} = 0$ for all $1 \le k \le i \le n$ because $p_{ki} \le t_k$. It follows that $o_{ki} = 0$ by constraints (EC.12d). Therefore, $o_{ki} = s_i p_{ki} = 0$.

(ii) If there exists $1 \le k^* \le n$ such that $t_{k^*} = 1$, then any other $t_k = 0$. It follows that $p_{ki} = 0$, and so $o_{ki} = s_i p_{ki} = 0$ for all $1 \le k \le i \le n$ and $k \ne k^*$. Then, for all $k^* \le i \le n$, inequalities (EC.21c)–(EC.21d) yield

$$s_i - \sum_{k=1}^{i} (o_{ki} - s_i^{\mathrm{L}} p_{ki}) \;=\; s_i - o_{k^*i} + s_i^{\mathrm{L}} p_{k^*i} \;\ge\; s_i^{\mathrm{L}}, \tag{EC.24a}$$

$$s_i - \sum_{k=1}^{i} (o_{ki} - s_i^{\mathrm{U}} p_{ki}) \;=\; s_i - o_{k^*i} + s_i^{\mathrm{U}} p_{k^*i} \;\le\; s_i^{\mathrm{U}}. \tag{EC.24b}$$

Hence, each s_i satisfies either inequality (EC.24a) or (EC.24b) at equality. We discuss the following two sub-cases to finish the proof.

Sub-case 1. If $p_{k^*i} = 0$, then $o_{k^*i} = 0$ by constraints (EC.12d). Therefore, $o_{k^*i} = s_i p_{k^*i} = 0$.

Sub-case 2. If $p_{k^*i} = 1$, then inequalities (EC.24a)–(EC.24b) imply $s_i = o_{k^*i}$. Therefore, $o_{k^*i} = s_i p_{k^*i}$.

□

Step 3: Finally, we prove Theorem EC.1 based on the previous two propositions.

Proof of Theorem EC.1 ($CG \supseteq \mathrm{conv}(G)$) By Proposition EC.6, since polyhedron CG consists of either trivial equalities/inequalities or valid inequalities of set G, we have $(t, q, s, p, o) \in CG$ if $(t, q, s, p, o) \in G$. It follows that $CG \supseteq \mathrm{conv}(G)$.

($CG \subseteq \mathrm{conv}(G)$) By Proposition EC.7, since each extreme point (t, q, s, p, o) of CG satisfies properties 1, 2, and 3, $(t, q, s, p, o) \in G$. By the Minkowski's Theorem on polyhedron, we have $x \in \mathrm{conv}(G)$ if $x \in CG$. It follows that $CG \subseteq \mathrm{conv}(G)$. This completes the proof. □

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