Appendix

EC.1. Technical Proofs

Proof of Theorem 1. Let $\delta(y)$ be the delta measure at y. For each $i=1,\ldots,m$, define

$$\tilde{P}^i=\sum_{j=1}^{n^i}\frac{L^i(y_j^i)}{\sum_{r=1}^{n^i}L^i(y_r^i)}\delta(y_j^i)$$

i.e., the distribution with point mass $L^i(y_j^i)/\sum_{r=1}^{n^i}L^i(y_r^i)$ on each y_j^i, where $L^i=dP_0^i/dQ^i$. We first show that as $n\to\infty$, the solution $(\tilde{P}^i)_{i=1,\ldots,m}$ is feasible for the optimization problems in (8) in an appropriate sense.

Consider Case 1. For each $l=1,\ldots,s^i$, by a change measure we have $E_{Q^i}|f_l^i(X^i)L(X^i)|=E_{P_0^i}|f_l^i(X^i)|<\infty$ by our assumption. Also note that $E_{Q^i}L^i=1$. Therefore, by the law of large numbers,

$$E_{\tilde{P}^i}[f_l^i(X^i)]=\frac{\sum_{j=1}^{n^i}L^i(y_j^i)f_l^i(y_j^i)}{\sum_{j=1}^{n^i}L^i(y_j^i)}=\frac{(1/n^i)\sum_{j=1}^{n^i}L^i(y_j^i)f_l^i(y_j^i)}{(1/n^i)\sum_{j=1}^{n^i}L^i(y_j^i)}\to E_{Q^i}[f_l^i(X^i)L(X^i)]\quad\text{a.s.}$$

Since $E_{Q^i}[f_l^i(X^i)L(X^i)]=E_{P_0^i}[f_l^i(X^i)]<\mu_j^i$ by our assumption, we have $E_{\tilde{P}^i}[f_l^i(X^i)]\leq\mu_l^i$ eventually as $n^i\to\infty$.

Consider Case 2. We have

$$\begin{aligned}d_\phi(\tilde{P}^i,\hat{P}_b^i)&=\sum_{j=1}^{n^i}\phi\left(\frac{L^i(y_j^i)/\sum_{r=1}^{n^i}L^i(y_r^i)}{L_b^i(y_j^i)/\sum_{r=1}^{n^i}L_b^i(y_r^i)}\right)\frac{L_b^i(y_j^i)}{\sum_{r=1}^{n^i}L_b^i(y_r^i)}\\&=\frac{1}{n^i}\sum_{j=1}^{n^i}\phi\left(\tilde{L}^i(y_j^i)\frac{(1/n^i)\sum_{r=1}^{n^i}L_b^i(y_r^i)}{(1/n^i)\sum_{r=1}^{n^i}L^i(y_r^i)}\right)\frac{L_b^i(y_j^i)}{(1/n^i)\sum_{r=1}^{n^i}L_b^i(y_r^i)}\end{aligned}$$

where $\tilde{L}^i=dP_0^i/dP_b^i$. Consider, for a given $\epsilon>0$,

$$\begin{aligned}&P(|d_\phi(\tilde{P}^i,\hat{P}_b^i)-d_\phi(P_0^i,P_b^i)|>\epsilon)\\&=P\left(\left|\frac{1}{n^i}\sum_{j=1}^{n^i}\phi\left(\tilde{L}^i(y_j^i)\frac{(1/n^i)\sum_{r=1}^{n^i}L_b^i(y_r^i)}{(1/n^i)\sum_{r=1}^{n^i}L^i(y_r^i)}\right)\frac{L_b^i(y_j^i)}{(1/n^i)\sum_{r=1}^{n^i}L_b^i(y_r^i)}-d_\phi(P_0^i,P_b^i)\right|>\epsilon\right)\\&\leq P\left(\left|\frac{1}{n^i}\sum_{j=1}^{n^i}\phi\left(\tilde{L}^i(y_j^i)\frac{(1/n^i)\sum_{r=1}^{n^i}L_b^i(y_r^i)}{(1/n^i)\sum_{r=1}^{n^i}L^i(y_r^i)}\right)\frac{L_b^i(y_j^i)}{(1/n^i)\sum_{r=1}^{n^i}L_b^i(y_r^i)}-\frac{1}{n^i}\sum_{j=1}^{n^i}\phi(\tilde{L}^i(y_j^i))\frac{L_b^i(y_j^i)}{(1/n^i)\sum_{r=1}^{n^i}L_b^i(y_r^i)}\right|>\frac{\epsilon}{2}\right)\\&\quad+P\left(\left|\frac{1}{n^i}\sum_{j=1}^{n^i}\phi(\tilde{L}^i(y_j^i))\frac{L_b^i(y_j^i)}{(1/n^i)\sum_{r=1}^{n^i}L_b^i(y_r^i)}-d_\phi(P_0^i,P_b^i)\right|>\frac{\epsilon}{2}\right)\end{aligned}\tag{EC.1}$$

We analyze the two terms in (EC.1). For any sufficiently small $\lambda > 0$, the first term is bounded from above by

$$P\left(\left|\frac{1}{n^i}\sum_{j=1}^{n^i}\phi\left(\tilde{L}^i(y_j^i)\frac{(1/n^i)\sum_{r=1}^{n^i}L_b^i(y_r^i)}{(1/n^i)\sum_{r=1}^{n^i}L^i(y_r^i)}\right)\frac{L_b^i(y_j^i)}{(1/n^i)\sum_{r=1}^{n^i}L_b^i(y_r^i)}-\frac{1}{n^i}\sum_{j=1}^{n^i}\phi(\tilde{L}^i(y_j^i))\frac{L_b^i(y_j^i)}{(1/n^i)\sum_{r=1}^{n^i}L_b^i(y_r^i)}\right|>\frac{\epsilon}{2}\right.$$
$$\left.;\left|\frac{1}{n^i}\sum_{r=1}^{n^i}L^i(y_r^i)-1\right|\leq\lambda,\left|\frac{1}{n^i}\sum_{r=1}^{n^i}L_b^i(y_r^i)-1\right|\leq\lambda\right)$$
$$+P\left(\left|\frac{1}{n^i}\sum_{j=1}^{n^i}\phi\left(\tilde{L}^i(y_j^i)\frac{(1/n^i)\sum_{r=1}^{n^i}L_b^i(y_r^i)}{(1/n^i)\sum_{r=1}^{n^i}L^i(y_r^i)}\right)\frac{L_b^i(y_j^i)}{(1/n^i)\sum_{r=1}^{n^i}L_b^i(y_r^i)}-\frac{1}{n^i}\sum_{j=1}^{n^i}\phi(\tilde{L}^i(y_j^i))\frac{L_b^i(y_j^i)}{(1/n^i)\sum_{r=1}^{n^i}L_b^i(y_r^i)}\right|>\frac{\epsilon}{2}\right.$$
$$\left.;\left|\frac{1}{n^i}\sum_{r=1}^{n^i}L^i(y_r^i)-1\right|>\lambda\text{ or }\left|\frac{1}{n^i}\sum_{r=1}^{n^i}L_b^i(y_r^i)-1\right|>\lambda\right)$$
$$\leq P\left(\frac{1}{n^i}\sum_{j=1}^{n^i}(|\phi(\tilde{L}^i(y_j^i))|+1)O(\lambda)\frac{L_b^i(y_j^i)}{(1/n^i)\sum_{r=1}^{n^i}L_b^i(y_r^i)}>\frac{\epsilon}{2};\left|\frac{1}{n^i}\sum_{r=1}^{n^i}L^i(y_r^i)-1\right|\leq\lambda,\left|\frac{1}{n^i}\sum_{r=1}^{n^i}L_b^i(y_r^i)-1\right|\leq\lambda\right)$$
$$+P\left(\left|\frac{1}{n^i}\sum_{r=1}^{n^i}L^i(y_r^i)-1\right|>\lambda\text{ or }\left|\frac{1}{n^i}\sum_{r=1}^{n^i}L_b^i(y_r^i)-1\right|>\lambda\right)\tag{EC.2}$$

where the first term in the last inequality follows from the continuity condition on ϕ, with $O(\lambda)$ being a deterministic positive function of λ that converges to 0 as $\lambda\to 0$. This first term is further bounded from above by

$$P\left(\frac{1}{n^i}\sum_{j=1}^{n^i}(|\phi(\tilde{L}^i(y_j^i))|+1)\frac{L_b^i(y_j^i)}{(1/n^i)\sum_{r=1}^{n^i}L_b^i(y_r^i)}O(\lambda)>\frac{\epsilon}{2}\right)\tag{EC.3}$$

By the law of large numbers, we have

$$\frac{1}{n^i}\sum_{j=1}^{n^i}(|\phi(\tilde{L}^i(y_j^i))|+1)L_b^i(y_j^i)\to E_{Q^i}[(|\phi(\tilde{L}^i(X^i))|+1)L_b^i(X^i)]=E_{P_b^i}|\phi(\tilde{L}^i(X^i))|+1\quad\text{a.s.}$$

by using our assumption $E_{P_b^i}|\phi(\tilde{L}^i(X^i))|<\infty$. Moreover, by the law of large numbers again, we have $(1/n^i)\sum_{r=1}^{n^i}L_b^i(y_r^i)\to 1$ a.s.. Thus,

$$\frac{1}{n^i}\sum_{j=1}^{n^i}(|\phi(\tilde{L}^i(y_j^i))|+1)\frac{L_b^i(y_j^i)}{(1/n^i)\sum_{r=1}^{n^i}L_b^i(y_r^i)}\to E_{P_b^i}|\phi(\tilde{L}^i(X^i))|+1\quad\text{a.s.}$$

When λ is chosen small enough relative to $\epsilon/2$, we have (EC.3) go to 0 as $n^i\to\infty$.

Since both $\frac{1}{n^i}\sum_{r=1}^{n^i}L^i(y_r^i)\to 1$ and $\frac{1}{n^i}\sum_{r=1}^{n^i}L_b^i(y_r^i)\to 1$ a.s., the second term in (EC.2) also goes to 0 as $n^i\to\infty$. This concludes that the first term in (EC.1) goes to 0.

For the second term in (EC.1), note that

$$\frac{1}{n^i}\sum_{j=1}^{n^i}\phi(\tilde{L}^i(y_j^i))L_b^i(y_j^i)\to E_{Q^i}[\phi(\tilde{L}^i(X^i))L_b^i(X^i)]=E_{P_b^i}[\phi(\tilde{L}^i(X^i))]=d_\phi(P_0^i,P_b^i)\quad\text{a.s.}$$

by the law of large numbers and the assumption that $E_{P_b^i}|\phi(\tilde{L}^i(X^i))|<\infty$. Moreover, since $(1/n^i)\sum_{r=1}^{n^i}L_b^i(y_r^i)\to 1$, we get

$$\frac{1}{n^i}\sum_{j=1}^{n^i}\phi(\tilde{L}^i(y_j^i))\frac{L_b^i(y_j^i)}{(1/n^i)\sum_{r=1}^{n^i}L_b^i(y_r^i)}\to d_\phi(P_0^i,P_b^i)\quad\text{a.s.}$$

Thus, the second term in (EC.1) goes to 0 as $n^i\to\infty$. Therefore, we conclude that $d_\phi(\tilde{P}^i,\hat{P}_b^i)\xrightarrow{p} d_\phi(P_0^i,P_b^i)$. Since $d_\phi(P_0^i,P_b^i)<\eta^i$ by our assumption, we have $P(d_\phi(\tilde{P}^i,\hat{P}_b^i)\leq\eta^i)\to 1$ as $n^i\to\infty$.

Next we consider the objective in (8). We show that $Z(\tilde{P}^1,\dots,\tilde{P}^m)-Z(P_0^1,\dots,P_0^m)=O_p(1/\sqrt{n})$, following the argument in the theory of differentiable statistical functionals (e.g., Serfling (2009), Chapter 6). For any λ between 0 and 1, we write

$$\begin{aligned}
&Z(P_0^1+\lambda(\tilde{P}^1-P_0^1),\dots,P_0^m+\lambda(\tilde{P}^m-P_0^m))\\
&=\int\cdots\int h(\mathbf{x}^1,\dots,\mathbf{x}^m)\prod_{i=1}^m\prod_{t=1}^{T^i}d[P_0^i+\lambda(\tilde{P}^i-P_0^i)](x_t^i)\\
&=\sum_{k=0}^{T}\lambda^k\sum_{u\in\mathcal{I}^k}\int\cdots\int h(\mathbf{x}^1,\dots,\mathbf{x}^m)\prod_{(i,t)\in(\mathcal{S}_u^k)^c}dP_0^i(x_t^i)\prod_{(i,t)\in\mathcal{S}_u^k}d(\tilde{P}^i-P_0^i)(x_t^i)
\end{aligned}$$

where $\{\mathcal{S}_u^k\}_{u\in\mathcal{I}^k}$ is the collection of all subsets of $\{(i,t):i=1,\dots,m,t=1,\dots,T^i\}$ with cardinality k, and $\mathcal{I}^k$ indexes all these subsets. Note that

$$\begin{aligned}
&\frac{d}{d\lambda}Z(P_0^1+\lambda(\tilde{P}^1-P_0^1),\dots,P_0^m+\lambda(\tilde{P}^m-P_0^m))\Big|_{\lambda=0^+}\\
&=\sum_{i=1}^m\sum_{t=1}^{T^i}\int\cdots\int h(\mathbf{x}^1,\dots,\mathbf{x}^m)\prod_{(j,s):(j,s)\neq(i,t)}dP_0^j(x_s^j)d(\tilde{P}^i-P_0^i)(x_t^i)\\
&=\sum_{i=1}^m\int\varphi^i(x;P_0^1,\dots,P_0^m)d(\tilde{P}^i-P_0^i)(x) \qquad\text{(EC.4)}
\end{aligned}$$

where

$$\varphi^i(x;P_0^1,\dots,P_0^m)=\sum_{t=1}^{T^i}E_{P_0^1,\dots,P_0^m}[h(\mathbf{X}^1,\dots,\mathbf{X}^m)|X_t^i=x] \qquad\text{(EC.5)}$$

By the definition of L^i, we can write (EC.4) as

$$\begin{aligned}&\sum_{i=1}^m\left(\frac{\int\varphi^i(x;P_0^1,\dots,P_0^m)L^i(x)d\hat{Q}^i(x)}{(1/n^i)\sum_{j=1}^{n^i}L^i(y_j^i)}-\int\varphi^i(x;P_0^1,\dots,P_0^m)L^i(x)dQ^i(x)\right)\\&=\sum_{i=1}^m\left(\frac{\int\varphi^i(x;P_0^1,\dots,P_0^m)L^i(x)d(\hat{Q}^i-Q^i)(x)}{(1/n^i)\sum_{j=1}^{n^i}L^i(y_j^i)}-\int\varphi^i(x;P_0^1,\dots,P_0^m)L^i(x)dQ^i(x)\right.\\&\left.\left(1-\frac{1}{(1/n^i)\sum_{j=1}^{n^i}L^i(y_j^i)}\right)\right)\end{aligned}\tag{EC.6}$$

where $\hat{Q}^i$ is the empirical distribution $(1/n^i)\sum_{j=1}^{n^i}\delta(y_j^i)$ on the n^i observations generated from Q^i.

