Electronic Companion
Spatial Pricing in Ride-Sharing Networks

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EC.1 Omitted Proofs from Section 3

Proof of Corollary 1

The corollary follows directly from Lemmas A.1 and A.2. In particular, note that the supply of drivers at any location i is lower bounded by:

$$x_i \geq \beta \sum_j \alpha_{ji}(1 - p_j^\star) \geq \beta \sum_j \alpha_{ji}\Big(1 - \frac{1}{2} - \frac{1-\beta^2}{2}\Big) \geq \kappa_i \frac{\beta^3}{2},$$

where the second inequality follows by using $p_j^\star \leq \frac{1+\lambda_j^\star - \beta^2}{2}$ from Lemma A.2 and $\lambda_j^\star \leq 1$ for all j (from Lemma A.1). Thus, the assumption that $\kappa_i > 1/\beta^3$ for the first part of the corollary implies that the supply of drivers at location i is greater than 1/2 and, consequently, location i has excess supply (note again by Lemmas A.1 and A.2, it holds that $p_i^\star \geq 1/2$, thus the maximum mass of riders that the platform finds optimal to serve at location i is 1/2).

For the second part of the corollary, assume by way of contradiction that there exists location i such that $\kappa_i(\mathbf{A}) < \beta$, but location i is not an entry point, i.e., $\delta_i + \sum_j y_{ji} = 0$. Then, the supply of drivers at location i is upper bounded by:

$$x_i \leq \beta \sum_j \alpha_{ji}(1 - p_j^\star) \leq \beta \sum_j \alpha_{ji}\Big(1 - \frac{1}{2}\Big) \leq \kappa_i \frac{\beta}{2}.$$

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Thus, since $\kappa_i(\mathbf{A}) < \beta$ we obtain that the supply of drivers at i is strictly less than $\beta^2/2$. Given that $p_i^\star \leq 1-\beta^2/2$ (from Proposition 3), this leads to a contradiction, since there has to be additional entry at location i to satisfy the demand induced by setting $p_i^\star \leq 1-\beta^2/2$ (otherwise, the platform can generate strictly higher profits by increasing $p_i^\star$ as doing so does not violate the feasibility constraints). Thus, we conclude that if $\kappa_i < \beta$, location i has to be an entry point. □

Proof of Proposition 4

Consider the optimal prices $\{p_i'\}_{i=1}^n$ corresponding to demand pattern $(\mathbf{A}', \mathbf{1})$. To establish that the platform can generate higher profits under demand pattern $(\mathbf{A}, \mathbf{1})$ than under $(\mathbf{A}', \mathbf{1})$ we consider the profits that correspond to prices $\{p_i'\}_{i=1}^n$ in the two demand patterns, assuming that the platform serves the induced demand in both cases. Given the latter, it's sufficient to compare the costs associated with serving the demand or, more specifically, compare the mass of drivers who do not get assigned to a ride (since the mass of drivers who get assigned to rides is the same in both cases).

First, note that the bounds on the optimal prices provided in Lemma A.2 apply to the locations in S_1 and S_2 regardless of whether the demand pattern is $(\mathbf{A}, \mathbf{1})$ or $(\mathbf{A}', \mathbf{1})$. This implies that the demand served at location i, which recall that we denote by d_i, satisfies:

$$\beta^2/2 \leq d_i = 1 - p_i \leq 1/2, \tag{1}$$

where we use Lemmas A.1 and A.2.

