Electronic Companion

EC.1. Summary of Notation

Notation is summarized in Table EC.1 through Table EC.4.

Table EC.1 Summary of notation—sets.

Sets	
$\mathcal{I}$	Set of patient classes, with $I = \lvert\mathcal{I}\rvert$
$\mathcal{T}$	Set of decision epochs, with $T = \lvert\mathcal{T}\rvert - 1$
$\mathcal{W}$	Set of evaluation windows, with $W = \lvert\mathcal{W}\rvert$
$\mathcal{T}_w$	Set of decision epochs in window $w \in \mathcal{W}$
$\mathcal{P}$	Set of patients arriving and seeking transplant in week 0

Table EC.2 Summary of notation—parameters.

Parameters	
c_i, e_i	Program- and SRTR-expected one-year post-transplant death probability for patients of class $i \in \mathcal{I}$
λ_i	Arrival rate of patient of type $i \in \mathcal{I}$
c_0^j, e_0^j	Program- and SRTR-expected death probability for patient $j \in \mathcal{P}$
α_w	Risk threshold for window $w \in \mathcal{W}$
D	Number of decision epochs per evaluation window
L_w, U_w	Lower and upper bound on the expected number of transplants in window $w \in \mathcal{W}$
ρ_w	Penalty for violating the chance constraint in window $w \in \mathcal{W}$
M_w	Big-M value for the chance constraint in window $w \in \mathcal{W}$
N_w^{init}	Constant number of patients added to the waitlist in window $w \in \mathcal{W}$ before $t = 0$
O_w^{init}	Constant number of patient transplanted in window $w \in \mathcal{W}$ who did not reach the one-year post transplant survival mark, and died before $t = 0$
E_w^{init}	SRTR-expected number of deaths from patients transplanted in window $w \in \mathcal{W}$ before $t = 0$
$f(\cdot)$	CMS p-value criteria function
$g(\cdot)$	Nonlinear function generically representing both CMS and OPTN regulations
L	Number of piecewise linear segments used to approximate the function g
K	Number of piecewise linear segments used to approximate the square root function in (MA)
m_ℓ, b_ℓ	Slope and y-intercept of segment $\ell \in \{1, \ldots, L\}$ used to approximate g
β_k, γ_k	Slope and y-intercept of segment $k \in \{1, \ldots, K\}$ used to approximate the square root function
$\widehat{\mu}_w$, $\widehat{\sigma}_w^2$	Expectation and variance of the random variable $\widehat{O}_w$ (see Table EC.4) for window $w \in \mathcal{W}$

Table EC.3 Summary of notation—decision variables.

Decision Variables	
z_j	Binary variable indicating whether or not patient $j \in \mathcal{P}$ is added to the waitlist
u_{it}	Probability that a patient of type $i \in \mathcal{I}$ is added to the waitlist in week $t \in \mathcal{T}$
y_w	Binary variable indicating whether program performs transplants in window $w \in \mathcal{W}$
x_w	Binary variable indicating whether the chance constraint is satisfied in window $w \in \mathcal{W}$

Table EC.4 Summary of notation—random variables.

Random Variables	
A_{it}^j	Bernoulli random variable indicating whether the j^{th} patient of type $i \in \mathcal{I}$ arriving in week $t \in \mathcal{T}$ is added to the waitlist
X_{it}^j	Bernoulli random variable indicating whether the j^{th} patient of type $i \in \mathcal{I}$ added to the waitlist in week $t \in \mathcal{T}$ fails to survive one year after transplantation
Y_{it}	Number of patients of type $i \in \mathcal{I}$ added to the waitlist in week $t \in \mathcal{T}$
Z_{it}	Number of patients of type $i \in \mathcal{I}$ arriving in week $t \in \mathcal{T}$ added to the waitlist
N_w	Number of patients added to the waitlist in window $w \in \mathcal{W}$
O_w	Observed number of deaths for patients transplanted in window $w \in \mathcal{W}$
E_w	SRTR-expected number of deaths for patients transplanted in window $w \in \mathcal{W}$
$\widehat{O}_w$	Number of patients transplanted in window $w \in \mathcal{W}$ before $t = 0$ who are still alive at $t = 0$, but die before reaching the one-year post-transplant mark
X_0^j	Bernoulli random variable indicating whether patient $j \in \mathcal{P}$ fails to survive for one year post-transplant