Suppose $\varphi^i(x;P_0^1,\dots,P_0^m)L^i(x)=0$ a.s., then $\int\varphi^i(x;P_0^1,\dots,P_0^m)L^i(x)d(\hat{Q}^i-Q^i)(x)=0$ a.s.. Otherwise, using the assumed boundedness of h, hence $\varphi^i(x;P_0^1,\dots,P_0^m)$, and L^i, we have, by the central limit theorem,

$$\sqrt{n^i}\left(\int\varphi^i(x;P_0^1,\dots,P_0^m)L^i(x)d(\hat{Q}^i-Q^i)(x)\right)\Rightarrow N(0,(\sigma^i)^2)$$

where $(\sigma^i)^2=Var_{Q^i}(\varphi^i(X^i;P_0^1,\dots,P_0^m)L^i(X^i))>0$ is finite. Since $(1/n^i)\sum_{j=1}^{n^i}L^i(y_j^i)\to 1$ a.s. by the law of large numbers, and that $\int\varphi^i(x;P_0^1,\dots,P_0^m)L^i(x)d\hat{Q}^i(x)$ is bounded, the second term in (EC.6) converges to 0 a.s.. Thus, by Slutsky's theorem, each summand in (EC.6) converges in distribution to $N(0,(\sigma^i)^2)$. Since for each i we have $n^i=nw^i$ for some fixed $w^i>0$, we conclude that (EC.6) equal $O_p(1/\sqrt{n})$.

Now consider

$$\begin{aligned}&\frac{d^2}{d\lambda^2}Z(P_0^1+\lambda(\tilde{P}^1-P_0^1),\dots,P_0^m+\lambda(\tilde{P}^m-P_0^m))\\&=\sum_{k=2}^T k(k-1)\lambda^{k-2}\sum_{u\in\mathcal{I}^k}\int\cdots\int h(\mathbf{x}^1,\dots,\mathbf{x}^m)\prod_{(i,t)\in(\mathcal{S}_u^k)^c}dP_0^i(x_t^i)\prod_{(i,t)\in\mathcal{S}_u^k}d(\tilde{P}^i-P_0^i)(x_t^i)\end{aligned}\tag{EC.7}$$

Fixing each $\mathcal{S}_u^k$, we define

$$h_{\mathcal{S}_u^k}(\mathbf{x}_{\mathcal{S}_u^k})=\int\cdots\int h(\mathbf{x}^1,\dots,\mathbf{x}^m)\prod_{(i,t)\in(\mathcal{S}_u^k)^c}dP_0^i(x_t^i)$$

where $\mathbf{x}_{\mathcal{S}_u^k}=(x_t^i)_{(i,t)\in\mathcal{S}_u^k}$. Next define

$$\begin{aligned}&\tilde{h}_{\mathcal{S}_u^k}(\mathbf{x}_{\mathcal{S}_u^k})\\&=h_{\mathcal{S}_u^k}(\mathbf{x}_{\mathcal{S}_u^k})-\sum_{(j,t)\in\mathcal{S}_u^k}\int h_{\mathcal{S}_u^k}(\mathbf{x}_{\mathcal{S}_u^k})dP_0^j(x_t^j)+\sum_{(j_1,t_1),(j_2,t_2)\in\mathcal{S}_u^k}\int\int h_{\mathcal{S}_u^k}(\mathbf{x}_{\mathcal{S}_u^k})dP_0^{j_1}(x_{t_1}^{j_1})dP_0^{j_2}(x_{t_2}^{j_2})-\cdots\\&\quad+(-1)^k\int\cdots\int h_{\mathcal{S}_u^k}(\mathbf{x}_{\mathcal{S}_u^k})dP_0^{j_1}(x_{t_1}^{j_1})\cdots dP_0^{j_k}(x_{t_k}^{j_k})\end{aligned}$$

where each summation above is over the set of all possible combinations of $(j,t)\in\mathcal{S}_u^k$ with increasing size. Direct verification shows that $\tilde h_{\mathcal{S}_u^k}$ has the property that

$$\int\cdots\int \tilde h_{\mathcal{S}_u^k}(\mathbf{x}_{\mathcal{S}_u^k})\prod_{(i,t)\in\mathcal{S}_u^k} dR^j(x_t^j)=\int\cdots\int h_{\mathcal{S}_u^k}(\mathbf{x}_{\mathcal{S}_u^k})\prod_{(i,t)\in\mathcal{S}_u^k} d(R^j(x_t^j)-P_0^j(x_t^j))$$

for any probability measures R^j's, and

$$\int \tilde h_{\mathcal{S}_u^k}(\mathbf{x}_{\mathcal{S}_u^k})dP_0^j(x_t^j)=0 \tag{EC.8}$$

for any $(j,t)\in\mathcal{S}_u^k$. Thus, (EC.7) is equal to

$$\sum_{k=2}^T k(k-1)\lambda^{k-2}\sum_{u\in\mathcal{I}^k}\int \tilde h_{\mathcal{S}_u^k}(\mathbf{x}_{\mathcal{S}_u^k})\prod_{(i,t)\in\mathcal{S}_u^k} d\tilde P^i(x_t^i)$$

Now, viewing $\tilde P^i$ as randomly generated from Q^i, consider

$$\begin{aligned}
&E_{Q^1,\dots,Q^m}\left(\sum_{k=2}^T k(k-1)\lambda^{k-2}\sum_{u\in\mathcal{I}^k}\int \tilde h_{\mathcal{S}_u^k}(\mathbf{x}_{\mathcal{S}_u^k})\prod_{(i,t)\in\mathcal{S}_u^k} d\tilde P^i(x_t^i)\right)^2\\
&=E_{Q^1,\dots,Q^m}\left[\left(\sum_{k=2}^T k(k-1)\lambda^{k-2}\sum_{u\in\mathcal{I}^k}\int \tilde h_{\mathcal{S}_u^k}(\mathbf{x}_{\mathcal{S}_u^k})\prod_{(i,t)\in\mathcal{S}_u^k} d\tilde P^i(x_t^i)\right)^2;\frac{1}{n^i}\sum_{r=1}^{n^i}L^i(Y_r^i)\geq 1-\epsilon \text{ for all } i=1,\dots,m\right]\\
&\quad+E_{Q^1,\dots,Q^m}\left[\left(\sum_{k=2}^T k(k-1)\lambda^{k-2}\sum_{u\in\mathcal{I}^k}\int \tilde h_{\mathcal{S}_u^k}(\mathbf{x}_{\mathcal{S}_u^k})\prod_{(i,t)\in\mathcal{S}_u^k} d\tilde P^i(x_t^i)\right)^2;\frac{1}{n^i}\sum_{r=1}^{n^i}L^i(Y_r^i)< 1-\epsilon\right.\\
&\quad\left.\text{for some } i=1,\dots,m\right]
\end{aligned}\tag{EC.9}$$

We analyze the two terms in (EC.9). Note that the first term can be written as

$$\begin{aligned}
&E_{Q^1,\dots,Q^m}\left[\left(\sum_{k=2}^T k(k-1)\lambda^{k-2}\sum_{u\in\mathcal{I}^k}\frac{1}{n^{i_1}n^{i_2}\cdots n^{i_k}}\sum_{j_1=1}^{n^{i_1}}\cdots\sum_{j_k=1}^{n^{i_k}}\tilde h_{\mathcal{S}_u^k}(Y_{j_1}^{i_1},\dots,Y_{j_k}^{i_k})\frac{L^{i_1}(Y_{j_1}^{i_1})\cdots L^{i_k}(Y_{j_k}^{i_k})}{\prod_{s=1}^k((1/n^{i_s})\sum_{r=1}^{n^{i_s}}L^{i_s}(Y_r^{i_s}))}\right)^2;\right.\\
&\left.\frac{1}{n^i}\sum_{r=1}^{n^i}L^i(Y_r^i)\geq 1-\epsilon \text{ for all } i=1,\dots,m\right]\\
&\leq\left(\sum_{k=2}^T k(k-1)\lambda^{k-2}\sum_{u\in\mathcal{I}^k}\frac{1}{n^{i_1}n^{i_2}\cdots n^{i_k}}\left(E_{Q^1,\dots,Q^m}\left[\left(\sum_{j_1=1}^{n^{i_1}}\cdots\sum_{j_k=1}^{n^{i_k}}\tilde h_{\mathcal{S}_u^k}(Y_{j_1}^{i_1},\dots,Y_{j_k}^{i_k})\right.\right.\right.\right.\\
&\left.\left.\left.\left.\frac{L^{i_1}(Y_{j_1}^{i_1})\cdots L^{i_k}(Y_{j_k}^{i_k})}{\prod_{s=1}^k((1/n^{i_s})\sum_{r=1}^{n^{i_s}}L^{i_s}(Y_r^{i_s}))}\right)^2;\frac{1}{n^i}\sum_{r=1}^{n^i}L^i(Y_r^i)\geq 1-\epsilon \text{ for all } i=1,\dots,m\right]\right)^{1/2}\right)^2
\end{aligned}\tag{EC.10}$$

by Minkowski's inequality, where we view Y_j^i's as the random variables constituting the observations generated from Q^i's. Since the expression $\prod_{s=1}^k((1/n^{i_s})\sum_{r=1}^{n^{i_s}} L^{i_s}(Y_r^{i_s}))$ inside the expectation in (EC.10) does not depend on the j_s's, (EC.10) is further bounded from above by

$$\left(\sum_{k=2}^T k(k-1)\lambda^{k-2}\sum_{u\in\mathcal{I}^k}\frac{1}{n^{i_1}n^{i_2}\cdots n^{i_k}}\left(E_{Q^1,\dots,Q^m}\left(\sum_{j_1=1}^{n^{i_1}}\cdots\sum_{j_k=1}^{n^{i_k}}\tilde{h}_{\mathcal{S}_u^k}(Y_{j_1}^{i_1},\dots,Y_{j_k}^{i_k})\frac{L^{i_1}(Y_{j_1}^{i_1})\cdots L^{i_k}(Y_{j_k}^{i_k})}{(1-\epsilon)^k}\right)^2\right)^{1/2}\right)^2$$

$$= \left(\sum_{k=2}^T \frac{k(k-1)\lambda^{k-2}}{(1-\epsilon)^k}\sum_{u\in\mathcal{I}^k}\frac{1}{n^{i_1}n^{i_2}\cdots n^{i_k}}\left(\sum_{j_1=1}^{n^{i_1}}\cdots\sum_{j_k=1}^{n^{i_k}}\sum_{j_1'=1}^{n^{i_1}}\cdots\sum_{j_k'=1}^{n^{i_k}}E_{Q^1,\dots,Q^m}\left[\tilde{h}_{\mathcal{S}_u^k}(Y_{j_1}^{i_1},\dots,Y_{j_k}^{i_k})L^{i_1}(Y_{j_1}^{i_1})\cdots L^{i_k}(Y_{j_k}^{i_k})\right.\right.\right.$$
$$\left.\left.\left.\tilde{h}_{\mathcal{S}_u^k}(Y_{j_1'}^{i_1},\dots,Y_{j_k'}^{i_k})L^{i_1}(Y_{j_1'}^{i_1})\cdots L^{i_k}(Y_{j_k'}^{i_k})\right]\right)^{1/2}\right)^2 \tag{EC.11}$$