For a location i in set $S_1(\mathbf{A})$ and demand pattern $(\mathbf{A}, \mathbf{1})$, we have that the excess supply of drivers at i is equal to:

$$\beta \sum_j \alpha_{ji} d_j - d_i \geq \beta^3/2 \sum_j \alpha_{ji} - 1/2 = \beta^3/2\kappa_i(\mathbf{A}) - 1/2,$$

where the first inequality follows from (1). Similarly, for $(\mathbf{A}', \mathbf{1})$ we have $\beta \sum_j \alpha'_{ji} d_j - d_i \geq \beta^3/2\kappa_i(\mathbf{A}') - 1/2$. Sets $S_1(\mathbf{A}) = S_1(\mathbf{A}')$ correspond to locations with excess supply under the two demand patterns. The difference in the excess supply of drivers at a location $i \in S_1(\mathbf{A})$ corresponding to the two demand patterns (under prices $\{p_i'\}_{i=1}^n$) can be bounded below from:

$$\left(\beta \sum_j \alpha'_{ji} d_j - d_i\right) - \left(\beta \sum_j \alpha_{ji} d_j - d_i\right) \geq \sum_j \alpha'_{ji} \frac{\beta^3}{2} - \sum_j \alpha_{ji} \frac{\beta}{2} = \frac{\beta}{2}\left(\beta^2 \kappa_i(\mathbf{A}') - \kappa_i(\mathbf{A})\right), \tag{2}$$

where again we use (1). Similarly, for the set of locations in $S_2(\mathbf{A}) = S_2(\mathbf{A}')$ we have that the difference of excess supply under the two demand patterns can be bounded as follows:

$$\left(\beta \sum_j \alpha_{ji} d_j - d_i\right)^+ - \left(\beta \sum_j \alpha'_{ji} d_j - d_i\right)^+ \leq \left(\sum_j \alpha_{ji} \frac{\beta}{2} - \sum_j \alpha'_{ji} \frac{\beta^3}{2}\right) = \frac{\beta}{2}\left(\kappa_i(\mathbf{A}) - \beta^2 \kappa_i(\mathbf{A}')\right)^+, \tag{3}$$

where the inequality follows from (1) and the fact that for any two real numbers a, b it holds that $a^+ - b^+ \leq (a-b)^+$. Finally, for any remaining location, i.e., $i \notin S_1(\mathbf{A}) \cup S_2(\mathbf{A})$, we have $\kappa_i(\mathbf{A}) < \beta$ and $\kappa_i(\mathbf{A}') < \beta$. By Corollary 1 it follows that i is an entry point under both demand patterns and, consequently, does not feature any excess supply. Expressions (2) and (3) along with the assumptions of the proposition establish the claim since the difference in the cost the platform has to incur in order to serve the demand induced by prices $\{p'_i\}_{i=1}^n$ under the two demand patterns is equal to:

$$\sum_{i \in S_1(\mathbf{A})} \left(\left(\sum_j \alpha'_{ji} d_j - d_i\right) - \left(\sum_j \alpha_{ji} d_j - d_i\right)\right) + \sum_{i \in S_2(\mathbf{A})} \left(\left(\sum_j \alpha'_{ji} d_j - d_i\right)^+ - \left(\sum_j \alpha_{ji} d_j - d_i\right)^+\right),$$

which, from the discussion above, is greater than zero. Thus, we conclude that the platform can generate (weakly) higher profits under $\mathbf{A}$ than under $\mathbf{A}'$, as claimed. □

Proof of Corollary 2

We consider demand patterns $(\mathbf{F}, \mathbf{1})$ for which $\kappa_i(\mathbf{F}) > 1/\beta^3$ or $\kappa_i(\mathbf{F}) < \beta$. Note that Lemma A.1 and Corollary 1 imply that $\lambda_i^\star = \beta$ for i such that $\kappa_i(\mathbf{F}) > 1/\beta^3$ and $\lambda_i^\star = 1$ for i such that $\kappa_i(\mathbf{F}) < \beta$. Thus, the bound provided in (34) can be rewritten as

