Improved Decision Rule Approximations for Multi-Stage Robust Optimization via Copositive Programming (Electronic Companion)

EC.1. Proof of Lemma 2

Proof. Fix $(\boldsymbol{z},\tau)\in\mathcal{K}$. For the sake of contradiction, suppose that $\tau=0$ but $\boldsymbol{z}\neq 0$. By Assumption 1, the set $\mathcal{U}^0$ is nonempty. Choose any $\boldsymbol{u}\in\mathcal{U}^0$, so that $\boldsymbol{u}\in\mathcal{K}$ and $u_{K+1}=1$. Then, for any nonnegative scalar $\rho\geq 0$, we have $\boldsymbol{w}(\rho):=\boldsymbol{u}+\rho(\boldsymbol{z}^\top,0)^\top\in\mathcal{K}$. Furthermore, $\boldsymbol{w}(\rho)\in\mathcal{U}^0$ as $[\boldsymbol{w}(\rho)]_{K+1}=1$. Since ρ can be arbitrarily large while $\boldsymbol{z}\neq 0$, we conclude that $\mathcal{U}^0$ is unbounded, contradicting the compactness condition of Assumption 1. Thus, the claim follows. $\square$

EC.2. Proof of Lemma 3

Proof. We prove the statement by showing that the dual problem (6) admits a Slater point. To this end, we set $\alpha_i=0$, $i\in[I]$. We then seek for a scalar λ that ensures

$$(\boldsymbol{z}^\top,\tau)\left(\lambda\mathbf{e}_{K+1}\mathbf{e}_{K+1}^\top-\widehat{\boldsymbol{C}}_0\right)(\boldsymbol{z}^\top,\tau)^\top=\lambda\tau^2-(\boldsymbol{z}^\top,\tau)\widehat{\boldsymbol{C}}_0(\boldsymbol{z}^\top,\tau)^\top>0$$

for all non-zero vector $(\boldsymbol{z},\tau)$ in $\mathcal{K}$. By Lemma 2, it suffices to consider the case where $\tau>0$, in which case we may divide the expression by τ^2. We thus require that $\lambda-((\boldsymbol{z}/\tau)^\top,1)\widehat{\boldsymbol{C}}_0((\boldsymbol{z}/\tau)^\top,1)^\top$ is strictly positive for all $(\boldsymbol{z},\tau)\in\mathcal{K}$, $\tau>0$. Since $(\boldsymbol{z},\tau)\in\mathcal{K}$, we have that $(\boldsymbol{z}/\tau,1)\in\mathcal{K}$, and, by construction, $(\boldsymbol{z}/\tau,1)\in\mathcal{U}^0$. In this case, the boundedness of $\mathcal{U}^0$ implies that there exists a constant $\lambda^\star$ such that $\lambda^\star>((\boldsymbol{z}/\tau)^\top,1)\widehat{\boldsymbol{C}}_0((\boldsymbol{z}/\tau)^\top,1)^\top$ for all $(\boldsymbol{z}/\tau,1)\in\mathcal{U}^0$. The claim thus follows since the point $(\lambda,\boldsymbol{\alpha})=(\lambda^\star,\mathbf{0})$ constitutes a Slater point for the problem (6). $\square$

EC.3. Proof of Theorem 1

Proof. Using Lemmas 1 and 3, we can reformulate the maximization problem in the objective function of $(\mathcal{L})$ as a copositive minimization problem. To this end, for any fixed decision rule coefficients $\boldsymbol{Y}\in\mathbb{R}^{N\times(K+1)}$, we consider the maximization problem given by

$$\sup_{\boldsymbol{u}\in\mathcal{U}}(\widehat{\boldsymbol{D}}\boldsymbol{u})^\top\boldsymbol{Y}\boldsymbol{u}. \tag{EC.1}$$

By Lemma 1, the problem can be reformulated as a linear program over the cone of completely positive matrices with respect to $\mathcal{K}$, as follows:

$$\begin{array}{ll}\sup & \dfrac{1}{2}\left(\widehat{\boldsymbol{D}}^\top\boldsymbol{Y}+\boldsymbol{Y}^\top\widehat{\boldsymbol{D}}\right)\bullet\boldsymbol{U}\\ \text{s.t.} & \mathbf{e}_{K+1}\mathbf{e}_{K+1}^\top\bullet\boldsymbol{U}=1\\ & \widehat{\boldsymbol{C}}_i\bullet\boldsymbol{U}=0\quad\forall i\in[I]\\ & \boldsymbol{U}\in\mathcal{S}^{K+1},\ \boldsymbol{U}\in\mathcal{CP}(\mathcal{K})\end{array} \tag{EC.2}$$