EC.2. Proofs of Formal Results

EC.2.1. Proof of Proposition 1

PROPOSITION 1. *Consider the model (M1) for a single evaluation window w satisfying $w \geq 6$ (cf. the indexing convention in Section 4). Under Assumption 3 and Assumption 4, there exists a stationary optimal solution u^* to (M1) (that is, $u^*_{it_1} = u^*_{it_2}$ for all $i \in \mathcal{I}$, $t_1, t_2 \in \mathcal{T}_w$).*

Proof of Proposition 1. Let $R^\ell(u) := O - m_\ell E - b_\ell$. Let $u^* \in [0,1]^{I \times D}$ be optimal for (M1). For each ℓ, Assumption 4 allows us to compute the characteristic function $\varphi_\ell(\,\cdot\,; u^*)$ of $R^\ell(u^*)$, given by

$$\varphi_\ell(s; u^*) = \exp\left[-\iota\beta s + \sum_{i\in\mathcal{I}} \xi_{i\ell}(s) \sum_{t=1}^{D} u^*_{it} \right]$$

where

$$\xi_{i\ell}(s) = \lambda_i \left[(1-c_i)\exp\left(-\iota m_\ell e_i s\right) + c_i \exp\left(\iota(1-m_\ell e_i)s\right) - 1 \right]$$

for $i \in \mathcal{I}$, and ι is the imaginary unit. From the form of $\varphi_\ell(s;u^*)$, we can immediately verify that $\varphi_\ell(s;u^*) = \varphi_\ell(s;\overline{u})$, where $\overline{u} \in [0,1]^{H\times D}$ is given by $\overline{u}_{ik} = \frac{1}{D}\sum_{t=1}^{D} u^*_{it}$ for all $i \in \mathcal{I}$, $k \in \{1,\dots,D\}$. This implies that $R^\ell(u^*)$ and $R^\ell(\overline{u})$ follow the same distribution for $\ell \in \{1,\dots,L\}$. Hence

$$\mathbb{P}\left[\min_{\ell\in\{1,\dots,L\}}\left\{R^\ell(\overline{u})\right\} \le 0\right] = \mathbb{P}\left[\min_{\ell\in\{1,\dots,L\}}\left\{R^\ell(u^*)\right\} \le 0\right] \ge 1-\alpha,$$

so $\overline{u}$ is feasible (checking the bounds $0 \le \overline{u}_{it} \le 1$ is trivial). Because $\mathbb{E}[Z_{it}]$ is assumed time homogeneous, direct calculation shows that both u^* and $\overline{u}$ produce the same objective value, and thus $\overline{u}$ is optimal. $\square$

EC.2.2. Proof of Theorem 1

THEOREM 1 *Let $\widetilde{u}$ be optimal for (M2) with risk parameter $\alpha_0 \in [0,1]$. Suppose that the planning horizon T is divisible by D, and that window $w=6$ begins in week $t=1$. Then for any initial data $\{(E_w^{\text{init}}, O_w^{\text{init}}, \widehat{O}_w)\}_{w=1}^5$ and risk parameters $\{\alpha_w\}_{w=1}^5$,*

$$\frac{z_T^*(\widetilde{\alpha})}{T} \le \sum_{i\in\mathcal{I}} \lambda_i \widetilde{u}_i,$$

where $\widetilde{\alpha}_w = \alpha_w$ for $w = 1,\dots,5$ and $\widetilde{\alpha}_w = \alpha_0$ otherwise. Moreover, for any initial data there exist risk parameters $\{\alpha_w\}_{w=1}^5$ for which this bound holds with equality.

Proof of Theorem 1. We consider the optimal solutions of three different models. Let $u^* \in [0,1]^{I\times T}$ denote the optimal solution to (M1) for full window set $\mathcal{W} = \{1,\dots,W\}$. Let $\widehat{u} \in [0,1]^{I\times T}$ denote the optimal solution to (M1) for only a single window $\widehat{w} \ge 6$ and risk parameter α_0. Finally, let $\widetilde{u} \in [0,1]^I$ denote the optimal solution to (M2) with risk parameter α_0. Then

$$\sum_{t\in\mathcal{T}_{\widehat{w}}}\sum_{i\in\mathcal{I}} \lambda_i u^*_{it} \le \sum_{t\in\mathcal{T}_{\widehat{w}}}\sum_{i\in\mathcal{I}} \lambda_i \widehat{u}_{it} = D\sum_{i\in\mathcal{I}} \lambda_i \widetilde{u}_i, \tag{EC.1}$$