$$\begin{aligned}
\Pi(\mathbf{D}, \mathbf{1}) - \Pi(\mathbf{F}, \mathbf{1}) &\leq \frac{\beta^2}{2} (\boldsymbol{\lambda}^\star)^T (\mathbf{1} - \mathbf{F}^T \mathbf{1}) \\
&= \frac{\beta^2}{2} \left(\sum_{i \mid \kappa_i(\mathbf{F}) > 1/\beta^3} \lambda_i^\star (1 - \kappa_i(\mathbf{F})) + \sum_{i \mid \kappa_i(\mathbf{F}) < \beta} \lambda_i^\star (1 - \kappa_i(\mathbf{F})) \right) \\
&= \frac{\beta^2}{2} \left(\beta \sum_{i \mid \kappa_i(\mathbf{F}) > 1/\beta^3} (1 - \kappa_i(\mathbf{F})) + \sum_{i \mid \kappa_i(\mathbf{F}) < \beta} (1 - \kappa_i(\mathbf{F})) \right) \\
&= \frac{\beta^2}{2} \left(\sum_{i \mid \kappa_i(\mathbf{F}) > 1/\beta^3} (1 - \kappa_i(\mathbf{F}))(\beta - 1) \right)
\end{aligned}$$

$$= \frac{\beta^2}{2}(1-\beta) \sum_{i \mid \kappa_i(\mathbf{F}) > 1/\beta^3} (\kappa_i(\mathbf{F}) - 1),$$

where the equality in the fourth line uses the fact that $\sum_i (1 - \kappa_i(\mathbf{F})) = 0$ or equivalently

$$\sum_{i \mid \kappa_i(\mathbf{F}) < \beta} (1 - \kappa_i(\mathbf{F})) = - \sum_{i \mid \kappa_i(\mathbf{F}) > 1/\beta^3} (1 - \kappa_i(\mathbf{F})).$$

This concludes the proof of the corollary.

□

Proof of Corollary 3

The corollary is a consequence of the fact that the expression for the profits corresponding to the platform's optimal prices, i.e., the value of the objective function in (17), is equal to the expression for the consumer surplus (up to a constant factor) induced under the same prices (this can be seen from Proposition A.1 and Definition 2). Then, invoking Theorem 1 directly yields the result. □

Proof of Proposition 6

Assume that the demand pattern across the n locations is given by $(\mathbf{A}^\xi, \mathbf{1})$. We provide a closed form characterization of the platform's optimal prices and profits as a function of ξ. Then, the proof of the proposition follows directly from this characterization. In particular, we show the following:

(a) If $\xi \in \left[0, \max\left(\frac{(n-1)}{2(1-\beta)\beta(n-2)}\left(\beta(1-2\beta) + \sqrt{\frac{\beta^2(n-1)+4\beta-4}{n-1}}\right), 0\right)\right]$, then optimal prices are given as:

$$p_1 = \frac{1}{2} \qquad \text{and} \qquad p_2 = \cdots = p_n = \frac{1}{2} + \frac{1 - \beta^2(1 - \xi + \xi/(n-1)) + \xi\beta(n-2)/(n-1)}{2} w,$$

where location 1 is the center of the star and locations $2, \cdots, n$ are the leaves. In addition, the platform's profits as a function of ξ for this range are equal to

$$\Pi(\mathbf{A}^\xi, \mathbf{1}) = \frac{n}{4} - (n-1)\left(\frac{1 - \beta^2(1 - \xi + \xi/(n-1)) + \xi\beta(n-2)/(n-1)}{2}\right)^2$$
$$- w\left(\frac{1}{2} - \frac{1 - \beta^2(1 - \xi + \frac{\xi}{n-1}) + \xi\beta\frac{n-2}{n-1}}{2}\right)\left((1-\beta^2)(n-1) - \beta^2 n\xi - (n-2)\xi\beta\right).$$