Note that

$$E_{Q^1,\dots,Q^m}\left[\tilde{h}_{\mathcal{S}_u^k}(Y_{j_1}^{i_1},\dots,Y_{j_k}^{i_k})L^{i_1}(Y_{j_1}^{i_1})\cdots L^{i_k}(Y_{j_k}^{i_k})\tilde{h}_{\mathcal{S}_u^k}(Y_{j_1'}^{i_1},\dots,Y_{j_k'}^{i_k})L^{i_1}(Y_{j_1'}^{i_1})\cdots L^{i_k}(Y_{j_k'}^{i_k})\right]=0 \tag{EC.12}$$

if any Y_j^i shows up only once among all those in both $\tilde{h}_{\mathcal{S}_u^k}(Y_{j_1}^{i_1},\dots,Y_{j_k}^{i_k})$ and $\tilde{h}_{\mathcal{S}_u^k}(Y_{j_1'}^{i_1},\dots,Y_{j_k'}^{i_k})$ in the expectation. To see this, suppose without loss of generality that $Y_{j_1}^{i_1}$ appears only once. Then we have

$$E_{Q^1,\dots,Q^m}\left[\tilde{h}_{\mathcal{S}_u^k}(Y_{j_1}^{i_1},\dots,Y_{j_k}^{i_k})L^{i_1}(Y_{j_1}^{i_1})\cdots L^{i_k}(Y_{j_k}^{i_k})\tilde{h}_{\mathcal{S}_u^k}(Y_{j_1'}^{i_1},\dots,Y_{j_k'}^{i_k})L^{i_1}(Y_{j_1'}^{i_1})\cdots L^{i_k}(Y_{j_k'}^{i_k})\right]$$
$$= E_{Q^1,\dots,Q^m}\left[E_{Q^1,\dots,Q^m}\left[\tilde{h}_{\mathcal{S}_u^k}(Y_{j_1}^{i_1},\dots,Y_{j_k}^{i_k})L^{i_1}(Y_{j_1}^{i_1})\middle|Y_j^{i_2},\dots,Y_{j_k}^{i_k},Y_{j_1'}^{i_1},\dots,Y_{j_k'}^{i_k}\right]L^{i_2}(Y_j^{i_2})\cdots L^{i_k}(Y_{j_k}^{i_k})\right.$$
$$\left.\tilde{h}_{\mathcal{S}_u^k}(Y_{j_1'}^{i_1},\dots,Y_{j_k'}^{i_k})L^{i_1}(Y_{j_1'}^{i_1})\cdots L^{i_k}(Y_{j_k'}^{i_k})\right]$$
$$= E_{Q^1,\dots,Q^m}\left[E_{P_0^{i_1}}\left[\tilde{h}_{\mathcal{S}_u^k}(Y_{j_1}^{i_1},\dots,Y_{j_k}^{i_k})\middle|Y_j^{i_2},\dots,Y_{j_k}^{i_k},Y_{j_1'}^{i_1},\dots,Y_{j_k'}^{i_k}\right]L^{i_2}(Y_j^{i_2})\cdots L^{i_k}(Y_{j_k}^{i_k})\right.$$
$$\left.\tilde{h}_{\mathcal{S}_u^k}(Y_{j_1'}^{i_1},\dots,Y_{j_k'}^{i_k})L^{i_1}(Y_{j_1'}^{i_1})\cdots L^{i_k}(Y_{j_k'}^{i_k})\right]$$
$$=0$$

since $E_{P_0^{i_1}}\left[\tilde{h}_{\mathcal{S}_u^k}(Y_{j_1}^{i_1},\dots,Y_{j_k}^{i_k})\middle|Y_j^{i_2},\dots,Y_{j_k}^{i_k},Y_{j_1'}^{i_1},\dots,Y_{j_k'}^{i_k}\right]=0$ by (EC.8).

The observation in (EC.12) implies that the summation in (EC.11)

$$\sum_{j_1=1}^{n^{i_1}}\cdots\sum_{j_k=1}^{n^{i_k}}\sum_{j_1'=1}^{n^{i_1}}\cdots\sum_{j_k'=1}^{n^{i_k}}E_{Q^1,\dots,Q^m}\left[\tilde{h}_{\mathcal{S}_u^k}(Y_{j_1}^{i_1},\dots,Y_{j_k}^{i_k})L^{i_1}(Y_{j_1}^{i_1})\cdots L^{i_k}(Y_{j_k}^{i_k})\tilde{h}_{\mathcal{S}_u^k}(Y_{j_1'}^{i_1},\dots,Y_{j_k'}^{i_k})L^{i_1}(Y_{j_1'}^{i_1})\cdots L^{i_k}(Y_{j_k'}^{i_k})\right]$$

contains only $O(n^k)$ non-zero summands. This is because in each non-zero summand only at most k distinct Y_j^i's can be present inside the expectation, and the cardinality of such combinations is $O(n^k)$. Note that each summand is bounded since h, hence $\tilde{h}_{\mathcal{S}_u^k}$, and L^i are all bounded by our assumptions. Hence (EC.11) is

$$\left(\sum_{k=2}^{T}\frac{k(k-1)\lambda^{k-2}}{(1-\epsilon)^k}\binom{T}{k}O\left(\frac{1}{n^{k/2}}\right)\right)^2=O\left(\frac{1}{n^2}\right) \tag{EC.13}$$

This shows that (EC.7) is $O_p(1/n)$ for any λ between 0 and 1. Therefore, by using Taylor's expansion, and the conclusion that (EC.6) is $O_p(1/\sqrt{n})$, we have

$$Z(\tilde{P}^1,\ldots,\tilde{P}^m)=Z(P_0^1,\ldots,P_0^m)+O_p\left(\frac{1}{\sqrt{n}}\right)=Z_0+O_p\left(\frac{1}{\sqrt{n}}\right) \tag{EC.14}$$

Note that we have shown previously that $P(\tilde{P}^i\in\hat{\mathcal{U}}^i)\to 1$ for any $i=1,\ldots,m$ in both Cases 1 and 2. Using this and (EC.14), for any given $\epsilon>0$, we can choose $M,N>0$ big enough such that

$$P(\sqrt{n}(\hat{Z}_*-Z_0)>M)\leq P(|\sqrt{n}(Z(\tilde{P}^1,\ldots,\tilde{P}^i)-Z_0)|>M)+\sum_{i=1}^m P(\tilde{P}^i\notin\hat{\mathcal{U}}^i)<\epsilon$$

and similarly

$$P(\sqrt{n}(Z_0-\hat{Z}^*)>M)\leq P(|\sqrt{n}(Z(\tilde{P}^1,\ldots,\tilde{P}^i)-Z_0)|>M)+\sum_{i=1}^m P(\tilde{P}^i\notin\hat{\mathcal{U}}^i)<\epsilon$$

for any $n>N$. This concludes that

$$\hat{Z}_*\leq Z_0+O_p\left(\frac{1}{\sqrt{n}}\right)\leq\hat{Z}^*$$

Proof of Theorem 3. To prove 1., consider first a mixture of $\mathbf{p}^i=(p_j^i)_{j=1,\ldots,n^i}$ with an arbitrary $\mathbf{q}^i\in\mathcal{P}_{n^i}$, in the form $(1-\epsilon)\mathbf{p}^i+\epsilon\mathbf{q}^i$. It satisfies

$$\frac{d}{d\epsilon}Z(\mathbf{p}^1,\ldots,\mathbf{p}^{i-1},(1-\epsilon)\mathbf{p}^i+\epsilon\mathbf{q}^i,\mathbf{p}^{i+1},\ldots,\mathbf{p}^m)\big|_{\epsilon=0}=\nabla^i Z(\mathbf{p})'(\mathbf{q}^i-\mathbf{p}^i)$$

by the chain rule. In particular, we must have

$$\psi_j^i(\mathbf{p})=\nabla^i Z(\mathbf{p})'(\mathbf{1}_j^i-\mathbf{p}^i)=\partial_j^i Z(\mathbf{p})-\nabla^i Z(\mathbf{p})'\mathbf{p}^i \tag{EC.15}$$

where $\partial_j^i Z(\mathbf{p})$ denotes partial derivative of Z with respect to p_j^i. Writing (EC.15) for all j together gives

$$\boldsymbol{\psi}^i(\mathbf{p}) = \nabla^i Z(\mathbf{p}) - (\nabla^i Z(\mathbf{p})'\mathbf{p}^i)\mathbf{1}^i$$

where $\mathbf{1}^i \in \mathbb{R}^{n^i}$ is a vector of 1. Therefore

$$\boldsymbol{\psi}^i(\mathbf{p})'(\mathbf{q}^i - \mathbf{p}^i) = (\nabla^i Z(\mathbf{p}) - (\nabla^i Z(\mathbf{p})'\mathbf{p}^i)\mathbf{1}^i)'(\mathbf{q}^i - \mathbf{p}^i) = \nabla^i Z(\mathbf{p})'(\mathbf{q}^i - \mathbf{p}^i)$$

since $\mathbf{q}^i, \mathbf{p}^i \in \mathcal{P}_{n^i}$. Summing up over i, (12) follows.

To prove 2., note that we have

$$\begin{aligned}
\psi_j^i(\mathbf{p}) &= \frac{d}{d\epsilon} Z(\mathbf{p}^1, \ldots, \mathbf{p}^{i-1}, (1-\epsilon)\mathbf{p}^i + \epsilon \mathbf{1}_j^i, \mathbf{p}^{i+1}, \ldots, \mathbf{p}^m)\Big|_{\epsilon=0} \\
&= \frac{d}{d\epsilon} E_{\mathbf{p}^1, \ldots, \mathbf{p}^{i-1}, (1-\epsilon)\mathbf{p}^i + \epsilon \mathbf{1}_j^i, \mathbf{p}^{i+1}, \ldots, \mathbf{p}^m}[h(\mathbf{X})]\Bigg|_{\epsilon=0} \\
&= E_{\mathbf{p}}[h(\mathbf{X}) s_j^i(\mathbf{X}^i)]
\end{aligned} \tag{EC.16}$$

where $s_j^i(\cdot)$ is the score function defined as

$$s_j^i(\mathbf{x}^i) = \sum_{t=1}^{T^i} \frac{d}{d\epsilon} \log((1-\epsilon)p^i(x_t^i) + \epsilon I(x_t^i = y_j^i))\Bigg|_{\epsilon=0}. \tag{EC.17}$$

Here $p^i(x_t^i) = p_j^i$ where j is chosen such that $x_t^i = y_j^i$. The last equality in (EC.16) follows from the fact that

$$\frac{d}{d\epsilon} \prod_{t=1}^{T^i} ((1-\epsilon)p^i(x_t^i) + \epsilon I(x_t^i = y_j^i))\Bigg|_{\epsilon=0} = \frac{d}{d\epsilon} \sum_{t=1}^{T^i} \log((1-\epsilon)p^i(x_t^i) + \epsilon I(x_t^i = y_j^i))\Bigg|_{\epsilon=0} \cdot \prod_{t=1}^{T^i} p^i(x_t^i)$$

Note that (EC.17) can be further written as

$$\sum_{t=1}^{T^i} \frac{-p^i(x_t^i) + I(x_t^i = y_j^i)}{p^i(x_t^i)} = -T^i + \sum_{t=1}^{T^i} \frac{I(x_t^i = y_j^i)}{p^i(x_t^i)} = -T^i + \sum_{t=1}^{T^i} \frac{I(x_t^i = y_j^i)}{p_j^i}$$

which leads to (13).

Proof of Lemma 1 We have

$$Var_{\mathbf{p}}(h(\mathbf{X}) s_j^i(\mathbf{X}^i)) \le E_{\mathbf{p}}(h(\mathbf{X}) s_j^i(\mathbf{X}^i))^2 \le M^2 E_{\mathbf{p}}(s_j^i(\mathbf{X}^i))^2 = M^2(Var_{\mathbf{p}}(s_j^i(\mathbf{X}^i)) + (E_{\mathbf{p}}[s_j^i(\mathbf{X}^i)])^2) \tag{EC.18}$$

Now note that by the definition of $s_j^i(\mathbf{X})$ in (14) we have $E_{\mathbf{p}}[s_j^i(\mathbf{X}^i)] = 0$ and

$$Var_{\mathbf{p}}(s_j^i(\mathbf{X}^i)) = \frac{T^i Var_{\mathbf{p}}(I(X_t^i = y_j^i))}{(p_j^i)^2} = \frac{T^i(1-p_j^i)}{p_j^i}$$

Hence, from (EC.18), we conclude that $Var_{\mathbf{p}}(h(\mathbf{X})s_j^i(\mathbf{X}^i)) \le M^2 T^i(1-p_j^i)/p_j^i$.

Proof of Proposition 3 Consider the Lagrangian relaxation

$$\max_{\alpha\ge 0,\lambda\in\mathbb{R}} \min_{\mathbf{p}^i\ge \mathbf{0}} \sum_{j=1}^{n^i} p_j^i \xi_j + \alpha\left(\sum_{j=1}^{n^i} p_{b,j}^i \phi\left(\frac{p_j^i}{p_{b,j}^i}\right) - \eta^i\right) + \lambda\left(\sum_{j=1}^{n^i} p_j^i - 1\right) \tag{EC.19}$$

$$= \max_{\alpha\ge 0,\lambda\in\mathbb{R}} -\alpha \sum_{j=1}^{n^i} p_{b,j}^i \max_{p_j^i\ge 0}\left\{-\frac{\xi_j+\lambda}{\alpha}\frac{p_j^i}{p_{b,j}^i} - \phi\left(\frac{p_j^i}{p_{b,j}^i}\right)\right\} - \alpha\eta^i - \lambda$$

$$= \max_{\alpha\ge 0,\lambda\in\mathbb{R}} -\alpha \sum_{j=1}^{n^i} p_{b,j}^i \phi^*\left(-\frac{\xi_j+\lambda}{\alpha}\right) - \alpha\eta^i - \lambda$$

In the particular case that $\alpha^* = 0$, the optimal value of (EC.19) is the same as

$$\max_{\lambda\in\mathbb{R}} \min_{\mathbf{p}^i\ge\mathbf{0}} \sum_{j=1}^{n^i} p_j^i\xi_j + \lambda\left(\sum_{j=1}^{n^i} p_j^i - 1\right)$$

whose inner minimization is equivalent to $\min_{\mathbf{p}^i\in\mathcal{P}^i} \sum_{j=1}^{n^i} p_j^i \xi_j = \min_{j\in\{1,\dots,n^i\}} \xi_j$. Among all solutions that lead to this objective value, we find the one that solves

$$\min_{p_j^i, j\in\mathcal{M}^i : \sum_{j\in\mathcal{M}^i} p_j^i = 1} \sum_{j\in\mathcal{M}^i} p_{b,j}^i \phi\left(\frac{p_j^i}{p_{b,j}^i}\right) \tag{EC.20}$$

Now note that by the convexity of ϕ and Jensen's inequality, for any $\sum_{j\in\mathcal{M}^i} p_j^i = 1$, we have

$$\sum_{j\in\mathcal{M}^i} p_{b,j}^i \phi\left(\frac{p_j^i}{p_{b,j}^i}\right) = \sum_{r\in\mathcal{M}^i} p_{b,r}^i \sum_{j\in\mathcal{M}^i} \frac{p_{b,j}^i}{\sum_{r\in\mathcal{M}^i} p_{b,r}^i} \phi\left(\frac{p_j^i}{p_{b,j}^i}\right) \ge \sum_{j\in\mathcal{M}^i} p_{b,j}^i \phi\left(\frac{1}{\sum_{j\in\mathcal{M}^i} p_{b,j}^i}\right) = \phi\left(\frac{1}{\sum_{j\in\mathcal{M}^i} p_{b,j}^i}\right) \tag{EC.21}$$

It is easy to see that choosing p_j^i in (EC.20) as q_j^i depicted in (22) achieves the lower bound in (EC.21), hence concluding the proposition.