where the inequality follows because $\{u^*_{it}\}_{i\in\mathcal{I},t\in\mathcal{T}_{\widehat{w}}}$ is feasible for the same model for which $\widehat{u}$ is optimal, and the equality follows from Proposition 1. Next, because T is evenly divisible by D and window $w=6$ begins in week $t=1$, there exists a set of windows $\widehat{\mathcal{W}} \subset \mathcal{W}$ such that $w \ge 6$ for all $w \in \widehat{\mathcal{W}}$, $|\widehat{\mathcal{W}}| = T/D$, and $\{\mathcal{T}_w\}_{w\in\widehat{\mathcal{W}}}$ partitions $\mathcal{T}$ (i.e., $\bigcup_{w\in\widehat{\mathcal{W}}} \mathcal{T}_w = \mathcal{T}$ and $\mathcal{T}_{w_1} \cap \mathcal{T}_{w_2} = \varnothing$ for all $w_1 \neq w_2 \in \widehat{\mathcal{W}}$). Consequently, we have that

$$z_T^*(\alpha) = \sum_{t\in\mathcal{T}} \sum_{i\in\mathcal{I}} \lambda_i u_{it}^* = \sum_{w\in\widehat{\mathcal{W}}} \sum_{t\in\mathcal{T}_w} \sum_{i\in\mathcal{I}} \lambda_i u_{it}^* \leq D \sum_{w\in\widehat{\mathcal{W}}} \sum_{i\in\mathcal{I}} \lambda_i \widetilde{u}_i = T \sum_{i\in\mathcal{I}} \lambda_i \widetilde{u}_i,$$

where the inequality follows from (EC.1). The result follows from dividing through by T.

To see that this bound is tight, take $\alpha_w = 1$ for $w = 1, \ldots, 5$. Then the solution $u_{it} = \widetilde{u}_i$ for all $i \in \mathcal{I}$, $t \in \mathcal{T}$ is feasible for (M1), and satisfies (EC.1) with equality.

□

EC.2.3. Proof of Theorem 2

The proof of Theorem 2 follows directly from the following lemma:

Lemma EC.1 (**Hillestad and Jacobsen** (**1980**)). *Given $A \in \mathbb{R}^{m\times n}$, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ and continuous, strictly concave $g : \mathbb{R}^n \to \mathbb{R}$, if the optimization problem $\max\{c^{\mathrm{T}}x \mid Ax \leq b, g(x) \leq 0\}$ has an optimal solution, then it has an optimal solution which lies on an edge of $F_A = \{x \in \mathbb{R}^n \mid Ax \leq b\}$.*

Theorem 2 *There exists an optimal solution u^* to the normal single-window model (M3) such that at most one patient class is accepted at a fractional rate, with all other classes either uniformly accepted or uniformly declined.*

Proof. Because $u \equiv 0$ is feasible for (M3), there exists an optimal solution for the same. The result then follows directly from Lemma EC.1 where the function g given by (M3-b) is strictly concave as the minimum of strictly concave functions. □

EC.2.4. Proof of Theorem 3

To prove Theorem 3, we derive a convex relaxation of the model (M4), then use the KKT conditions to obtain our final result. By replacing the values u_i in (M4-d) with u_i^2, the concave term $\Phi^{-1}(1 - \alpha)\sqrt{\sigma^2}$ becomes convex, yielding the model:

$$\max\ D\lambda^{\mathrm{T}}u \tag{EC.2a}$$

$$\text{s.t. } \mathcal{F}(u) = \eta^{\mathrm{T}} u + \varphi \|N^{1/2} u\|_2 \leq 0 \tag{EC.2b}$$

$$0 \leq u_i \leq 1 \text{ for } i \in \mathcal{I} \tag{EC.2c}$$

where the vector $\eta \in \mathbb{R}^I$ is given by $\eta_i = D\lambda_i(c_i - m_L e_i)$, the matrix $N \in \mathbb{R}^{I \times I}$ is given by $N = \mathrm{diag}(\nu)$ for $\nu_i = D\lambda_i\left[(c_i - 1.5e_i)^2 + c_i(1 - c_i)\right] > 0$, and $\varphi \equiv \Phi^{-1}(1 - \alpha)$.