(b) If $\xi \in \left[\max\left(\frac{(n-1)}{2(1-\beta)\beta(n-2)}\left(\beta(1-2\beta)+\sqrt{\frac{\beta^2(n-1)+4\beta-4}{n-1}}\right),0\right), \frac{\beta(n-1)-1}{\beta(n-2)}\right]$, then optimal prices are given as:

$$p_2 = \cdots = p_n = \frac{1}{2} + \frac{\beta Z(1+\beta Z+\beta w)+w(n-1)-w\beta\xi(n-2)}{2(n-1)+2\beta^2 Z^2},$$

where $Z = (\xi(n-2)-(n-1))$ and

$$p_1 = 1-\beta((1-\xi)(n-1)+\xi)(1-p_2).$$

In addition, the platform's profits for this range are equal to

$$p_1(1-p_1)+(n-1)p_2(1-p_2)-w(1-\beta)\left((n-1)(1-p_2)+(1-p_1)\right).$$

(c) Finally, if $\xi \in \left[\frac{\beta(n-1)-1}{\beta(n-2)}, 1\right]$, then optimal prices are all equal, i.e.,

$$p_1 = \cdots = p_n = \frac{1}{2} + \frac{(1-\beta)w}{2}.$$

The platform's profits are equal to

$$n\left(\frac{1}{2}-\frac{(1-\beta)w}{2}\right)^2.$$

Proof. First, recall that in any optimal solution we must have $d_i = (1-p_i)$. Thus, we can rewrite problem (6) as follows

$$\begin{aligned} \max_{\{d_i,\delta_i,y_{ij}\}_{i,j=1}^n} \quad & \sum_i (1-d_i)\,d_i - w\sum_i \delta_i \\ s.t. \quad & \sum_j y_{ij} = \beta\Big[\sum_j \alpha_{ji}d_j + \sum_j y_{ji}\Big] + \delta_i - d_i \text{ for all } i \\ & \delta_i, y_{ij} \geq 0, \text{ for all } i,j. \end{aligned} \tag{4}$$

Note that in Problem (4) we relax the constraints $\mathbf{p} \geq \mathbf{0}$ and $\mathbf{p} \leq \mathbf{1}$ (equivalently, $\mathbf{d} \leq \mathbf{1}$ and $\mathbf{d} \geq \mathbf{0}$), which is without loss of optimality, since the resulting optimal prices do not violate the constraints, as we argue subsequently. This is a convex optimization problem with affine constraints, so the

Karush–Kuhn–Tucker conditions are both necessary and sufficient for optimality. In particular, let $\gamma_i(i = 1, \cdots, n)$ and $\omega_{ij}(i, j = 1, \cdots, n)$, denote the dual variables corresponding to the inequality constraints $-\delta_i \leq 0$ and $-y_{ij} \leq 0$ respectively, and $\lambda_i(i = 1, \cdots, n)$ denote those corresponding to the equality constraints in optimization problem (4). Then, the corresponding KKT conditions can be written as:

(i) $1 - 2d_i + \beta \sum_j \lambda_j \alpha_{ij} - \lambda_i = 0$ for all i,

(ii) $-w + \lambda_i + \gamma_i = 0$ for all i,

(iii) $-\lambda_i + \beta\lambda_j + \omega_{ij} = 0$ for all i, j,

(iv) $\lambda_i \left(\beta\left[\sum_j \alpha_j d_j + \sum_j y_{ji}\right] + \delta_i - d_i - \sum_j y_{ij}\right) = 0$,

(v) $\gamma_i\delta_i = 0 = \omega_{ij}y_{ij}$, for all i, j,

along with primal feasibility and the non-negativity of γ_i, ω_{ij}. Using these conditions, we establish the optimality of Cases (a) and (b) by constructing a pair of primal-dual solutions. The optimality of Case (c) follows directly from Proposition 2.