Proof of Proposition 4 Consider the Lagrangian for the optimization (18)

$$\min_{\mathbf{p}^i\in\mathcal{P}^i} \sum_{j=1}^{n^i} \xi_j p_j^i + \alpha\left(\sum_{j=1}^{n^i} p_j^i \log\frac{p_j^i}{p_{b,j}^i} - \eta^i\right) \tag{EC.22}$$

By Theorem 1, P.220 in Luenberger (1969), suppose that one can find $\alpha^* \geq 0$ such that $\mathbf{q}^i = (q_j^i)_{j=1,\ldots,n^i} \in \mathcal{P}_{n^i}$ minimizes (EC.22) for $\alpha = \alpha^*$ and moreover that $\alpha^*\left(\sum_{j=1}^{n^i} q_j^i \log \frac{q_j^i}{p_{b,j}^i} - \eta^i\right) = 0$, then $\mathbf{q}^i$ is optimal for (18).

Suppose $\alpha^* = 0$, then the minimizer of (EC.22) can be any probability distributions that have masses concentrated on the set of indices in $\mathcal{M}^i$. Any one of these distributions that lies in $\hat{\mathcal{U}}^i$ will be an optimal solution to (18). To check whether any of them lies in $\hat{\mathcal{U}}^i$, consider the one that has the minimum $d_\phi(\mathbf{q}^i, \mathbf{p}_b^i)$ and see whether it is less than or equal to η^i. In other words, we want to find $\min_{p_j^i, j\in\mathcal{M}^i : \sum_{j\in\mathcal{M}^i} p_j^i = 1} \sum_{j\in\mathcal{M}^i} p_j^i \log(p_j^i / p_{b,j}^i)$. The optimal solution to this minimization is $p_{b,j}^i / \sum_{j\in\mathcal{M}^i} p_{b,j}^i$ for $j \in \mathcal{M}^i$, which gives an optimal value $-\log \sum_{j\in\mathcal{M}^i} p_{b,j}^i$. Thus, if $-\log \sum_{j\in\mathcal{M}^i} p_{b,j}^i \leq \eta^i$, we find an optimal solution $\mathbf{q}^i$ to (18) given by (23).

In the case that $\alpha^* = 0$ does not lead to an optimal solution, or equivalently $-\log \sum_{j\in\mathcal{M}^i} p_{b,j}^i > \eta^i$, we consider $\alpha^* > 0$. We write the objective value of (EC.22) with $\alpha = \alpha^*$ as

$$\sum_{j=1}^{n^i} \xi_j p_j^i + \alpha^* \sum_{j=1}^{n^i} p_j^i \log \frac{p_j^i}{p_{b,j}^i} - \alpha^* \eta^i \tag{EC.23}$$

By Jensen's inequality,

$$\sum_{j=1}^{n^i} p_j^i e^{-\xi_j/\alpha^* - \log(p_j^i/p_{b,j}^i)} \geq e^{-\sum_{j=1}^{n^i} \xi_j p_j^i/\alpha^* - \sum_{j=1}^{n^i} p_j^i \log(p_j^i/p_{b,j}^i)}$$

giving

$$\sum_{j=1}^{n^i} \xi_j p_j^i + \alpha^* \sum_{j=1}^{n^i} p_j^i \log \frac{p_j^i}{p_{b,j}^i} \geq -\alpha^* \log \sum_{j=1}^{n^i} p_{b,j}^i e^{-\xi_j/\alpha^*} \tag{EC.24}$$

It is easy to verify that putting p_j^i as

$$q_j^i = \frac{p_{b,j}^i e^{-\xi_j/\alpha^*}}{\sum_{r=1}^{n^i} p_{b,r}^i e^{-\xi_r/\alpha^*}}$$

gives the lower bound in (EC.24). Thus q_j^i minimizes (EC.23). Moreover, $\alpha^* > 0$ can be chosen such that

$$\sum_{j=1}^{n^i} q_j^i \log \frac{q_j^i}{p_{b,j}^i} = -\frac{\sum_{j=1}^{n^i} \xi_j p_{b,j}^i e^{-\xi_j/\alpha^*}}{\alpha^* \sum_{j=1}^{n^i} p_{b,j}^i e^{-\xi_j/\alpha^*}} - \log \sum_{j=1}^{n^i} p_{b,j}^i e^{-\xi_j/\alpha^*} = \eta^i$$

Letting $\beta = -1/\alpha^*$, we obtain (24) and (25). Note that (25) must bear a negative root because of the following. Note that the left hand side of (25) is continuous, and goes to 0 when $\beta \to 0$. Defining $\xi_* = \min\{\xi_j : j = 1,\dots,n^i\}$, we have, as $\beta \to -\infty$, $\varphi^i_{\boldsymbol{\xi}}(\beta) = \log\sum_{j=1}^{n^i} p^i_{b,j} e^{\beta\xi_j} = \log\left(\sum_{j\in\mathcal{M}^i} p^i_{b,j} e^{\beta\xi_*}(1+\sum_{j\notin\mathcal{M}^i} p^i_{b,j} e^{\beta(\xi_j-\xi_*)}/\sum_{j\in\mathcal{M}^i} p^i_{b,j})\right) = \beta\xi_* + \log\sum_{j\in\mathcal{M}^i} p^i_{b,j} + O(e^{c_1\beta})$ for some positive constant c_1, and $\varphi^{i\prime}_{\boldsymbol{\xi}}(\beta) = \sum_{j=1}^{n^i}\xi_j p^i_{b,j} e^{\beta\xi_j}/\sum_{j=1}^{n^i} p^i_{b,j} e^{\beta\xi_j} = \xi_*(1+\sum_{j\notin\mathcal{M}^i}\xi_j p^i_{b,j} e^{\beta(\xi_j-\xi_*)}/\sum_{j\in\mathcal{M}^i} p^i_{b,j})/(1+\sum_{j\notin\mathcal{M}^i} p^i_{b,j} e^{\beta(\xi_j-\xi_*)}/\sum_{j\in\mathcal{M}^i} p^i_{b,j}) = \xi_* + O(e^{c\beta})$ for some positive constant c_2. So $\beta\varphi^{i\prime}_{\boldsymbol{\xi}}(\beta) - \varphi^i_{\boldsymbol{\xi}}(\beta) = -\log\sum_{j\in\mathcal{M}^i} p^i_{b,j} + O(e^{(c_1\wedge c_2)\beta}) > \eta^i$ when β is negative enough.

Proof of Theorem 4 The proof is an adaptation of Blum (1954). Recall that $\mathbf{p}_k = \text{vec}(\mathbf{p}^i_k : i = 1,\dots,m)$ where we write each component of $\mathbf{p}_k$ as $p^i_{k,j}$. Let $N = \sum_{i=1}^m n^i$ be the total counts of support points. Since $h(\mathbf{X})$ is bounded a.s., we have $|h(\mathbf{X})| \le M$ a.s. for some M. Without loss of generality, we assume that $Z(\mathbf{p}) \ge 0$ for all $\mathbf{p}$. Also note that $Z(\mathbf{p})$, as a high-dimensional polynomial, is continuous everywhere in $\hat{\mathcal{U}}$.

For notational convenience, we write $\mathbf{d}_k = \mathbf{q}(\mathbf{p}_k) - \mathbf{p}_k$ and $\hat{\mathbf{d}}_k = \hat{\mathbf{q}}(\mathbf{p}_k) - \mathbf{p}_k$, i.e. $\mathbf{d}_k$ is the k-th step best feasible direction given the exact gradient estimate, and $\hat{\mathbf{d}}_k$ is the one with estimated gradient.

Now, given $\mathbf{p}_k$, consider the iterative update $\mathbf{p}_{k+1} = (1-\epsilon_k)\mathbf{p}_k + \epsilon_k\hat{\mathbf{q}}(\mathbf{p}_k) = \mathbf{p}_k + \epsilon_k\hat{\mathbf{d}}_k$. We have, by Taylor series expansion,

$$Z(\mathbf{p}_{k+1}) = Z(\mathbf{p}_k) + \epsilon_k\nabla Z(\mathbf{p}_k)'\hat{\mathbf{d}}_k + \frac{\epsilon_k^2}{2}\hat{\mathbf{d}}_k'\nabla^2 Z(\mathbf{p}_k + \theta_k\epsilon_k\hat{\mathbf{d}}_k)\hat{\mathbf{d}}_k$$

for some θ_k between 0 and 1. By Theorem 3, we can rewrite the above as

$$Z(\mathbf{p}_{k+1}) = Z(\mathbf{p}_k) + \epsilon_k\boldsymbol{\psi}(\mathbf{p}_k)'\hat{\mathbf{d}}_k + \frac{\epsilon_k^2}{2}\hat{\mathbf{d}}_k'\nabla^2 Z(\mathbf{p}_k + \theta_k\epsilon_k\hat{\mathbf{d}}_k)\hat{\mathbf{d}}_k \qquad \text{(EC.25)}$$

Consider the second term in the right hand side of (EC.25). We can write

$$\begin{aligned}\boldsymbol{\psi}(\mathbf{p}_k)'\hat{\mathbf{d}}_k &= \hat{\boldsymbol{\psi}}(\mathbf{p}_k)'\hat{\mathbf{d}}_k + (\boldsymbol{\psi}(\mathbf{p}_k) - \hat{\boldsymbol{\psi}}(\mathbf{p}_k))'\hat{\mathbf{d}}_k \\ &\le \hat{\boldsymbol{\psi}}(\mathbf{p}_k)'\mathbf{d}_k + (\boldsymbol{\psi}(\mathbf{p}_k) - \hat{\boldsymbol{\psi}}(\mathbf{p}_k))'\hat{\mathbf{d}}_k \quad \text{by the definition of } \hat{\mathbf{d}}_k \\ &= \boldsymbol{\psi}(\mathbf{p}_k)'\mathbf{d}_k + (\hat{\boldsymbol{\psi}}(\mathbf{p}_k) - \boldsymbol{\psi}(\mathbf{p}_k))'\mathbf{d}_k + (\boldsymbol{\psi}(\mathbf{p}_k) - \hat{\boldsymbol{\psi}}(\mathbf{p}_k))'\hat{\mathbf{d}}_k \\ &= \boldsymbol{\psi}(\mathbf{p}_k)'\mathbf{d}_k + (\hat{\boldsymbol{\psi}}(\mathbf{p}_k) - \boldsymbol{\psi}(\mathbf{p}_k))'(\mathbf{d}_k - \hat{\mathbf{d}}_k) \qquad \text{(EC.26)}\end{aligned}$$

Hence (EC.25) and (EC.26) together imply

$$Z(\mathbf{p}_{k+1})\leq Z(\mathbf{p}_k)+\epsilon_k\boldsymbol{\psi}(\mathbf{p}_k)'\mathbf{d}_k+\epsilon_k(\hat{\boldsymbol{\psi}}(\mathbf{p}_k)-\boldsymbol{\psi}(\mathbf{p}_k))'(\mathbf{d}_k-\hat{\mathbf{d}}_k)+\frac{\epsilon_k^2}{2}\hat{\mathbf{d}}_k'\nabla^2 Z(\mathbf{p}_k+\theta_k\epsilon_k\hat{\mathbf{d}}_k)\hat{\mathbf{d}}_k$$

Let $\mathcal{F}_k$ be the filtration generated by $\mathbf{p}_1,\ldots,\mathbf{p}_k$. We then have

$$\begin{aligned}E[Z(\mathbf{p}_{k+1})|\mathcal{F}_k]\leq{}& Z(\mathbf{p}_k)+\epsilon_k\boldsymbol{\psi}(\mathbf{p}_k)'\mathbf{d}_k+\epsilon_k E[(\hat{\boldsymbol{\psi}}(\mathbf{p}_k)-\boldsymbol{\psi}(\mathbf{p}_k))'(\mathbf{d}_k-\hat{\mathbf{d}}_k)|\mathcal{F}_k]\\&+\frac{\epsilon_k^2}{2}E[\hat{\mathbf{d}}_k'\nabla^2 Z(\mathbf{p}_k+\theta_k\epsilon_k\hat{\mathbf{d}}_k)\hat{\mathbf{d}}_k|\mathcal{F}_k]\end{aligned}\tag{EC.27}$$