Letting λ and $\boldsymbol{\alpha}$ be the dual variables corresponding to the constraints $\mathbf{e}_{K+1}\mathbf{e}_{K+1}^\top \bullet \boldsymbol{U} = 1$ and $\widehat{\boldsymbol{C}}_i \bullet \boldsymbol{U} = 0$, $i \in [I]$, respectively, the dual problem is written as:

$$\begin{array}{ll} \inf & \lambda \\ \text{s.t.} & \lambda \mathbf{e}_{K+1}\mathbf{e}_{K+1}^\top - \dfrac{1}{2}\left(\widehat{\boldsymbol{D}}^\top \boldsymbol{Y} + \boldsymbol{Y}^\top \widehat{\boldsymbol{D}}\right) + \sum_{i=1}^{I} \alpha_i \widehat{\boldsymbol{C}}_i \in \mathcal{COP}(\mathcal{K}) \\ & \lambda \in \mathbb{R},\ \boldsymbol{\alpha} \in \mathbb{R}^I. \end{array} \tag{EC.3}$$

In view of Lemma 3, strong duality holds for the primal and dual pair, i.e., the optimal value of problem (EC.1) coincides with that of problem (EC.3). Replacing the maximization problem in ($\mathcal{L}$) with the minimization problem in (EC.3) yields the objective function and the first constraint in (9).

Next, using standard techniques from robust optimization, we reformulate the semi-infinite constraints in ($\mathcal{L}$) into a finite constraint system. By substituting the definition of problem parameters $\boldsymbol{\mathcal{A}}(\boldsymbol{u})$, $\boldsymbol{\mathcal{B}}(\boldsymbol{u})$, and $\boldsymbol{h}(\boldsymbol{u})$, and using the definitions in (7), we can simplify the semi-infinite constraints in ($\mathcal{L}$) to the constraints

$$\boldsymbol{u}^\top \widehat{\boldsymbol{\Theta}}_j \boldsymbol{x} + \boldsymbol{u}^\top \widehat{\boldsymbol{\Lambda}}_j \boldsymbol{Y}\boldsymbol{u} \geq \widehat{\boldsymbol{h}}_j^\top \boldsymbol{u} \quad \forall \boldsymbol{u} \in \mathcal{U} \quad \forall j \in [J],$$

where $\widehat{\boldsymbol{\Theta}}_j$ and $\widehat{\boldsymbol{\Lambda}}_j$ are defined as in (7). For any fixed $(\boldsymbol{x}, \boldsymbol{Y}) \in \mathbb{R}^M \times \mathbb{R}^{N\times(K+1)}$, we consider the j-th constraint separately, which can equivalently be stated as

$$\inf_{\boldsymbol{u}\in\mathcal{U}} \left(\boldsymbol{u}^\top \widehat{\boldsymbol{\Theta}}_j \boldsymbol{x} + \boldsymbol{u}^\top \widehat{\boldsymbol{\Lambda}}_j \boldsymbol{Y}\boldsymbol{u} - \widehat{\boldsymbol{h}}_j^\top \boldsymbol{u}\right) \geq 0. \tag{EC.4}$$

By Lemma 1, the minimization problem on the left-hand side of (EC.4) can be reformulated as the following linear program over the cone of completely positive matrices:

$$\begin{array}{ll} \inf & \boldsymbol{\Omega}_j(\boldsymbol{x}, \boldsymbol{Y}) \bullet \boldsymbol{U}_j \\ \text{s.t.} & \mathbf{e}_{K+1}\mathbf{e}_{K+1}^\top \bullet \boldsymbol{U}_j = 1 \\ & \widehat{\boldsymbol{C}}_i \bullet \boldsymbol{U}_j = 0 \quad \forall i \in [I] \\ & \boldsymbol{U}_j \in \mathcal{S}^{K+1},\ \boldsymbol{U}_j \in \mathcal{CP}(\mathcal{K}). \end{array} \tag{EC.5}$$

Letting $\pi_j \in \mathbb{R}$ and $\boldsymbol{\beta}_j \in \mathbb{R}^I$ be the dual variables corresponding to the constraints $\mathbf{e}_{K+1}\mathbf{e}_{K+1}^\top \bullet \boldsymbol{U}_j = 1$ and $\widehat{\boldsymbol{C}}_i \bullet \boldsymbol{U}_j = 0$, $i \in [I]$, respectively, the dual problem is given by

$$\begin{array}{ll} \sup & \pi_j \\ \text{s.t.} & \boldsymbol{\Omega}_j(\boldsymbol{x}, \boldsymbol{Y}) - \pi_j \mathbf{e}_{K+1}\mathbf{e}_{K+1}^\top - \sum_{i=1}^{I} [\boldsymbol{\beta}_j]_i \widehat{\boldsymbol{C}}_i \in \mathcal{COP}(\mathcal{K}) \\ & \pi_j \in \mathbb{R},\ \boldsymbol{\beta}_j \in \mathbb{R}^I. \end{array} \tag{EC.6}$$

If the conditions in Assumption 1 hold, then by Lemmas 1 and 3, the optimal value of the left-hand side problem in (EC.4) coincides with that of problem (EC.6). The emerging constraint is satisfied if and only if there exists $\pi_j \geq 0$ and $\boldsymbol{\beta}_j \in \mathbb{R}^I$ such that

$$\boldsymbol{\Omega}_j(\boldsymbol{x}, \boldsymbol{Y}) - \pi_j \mathbf{e}_{K+1}\mathbf{e}_{K+1}^\top - \sum_{i=1}^{I} [\boldsymbol{\beta}_j]_i \widehat{\boldsymbol{C}}_i \in \mathcal{COP}(\mathcal{K}).$$