Case (a): First, we provide values for the primal variables. Using the expressions for p_1 and $p_2, \cdots, p_n$ as stated in Case (a) above, we have $d_1 = \frac{1}{2}$ and $d_i = 1 - p_i$ for $i \geq 2$. In addition, we let $\delta_1 = 0$ and

$$\delta_i = d_i - \beta\frac{x_1}{n-1} - \beta d_i \xi \frac{n-2}{n-1} = \left(1 - \beta\xi\frac{n-2}{n-1} - \beta^2\left((1-\xi) + \frac{\xi}{n-1}\right)\right)d_i.$$

In addition, we let $y_{i1} = 0$, $y_{ij} = 0$ and $y_{1i} = \beta((1-\xi) + \xi/(n-1))d_i - \frac{1}{2(n-1)}$ for $2 \leq i, j \leq n$. Note that, under the conditions of Case (a), this is a feasible solution. In particular, note that the supply of drivers reaching the center of the star is at least $1/2$. Then, it is straightforward to see that if we let $\lambda_i = w$ for $i \geq 2$, $\lambda_1 = \beta w$, $\gamma_1 = (1-\beta)w$, $\gamma_i = 0$ for $i \geq 2$, and $\omega_{ij} = \lambda_i - \beta\lambda_j$, for i, j, the KKT conditions are satisfied.

Case (b): Similarly, we provide values for the primal variables. Using the expressions for p_1 and $p_2, \cdots, p_n$ as stated in Case (b) above as well as the expression for Z, we have that if $d_1 = 1 - p_1$ and $d_i = 1 - p_i$ for $i \geq 2$, then $d_1 = -\beta Z d_2$. Noting that the entry is only at the leaves (and thus

$\delta_2, \cdots, \delta_n > 0$), we obtain $\gamma_2 = \cdots = \gamma_n = 0$ and $\lambda_2 = \cdots = \lambda_n = w$. Using the definition of λ_i for $i \geq 2$, we note that λ_1 must be equal to $1 - 2d_1 + \beta(n-1)\lambda_i\alpha_{ij} = 1 + 2\beta Z d_i + \beta w$ for $i \geq 2$. To show optimality, we simply need to check that the equality $2d_i = 1 + \beta(n-2)\lambda_i\alpha_{ij} + \beta\lambda_1\alpha_{i1} - \lambda_i$ is satisfied for all $i, j \neq 1$ (all other conditions are satisfied). Note that $\alpha_{i1} = \xi/(n-1) + (1-\xi) = -Z/(n-1)$ for $i \neq 1$; thus, we can rewrite the right hand side as:

$$1 + \beta(n-2)\lambda_i\alpha_{ij} + \beta\lambda_1\alpha_{i1} - \lambda_i = 1 + \beta(n-2)w\frac{\xi}{n-1} - \beta\frac{Z}{n-1}(1 + 2\beta Z d_i + \beta w) - w,$$

for $i, j \neq 1$. Multiplying by $(n-1)$ and rearranging terms yields

$$\left(2(n-1) + 2\beta^2 Z^2\right) d_i = (n-1) + \left(\xi\beta(n-2) - \beta^2 Z - (n-1)\right) w - \beta Z.$$

By adding and subtracting $\beta^2 Z^2$ from the left-hand side, we obtain the desired expression. Therefore, we have shown that the solution induced by these prices is optimal. □

EC.2 Proofs from Sections 4 and 5

Proof of Corollary 4

The claim follows directly from Proposition 2, which establishes that when the underlying demand pattern is balanced, the platform maximizes its profits by setting the same price at all locations, i.e., $p_i^\star = p^\star = 1/2 + (1-\beta)w/2$ for all i, and, in addition, the optimal solution can be supported by the same compensation for drivers at all locations, i.e., $c_i^\star = c^\star = (1-\beta)w$, for all i. Thus, the platform maximizes its profits by setting $p_i^\star = p^\star = 1/2 + (1-\beta)w/2$ for all i and using fixed commission rate $\gamma^\star = c^\star/p^\star$, which implies that for every ride a driver completes, she earns $\gamma^\star p^\star = c^\star$. □

Proof of Proposition 7

Before establishing the proposition, we state and prove two lemmas.