We analyze (EC.27) term by term. First, since $Z(\mathbf{p})$ is a high-dimensional polynomial and $\hat{\mathcal{U}}$ is a bounded set, the largest eigenvalue of the Hessian matrix $\nabla^2 Z(\mathbf{p})$, for any $\mathbf{p}\in\hat{\mathcal{U}}$, is uniformly bounded by a constant $H>0$. Hence

$$E[\hat{\mathbf{d}}_k'\nabla^2 Z(\mathbf{p}_k+\theta_k\epsilon_k\hat{\mathbf{d}}_k)\hat{\mathbf{d}}_k|\mathcal{F}_k]\leq HE[\|\hat{\mathbf{d}}_k\|^2|\mathcal{F}_k]\leq V<\infty\tag{EC.28}$$

for some $V>0$. Now

$$E[(\hat{\boldsymbol{\psi}}(\mathbf{p}_k)-\boldsymbol{\psi}(\mathbf{p}_k))'(\mathbf{d}_k-\hat{\mathbf{d}}_k)|\mathcal{F}_k]\tag{EC.29}$$

$$\begin{aligned}&\leq\sqrt{E[\|\hat{\boldsymbol{\psi}}(\mathbf{p}_k)-\boldsymbol{\psi}(\mathbf{p}_k)\|^2|\mathcal{F}_k]E[\|\mathbf{d}_k-\hat{\mathbf{d}}_k\|^2|\mathcal{F}_k]}\quad\text{by Cauchy-Schwarz inequality}\\&\leq\sqrt{E[\|\hat{\boldsymbol{\psi}}(\mathbf{p}_k)-\boldsymbol{\psi}(\mathbf{p}_k)\|^2|\mathcal{F}_k]E[2(\|\mathbf{d}_k\|^2+\|\hat{\mathbf{d}}_k\|^2)|\mathcal{F}_k]}\quad\text{by parallelogram law}\\&\leq\sqrt{8mE[\|\hat{\boldsymbol{\psi}}(\mathbf{p}_k)-\boldsymbol{\psi}(\mathbf{p}_k)\|^2|\mathcal{F}_k]}\quad\text{since }\|\mathbf{d}_k\|^2,\|\hat{\mathbf{d}}_k\|^2\leq 2m\text{ by using the fact that }\mathbf{p}_k,\mathbf{q}(\mathbf{p}_k),\hat{\mathbf{q}}(\mathbf{p}_k)\in\mathcal{P}\\&\leq\sqrt{\frac{8mM^2T}{R_k}\sum_{i,j}\frac{1-p_{k,j}^i}{p_{k,j}^i}}\quad\text{by Lemma 1}\\&\leq M\sqrt{\frac{8mTN}{R_k\min_{i,j}p_{k,j}^i}}\end{aligned}\tag{EC.30}$$

Note that by iterating the update rule $(1-\epsilon_k)\mathbf{p}_k+\epsilon_k\mathbf{q}_k$, we have

$$\min_{i,j}p_{k,j}^i\geq\prod_{j=1}^{k-1}(1-\epsilon_j)\delta$$

where $\delta=\min_{i,j}p_{1,j}^i>0$. We thus have (EC.30) less than or equal to

$$M\sqrt{\frac{8mTN}{\delta R_k}}\prod_{j=1}^{k-1}(1-\epsilon_j)^{-1/2}\tag{EC.31}$$

Therefore, noting that $\boldsymbol{\psi}(\mathbf{p}_k)'\mathbf{d}_k \leq 0$ by the definition of $\mathbf{d}_k$, from (EC.27) we have

$$E[Z(\mathbf{p}_{k+1}) - Z(\mathbf{p}_k)|\mathcal{F}_k] \leq \epsilon_k M\sqrt{\frac{8mTN}{\delta R_k}}\prod_{j=1}^{k-1}(1-\epsilon_j)^{-1/2} + \frac{\epsilon_k^2 V}{2} \tag{EC.32}$$

and hence

$$\sum_{k=1}^{\infty} E[E[Z(\mathbf{p}_{k+1}) - Z(\mathbf{p}_k)|\mathcal{F}_k]^+] \leq M\sqrt{\frac{8mTN}{\delta}}\sum_{k=1}^{\infty}\frac{\epsilon_k}{\sqrt{R_k}}\prod_{j=1}^{k-1}(1-\epsilon_j)^{-1/2} + \sum_{k=1}^{\infty}\frac{\epsilon_k^2 V}{2}$$

By Assumptions 1 and 2, and Lemma EC.1 (depicted after this proof), we have $Z(\mathbf{p}_k)$ converge to an integrable random variable.

Now take expectation on (EC.27) further to get

$$\begin{aligned} E[Z(\mathbf{p}_{k+1})] \leq\ & E[Z(\mathbf{p}_k)] + \epsilon_k E[\boldsymbol{\psi}(\mathbf{p}_k)'\mathbf{d}_k] + \epsilon_k E[(\hat{\boldsymbol{\psi}}(\mathbf{p}_k) - \boldsymbol{\psi}(\mathbf{p}_k))'(\mathbf{d}_k - \hat{\mathbf{d}}_k)] \\ & + \frac{\epsilon_k^2}{2} E[\hat{\mathbf{d}}_k' \nabla^2 Z(\mathbf{p}_k + \theta_k\epsilon_k\hat{\mathbf{d}}_k)\hat{\mathbf{d}}_k] \end{aligned}$$

and telescope to get

$$\begin{aligned} E[Z(\mathbf{p}_{k+1})] \leq\ & E[Z(\mathbf{p}_1)] + \sum_{j=1}^{k}\epsilon_j E[\boldsymbol{\psi}(\mathbf{p}_j)'\mathbf{d}_j] + \sum_{j=1}^{k}\epsilon_j E[(\hat{\boldsymbol{\psi}}(\mathbf{p}_j) - \boldsymbol{\psi}(\mathbf{p}_j))'(\mathbf{d}_j - \hat{\mathbf{d}}_j)] \\ & + \sum_{j=1}^{k}\frac{\epsilon_j^2}{2} E[\hat{\mathbf{d}}_j' \nabla^2 Z(\mathbf{p}_j + \theta_j\epsilon_j\hat{\mathbf{d}}_j)\hat{\mathbf{d}}_j] \end{aligned} \tag{EC.33}$$

Now take the limit on both sides of (EC.33). Note that $E[Z(\mathbf{p}_{k+1})] \to E[Z_\infty]$ for some integrable Z_∞ by dominated convergence theorem. Also $Z(\mathbf{p}_1) < \infty$, and by (EC.28) and (EC.31) respectively, we have

$$\lim_{k\to\infty}\sum_{j=1}^{k}\frac{\epsilon_j^2}{2}E[\hat{\mathbf{d}}_j'\nabla Z(\mathbf{p}_j + \theta_j\epsilon_j\hat{\mathbf{d}}_j)\hat{\mathbf{d}}_j] \leq \sum_{j=1}^{\infty}\frac{\epsilon_j^2 V}{2} < \infty$$

and

$$\lim_{k\to\infty}\sum_{j=1}^{k}\epsilon_j E[(\hat{\boldsymbol{\psi}}(\mathbf{p}_j) - \boldsymbol{\psi}(\mathbf{p}_j))'(\mathbf{d}_j - \hat{\mathbf{d}}_j)] \leq M\sqrt{\frac{8mTN}{\delta}}\sum_{j=1}^{\infty}\frac{\epsilon_j}{\sqrt{R_j}}\prod_{i=1}^{j-1}(1-\epsilon_i)^{-1/2} < \infty$$

Therefore, from (EC.33), and since $E[\boldsymbol{\psi}(\mathbf{p}_j)'\mathbf{d}_j] \leq 0$, we must have $\sum_{j=1}^{k}\epsilon_j E[\boldsymbol{\psi}(\mathbf{p}_j)'\mathbf{d}_j]$ converges a.s., which implies that $\limsup_{k\to\infty} E[\boldsymbol{\psi}(\mathbf{p}_k)'\mathbf{d}_k] = 0$. So there exists a subsequence k_i such that $\lim_{i\to\infty} E[\boldsymbol{\psi}(\mathbf{p}_{k_i})'\mathbf{d}_{k_i}] = 0$. This in turn implies that $\boldsymbol{\psi}(\mathbf{p}_{k_i})'\mathbf{d}_{k_i} \xrightarrow{p} 0$. Then, there exists a further subsequence l_i such that $\boldsymbol{\psi}(\mathbf{p}_{l_i})'\mathbf{d}_{l_i} \to 0$ a.s..

Consider part 1 of the theorem. Let $S^* = \{\mathbf{p} \in \mathcal{P} : g(\mathbf{p}) = 0\}$. Since $g(\cdot)$ is continuous, we have $D(\mathbf{p}_{l_i}, S^*) \to 0$ a.s.. Since $Z(\cdot)$ is continuous, we have $D(Z(\mathbf{p}_{l_i}), \mathcal{Z}^*) \to 0$ a.s.. But since we have proven that $Z(\mathbf{p}_k)$ converges a.s., we have $D(Z(\mathbf{p}_k), \mathcal{Z}^*) \to 0$ a.s.. This gives part 1 of the theorem.

Now consider part 2. By Assumption 3, since $\mathbf{p}^*$ is the only $\mathbf{p}$ such that $g(\mathbf{p}) = 0$ and $g(\cdot)$ is continuous, we must have $\mathbf{p}_{l_i} \to \mathbf{p}^*$ a.s.. Since $Z(\cdot)$ is continuous, we have $Z(\mathbf{p}_{l_i}) \to Z(\mathbf{p}^*)$. But since $Z(\mathbf{p}_k)$ converges a.s. as shown above, we must have $Z(\mathbf{p}_k) \to Z(\mathbf{p}^*)$. Then by Assumption 3 again, since $\mathbf{p}^*$ is the unique optimizer, we have $\mathbf{p}_k \to \mathbf{p}^*$ a.s.. This concludes part 2 of the theorem.

Lemma EC.1 **(Adapted from Blum (1954))**. *Consider a sequence of integrable random variable $Y_k, k = 1, 2, \dots$. Let $\mathcal{F}_k$ be the filtration generated by $Y_1, \dots, Y_k$. Assume*

$$\sum_{k=1}^{\infty} E[E[Y_{k+1} - Y_k | \mathcal{F}_k]^+] < \infty$$

where x^+ denotes the positive part of x, i.e. $x^+ = x$ if $x \geq 0$ and 0 if $x < 0$. Moreover, assume that Y_k is bounded uniformly from above. Then $Y_k \to Y_\infty$ a.s., where Y_∞ is an integrable random variable.

The lemma follows from Blum (1954), with the additional conclusion that Y_∞ is integrable, which is a direct consequence of the martingale convergence theorem.

Theorem EC.1 **(Conditions in Theorem 5)**. *Conditions 1-9 needed in Theorem 5 are:*

1.

$$k_0 \geq 2a \left(\frac{4KMTm}{c^2\tau^2} + \frac{KL\vartheta}{c\tau} \right)$$

2.

$$-\left(1 - \frac{2KL\vartheta}{c\tau} - \frac{2a\varrho K}{c^2\tau^2 k_0}\right)\nu + \frac{2aKL\vartheta\varrho}{c\tau k_0^{1+\gamma}} + \frac{\varrho}{k_0^\gamma} + \frac{2K\nu^2}{c^2\tau^2} \leq 0$$

3.

$$\frac{2KL\vartheta}{c\tau} + \frac{2K\nu}{c^2\tau^2} < 1$$

4.

$$\frac{a}{k_0}\left(1-\frac{2KL\vartheta}{c\tau}-\frac{2K\nu}{c^2\tau^2}\right)<1$$

5.

$$k_0 \geq \frac{a\rho}{\rho-1}$$

6.

$$\beta > \rho a + 2\gamma + 2$$

7.

$$\prod_{j=1}^{k_0-1}(1-\epsilon_j)^{-1}\frac{M^2TN}{\vartheta^2\delta b}\frac{1}{(\beta-\rho a-1)(k_0-1)^{\beta-1}} + \prod_{j=1}^{k_0-1}(1-\epsilon_j)^{-1/2}\frac{M}{\varrho}\sqrt{\frac{8mTN}{\delta b}}\frac{1}{((\beta-\rho a)/2-\gamma-1)(k_0-1)^{\beta/2-\gamma-1}}<\varepsilon$$

where $N=\sum_{i=1}^m n^i$ *is the total count of all support points.*

8. $K>0$ *is a constant such that* $|\mathbf{x}'\nabla^2 Z(\mathbf{p})\mathbf{y}|\leq K\|\mathbf{x}\|\|\mathbf{y}\|$ *for any* $x,y\in\mathbb{R}^n$ *and* $\mathbf{p}\in\mathcal{A}$ *(which must exist because* $Z(\cdot)$ *is a polynomial defined over a bounded set).*

9. $\delta=\min_{\substack{i=1,\ldots,m\\ j=1,\ldots,n^i}} p_{1,j}^i>0$

Proof of Theorem 5 We adopt the notation as in the proof of Theorem 4. In addition, for convenience, we write $\boldsymbol{\psi}_k=\boldsymbol{\psi}(\mathbf{p}_k)$, $\hat{\boldsymbol{\psi}}_k=\hat{\boldsymbol{\psi}}(\mathbf{p}_k)$, $\mathbf{q}_k=\mathbf{q}(\mathbf{p}_k)$, $\hat{\mathbf{q}}_k=\hat{\mathbf{q}}(\mathbf{p}_k)$, $g_k=g(\mathbf{p}_k)=-\boldsymbol{\psi}(\mathbf{p}_k)'\mathbf{d}_k$, $\nabla Z_k=\nabla Z(\mathbf{p}_k)$, and $\nabla^2 Z_k=\nabla^2 Z(\mathbf{p}_k)$. Note that $\mathbf{p}_{k+1}=\mathbf{p}_k+\epsilon_k\hat{\mathbf{d}}_k$.