Combining the result for all J constraints yields the finite constraint system

$$\boldsymbol{\Omega}_j(\boldsymbol{x}, \boldsymbol{Y}) - \pi_j \mathbf{e}_{K+1}\mathbf{e}_{K+1}^\top - \sum_{i=1}^{I} [\boldsymbol{\beta}_j]_i \widehat{\boldsymbol{C}}_i \in \mathcal{COP}(\mathcal{K}),\ \pi_j \geq 0 \quad \forall j \in [J],$$

which completes the proof. □

EC.4. Proof of Theorem 2

Proof. The proof parallels that of Theorem 1. Using Lemma 1, we reformulate the maximization problem $\sup_{\boldsymbol{u}\in\mathcal{U}} \sum_{n=1}^{N} \widehat{d}_n \boldsymbol{u}^\top \boldsymbol{Q}_n \boldsymbol{u}$ in the objective function of $(\mathcal{Q})$ into a copositive minimization problem given by

$$\begin{aligned} &\inf\ \lambda \\ &\text{s.t.}\ \lambda \mathbf{e}_{K+1}\mathbf{e}_{K+1}^\top - \sum_{n=1}^{N} \widehat{d}_n \boldsymbol{Q}_n - \sum_{i=1}^{I} \alpha_i \widehat{\boldsymbol{C}}_i \in \mathcal{COP}(\mathcal{K}) \\ &\quad\ \ \lambda \in \mathbb{R},\ \boldsymbol{\alpha} \in \mathbb{R}^I. \end{aligned}$$

Then, replacing the maximization problem in $(\mathcal{Q})$ with the above minimization problem yields the objective function and the first constraint in (11).

Next, we can reformulate the constraint

$$\boldsymbol{u}^\top \widehat{\boldsymbol{\Theta}}_j \boldsymbol{x} + \sum_{n=1}^{N} \widehat{b}_{jn} \boldsymbol{u}^\top \boldsymbol{Q}_n \boldsymbol{u} \geq \widehat{\boldsymbol{h}}_j^\top \boldsymbol{u} \quad \forall \boldsymbol{u} \in \mathcal{U}$$

corresponding to the j-th semi-infinite constraint in $(\mathcal{Q})$ into the equivalent constraints

$$\boldsymbol{\Gamma}_j(\boldsymbol{x}, \boldsymbol{Q}_1, \ldots, \boldsymbol{Q}_N) - \pi_j \mathbf{e}_{K+1}\mathbf{e}_{K+1}^\top - \sum_{i=1}^{I} [\boldsymbol{\beta}_j]_i \widehat{\boldsymbol{C}}_i \in \mathcal{COP}(\mathcal{K}),\ \pi_j \geq 0.$$

Combining this result for all J semi-infinite constraints yields the second constraint system in (11). This completes the proof. □

EC.5. Proof of Lemma 4

Proof. For any $\boldsymbol{u} \in \mathcal{K}$, the second-order cone constraint $\widehat{\boldsymbol{R}}\boldsymbol{u} \in \mathcal{SOC}(K_r)$ in the description of $\mathcal{K}$ stipulates that

$$\mathbf{e}_{K_r}^\top \widehat{\boldsymbol{R}}\boldsymbol{u} \geq \sqrt{(\mathbf{e}_1^\top \widehat{\boldsymbol{R}}\boldsymbol{u})^2 + \cdots + (\mathbf{e}_{K_r-1}^\top \widehat{\boldsymbol{R}}\boldsymbol{u})^2}. \tag{EC.7}$$

Squaring both sides of the inequality yields

$$\begin{aligned} \boldsymbol{u}^\top \widehat{\boldsymbol{R}}^\top \mathbf{e}_{K_r}\mathbf{e}_{K_r}^\top \widehat{\boldsymbol{R}}\boldsymbol{u} \geq \boldsymbol{u}^\top \widehat{\boldsymbol{R}}^\top \mathbf{e}_1\mathbf{e}_1^\top \widehat{\boldsymbol{R}}\boldsymbol{u} + \cdots + \boldsymbol{u}^\top \widehat{\boldsymbol{R}}^\top \mathbf{e}_{K_r-1}\mathbf{e}_{K_r-1}^\top \widehat{\boldsymbol{R}}\boldsymbol{u} \quad &\iff \\ \boldsymbol{u}^\top \left(\widehat{\boldsymbol{R}}^\top \mathbf{e}_{K_r}\mathbf{e}_{K_r}^\top \widehat{\boldsymbol{R}} - \sum_{\ell=1}^{K_r-1} \widehat{\boldsymbol{R}}^\top \mathbf{e}_\ell \mathbf{e}_\ell^\top \widehat{\boldsymbol{R}} \right) \boldsymbol{u} \geq 0 \quad &\iff \\ \boldsymbol{u}^\top \widehat{\boldsymbol{S}}\boldsymbol{u} \geq 0 \end{aligned}$$