Lemma EC.1. *Consider a demand pattern $(\mathbf{A}, \boldsymbol{\theta})$, and assume that $w = 1$. Let $\{p_i^\star, \delta_i^\star, y_{ij}^\star\}_{i,j=1}^n$ be an optimal solution to Problem (6). Then, if $\delta_i^\star + \sum_j y_{ji}^\star > 0$ for all i, the solution to Problem (6) can be implemented using a commission rate that is fixed across the network's locations with the same prices*

$\{p_i^\star\}_{i=1}^n$ *and* $\gamma^\star = \frac{2(1-\beta)}{2-\beta}$.

Proof. Let $\{p_i^\star, \delta_i^\star, y_{ij}^\star\}_{i,j=1}^n$ be an optimal solution to Problem (6). Recall that for any arbitrary $\boldsymbol{\theta}$ the vector of optimal prices $\{p_i^\star\}_{i=1}^n$ and the vector of optimal dual variables $\boldsymbol{\lambda}^\star$ must satisfy:

$$\mathbf{p}^\star = \frac{\mathbf{1} + \boldsymbol{\lambda}^\star - \beta \mathbf{A} \boldsymbol{\lambda}^\star}{2},$$

as given in Equation (19). Furthermore, as all locations are entry points, we must have $\lambda_i^\star = 1$ for all i (see Lemma A.1). Given that $\mathbf{A}$ is row-stochastic, we have $p_i^\star = 1 - \beta/2$ for all i. In addition, since all locations are entry points, we obtain that the supply in the optimal solution must satisfy $x_i^\star = (1 - p_i^\star)\theta_i$ at all locations.

To complete the proof, we show that the optimal solution to Problem (6) is in fact an equilibrium under the same vector of prices $\{p_i^\star\}_{i=1}^n$and fixed commission rate $\gamma^\star$ as defined in the statement of the lemma. To that end, consider the recursion given by

$$V_i = \frac{(1-p_i^\star)\theta_i}{x_i^\star}\gamma p_i^\star + \frac{(1-p_i^\star)\theta_i}{x_i^\star}\beta \sum_j \alpha_{ij} V_j + \beta \left(1 - \frac{(1-p_i^\star)\theta_i}{x_i^\star}\right) \overline{V} \text{ for all } i. \tag{5}$$

Using the fact that $p_i^\star = 1 - \beta/2$ and $x_i^\star = (1 - p_i^\star)\theta_i$ at all locations, we can rewrite the expression above as $V = \gamma \left(1 - \frac{\beta}{2}\right) \mathbf{1} + \beta \mathbf{A} V$. Using $\gamma = \gamma^\star$, it is straightforward to see that $V_i = 1$ for all i is in fact a solution to the above system. Therefore, the fixed commission rate $\gamma^\star$ and the vector of prices $\mathbf{p}^\star$ given above constitute an equilibrium as claimed. □

Lemma EC.2. *Consider a network with two locations, demand pattern* $(\mathbf{A}, \boldsymbol{\theta})$, *and* $w = 1$. *Then, the optimal solution to Problem* (6) *can be implemented using a fixed commission rate.*

Proof. Let $\{p_i^\star, \delta_i^\star, y_{ij}^\star\}_{i,j=1}^2$ be an optimal solution to Problem (6). We will show that there exists a $\gamma^\star$ such that the profit of the platform using a fixed commission rate with parameters $\{p_i^\star\}_{i=1}^n$and $\gamma = \gamma^\star$ is equal to the optimal profit corresponding to the solution to Problem (6). Note that if $\delta_i^\star = 0$ for all i, then the claim follows trivially since the platform provides no service. Thus, for the remainder of the proof we assume that there exists location i with $\delta_i^\star > 0$. Since the network has only two locations, we know that it must be the case that either one or both are entry points in the optimal solution to Problem (6). If both locations are entry points, the result follows by Lemma EC.1. Therefore, it suffices to show the result for the case in which only one location is an entry point.