First, by the proof of Theorem 4, given any ν and $\tilde{k}_0$, almost surely there must exists a $k_0\geq\tilde{k}_0$ such that $g_{k_0}\leq\nu$. If the optimal solution is reached and is kept there, then $g_k=0$ from thereon and the algorithm reaches and remains at optimum at finite time, hence there is nothing to prove. So let us assume that $0<g_{k_0}\leq\nu$. Moreover, let us assume that ν is chosen small enough so that for any $\mathbf{p}$ with $g(\mathbf{p})\leq\nu$ and $\mathbf{p}>\mathbf{0}$, we have $\boldsymbol{\psi}(\mathbf{p})\in\mathcal{N}_{\Delta-\vartheta}(\boldsymbol{\psi}(\mathbf{p}^*))$ (which can be done since $g(\cdot)$ is assumed continuous by Assumption 3 and $\boldsymbol{\psi}(\mathbf{p})$ is continuous for any $\mathbf{p} > \mathbf{0}$ by the construction in Theorem 3).

We consider the event

$$\mathcal{E} = \bigcup_{k=k_0}^{\infty} \mathcal{E}_k \cup \bigcup_{k=k_0}^{\infty} \mathcal{E}_k'$$

where

$$\mathcal{E}_k = \{\|\hat{\boldsymbol{\psi}}_k - \boldsymbol{\psi}_k\| > \vartheta\}$$

and

$$\mathcal{E}_k' = \left\{|(\hat{\boldsymbol{\psi}}_k - \boldsymbol{\psi}_k)'(\hat{\mathbf{d}}_k - \mathbf{d}_k)| > \frac{\varrho}{k^\gamma}\right\}$$

Note that by the Markov inequality,

$$P(\mathcal{E}_k) \leq \frac{E\|\hat{\boldsymbol{\psi}}_k - \boldsymbol{\psi}_k\|^2}{\vartheta^2} \leq \frac{M^2T}{\vartheta^2 R_k} \sum_{i,j} \frac{1-p_{k,j}^i}{p_{k,j}^i} \leq \frac{M^2TN}{\vartheta^2 R_k \delta} \prod_{j=1}^{k-1} (1-\epsilon_j)^{-1}$$

where the second inequality follows from Lemma 1 and the last inequality follows as in the derivation in (EC.30) and (EC.31). On the other hand, we have

$$P(\mathcal{E}_k') \leq \frac{k^\gamma E|(\hat{\boldsymbol{\psi}}_k - \boldsymbol{\psi}_k)'(\hat{\mathbf{d}}_k - \mathbf{d}_k)|}{\varrho} \leq \frac{k^\gamma M}{\varrho} \sqrt{\frac{8mTN}{\delta R_k}} \prod_{j=1}^{k-1} (1-\epsilon_j)^{-1/2} \tag{EC.34}$$

by following the derivation in (EC.30) and (EC.31). Therefore,

$$\begin{aligned} P(\mathcal{E}) &\leq \sum_{k=k_0}^{\infty} P(\mathcal{E}_k) + \sum_{k=k_0}^{\infty} P(\mathcal{E}_k') \\ &\leq \frac{M^2TN}{\vartheta^2\delta} \sum_{k=k_0}^{\infty} \frac{1}{R_k} \prod_{j=1}^{k-1}(1-\epsilon_j)^{-1} + \frac{M}{\varrho}\sqrt{\frac{8mTN}{\delta}} \sum_{k=k_0}^{\infty} \frac{k^\gamma}{\sqrt{R_k}} \prod_{j=1}^{k-1} (1-\epsilon_j)^{-1/2} \\ &= \prod_{j=1}^{k_0-1}(1-\epsilon_j)^{-1} \frac{M^2TN}{\vartheta^2\delta} \sum_{k=k_0}^{\infty} \frac{1}{R_k} \prod_{j=k_0}^{k-1} (1-\epsilon_j)^{-1} + \prod_{j=1}^{k_0-1}(1-\epsilon_j)^{-1/2} \frac{M}{\varrho}\sqrt{\frac{8mTN}{\delta}} \sum_{k=k_0}^{\infty} \frac{k^\gamma}{\sqrt{R_k}} \prod_{j=k_0}^{k-1}(1-\epsilon_j)^{-1/2} \end{aligned} \tag{EC.35}$$

Now recall that $\epsilon_k = a/k$. Using the fact that $1 - x \geq e^{-\rho x}$ for any $0 \leq x \leq (\rho-1)/\rho$ and $\rho > 1$, we have, for any

$$\frac{a}{k} \leq \frac{\rho - 1}{\rho}$$

or equivalently

$$k \geq \frac{a\rho}{\rho - 1}$$

we have

$$1 - \epsilon_k = 1 - \frac{a}{k} \geq e^{-\rho a/k}$$

Hence choosing k_0 satisfying Condition 5, we get

$$\prod_{j=k_0}^{k-1} (1-\epsilon_j)^{-1} \leq e^{\rho a \sum_{k_0}^{k-1} 1/j} \leq \left(\frac{k-1}{k_0-1}\right)^{\rho a} \tag{EC.36}$$

Therefore, picking $R_k = bk^\beta$ and using (EC.36), we have (EC.35) bounded from above by

$$\begin{aligned}
&\prod_{j=1}^{k_0-1} (1-\epsilon_j)^{-1} \frac{M^2TN}{\vartheta^2 \delta b} \sum_{k=k_0}^{\infty} \frac{1}{(k_0-1)^{\rho a} k^{\beta - \rho a}} + \prod_{j=1}^{k_0-1} (1-\epsilon_j)^{-1/2} \frac{M}{\varrho} \sqrt{\frac{8mTN}{\delta b}} \sum_{k=k_0}^{\infty} \frac{1}{(k_0-1)^{\rho a/2} k^{(\beta-\rho a)/2-\gamma}} \\
&\leq \prod_{j=1}^{k_0-1} (1-\epsilon_j)^{-1} \frac{M^2TN}{\vartheta^2 \delta b} \frac{1}{(\beta - \rho a - 1)(k_0-1)^{\beta-1}} \\
&\quad + \prod_{j=1}^{k_0-1} (1-\epsilon_j)^{-1/2} \frac{M}{\varrho} \sqrt{\frac{8mTN}{\delta b}} \frac{1}{((\beta-\rho a)/2 - \gamma - 1)(k_0-1)^{\beta/2-\gamma-1}}
\end{aligned} \tag{EC.37}$$

if Condition 6 holds. Then Condition 7 guarantees that $P(\mathcal{E}) < \varepsilon$.

The rest of the proof will show that under the event $\mathcal{E}^c$, we must have the bound (26), hence concluding the theorem. To this end, we first set up a recursive representation of g_k. Consider

$$\begin{aligned}
g_{k+1} &= -\boldsymbol{\psi}_{k+1}' \mathbf{d}_{k+1} = -\boldsymbol{\psi}_{k+1}'(\mathbf{q}_{k+1} - \mathbf{p}_{k+1}) \\
&= -\boldsymbol{\psi}_k'(\mathbf{q}_{k+1} - \mathbf{p}_{k+1}) + (\boldsymbol{\psi}_k - \boldsymbol{\psi}_{k+1})'(\mathbf{q}_{k+1} - \mathbf{p}_{k+1}) \\
&= -\boldsymbol{\psi}_k'(\mathbf{q}_{k+1} - \mathbf{p}_k) + \boldsymbol{\psi}_k'(\mathbf{p}_{k+1} - \mathbf{p}_k) + (\boldsymbol{\psi}_k - \boldsymbol{\psi}_{k+1})'(\mathbf{q}_{k+1} - \mathbf{p}_{k+1}) \\
&\leq g_k + \epsilon_k \boldsymbol{\psi}_k' \hat{\mathbf{d}}_k + (\boldsymbol{\psi}_k - \boldsymbol{\psi}_{k+1})' \mathbf{d}_{k+1} \quad \text{ by the definition of } g_k,\ \hat{\mathbf{d}}_k \text{ and } \mathbf{d}_{k+1} \\
&\leq g_k - \epsilon_k g_k + \epsilon_k (\hat{\boldsymbol{\psi}}_k - \boldsymbol{\psi}_k)'(\mathbf{d}_k - \hat{\mathbf{d}}_k) + (\boldsymbol{\psi}_k - \boldsymbol{\psi}_{k+1})' \mathbf{d}_{k+1} \quad \text{ by (EC.26)} \\
&= (1-\epsilon_k) g_k + (\nabla Z_k - \nabla Z_{k+1})' \mathbf{d}_{k+1} + \epsilon_k (\hat{\boldsymbol{\psi}}_k - \boldsymbol{\psi}_k)'(\mathbf{d}_k - \hat{\mathbf{d}}_k)
\end{aligned} \tag{EC.38}$$

Now since $\nabla Z(\cdot)$ is continuously differentiable, we have $\nabla Z_{k+1} = \nabla Z_k + \epsilon_k \nabla^2 Z(\mathbf{p}_k + \tilde{\theta}_k \hat{\mathbf{d}}_k) \hat{\mathbf{d}}_k$ for some $\tilde{\theta}_k$ between 0 and 1. Therefore (EC.38) is equal to

$$(1-\epsilon_k) g_k - \epsilon_k \hat{\mathbf{d}}_k' \nabla^2 Z(\mathbf{p}_k + \tilde{\theta}_k \hat{\mathbf{d}}_k) \mathbf{d}_{k+1} + \epsilon_k (\hat{\boldsymbol{\psi}}_k - \boldsymbol{\psi}_k)'(\mathbf{d}_k - \hat{\mathbf{d}}_k)$$

$$\leq (1-\epsilon_k)g_k + \epsilon_k K\|\hat{\mathbf{d}}_k\|\|\mathbf{d}_{k+1}\| + \epsilon_k(\hat{\boldsymbol{\psi}}_k - \boldsymbol{\psi}_k)'(\mathbf{d}_k - \hat{\mathbf{d}}_k) \quad \text{by Condition 8}$$

$$\leq (1-\epsilon_k)g_k + \epsilon_k K\|\mathbf{d}_k\|\|\mathbf{d}_{k+1}\| + \epsilon_k K\|\hat{\mathbf{d}}_k - \mathbf{d}_k\|\|\mathbf{d}_{k+1}\| + \epsilon_k(\hat{\boldsymbol{\psi}}_k - \boldsymbol{\psi}_k)'(\mathbf{d}_k - \hat{\mathbf{d}}_k)$$

by the triangle inequality

$$\leq (1-\epsilon_k)g_k + \epsilon_k K\frac{g_k g_{k+1}}{c^2\|\boldsymbol{\psi}_k\|\|\boldsymbol{\psi}_{k+1}\|} + \epsilon_k KL\|\hat{\boldsymbol{\psi}}_k - \boldsymbol{\psi}_k\|\frac{g_{k+1}}{c\|\boldsymbol{\psi}_{k+1}\|} + \epsilon_k(\hat{\boldsymbol{\psi}}_k - \boldsymbol{\psi}_k)'(\mathbf{d}_k - \hat{\mathbf{d}}_k)$$

by using Assumption 4 with the fact that $g_k \leq \nu$ and hence $\boldsymbol{\psi}_k, \hat{\boldsymbol{\psi}}_k \in \mathcal{N}_\Delta(\boldsymbol{\psi}(\mathbf{p}^*))$, and also Assumption 5. The fact $g_k \leq \nu$ will be proved later by induction.

$$\leq (1-\epsilon_k)g_k + \epsilon_k\frac{K}{c^2\tau^2}g_k g_{k+1} + \epsilon_k\frac{KL}{c\tau}\|\hat{\boldsymbol{\psi}}_k - \boldsymbol{\psi}_k\|g_{k+1} + \epsilon_k(\hat{\boldsymbol{\psi}}_k - \boldsymbol{\psi}_k)'(\mathbf{d}_k - \hat{\mathbf{d}}_k) \tag{EC.39}$$

by Assumption 6

Now under the event $\mathcal{E}^c$, and noting that $\epsilon = a/k$, (EC.39) implies that

$$g_{k+1} \leq \left(1 - \frac{a}{k}\right)g_k + \frac{aK}{c^2\tau^2 k}g_k g_{k+1} + \frac{aKL\vartheta}{c\tau k}g_{k+1} + \frac{a\varrho}{k^{1+\gamma}}$$

or

$$\left(1 - \frac{aK}{c^2\tau^2 k}g_k - \frac{aKL\vartheta}{c\tau k}\right)g_{k+1} \leq \left(1 - \frac{a}{k}\right)g_k + \frac{a\varrho}{k^{1+\gamma}}$$

We claim that $|g_k| = |\boldsymbol{\psi}_k'\mathbf{d}_k| \leq 4MTm$, which can be seen by writing

$$\begin{aligned}\psi_j^i(\mathbf{p}) = E_{\mathbf{p}}[h(\mathbf{X})s_j^i(\mathbf{X}^i)] &= \sum_{t=1}^{T^i} E_{\mathbf{p}}\left[h(\mathbf{X})\frac{I(X_t^i = y_j^i)}{p_j^i}\right] - T^i E_{\mathbf{p}}[h(\mathbf{X})] \\ &= \sum_{t=1}^{T^i} E_{\mathbf{p}}[h(\mathbf{X})|X_t = y_j^i] - T^i E_{\mathbf{p}}[h(\mathbf{X})]\end{aligned} \tag{EC.40}$$

so that $|\psi_j^i(\mathbf{p})| \leq 2MT^i$ for any $\mathbf{p}$ and i. Using this and the fact that $1/(1-x) \leq 1 + 2x$ for any $0 \leq x \leq 1/2$, we have, for

$$\frac{4aKMTm}{c^2\tau^2 k} + \frac{aKL\vartheta}{c\tau k} \leq \frac{1}{2} \tag{EC.41}$$

we must have

$$g_{k+1} \leq \left(1 + \frac{2aK}{c^2\tau^2 k}g_k + \frac{2aKL\vartheta}{c\tau k}\right)\left(\left(1 - \frac{a}{k}\right)g_k + \frac{a\varrho}{k^{1+\gamma}}\right) \tag{EC.42}$$