Thus, the claim follows. □

EC.6. Proof of Proposition 1

Proof. For any $\boldsymbol{V} \in \mathcal{IA}(\mathcal{K})$, we need to show that $\boldsymbol{u}^\top \boldsymbol{V}\boldsymbol{u} \geq 0$ for all $\boldsymbol{u} \in \mathcal{K}$. To this end, fix any $\boldsymbol{V} \in \mathcal{IA}(\mathcal{K})$ and $\boldsymbol{u} \in \mathcal{K}$. By construction, we have

$$
\begin{aligned}
&\boldsymbol{u}^\top \left(\boldsymbol{W} + \tau\widehat{\boldsymbol{S}} + \widehat{\boldsymbol{P}}^\top \boldsymbol{\Sigma}\widehat{\boldsymbol{P}} + \boldsymbol{\Psi}\right)\boldsymbol{u} \\
&= \boldsymbol{u}^\top \boldsymbol{W}\boldsymbol{u} + \tau\boldsymbol{u}^\top \widehat{\boldsymbol{S}}\boldsymbol{u} + \boldsymbol{u}^\top \widehat{\boldsymbol{P}}^\top \boldsymbol{\Sigma}\widehat{\boldsymbol{P}}\boldsymbol{u} + \boldsymbol{u}^\top \left(\frac{1}{2}\widehat{\boldsymbol{P}}^\top \boldsymbol{\Phi}\widehat{\boldsymbol{R}} + \frac{1}{2}\widehat{\boldsymbol{R}}^\top \boldsymbol{\Phi}^\top \widehat{\boldsymbol{P}}\right)\boldsymbol{u} \\
&= \boldsymbol{u}^\top \boldsymbol{W}\boldsymbol{u} + \tau\boldsymbol{u}^\top \widehat{\boldsymbol{S}}\boldsymbol{u} + \boldsymbol{u}^\top \widehat{\boldsymbol{P}}^\top \boldsymbol{\Sigma}\widehat{\boldsymbol{P}}\boldsymbol{u} + \boldsymbol{u}^\top \widehat{\boldsymbol{P}}^\top \boldsymbol{\Phi}\widehat{\boldsymbol{R}}\boldsymbol{u}.
\end{aligned}
$$

We next analyze each of the four summands separately:

1. Since $\boldsymbol{W} \succeq \boldsymbol{0}$, we have $\boldsymbol{u}^\top \boldsymbol{W}\boldsymbol{u} \geq 0$.
2. Since $\tau \geq 0$ and by Lemma 4, we have $\tau\boldsymbol{u}^\top \widehat{\boldsymbol{S}}\boldsymbol{u} \geq 0$.
3. Since $\widehat{\boldsymbol{P}}\boldsymbol{u} \geq \boldsymbol{0}$ and $\boldsymbol{\Sigma} \geq \boldsymbol{0}$, we have $(\widehat{\boldsymbol{P}}\boldsymbol{u})^\top \boldsymbol{\Sigma}(\widehat{\boldsymbol{P}}\boldsymbol{u}) \geq 0$.
4. Since $\widehat{\boldsymbol{R}}\boldsymbol{u}$ and the vectors Rows($\boldsymbol{\Phi}$) belong to $\mathcal{SOC}(K_r)$, we have $\boldsymbol{\Phi}\widehat{\boldsymbol{R}}\boldsymbol{u} \geq \boldsymbol{0}$ (as a second-order cone is self-dual). This further implies that $\boldsymbol{u}^\top \widehat{\boldsymbol{P}}^\top \boldsymbol{\Phi}\widehat{\boldsymbol{R}}\boldsymbol{u} = (\widehat{\boldsymbol{P}}\boldsymbol{u})^\top (\boldsymbol{\Phi}\widehat{\boldsymbol{R}}\boldsymbol{u}) \geq 0$ as $\widehat{\boldsymbol{P}}\boldsymbol{u} \geq \boldsymbol{0}$.

This completes the proof. □

EC.7. Proof of Proposition 6

Proof. For a clear proof, we begin with the case when the piecewise linear lifting is only applied to the first coordinate axis u_1, where the breakpoints are given by $h_1 = \underline{u}_1 < h_2 < \ldots < h_L < \overline{u}_1$. In this case the lifted set $\mathcal{U}'$ and the set $\mathcal{U}^*$ respectively simplify to

$$
\mathcal{U}' = \left\{ (\boldsymbol{w}, \boldsymbol{u}) \in \mathbb{R}^L \times \mathcal{U} : \begin{array}{ll} \boldsymbol{z} \in \mathbb{R}_+^{L+1} & \\ w_\ell = z_\ell - z_{\ell+1} & \ell \in [L] \\ z_1 = u_1 - \underline{u}_1,\ z_{L+1} = 0 & \\ z_\ell \geq u_1 - h_\ell,\ \overline{u}_1 \geq z_\ell & \ell \in [L+1] \\ z_\ell(z_\ell - u_1 + h_\ell) = 0 & \ell \in [L+1] \end{array} \right\}, \tag{EC.8}
$$