Proposition EC.1. *Suppose $\varphi > 0$ (that is, $\alpha < 1/2$). Then the model (EC.2) is convex. Furthermore, the feasible region of (EC.2) is a relaxation of the feasible region of (M4).*

Proof. To see that (EC.2) is convex, note that if $\varphi > 0$, then model (EC.2) is a second-order conic program, and hence convex. To see that (EC.2) is a relaxation of (M4), note simply that $\|N^{1/2}u\|_2 = (\sum_{i \in \mathcal{I}} \nu_i u_i^2)^{1/2} \leq (\sum_{i \in \mathcal{I}} \nu_i u_i)^{1/2}$ for $u_i \in [0, 1]$. The result follows immediately. $\square$

Beacuse the model (EC.2) is convex, the KKT conditions are sufficient for optimality. That is, if there exists a solution u to (EC.2), multipliers $\mu \in \mathbb{R}$, $\sigma^+, \sigma^- \in \mathbb{R}^I$ and a vector $z \in \partial\mathcal{F}(u)$ (the subdifferential of $\mathcal{F}$ at u) such that

$$D\lambda_i = \mu z_i + \sigma_i^+ - \sigma_i^- \text{ for all } i \in \mathcal{I} \qquad \text{(Stationarity)} \tag{EC.3a}$$

$$\mu \mathcal{F}(u) = 0 \qquad \text{(Complementary Slackness \#1)} \tag{EC.3b}$$

$$\sigma_i^+(u_i - 1) = 0 \text{ for all } i \in \mathcal{I} \qquad \text{(Complementary Slackness \#2)} \tag{EC.3c}$$

$$\sigma_i^- u_i = 0 \text{ for all } i \in \mathcal{I} \qquad \text{(Complementary Slackness \#3)} \tag{EC.3d}$$

$$u \in [0, 1]^I \text{ and } \mathcal{F}(u) \leq 0 \qquad \text{(Primal Feasibility)} \tag{EC.3e}$$

$$\mu \geq 0, \sigma_i^+ \geq 0, \sigma_i^- \geq 0 \text{ for all } i \in \mathcal{I} \qquad \text{(Dual Feasibility)} \tag{EC.3f}$$

then u is optimal for (EC.2). We note that for $u \neq 0$, $\partial\mathcal{F}(u)$ is simply the gradient of $\mathcal{F}$, and $\partial\mathcal{F}(0) = \{\eta + \varphi z \mid \|N^{-1/2} z\| \leq 1\}$. The following lemma is a statement of the KKT conditions (EC.3) for (EC.2) at $u \equiv 0$:

Lemma EC.2. *Suppose there exists a scalar $\mu \geq 0$, vector $\sigma \in \mathbb{R}_+^I$ and a vector $z \in \partial\mathcal{F}(0)$ (the subdifferential of $\mathcal{F}$ at 0) such that $D\lambda_i = \mu z_i - \sigma_i$ for all $i \in \mathcal{I}$. Then $u \equiv 0$ is optimal for (M4).*

Proof. Suppose such a μ, σ and z exist. Then $u \equiv 0, \mu, \sigma$ satisfy the KKT conditions (EC.3) for (EC.2).Because the model (EC.2) is convex, the KKT conditions are sufficient, and thus $u \equiv 0$ is optimal for (EC.2). By Proposition EC.1, $u \equiv 0$ must also be optimal for (M4). □

THEOREM 3 *If the optimal solution u to (M4) is non-zero, then $\alpha \geq 1 - \Phi(\sqrt{D \sum_{i \in \mathcal{I}^-} \lambda_i/(1+\beta_i^2)})$, where $\mathcal{I}^- := \{i \in \mathcal{I} : c_i < 1.5e_i\}$.*

Proof (of Theorem 3). We prove the contrapositive. Suppose that

$$\sqrt{D \sum_{i \in \mathcal{I}^-} \lambda_i \frac{(c_i - 1.5e_i)^2}{(c_i - 1.5e_i)^2 + c_i(1-c_i)}} < \Phi^{-1}(1-\alpha).$$

This may be expressed in more compact notation as $\|N^{-1/2}\hat{y}\|_2 < 1$, where $\hat{y} = -\min\{\eta, 0\}/\varphi$, and the $\min\{\cdot\}$ is taken component-wise. We have that $\eta + \varphi\hat{y} = \eta - \min\{\eta, 0\} = \max\{\eta, 0\} \geq 0$, and hence, there exists a sufficiently small perturbation vector $\varepsilon \in \mathbb{R}^I$ such that $\eta + \varphi\tilde{y} > 0$ and $\|N^{-1/2}\tilde{y}\|_2 \leq 1$ for $\tilde{y} = \hat{y} + \varepsilon$ (note in particular that $\eta + \varphi\tilde{y} \in \partial\mathcal{F}(0)$). Next, define

$$\mu \equiv \max_{i \in \mathcal{I}} \left\{ \frac{D\lambda_i}{\eta_i + \varphi\tilde{y}_i} \right\},$$

and, for all $i \in \mathcal{I}$, $\sigma_i \equiv \mu(\eta_i + \varphi\tilde{y}_i) - D\lambda_i$. By construction, $\sigma \in \mathbb{R}^I_+$, $\mu \geq 0$, $\eta + \varphi\tilde{y} \in \partial\mathcal{F}(0)$, and $D\lambda + \sigma = \mu(\eta + \varphi\tilde{y})$. By Lemma EC.2, $u^* \equiv 0$ is optimal for (EC.2). By Proposition EC.1, it is also optimal for (M4), as was to be shown. □