Note that (EC.41) holds if

$$k \geq 2a\left(\frac{4KMTm}{c^2\tau^2} + \frac{KL\vartheta}{c\tau}\right)$$

which is Condition 1 in the theorem. Now (EC.42) can be written as

$$\begin{aligned}
g_{k+1} &\leq \left(1-\frac{a}{k}+\frac{2aKL\vartheta}{c\tau k}+\frac{2a^2K\varrho}{c^2\tau^2k^{2+\gamma}}\right)g_k+\frac{a\varrho}{k^{1+\gamma}}+\frac{2a^2KL\vartheta\varrho}{c\tau k^{2+\gamma}}-\frac{2a^2KL\vartheta}{c\tau k^2}g_k+\frac{2aK}{c^2\tau^2k}\left(1-\frac{a}{k}\right)g_k^2 \\
&\leq \left(1-\frac{a}{k}+\frac{2aKL\vartheta}{c\tau k}+\frac{2a^2K\varrho}{c^2\tau^2k^{2+\gamma}}\right)g_k+\frac{a\varrho}{k^{1+\gamma}}+\frac{2a^2KL\vartheta\varrho}{c\tau k^{2+\gamma}}+\frac{2aK}{c^2\tau^2k}\left(1-\frac{a}{k}\right)g_k^2 \qquad \text{(EC.43)}
\end{aligned}$$

We argue that under Condition 2, we must have $g_k \leq \nu$ for all $k \geq k_0$. This can be seen by induction using (EC.43). By our setting at the beginning of this proof we have $g_{k_0} \leq \nu$. Suppose $g_k \leq \nu$ for some k. We then have

$$\begin{aligned}
g_{k+1} &\leq \left(1-\frac{a}{k}+\frac{2aKL\vartheta}{c\tau k}+\frac{2a^2K\varrho}{c^2\tau^2k^{2+\gamma}}\right)\nu+\frac{a\varrho}{k^{1+\gamma}}+\frac{2a^2KL\vartheta\varrho}{c\tau k^{2+\gamma}}+\frac{2aK}{c^2\tau^2k}\left(1-\frac{a}{k}\right)\nu^2 \\
&\leq \nu+\frac{a}{k}\left(\left(-1+\frac{2KL\vartheta}{c\tau}+\frac{2aK\varrho}{c^2\tau^2k^{1+\gamma}}\right)\nu+\frac{\varrho}{k_0^{\gamma}}+\frac{2aKL\vartheta\varrho}{c\tau k_0^{1+\gamma}}+\frac{2K\nu^2}{c^2\tau^2}\right) \\
&\leq \nu \qquad \text{(EC.44)}
\end{aligned}$$

by Condition 2. This concludes our claim.

Given that $g_k \leq \nu$ for all $k \geq k_0$, (EC.42) implies that

$$\begin{aligned}
g_{k+1} &\leq \left(1-\frac{a}{k}\left(1-\frac{2KL\vartheta}{c\tau}\right)-\frac{2a^2KL\vartheta}{c\tau k^2}+\frac{2aK\nu}{c^2\tau^2k}\left(1-\frac{a}{k}\right)\right)g_k+\frac{a\varrho}{k^{1+\gamma}}+\frac{a^2\varrho}{k^{2+\gamma}}\left(\frac{2K\nu}{c^2\tau^2}+\frac{2KL\vartheta}{c\tau}\right) \\
&\leq \left(1-\frac{a}{k}\left(1-\frac{2KL\vartheta}{c\tau}-\frac{2K\nu}{c^2\tau^2}\right)\right)g_k+\frac{a\varrho}{k^{1+\gamma}}+\frac{a^2\varrho}{k^{2+\gamma}}\left(\frac{2K\nu}{c^2\tau^2}+\frac{2KL\vartheta}{c\tau}\right) \\
&\leq \left(1-\frac{C}{k}\right)g_k+\frac{G}{k^{1+\gamma}} \qquad \text{(EC.45)}
\end{aligned}$$

where

$$C = a\left(1-\frac{2KL\vartheta}{c\tau}-\frac{2K\nu}{c^2\tau^2}\right)$$

and

$$G = a\varrho+\frac{a^2\varrho}{k_0}\left(\frac{2K\nu}{c^2\tau^2}+\frac{2KL\vartheta}{c\tau}\right)$$

Now note that Conditions 3 and 4 imply $C>0$ and $1-C/k>0$ respectively. By recursing the relation (EC.45), we get

$$g_{k+1} \leq \prod_{j=k_0}^{k}\left(1-\frac{C}{j}\right)g_{k_0}+\sum_{j=k_0}^{k}\prod_{i=j+1}^{k}\left(1-\frac{C}{i}\right)\frac{G}{j^{1+\gamma}}$$

$$
\begin{aligned}
&\leq e^{-C\sum_{j=k_0}^{k}1/j}g_{k_0}+\sum_{j=k_0}^{k}e^{-C\sum_{i=j+1}^{k}1/i}\frac{G}{j^{1+\gamma}}\\
&\leq \left(\frac{k_0}{k+1}\right)^C g_{k_0}+\sum_{j=k_0}^{k}\left(\frac{j+1}{k+1}\right)^C\frac{G}{j^{1+\gamma}}\\
&\leq \left(\frac{k_0}{k+1}\right)^C g_{k_0}+\left(1+\frac{1}{k_0}\right)^C G\times\begin{cases}\frac{1}{(C-\gamma)(k+1)^{\gamma}} & \text{if } 0<\gamma<C\\ \frac{1}{(\gamma-C)(k_0-1)^{\gamma-C}(k+1)^C} & \text{if } \gamma>C\\ \frac{\log(k/(k_0-1))}{(k+1)^C} & \text{if } \gamma=C\end{cases}
\end{aligned}
$$

which gives (26). This concludes the proof.

Proof of Corollary 1 We use the notations in the proof of Theorem 5. Our analysis starts from (EC.25), namely

$$Z_{k+1}=Z_k+\epsilon_k\boldsymbol{\psi}_k'\hat{\mathbf{d}}_k+\frac{\epsilon_k^2}{2}\hat{\mathbf{d}}_k'\nabla^2 Z(\mathbf{p}_k+\theta_k\epsilon_k\hat{\mathbf{d}}_k)\hat{\mathbf{d}}_k$$

for some $\tilde{\theta}_k$ between 0 and 1. Using the fact that $\boldsymbol{\psi}_k'\hat{\mathbf{d}}_k\geq\boldsymbol{\psi}_k'\mathbf{d}_k$ by the definition of $\mathbf{d}_k$, we have

$$
\begin{aligned}
Z_{k+1}&\geq Z_k+\epsilon_k\boldsymbol{\psi}_k'\mathbf{d}_k+\frac{\epsilon_k^2}{2}\hat{\mathbf{d}}_k'\nabla^2 Z(\mathbf{p}_k+\theta_k\epsilon_k\hat{\mathbf{d}}_k)\hat{\mathbf{d}}_k\\
&=Z_k-\epsilon_k g_k+\frac{\epsilon_k^2}{2}\hat{\mathbf{d}}_k'\nabla^2 Z(\mathbf{p}_k+\theta_k\epsilon_k\hat{\mathbf{d}}_k)\hat{\mathbf{d}}_k
\end{aligned}
$$

Now, using (26), Condition 8 in Theorem 5 and $\|\hat{\mathbf{d}}_k\|^2\leq 2$, we have

$$
\begin{aligned}
Z_{k+1}&\geq Z_k-\epsilon_k\left(\frac{A}{k^C}+B\times\left\{\begin{array}{ll}\frac{1}{(C-\gamma)k^{\gamma}} & \text{if } 0<\gamma<C\\ \frac{1}{(\gamma-C)(k_0-1)^{\gamma-C}k^C} & \text{if } \gamma>C\\ \frac{\log((k-1)/(k_0-1))}{k^C} & \text{if } \gamma=C\end{array}\right\}\right)-\epsilon_k^2 K\\
&=Z_k-\frac{aA}{k^{1+C}}-aB\times\left\{\begin{array}{ll}\frac{1}{(C-\gamma)k^{1+\gamma}} & \text{if } 0<\gamma<C\\ \frac{1}{(\gamma-C)(k_0-1)^{\gamma-C}k^{1+C}} & \text{if } \gamma>C\\ \frac{\log((k-1)/(k_0-1))}{k^{1+C}} & \text{if } \gamma=C\end{array}\right\}-\frac{a^2K}{k^2} \qquad \text{(EC.46)}
\end{aligned}
$$

Now iterating (EC.46) from k to l, we have

$$Z_l\geq Z_k-\sum_{j=k}^{l-1}\frac{aA}{j^{1+C}}-aB\times\left\{\begin{array}{ll}\frac{1}{(C-\gamma)}\sum_{j=k}^{l-1}\frac{1}{j^{1+\gamma}} & \text{if } 0<\gamma<C\\ \frac{1}{(\gamma-C)(k_0-1)^{\gamma-C}}\sum_{j=k}^{l-1}\frac{1}{j^{1+C}} & \text{if } \gamma>C\\ \sum_{j=k}^{l-1}\frac{\log((j-1)/(k_0-1))}{j^{1+C}} & \text{if } \gamma=C\end{array}\right\}-a^2K\sum_{j=k}^{l-1}\frac{1}{j^2}$$

and letting $l\to\infty$, we get

$$Z^*\geq Z_k-\sum_{j=k}^{\infty}\frac{aA}{j^{1+C}}-aB\times\left\{\begin{array}{ll}\frac{1}{(C-\gamma)}\sum_{j=k}^{\infty}\frac{1}{j^{1+\gamma}} & \text{if } 0<\gamma<C\\ \frac{1}{(\gamma-C)(k_0-1)^{\gamma-C}}\sum_{j=k}^{\infty}\frac{1}{j^{1+C}} & \text{if } \gamma>C\\ \sum_{j=k}^{\infty}\frac{\log((j-1)/(k_0-1))}{j^{1+C}} & \text{if } \gamma=C\end{array}\right\}-a^2K\sum_{j=k}^{\infty}\frac{1}{j^2} \tag{EC.47}$$

where the convergence to Z^* is guaranteed by Theorem 4. Note that (EC.47) implies that

$$\begin{aligned}Z^*&\geq Z_k-\frac{aA}{C(k-1)^C}-aB\times\left\{\begin{array}{ll}\frac{1}{(C-\gamma)\gamma(k-1)^{\gamma}} & \text{if } 0<\gamma<C\\ \frac{1}{(\gamma-C)(k_0-1)^{\gamma-C}C(k-1)^C} & \text{if } \gamma>C\\ \frac{\log((k-1)/(k_0-1))}{C(k-1)^C} & \text{if } \gamma=C\end{array}\right\}-\frac{a^2K}{k-1}\\ &\geq Z_k-\frac{D}{k-1}-\frac{E}{(k-1)^C}-F\times\left\{\begin{array}{ll}\frac{1}{(C-\gamma)\gamma(k-1)^{\gamma}} & \text{if } 0<\gamma<C\\ \frac{1}{(\gamma-C)(k_0-1)^{\gamma-C}C(k-1)^C} & \text{if } \gamma>C\\ \frac{\log((k-1)/(k_0-1))}{C(k-1)^C} & \text{if } \gamma=C\end{array}\right.\end{aligned}$$

where $D=a^2K$, $E=aA/C$ and $F=aB$. This gives (28).

Proof of Lemma 2 Consider first a fixed a. When $a(1-\omega)>1$, (29) reduces to $\frac{\beta-\rho a-\zeta-2}{2(\beta+1)}\wedge\frac{1}{\beta+1}$. Since $\frac{\beta-\rho a-\zeta-2}{2(\beta+1)}$ is increasing in β and $\frac{1}{\beta+1}$ is decreasing in β, the maximizer of $\frac{\beta-\rho a-\zeta-2}{2(\beta+1)}\wedge\frac{1}{\beta+1}$ occurs at the intersection of $\frac{\beta-\rho a-\zeta-2}{2(\beta+1)}$ and $\frac{1}{\beta+1}$, which is $\beta=\rho a+\zeta+4$. The associated value of (29) is $\frac{1}{\rho a+\zeta+5}$.

When $a(1-\omega)\leq 1$, (29) reduces to $\frac{a(1-\omega)}{\beta+1}\wedge\frac{\beta-\rho a-\zeta-2}{2(\beta+1)}$. By a similar argument, the maximizer is $\beta=a(2-2\omega+\rho)+\zeta+2$, with the value of (29) equal to $\frac{a(1-\omega)}{a(2-2\omega+\rho)+\zeta+3}$.

Thus, overall, given a, the optimal choice of β is $\beta=\rho a+\zeta+2+2((a(1-\omega))\wedge 1)$, with the value of (29) given by $\frac{(a(1-\omega))\wedge 1}{\rho a+\zeta+3+2((a(1-\omega))\wedge 1)}$. When $a(1-\omega)>1$, the value of (29) is $\frac{1}{\rho a+\zeta+5}$ which is decreasing in a, whereas when $a(1-\omega)\leq 1$, the value of (29) is $\frac{a(1-\omega)}{a(2-2\omega+\rho)+\zeta+3}$ which is increasing in a. Thus the maximum occurs when $a(1-\omega)=1$, or $a=\frac{1}{1-\omega}$. The associated value of (29) is $\frac{1}{\rho/(1-\omega)+\zeta+5}$.