$$
\mathcal{U}^* := \left\{ (\boldsymbol{w}, \boldsymbol{u}) \in \mathbb{R}^L \times \mathcal{U} : \begin{array}{l} \boldsymbol{z} \in \mathbb{R}_+^{L+1},\ \boldsymbol{p} \in \mathbb{R}_+^{L+1},\ \boldsymbol{r} \in \mathbb{R}_+^{L},\ U \in \mathbb{R}_+ \\ \boldsymbol{Z} \in \mathcal{S}^{L+1}, \boldsymbol{W} \in \mathcal{S}^L,\ \boldsymbol{Q} \in \mathbb{R}^{L\times(L+1)} \\ \boldsymbol{B}\boldsymbol{z} = \boldsymbol{w},\ z_1 = u_1 - \underline{u}_1,\ z_{L+1} = 0 \\ \boldsymbol{z} \geq u_1\mathbf{e} - \boldsymbol{h},\ \overline{u}_1\mathbf{e} \geq \boldsymbol{z} \\ \operatorname{diag}(\boldsymbol{Z}) - \boldsymbol{p} + \boldsymbol{h} \circ \boldsymbol{z} = \boldsymbol{0} \\ \boldsymbol{H} \begin{pmatrix} \boldsymbol{W} & \boldsymbol{Q} & \boldsymbol{r} & \boldsymbol{w} \\ \boldsymbol{Q}^\top & \boldsymbol{Z} & \boldsymbol{p} & \boldsymbol{z} \\ \boldsymbol{r}^\top & \boldsymbol{p}^\top & U & u_1 \\ \boldsymbol{w}^\top & \boldsymbol{z}^\top & u_1 & 1 \end{pmatrix} \boldsymbol{H}^\top \geq \boldsymbol{0},\ \begin{pmatrix} \boldsymbol{W} & \boldsymbol{Q} & \boldsymbol{r} & \boldsymbol{w} \\ \boldsymbol{Q}^\top & \boldsymbol{Z} & \boldsymbol{p} & \boldsymbol{z} \\ \boldsymbol{r}^\top & \boldsymbol{p}^\top & U & u_1 \\ \boldsymbol{w}^\top & \boldsymbol{z}^\top & u_1 & 1 \end{pmatrix} \succeq \boldsymbol{0} \end{array} \right\}, \tag{EC.9}
$$

where the matrices $\boldsymbol{B} \in \mathbb{R}^{L\times(L+1)}$ and $\boldsymbol{H} \in \mathbb{R}^{(4L+2)\times(2L+3)}$ are defined as

$$
\boldsymbol{B} = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & 1 & -1 & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & 1 & -1 \end{pmatrix} \quad \text{and} \quad \boldsymbol{H} = \begin{pmatrix} \boldsymbol{0} & \mathbb{I} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \mathbb{I} & -\mathbf{e} & \boldsymbol{h} \\ -\mathbb{I} & \boldsymbol{B} & \boldsymbol{0} & \boldsymbol{0} \\ \mathbb{I} & -\boldsymbol{B} & \boldsymbol{0} & \boldsymbol{0} \end{pmatrix},
$$

respectively. One can verify that a point from $\mathcal{U}'$ is also a member of $\mathcal{U}^*$ since the constraints in the definition of $\mathcal{U}^*$ consist of a SDP relaxation of those from $\mathcal{U}'$; see e.g., Burer (2012). Furthermore, the outer approximation $\mathcal{U}^{**}$ simplifies to:

$$\mathcal{U}^{**} = \left\{ (\boldsymbol{w}, \boldsymbol{u}) \in \mathbb{R}^L \times \mathcal{U} : \begin{array}{l} u_1 - \underline{u}_1 = \sum_{\ell \in [L]} w_\ell \\ h_2 - \underline{u}_1 \geq w_1 \\ (h_{\ell+1} - h_\ell) w_{\ell-1} \geq (h_\ell - h_{\ell-1}) w_\ell \ \forall \ell \in [L] \setminus \{1\} \end{array} \right\}. \tag{EC.10}$$

We now establish that $\mathcal{U}^* \subseteq \mathcal{U}^{**}$. First, the constraints $\boldsymbol{B}\boldsymbol{z} = \boldsymbol{w}$, $z_1 = u_1 - \underline{u}_1$, and $z_{L+1} = 0$ in $\mathcal{U}^*$ imply that

$$\sum_{\ell \in [L]} w_\ell = z_1 - z_2 + \left[\sum_{\ell \in \{2,\ldots,L\}} z_\ell - z_{\ell+1} \right] = u_1 - \underline{u}_1.$$

Next, since $z_2 \geq u_1 - h_2$, we have that $w_1 = z_1 - z_2 \leq u_1 - \underline{u}_1 - u_1 + h_2 = h_2 - \underline{u}_1$. Thus, the first two constraints in $\mathcal{U}^{**}$ are implied by $\mathcal{U}^*$. It remains to show that the final system of inequalities in $\mathcal{U}^{**}$ are also implied by the constraints in $\mathcal{U}^*$. By expanding the matrix product in the penultimate constraint of $\mathcal{U}^*$, we find that $\boldsymbol{Q} = \boldsymbol{B}\boldsymbol{Z}$, and the following constraints hold:

$$-\boldsymbol{r} + \boldsymbol{w} \circ \boldsymbol{h} = -\boldsymbol{B}\boldsymbol{p} + (\boldsymbol{B}\boldsymbol{z}) \circ \boldsymbol{h}, \quad -\boldsymbol{p}\mathbf{e}^\top + \boldsymbol{Z} + \boldsymbol{z}\boldsymbol{h}^\top \geq \mathbf{0}, \quad -\boldsymbol{r}\mathbf{e}^\top + \boldsymbol{B}\boldsymbol{Z} + \boldsymbol{w}\boldsymbol{h}^\top \geq \mathbf{0}.$$