EC.2.5. Proof of Proposition 2

PROPOSITION 2 *Suppose that all of the following hold:*

1. *The piecewise linear approximation of the square root function satisfies $\widetilde{\sigma}_{w\ell} \geq \sigma_{w\ell}$ for all $\ell \in \{1, \ldots, L\}$ and $w \in \mathcal{W}$.*
2. *The function g^{PWL} is chosen so that $g^{\text{PWL}}(E_w) \leq g(E_w)$ for all $E_w \geq 0$.*
3. *The random variables $O_w - m_\ell E_w - b_\ell$ are normally distributed with mean $\mu_{w\ell}$ and variance $\sigma^2_{w\ell}$ for all $\ell \in \{1, \ldots, L\}$ and $w \in \mathcal{W}$.*

Then any optimal solution to the approximation model (MA) with $x_w = 0$ for all $w \in \mathcal{W}$ is a feasible (potentially suboptimal) solution to the model (M0).

Proof. Let (z^*, u^*, y^*, x^*) be an optimal solution to (MA), with $x^*_w = 0$ for all $w \in \mathcal{W}$. It suffices to show that (z^*, u^*) satisfy the constraints (M0-b) for all $w \in \mathcal{W}$. We have that, for any $w \in \mathcal{W}$,

$$\begin{aligned}
&\min_{\ell \in \{1,\dots,L\}} \left\{ \mu_{w\ell} + \Phi^{-1}(1-\alpha)\widetilde{\sigma}_{w\ell} \right\} \le 0 \\
\implies &\min_{\ell \in \{1,\dots,L\}} \left\{ \mu_{w\ell} + \Phi^{-1}(1-\alpha)\sigma_{w\ell} \right\} \le 0 && \text{(By condition 1)} \\
\implies &\max_{\ell \in \{1,\dots,L\}} \mathbb{P}\big[O_w - m_\ell E_w - b_\ell \le 0\big] \ge 1-\alpha_w && \text{(By condition 3)} \\
\implies &\mathbb{P}\left[\min_{\ell \in \{1,\dots,L\}} \{O_w - m_\ell E_w - b_\ell\} \le 0\right] \ge 1-\alpha_w && \text{(By (6))} \\
\implies &\mathbb{P}\left[O_w \le \max_{\ell \in \{1,\dots,L\}} \{m_\ell E_w + b_\ell\}\right] \ge 1-\alpha_w \\
\implies &\mathbb{P}\Big[O_w \le g(E_w)\Big] \ge 1-\alpha_w && \text{(By condition 2),}
\end{aligned}$$

completing the proof. □

EC.3. Additional Results

We now demonstrate that, in the limit $\alpha \to 0$, the optimal objective value of (M3) goes to zero. This result is illustrated in Figure EC.2.

PROPOSITION EC.2. *Let $z^*(\alpha)$ denote the optimal objective value of (M3) as a function of $\alpha \in (0,1)$. Suppose that $b_\ell \ge 0$ for all $\ell = 1,\dots,L$. Then $\lim_{\alpha \downarrow 0} z^*(\alpha) = 0$.*

Althought the assumption $b_\ell \ge 0$ for all ℓ in Proposition EC.2 can be relaxed slightly, we note that the choices of b_ℓ in Section 6 satisfy this condition as stated, and thus we leave it for simplicity. For clarity, we introduce the notation $r_{i\ell} \equiv c_i - m_\ell e_i$ and $s_{i\ell} \equiv r_{i\ell}^2 + c_i(1-c_i)$ for all $i \in \mathcal{I}$ and $\ell \in \{1,\dots,L\}$.