To that end, assume without loss of generality that $\delta_1 > 0$, i.e., location 1 is an entry point. First, we show that there exists a γ such that the expected earnings of the drivers when prices, entry, and relocation are given by $\{p_i^\star, \delta_i^\star, y_{ij}^\star\}_{i,j=1}^2$ satisfy $V_1 = 1$ and $V_2 \leq 1$, that is, they satisfy the equilibrium conditions under a fixed commission rate. Second, we establish that this implies that the profits under a fixed commission rate are equal to those of the optimal solution to Problem (6).

Let $q_i^\star = \frac{\theta_i(1-p_i^\star)}{x_i^\star}$ denote the probability of accepting a ride at location i, where $x_i^\star$ is as defined by Equation (1). Noting that $q_1^\star = 1$ (since Lemma A.1 implies that when $\delta_i^\star > 0$ for some location i, then $y_{ij}^\star = 0$ for all j and $q_i^\star = 1$), we obtain the following for a fixed $\gamma \in (0, 1)$:

$$V_1 = \gamma p_1^\star + \beta\alpha_{11}V_1 + \beta\alpha_{12}V_2 \tag{6}$$

$$V_2 = \gamma q_2^\star p_2^\star + \beta(1 - q_2^\star\alpha_{22})V_1 + \beta q_2^\star\alpha_{22}V_2. \tag{7}$$

Next, we show that $p_1^\star \geq p_2^\star$ and, as a consequence, $0 \leq (p_1^\star - p_2^\star) \leq p_1^\star - q_2^\star p_2^\star$. Assume by way of contradiction that $p_2^\star > p_1^\star$. Recall that the vector of optimal prices $\mathbf{p}^\star$ and the vector of optimal dual variables $\boldsymbol{\lambda}^\star$ must satisfy

$$\mathbf{p}^\star = \frac{\mathbf{1} + \boldsymbol{\lambda}^\star - \beta\mathbf{A}\boldsymbol{\lambda}^\star}{2},$$

from Equation (19). By subtracting the equation for $p_1^\star$ from that of $p_2^\star$, we must have:

$$\begin{aligned} 0 \ < \ 2(p_2^\star - p_1^\star) \quad &= \quad \lambda_2^\star - \lambda_1^\star + \beta\lambda_2^\star(\alpha_{12} - \alpha_{22}) + \beta\lambda_1^\star(\alpha_{11} - \alpha_{21}) \\ &= \quad (\lambda_2^\star - \lambda_1^\star) + \beta(\lambda_2^\star - \lambda_1^\star)(\alpha_{12} - \alpha_{22}) \\ &= \quad (\lambda_2^\star - \lambda_1^\star)(1 + \beta(\alpha_{12} - \alpha_{22})), \end{aligned}$$

where the second equality follows from the fact that A is row-stochastic and thus $\alpha_{11} = 1 - \alpha_{12}$ and $\alpha_{21} = 1 - \alpha_{22}$. Note that $(1 + \beta(\alpha_{12} - \alpha_{22})) > 0$ for $\beta < 1$. In addition, recall that at any optimal solution we must have $\beta \leq \lambda_i^\star \leq 1$ for all i, and $\lambda_i^\star = 1$ if i is an entry entry point (see Lemma A.1). Therefore, we have that $(\lambda_2^\star - \lambda_1^\star) \leq 0$ and thus $(\lambda_2^\star - \lambda_1^\star)(1 + \beta(\alpha_{12} - \alpha_{22})) \leq 0$, which is a contradiction implying that $p_1^\star \geq p_2^\star$.