Next, we perform the substitutions $\boldsymbol{p} = \operatorname{diag}(\boldsymbol{Z}) + \boldsymbol{h} \circ \boldsymbol{z}$, $\boldsymbol{w} = \boldsymbol{B}\boldsymbol{z}$ and $\boldsymbol{Q} = \boldsymbol{B}\boldsymbol{Z}$ to all occurrences of $\boldsymbol{p}$, $\boldsymbol{w}$, and $\boldsymbol{Q}$, respectively, in the above constraint system. We then get $\boldsymbol{r} = \boldsymbol{B}(\operatorname{diag}(\boldsymbol{Z}) + \boldsymbol{h} \circ \boldsymbol{z})$, and by further substituting this value, we arrive at the equivalent constraint system

$$-(\operatorname{diag}(\boldsymbol{Z}) + \boldsymbol{h} \circ \boldsymbol{z})\mathbf{e}^\top + \boldsymbol{Z} + \boldsymbol{z}\boldsymbol{h}^\top \geq \mathbf{0}, \ -\boldsymbol{B}(\operatorname{diag}(\boldsymbol{Z}) + \boldsymbol{h} \circ \boldsymbol{z})\mathbf{e}^\top + \boldsymbol{B}\boldsymbol{Z} + \boldsymbol{B}\boldsymbol{z}\boldsymbol{h}^\top \geq \mathbf{0}.$$

For $\ell \in \{2, \ldots, L\}$, one can show that these constraints further imply the following system of linear inequalities:

$$\begin{array}{r} \mathbf{e}_{\ell-1}\mathbf{e}_{\ell+1}^\top \bullet \boldsymbol{Z} - \mathbf{e}_{\ell+1}\mathbf{e}_{\ell+1}^\top \bullet \boldsymbol{Z} + z_{\ell+1}(h_{\ell-1} - h_{\ell+1}) \geq 0 \\ \mathbf{e}_{\ell}\mathbf{e}_{\ell+1}^\top \bullet \boldsymbol{Z} - \mathbf{e}_{\ell+1}\mathbf{e}_{\ell+1}^\top \bullet \boldsymbol{Z} + z_{\ell+1}(h_\ell - h_{\ell+1}) \geq 0 \\ \mathbf{e}_{\ell-1}\mathbf{e}_{\ell+1}^\top \bullet \boldsymbol{Z} + \mathbf{e}_{\ell}\mathbf{e}_{\ell}^\top \bullet \boldsymbol{Z} - \mathbf{e}_{\ell-1}\mathbf{e}_{\ell-1}^\top \bullet \boldsymbol{Z} - \mathbf{e}_{\ell}\mathbf{e}_{\ell+1}^\top \bullet \boldsymbol{Z} + z_{\ell-1}(h_{\ell+1} - h_{\ell-1}) + z_\ell(h_\ell - h_{\ell+1}) \geq 0 \\ \mathbf{e}_{\ell-1}\mathbf{e}_{\ell}^\top \bullet \boldsymbol{Z} + \mathbf{e}_{\ell+1}\mathbf{e}_{\ell+1}^\top \bullet \boldsymbol{Z} - \mathbf{e}_{\ell}\mathbf{e}_{\ell}^\top \bullet \boldsymbol{Z} - \mathbf{e}_{\ell-1}\mathbf{e}_{\ell+1}^\top \bullet \boldsymbol{Z} + z_{\ell+1}(h_{\ell+1} - h_{\ell-1}) + z_\ell(h_{\ell-1} - h_\ell) \geq 0 \\ \mathbf{e}_{\ell}\mathbf{e}_{\ell+1}^\top \bullet \boldsymbol{Z} - \mathbf{e}_{\ell}\mathbf{e}_{\ell}^\top \bullet \boldsymbol{Z} + z_\ell(h_{\ell+1} - h_\ell) \geq 0. \end{array} \tag{EC.11}$$

We further relax the large semidefinite constraint in $\mathcal{U}^*$ into $\mathcal{O}(L)$ semidefinite constraints involving 3×3 matrices, as follows:

$$\boldsymbol{M}_\ell := \begin{pmatrix} \mathbf{e}_{\ell-1}\mathbf{e}_{\ell-1}^\top \bullet \boldsymbol{Z} & \mathbf{e}_{\ell-1}\mathbf{e}_{\ell}^\top \bullet \boldsymbol{Z} & \mathbf{e}_{\ell-1}\mathbf{e}_{\ell+1}^\top \bullet \boldsymbol{Z} \\ \mathbf{e}_{\ell-1}\mathbf{e}_{\ell}^\top \bullet \boldsymbol{Z} & \mathbf{e}_{\ell}\mathbf{e}_{\ell}^\top \bullet \boldsymbol{Z} & \mathbf{e}_{\ell}\mathbf{e}_{\ell+1}^\top \bullet \boldsymbol{Z} \\ \mathbf{e}_{\ell-1}\mathbf{e}_{\ell+1}^\top \bullet \boldsymbol{Z} & \mathbf{e}_{\ell}\mathbf{e}_{\ell+1}^\top \bullet \boldsymbol{Z} & \mathbf{e}_{\ell+1}\mathbf{e}_{\ell+1}^\top \bullet \boldsymbol{Z} \end{pmatrix} \succeq \mathbf{0} \qquad \forall \ell \in [L+1]. \tag{EC.12}$$