Proof (of Proposition EC.2). Let $\alpha^* \in (0,1)$ satisfy

$$\Phi^{-1}(1-\alpha^*)\sqrt{\hat{s}} + \min\{\hat{r}, 0\}\sqrt{D\sum_{i\in\mathcal{I}}\lambda_i} \;=\; 0.$$

where $\hat{r} = \min_{i,\ell}\{r_{i\ell}\}$ and $\hat{s} = \min_{i,\ell}\{s_{i\ell}\}$. Then, because $\hat{s} > 0$, we have that

$$\Phi^{-1}(1-\alpha)\sqrt{\hat{s}} + \min\{\hat{r}, 0\}\sqrt{D\sum_{i\in\mathcal{I}}\lambda_i} \;>\; 0, \tag{EC.4}$$

for all $\alpha \in (0, \alpha^*)$. Fix such an α, and let u^α be the corresponding optimal solution to (M3). If u^α is the zero vector, then $z^*(\alpha) = 0$. Otherwise, because u^α is feasible, we have that

$$\begin{aligned}
0 &\geq \min_\ell \left\{ -b_\ell + \sum_{i\in\mathcal{I}} r_{il}(D\lambda_i u_i^\alpha) + \Phi^{-1}(1-\alpha)\sqrt{\sum_{i\in\mathcal{I}} s_{i\ell}(D\lambda_i u_i^\alpha)} \right\} \\
&\geq \min_\ell\{-b_\ell\} + \min_\ell \left\{ \sum_{i\in\mathcal{I}} r_{i\ell}(D\lambda_i u_i^\alpha) \right\} + \min_\ell \left\{ \Phi^{-1}(1-\alpha)\sqrt{\sum_{i\in\mathcal{I}} s_{i\ell}(D\lambda_i u_i^\alpha)} \right\} \\
&= \min_\ell\{-b_\ell\} + \min_\ell \left\{ \sum_{i\in\mathcal{I}} r_{i\ell}(D\lambda_i u_i^\alpha) \right\} + \Phi^{-1}(1-\alpha)\sqrt{\min_\ell \left\{ \sum_{i\in\mathcal{I}} s_{i\ell}(D\lambda_i u_i^\alpha) \right\}} \\
&\geq \min_\ell\{-b_\ell\} + \min_\ell \left\{ \min_i\{r_{i\ell}\} \sum_{i\in\mathcal{I}} (D\lambda_i u_i^\alpha) \right\} + \Phi^{-1}(1-\alpha)\sqrt{\min_\ell \left\{ \min_i\{s_{i\ell}\} \sum_{i\in\mathcal{I}} (D\lambda_i u_i^\alpha) \right\}} \\
&= \min_\ell\{-b_\ell\} + \hat{r}\sum_{i\in\mathcal{I}} D\lambda_i u_i^\alpha + \Phi^{-1}(1-\alpha)\sqrt{\hat{s}\sum_{i\in\mathcal{I}} D\lambda_i u_i^\alpha} \\
&= \min_\ell\{-b_\ell\} + \hat{r} z^*(\alpha) + \Phi^{-1}(1-\alpha)\sqrt{\hat{s} z^*(\alpha)}.
\end{aligned}$$

Re-arranging this expression gives that

$$\Phi^{-1}(1-\alpha)\sqrt{\hat{s}} \leq -\frac{\min_\ell\{-b_\ell\}}{\sqrt{z^*(\alpha)}} - \hat{r}\sqrt{z^*(\alpha)} \leq -\frac{\min_\ell\{-b_\ell\}}{\sqrt{z^*(\alpha)}} - \min\{\hat{r}, 0\}\sqrt{\sum_{i\in\mathcal{I}} D\lambda_i}.$$

Re-arranging, using the condition (EC.4) and the fact that $b_\ell \geq 0$ for all ℓ gives

$$z^*(\alpha) \leq \frac{-\min_\ell\{-b_\ell\}}{\Phi^{-1}(1-\alpha)\sqrt{\hat{s}} + \min\{\hat{r}, 0\}\sqrt{\sum_i D\lambda_i}}. \tag{EC.5}$$

Because the relation (EC.5) holds for all $\alpha \in (0, \alpha^*)$, and $z^*(\alpha) \geq 0$ for all $\alpha \in (0, 1)$, we have that

$$0 \leq \lim_{\alpha\downarrow 0} z^*(\alpha) \leq \lim_{\alpha\downarrow 0} \frac{-\min_\ell\{-b_\ell\}}{\Phi^{-1}(1-\alpha)\sqrt{\hat{s}} + \min\{\hat{r}, 0\}\sqrt{\sum_i D\lambda_i}} = 0,$$