REMARK EC.1. Suppose that Assumption 4 is replaced by letting

$$\|\mathbf{v}(\boldsymbol{\xi}_1)-\mathbf{v}(\boldsymbol{\xi}_2)\|\leq L\|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2\|$$

hold for any $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{R}^N$. Then, in the proof of Theorem 5, the inequality (EC.34) can be replaced by

$$\begin{aligned}
P(\mathcal{E}_k') &\leq \frac{k^\gamma E|(\hat{\boldsymbol{\psi}}_k - \boldsymbol{\psi}_k)'(\hat{\mathbf{d}}_k - \mathbf{d}_k)|}{\varrho} \\
&\leq \frac{k^\gamma}{\varrho}\sqrt{E[\|\hat{\boldsymbol{\psi}}_k - \boldsymbol{\psi}_k\|^2]E[\|\mathbf{d}_k - \hat{\mathbf{d}}_k\|^2]} \quad \text{by the Cauchy-Schwarz inequality} \\
&\leq \frac{k^\gamma L}{\varrho}E[\|\hat{\boldsymbol{\psi}}_k - \boldsymbol{\psi}_k\|^2] \quad \text{by the relaxed Assumption 4} \\
&\leq \frac{LM^2TNk^\gamma}{R_k\varrho\delta}\prod_{j=1}^{k-1}(1-\epsilon_j)^{-1} \quad \text{by following the derivation in (EC.30) and (EC.31)}
\end{aligned}$$

Consequently, equation (EC.37) becomes

$$\prod_{j=1}^{k_0-1}(1-\epsilon_j)^{-1}\frac{M^2TN}{\delta b}\left(\frac{1}{\vartheta^2(\beta-\rho a-1)(k_0-1)^{\beta-1}} + \frac{L}{\varrho(\beta-\gamma-\rho a-1)(k_0-1)^{\beta-\gamma-1}}\right)$$

if Condition 6 is replaced by

$$\beta > \gamma + \rho a + 1$$

Correspondingly, Condition 7 needs to be replaced by

$$\prod_{j=1}^{k_0-1}(1-\epsilon_j)^{-1}\frac{M^2TN}{\delta b}\left(\frac{1}{\vartheta^2(\beta-\rho a-1)(k_0-1)^{\beta-1}} + \frac{L}{\varrho(\beta-\gamma-\rho a-1)(k_0-1)^{\beta-\gamma-1}}\right) < \varepsilon$$

The results in Theorem 5 and Corollary 1 then retain. Under these modified Conditions 6 and 7, discussion point 3(b) in Section 6.2 then gives $\beta = \gamma + \rho a + 1 + \zeta$ for some $\zeta > 0$ and $\gamma = \beta - \rho a - \zeta - 1$. In discussion point 4, the convergence rate in terms of replications becomes $1/W^{((a(1-\omega))\wedge(\beta-\rho a-\zeta-1)\wedge 1)/(\beta+1)}$. By maximizing

$$\frac{(a(1-\omega))\wedge(\beta-\rho a-\zeta-1)\wedge 1}{\beta+1} \tag{EC.48}$$

like in (29) by Lemma 2 (see Lemma EC.2 right after this remark), we get

$$a = \frac{1}{1-\omega}, \quad \beta = \frac{\rho}{1-\omega} + \zeta + 2$$

and the optimal value is

$$\frac{1}{\rho/(1-\omega)+\zeta+3}$$

So, following the argument there, we choose ϑ and ν, and hence ω, to be small, and we choose ρ to be close to 1. This gives rise to the approximate choice that $a\approx 1+\omega$ and $\beta\approx 3+\zeta+\omega$. The convergence rate is then $O(W^{-1/(4+\zeta+\omega)})$, leading to our claim in Section 6.2 that the complexity can improve to $O(1/\epsilon^{4+\zeta+\omega})$ if Assumption 4 is relaxed.

Lemma EC.2. *The maximizer of* (EC.48) *is given by*

$$a=\frac{1}{1-\omega},\quad \beta=\frac{\rho}{1-\omega}+\zeta+2$$

and the optimal value is

$$\frac{1}{\rho/(1-\omega)+\zeta+3}$$

Proof of Lemma EC.2 Consider first a fixed a. When $a(1-\omega)>1$, (EC.48) reduces to $\frac{\beta-\rho a-\zeta-1}{\beta+1}\wedge\frac{1}{\beta+1}$. Since $\frac{\beta-\rho a-\zeta-1}{\beta+1}$ is increasing in β and $\frac{1}{\beta+1}$ is decreasing in β, the maximizer of $\frac{\beta-\rho a-\zeta-1}{\beta+1}\wedge\frac{1}{\beta+1}$ occurs at the intersection of $\frac{\beta-\rho a-\zeta-1}{\beta+1}$ and $\frac{1}{\beta+1}$, which is $\beta=\rho a+\zeta+2$. The associated value of (EC.48) is $\frac{1}{\rho a+\zeta+3}$.

When $a(1-\omega)\le 1$, (EC.48) reduces to $\frac{a(1-\omega)}{\beta+1}\wedge\frac{\beta-\rho a-\zeta-1}{\beta+1}$. By a similar argument, the maximizer is $\beta=a(1-\omega+\rho)+\zeta+1$, with the value of (EC.48) equal to $\frac{a(1-\omega)}{a(1-\omega+\rho)+\zeta+2}$.

Thus, overall, given a, the optimal choice of β is $\beta=\rho a+\zeta+1+(a(1-\omega))\wedge 1$, with the value of (EC.48) given by $\frac{(a(1-\omega))\wedge 1}{\rho a+\zeta+2+(a(1-\omega))\wedge 1}$. When $a(1-\omega)>1$, the value of (EC.48) is $\frac{1}{\rho a+\zeta+3}$ which is decreasing in a, whereas when $a(1-\omega)\le 1$, the value of (29) is $\frac{a(1-\omega)}{a(1-\omega+\rho)+\zeta+2}$ which is increasing in a. Thus the maximum occurs when $a(1-\omega)=1$, or $a=\frac{1}{1-\omega}$. The associated value of (EC.48) is $\frac{1}{\rho/(1-\omega)+\zeta+3}$.

EC.2. Additional Details of the Numerical Results

EC.2.1. Multi-start Initialization

The results in Section 7.1 are implemented with an initialization that assigns equal probabilities to the support points. To test the procedure under different initializations, we repeat ten runs of the FWSA algorithm where the initial probability masses for the support points (held constant for all runs) are sampled uniformly independently with appropriate normalization. Figure EC.1 provides a box-plot of the identified optima. The sample size for moment constraint generation is $N_s = 50$ and the discretization support size is $n = 30$. The returned optimal solutions for each of the minimization and maximization formulations all agree up to the first two digits (the box plot shows the small spread of the max values, while the min values are very clustered and they appear to all overlap at the same point). This indicates that the formulations have a unique global optimal solution or similar local optimal solutions. Note that the bounds generated from this setting are quite loose with a small N_s.

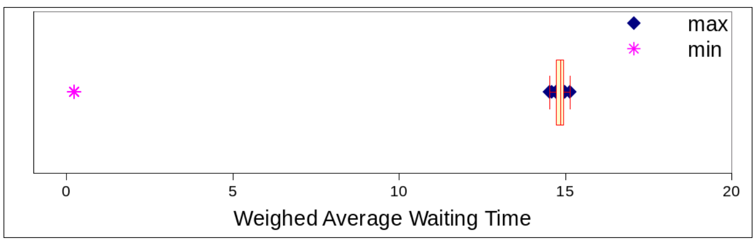


Figure EC.1 Returned optimal solutions from 10 runs on $n = 30$, $M = 50$, exponential for discretization

EC.2.2. Details of the Benchmark Steady-State Formulation in Section 7.2

We consider the depicted $Z(\mathbf{p})$ in Section 7.2. As T grows, the average waiting time converges to the corresponding steady-state value, which, when the traffic intensity $\rho_{\mathbf{p}} = E_{\mathbf{p}}[X_t]$ is less than 1, is given in closed form by the Pollaczek-Khinchine formula (Asmussen (2008)) as:

$$Z_\infty(\mathbf{p}) = \frac{\rho_{\mathbf{p}} E_{\mathbf{p}}[X_1] + Var_{\mathbf{p}}(X_1)}{2(1-\rho_{\mathbf{p}})}.$$

So, when T is large, an approximation Z_∞^* to the worst-case performance estimate can be obtained by replacing $Z(\mathbf{p})$ with $Z_\infty(\mathbf{p})$. (In experiments, a choice of $T = 500$ seems to show close agreement.) With $E_{\mathbf{p}}[X_1] = \sum p_j y_j$ and $E_{\mathbf{p}}[X_1^2] = \sum p_j y_j^2$, the steady-state approximation to (32) is given by **(SS)** below, which is equivalent to **(SS′)** via variable substitutions (see p.191 in Boyd and Vandenberghe (2004)):

$$
\begin{aligned}
\min_{\mathbf{p}} \quad & \frac{\sum_j p_j y_j^2}{2(1-\sum_j p_j y_j)} && \textbf{(SS)} \\
s.t. \quad & \sum_j p_j \log\left(\frac{p_j}{p_{b,j}}\right) \leq \eta \\
& \sum_j p_j = 1 \\
& 0 \leq p_j \leq 1, \quad \forall j = 1,\ldots,n
\end{aligned}
\quad \Longrightarrow \quad
\begin{aligned}
\min_{\mathbf{p}} \quad & \sum_j w_j {y_j}^2 && \textbf{(SS}'\textbf{)} \\
s.t. \quad & \sum_j w_j \log\left(\frac{w_j}{t p_{b,j}}\right) \leq \eta t \\
& 2t - 2\sum_j w_j y_j = 1 \\
& \sum_j w_j = t \\
& 0 \leq w_j \leq t \quad \forall j = 1,\ldots,n
\end{aligned}
$$

EC.2.3. Shape of the Obtained Optimal Distributions in Section 7.2

Continuing with the example in Section 7.2, Figure EC.2 shows the form of the optimal distributions $\mathbf{p}^*$ identified by the FWSA algorithm for the minimization (Figure EC.2a) and maximization (Figure EC.2b) problems under (32). The optimal distributions follow a similar bimodal structure as the baseline distribution $\mathbf{p}_b$. The maximization version assigns probability masses in an unequal manner to the two modes in order to drive up both the mean and the variance of $\mathbf{p}$, as **(SS)** (in Appendix EC.2.2) leads us to expect, whereas the minimization version on the other hand makes the mass allocation more equal in order to minimize the mean and the variance of $\mathbf{p}$ while maintaining the maximum allowed KL divergence.

References

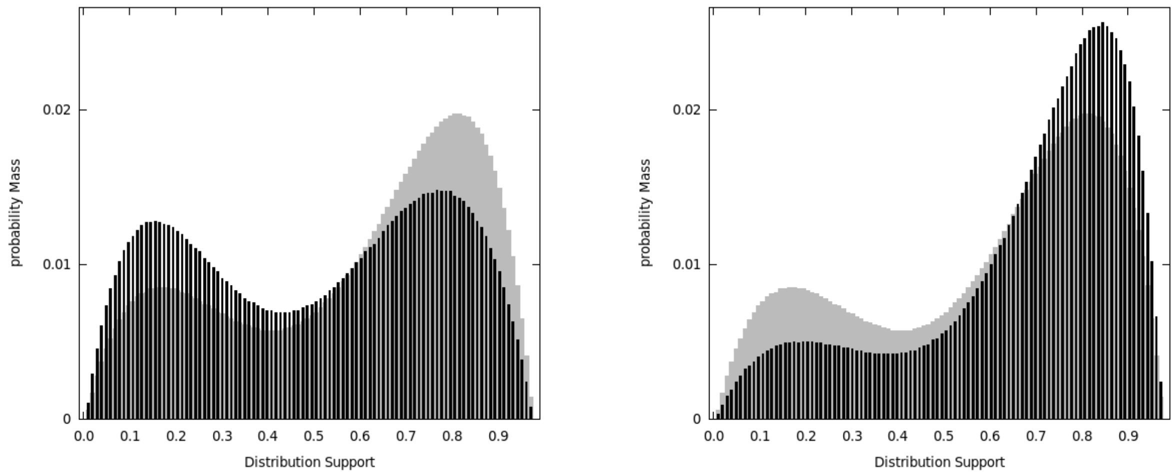
Asmussen, S. (2008), *Applied Probability and Queues*, Vol. 51, Springer Science & Business Media, New York.

Blum, J. R. (1954), Multidimensional stochastic approximation methods, *The Annals of Mathematical Statistics*, 25(4):737–744.

Boyd, S., Vandenberghe, L. (2004), *Convex Optimization*, Cambridge University Press, Cambridge, UK.

Luenberger, D. G. (1969), *Optimization by Vector Space Methods*, John Wiley & Sons, New York, NY.

Serfling, R. J. (2009), *Approximation Theorems of Mathematical Statistics*, Vol. 162, John Wiley & Sons, Hoboken, NJ.



(a) (min) $\mathbf{p}_b$ from beta-mixture

(b) (max) $\mathbf{p}_b$ from beta-mixture

Figure EC.2 Optimal solutions $\mathbf{p}^*$ identified by the FWSA algorithm with $n = 100$ and $\eta = 0.05$, setting $a = 1.5, \beta = 2.75$. The gray bars represent the baseline p.m.f. $\mathbf{p}_b$.