We now show that the relaxations (EC.11) and (EC.12) are sufficient to imply that

$$(h_{\ell+1} - h_\ell) w_{\ell-1} \geq (h_\ell - h_{\ell-1}) w_\ell \Longleftrightarrow (h_{\ell+1} - h_\ell) z_{\ell-1} + (h_\ell - h_{\ell-1}) z_{\ell+1} \geq (h_{\ell+1} - h_{\ell-1}) w_\ell, \tag{EC.13}$$

where the equivalence follows from the substitutions $w_{\ell-1} = z_{\ell-1} - z_\ell$ and $w_\ell = z_\ell - z_{\ell+1}$. In order to arrive the desired implication, we require that the optimal value of the following optimization problem is greater than or equal to 0:

$$\begin{array}{ll} \inf & (h_{\ell+1} - h_\ell) z_{\ell-1} + (h_\ell - h_{\ell-1}) z_{\ell+1} - (h_{\ell+1} - h_{\ell-1}) w_\ell \\ \text{s.t.} & \boldsymbol{M}_\ell, z_{\ell-1}, z_\ell, \text{ and } z_{\ell+1} \text{ satisfy (EC.11) and (EC.12).} \end{array} \tag{EC.14}$$

By weak duality, the optimal value of this problem is lower bounded by the maximization problem

$$\begin{array}{ll} \sup & 0 \\ \text{s.t.} & (h_{\ell+1} - h_{\ell-1}) c = (h_{\ell+1} - h_\ell) \\ & (h_\ell - h_{\ell-1}) d + (h_{\ell+1} - h_\ell) c = (h_{\ell+1} - h_{\ell-1}) + (h_{\ell+1} - h_\ell) e \\ & (h_\ell - h_{\ell-1}) + (h_{\ell+1} - h_{\ell-1}) a + (h_{\ell+1} - h_\ell) b = (h_{\ell+1} - h_{\ell-1}) d \\ & \begin{pmatrix} c & -\frac{d}{2} & \frac{d-a-c}{2} \\ -\frac{d}{2} & -c+d+e & \frac{-b+c-e}{2} \\ \frac{d-a-c}{2} & \frac{-b+c-e}{2} & a+b-d \end{pmatrix} \succeq 0 \\ & (a,b,c,d,e) \in \mathbb{R}^5_+. \end{array}$$

One can verify that the solution $(a,b,c,d,e) \in \mathbb{R}^5_+$ satisfying $a = c = \frac{h_{\ell+1} - h_\ell}{h_{\ell+1} - h_{\ell-1}}$, $b = \frac{h_{\ell+1} - h_{\ell-1}}{h_{\ell+1} - h_\ell}$, $d = 2$, and $e = \frac{(h_\ell - h_{\ell-1})^2}{(h_{\ell+1} - h_{\ell-1})(h_{\ell+1} - h_\ell)}$ is feasible to the dual problem. Thus, the optimal value of the primal problem (EC.14) is bounded below by 0, which verifies that the constraints (EC.11) and (EC.12) imply (EC.13).

The above analysis also holds to the general case when the piecewise linear lifting is applied to all coordinate axis $\boldsymbol{u}$. In summary, we have shown that the containment $\mathcal{U}^* \subseteq \mathcal{U}^{**}$ holds. □

EC.8. Proof of Theorem 6

Proof. For a clear proof, we begin with the case when the piecewise linear lifting is only applied to the first coordinate axis u_1, where the breakpoints are given by $h_1 = \underline{u}_1 < h_2 < \ldots < h_L < \overline{u}_1$. In this case, the lifted set $\mathcal{U}'$ simplifies to the one shown in (EC.8).

We apply linear decision rules on the lifted uncertain parameters, which gives rise to the following semi-infinite linear program:

$$\begin{array}{ll} \inf & \boldsymbol{c}^\top \boldsymbol{x} + \sup_{(\boldsymbol{w},\boldsymbol{u}) \in \mathcal{U}'} \widehat{\boldsymbol{d}}^\top \boldsymbol{Y} (\boldsymbol{w}^\top, \boldsymbol{u}^\top)^\top \\ \text{s.t.} & \boldsymbol{\mathcal{A}}'(\boldsymbol{v}) \boldsymbol{x} + \widehat{\boldsymbol{B}} \boldsymbol{Y} (\boldsymbol{w}^\top, \boldsymbol{u}^\top)^\top \geq \boldsymbol{h}'(\boldsymbol{v}) \quad \forall \boldsymbol{v} := (\boldsymbol{w}, \boldsymbol{u}) \in \mathcal{U}' \\ & \boldsymbol{x} \in \mathcal{X}, \ \boldsymbol{Y} \in \mathbb{R}^{N \times (L+K+1)}. \end{array} \tag{EC.15}$$