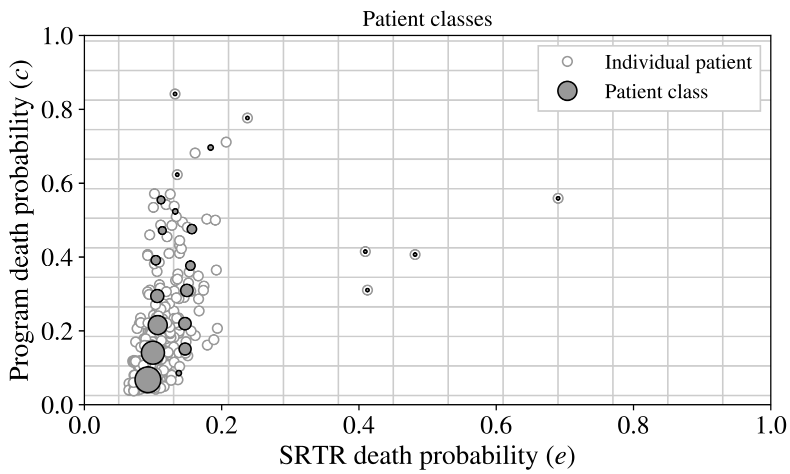
as was to be shown. □

EC.4. Convex PWL Approximation to OPTN and CMS Criteria

The models presented throughout this paper require a convex piecewise linear approximation of the criteria function g^{CMS} and g^{OPTN}. Recall that $g^{\text{CMS}}(x) = \max\{x + 3, 1.5x, f^{-1}(x)\}$ where the function f is given by

$$f(O) \equiv O\left(1 - \frac{1}{9\cdot O} - \frac{1.96}{3\sqrt{O}}\right)^3.$$

Figure EC.1 Patient classes created using the technique described in Section 6.1. The light gray grid partitions the unit square into square bins of width 0.08, and each patient pair (e, c) is indicated by an open circle. Filled circles indicate the resulting patient classes, with the size of the circle proportional to the (log of) the arrival rate λ_i.



To obtain a piecewise linear approximation, we replace the nonlinear segment f^{-1} with a linear interpolation between the two circled points in Figure 2 (i.e., between the intersection of the curve $f^{-1}(x)$ and the line $x+3$, and the intersection of the curve $f^{-1}(x)$ and the line $1.5x$). The resulting approximation to g^{CMS} is $\widehat{g}^{\text{CMS}}(x) = \max\{x+3, 1.364x+2.579, 1.5x\}$ (cf. condition 2 of Proposition 2).

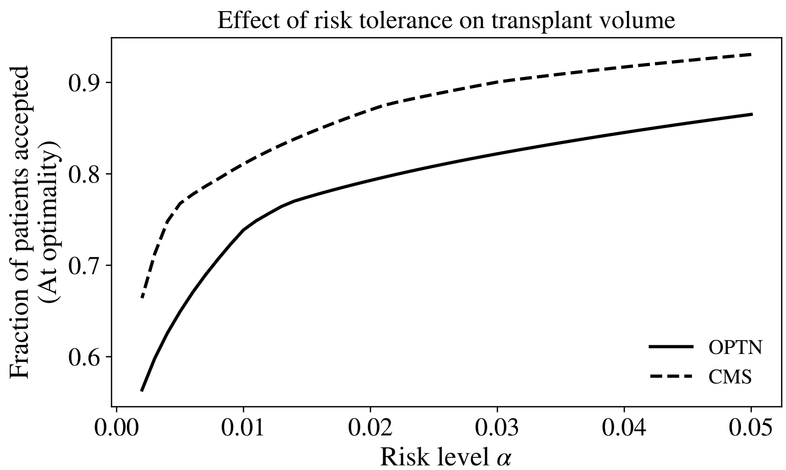
To approximate to the function g^{OPTN}, we use a line of best fit, resulting in the approximation $\widehat{g}^{\text{OPTN}}(x) = 1.298x + 2.265$. Note that this approximation is affine, and does not satisfy condition 2 of Proposition 2 for all $x \geq 0$. Nevertheless, we will see in Section EC.5.2 that the resulting approximation is still conservative in practice.

EC.5. Sensitivity Analysis

EC.5.1. Risk Parameter Selection

We explore the sensitivity of optimal program behavior to the choice of risk parameter α_w. Using the patient classes from Section 6.1, we solve the model (MA) for a single window $w \geq 6$, with $L_w = 0$, $\rho_w = 10^3$ and varying values of α_w. As proven in Proposition EC.2, the optimal fraction

Figure EC.2 Sensitivity of optimal acceptance fraction to risk parameter selection. As proven in the electronic companion, the optimal fraction of accepted patients goes to zero as $\alpha_w \downarrow 0$. We observe that, for the patient classes in Figure EC.1, the fraction of patients accepted to the waitlist at optimality is relatively level until approximately $\alpha = 1\%$, at which point it begins to drop significantly.



of accepted patients goes to zero as $\alpha_w \to 0$, consistent with intuition. We see in Figure EC.2 that this decline happens largely for $\alpha_w \leq 2\%$. This result suggests that a transplant program can only reasonably control its risk of false flagging up to approximately 2%, at which point it must cut transplant volume drastically. Potential modifications to OPTN/CMS-style regulation could focus on decreasing this number for transplant programs.

EC.5.2. Poisson Arrivals

We assess the sensitivity of our model to the Poisson arrivals assumption (Assumption 4). As a byproduct of this analysis, we also demonstrate the effect of using piecewise linear approximations for the true penalization curves g (Figure 2). We begin by randomly perturbing a Poisson distribution (described in detail below). We next solve the model (MA) using the unperturbed Poisson distribution to obtain an optimal acceptance policy u^*, and use the perturbed distribution and u^* to generate $n = 50,000$ pairs (E, O) of expected and observed deaths using the statistical model introduced in Section 4 (excluding Assumption 4). We then record the fraction of the $50,000$ sample windows which resulted in a flag under both the true criteria function g and the piecewise linear criteria function g^{PWL}. We repeat this process for 100 different random perturbations to the Poisson distribution, and report the mean, minimum and maximum flagging rates for these 100 trials. We repeat this entire process for both the CMS and OPTN criteria, and for risk values $\alpha \in \{0.97, 0.98, 0.99\}$.

We create perturbed distributions as follows. For each patient class $i \in \mathcal{I}$, we truncate a Poisson distribution with mean λ_i at the 10^{th} spot—i.e., we assume that no more than 10 patients from class i will arrive each week. Let $p^i \in \mathbb{R}^{11}$ denote the resulting vector of probabilities, so that p^i_j is the probability that j patients of class i arrive in a single week. The perturbed distribution $\tilde{p}^i$ is sampled uniformly from the set of vectors $p \in [0,1]^{11}$ satisfying

$$\sum_{j=0}^{10} p_j = 1, \tag{EC.6a}$$

$$\sum_{j=0}^{10} j \cdot p_j = \lambda_i, \tag{EC.6b}$$

$$p_0 \geq p^i_0, \tag{EC.6c}$$

$$\|p - p^i\| \leq \varepsilon = 0.1. \tag{EC.6d}$$

Constraint (EC.6a) ensures that $\tilde{p}^i$ is a probability distribution, while (EC.6b) ensures that $\tilde{p}^i$ has expectation λ_i (the same as p^i). Constraint (EC.6c) is based on the empirical observation (Section 4) that, in practice, the Poisson distribution underestimates the probability of zero patient arrivals in a given week. Finally, (EC.6d) ensures that the sampled distribution is not too different from the truncated Poisson distribution, consistent with empirical data.

The results in Table EC.5 show that, by removing the Poisson assumption, the observed flag rate stays within 0.4% of the target value $1-\alpha$ for the CMS criteria, and 0.6% for the OPTN criteria. These results indicate that the combined effect of the Poisson assumption and piecewise linearization used in (MA) is quite small—on the order of one half percent. We note also that the observed flag rates under the PWL criteria are higher than the true criteria because the PWL criteria were chosen conservatively, so that $g(E) \geq g^{\text{PWL}}(E)$ for a majority of the E values of interest.

Table EC.5 Sensitivity of program flagging rate to the Poisson arrivals assumption (Assumption 4). All values are percentages. Reported values are averaged over 100 different sampled distributions. For each sampled distribution, the flagging rate was estimated from $50,000$ trials.

		OPTN		CMS	
	Target $(1-\alpha)$	Mean	(Min, Max)	Mean	(Min, Max)
True	1	1.33	(1.17, 1.45)	1.17	(1.08, 1.34)
	2	2.40	(2.25, 2.56)	2.24	(2.08, 2.39)
	3	3.46	(3.26, 3.64)	3.25	(3.08, 3.42)
PWL	1	1.44	(1.27, 1.56)	1.40	(1.29, 1.55)
	2	2.55	(2.38, 2.70)	2.45	(2.29, 2.60)
	3	3.62	(3.40, 3.79)	3.45	(3.28, 3.68)