Consider the worst-case maximization problem in the objective function of (EC.15). For a fixed decision rule coefficient matrix $\boldsymbol{Y}$, let us denote its optimal value by $v(\boldsymbol{Y})$. That is,

$$v(\boldsymbol{Y}) = \sup_{(\boldsymbol{w},\boldsymbol{u}) \in \mathcal{U}'} \widehat{\boldsymbol{d}}^\top \boldsymbol{Y} (\boldsymbol{w}^\top, \boldsymbol{u}^\top)^\top. \tag{EC.16}$$

Replacing the set $\mathcal{U}'$ with the outer approximation given by (EC.10) yields the upper bound $v^{**}(\boldsymbol{Y}) \geq v(\boldsymbol{Y})$. A tractable finite reformulation can then be derived by virtue of the standard dualization technique in robust optimization.

Alternatively, by applying Proposition 3 to the lifted set $\mathcal{U}'$ in (EC.8) and using Lemma 1, we arrive at the equivalent completely positive program

$$
\begin{array}{rl}
v(\boldsymbol{Y}) = \sup & \widehat{\boldsymbol{d}}^\top \boldsymbol{Y}(\boldsymbol{w}^\top, \boldsymbol{u}^\top)^\top \\
\text{s.t.} & \mathbf{e}_{K+1}\mathbf{e}_{K+1}^\top \bullet \boldsymbol{U}' = 1 \\
& \mathbf{e}_\ell \mathbf{e}_\ell^\top \bullet \boldsymbol{Z}' - \mathbf{e}_\ell \mathbf{e}_1^\top \bullet \boldsymbol{P}' + h_\ell \mathbf{e}_\ell \mathbf{e}_{K+1}^\top \bullet \boldsymbol{P}' = 0 \quad \forall \ell \in [L+1] \\
& \begin{pmatrix} \boldsymbol{W}' & \boldsymbol{Q}' & \boldsymbol{R}' \\ (\boldsymbol{Q}')^\top & \boldsymbol{Z}' & \boldsymbol{P}' \\ (\boldsymbol{R}')^\top & (\boldsymbol{P}')^\top & \boldsymbol{U}' \end{pmatrix} \in \mathcal{CP}(\mathcal{K}'), \ \boldsymbol{u} = \boldsymbol{U}'\mathbf{e}_{K+1}, \ \boldsymbol{w} = \boldsymbol{R}'\mathbf{e}_{K+1} \\
& \boldsymbol{U}' \in \mathcal{S}^{K+1}, \ \boldsymbol{Z}' \in \mathcal{S}^{L+1}, \ \boldsymbol{W}' \in \mathcal{S}^L \\
& \boldsymbol{P}' \in \mathbb{R}^{(L+1)\times(K+1)}, \ \boldsymbol{R}' \in \mathbb{R}^{L\times(K+1)}, \ \boldsymbol{Q}' \in \mathbb{R}^{L\times L+1},
\end{array}
\tag{EC.17}
$$

where the cone $\mathcal{K}'$ is defined as

$$
\mathcal{K}' := \left\{ (\boldsymbol{w}, \boldsymbol{z}, \boldsymbol{u}) \in \mathbb{R}^L \times \mathbb{R}_+^{L+1} \times \mathcal{K} : \begin{array}{ll} w_\ell = z_\ell - z_{\ell+1} & \ell \in [L] \\ z_1 = u_1 - \underline{u}_1, \ z_{L+1} = 0 & \\ z_\ell \geq u_1 - h_\ell u_{K+1}, \ \overline{u}_1 u_{K+1} \geq z_\ell & \ell \in [L+1] \end{array} \right\}.
$$

An upper bound to $v(\boldsymbol{Y})$ is then obtained by replacing the completely positive cone $\mathcal{CP}(\mathcal{K}')$ in (EC.17) with a valid semidefinite-representable outer approximation. To this end, we further loosen the relaxation by considering only those constraints that are independent across dimensions. We then obtain an outer approximation defined by (EC.9) to the feasible set of decision variables $\boldsymbol{u}$ and $\boldsymbol{w}$ in (EC.17). Using $\mathcal{U}^*$ to replace $\mathcal{U}$ in (EC.16), we arrive at another upper bound $v^*(\boldsymbol{Y}) \geq v(\boldsymbol{Y})$. As the resulting maximization problem admits a Slater point, a tractable finite reformulation can then be obtained by applying standard conic duality. By Proposition 6, we establish that $\mathcal{U}^* \subseteq \mathcal{U}^{**}$, which, in turn, establishes that $v^*(\boldsymbol{Y}) \leq v^{**}(\boldsymbol{Y})$.

The above analysis also holds to the general case when the piecewise linear lifting is applied to all coordinate axis $\boldsymbol{u}$. Thus, the claim follows. $\square$