Subtracting the Expression (7) from (6) yields

$$\begin{aligned} 0 \leq \gamma\left(p_1^\star - q_2^\star p_2^\star\right) &= V_1 - \beta\alpha_{11}V_1 - \beta\alpha_{12}V_2 - V_2 + \beta(1-q_2^\star\alpha_{22})V_1 + \beta q_2^\star\alpha_{22}V_2 \\ &= V_1\left(1 - \beta\alpha_{11} + \beta(1-q_2^\star\alpha_{22})\right) - V_2\left(1+\beta\alpha_{12} - \beta q_2^\star\alpha_{22}\right) \\ &= \left(V_1 - V_2\right)\left(1+\beta-\beta\alpha_{11} - \beta q_2^\star\alpha_{22}\right), \end{aligned}$$

where in the last equality we used the fact that $\mathbf{A}$ is row-stochastic and thus $\alpha_{12} = 1-\alpha_{11}$. Note that this implies that $V_1 \geq V_2$ for *any* fixed γ, provided that the rest of $\{p_i^\star, \delta_i^\star, y_{ij}^\star\}_{i,j=1}^n$ remain fixed. We conclude the first step by noting that for $\{x_i^\star, \delta_i^\star, y_{ij}^\star\}_{i,j=1}^n$ to be an equilibrium under a fixed commission rate with $\{p_i^\star\}_{i=1}^n$and $\gamma = \gamma^\star$ it suffices to set $\gamma^\star$ so that $V_1 = 1$.

The second step involves establishing that the profits corresponding to the two solutions, i.e., the optimal solution to Problem (6) and its implementation using a fixed commission rate, are equal. This follows from noting that Expressions (6) and (7) when $V_1 = 1$ imply that

$$\begin{aligned} \gamma p_1^\star(1-p_1^\star)\theta_1 &= (1-p_1^\star)\theta_1 - \beta\alpha_{11}(1-p_1^\star)\theta_1 - \beta\alpha_{12}V_2(1-p_1^\star)\theta_1 \\ \gamma p_2^\star(1-p_2^\star)\theta_2 &= (V_2-\beta)x_2^\star + \beta\alpha_{22}(1-p_2^\star)\theta_2 - \beta\alpha_{22}(1-p_2^\star)\theta_2 V_2, \end{aligned}$$

which, in turn, using the fact that $x_2^\star = \beta(\alpha_{12}(1-p_1^\star)\theta_1 + \alpha_{22}(1-p_2^\star)\theta_2)$ yields:

$$\gamma(p_1^\star(1-p_1^\star)\theta_1 + p_2^\star(1-p_2^\star)\theta_2) = (1-p_1^\star)\theta_1 - \beta\alpha_{11}(1-p_1^\star)\theta_1 - \beta x_2^\star + \beta\alpha_{22}(1-p_2^\star)\theta_2 = \delta_1^\star,$$

where the last equality following from the first constraint in Problem 6. Finally, given that the profits corresponding to the two solutions can be written as $\sum_i p_i^\star(1-p_i^\star)\theta_i - \delta_1^\star$ and $\sum_i p_i^\star(1-p_i^\star)\theta_i - \gamma^\star\left(\sum_i p_i^\star(1-p_i^\star)\theta_i\right)$ respectively, we conclude that the two solutions lead to equal profits. □

Proof of Proposition 7: We reduce the platform's pricing problem with a two-type demand pattern to an equivalent pricing problem in a network with only two locations such that each location aggregates all locations belonging to the same type. We show that the optimal solution in this two-location network can be constructed using the optimal solution to the original problem. We then exploit the fact that there exists a $\gamma^\star$ such that the optimal solution for the two-location network can be implemented using a fixed commission rate (Lemma EC.2) to finally argue that $\{x_i^\star, \delta_i^\star, y_{ij}^\star\}_{i,j=1}^n$

is an equilibrium under a fixed commission rate with $\{p_i^\star\}_{i=1}^n$and $\gamma = \gamma^\star$ that achieves the optimal profit.

In particular, we define a network with two locations for which $\hat{\boldsymbol{\theta}}$ and $\hat{\mathbf{A}}$ are defined as follows:

- $\hat{\theta}_1 = \sum_{i\in\mathcal{N}_1} \theta_i$ and $\hat{\theta}_2 = \sum_{j\in\mathcal{N}_2} \theta_j$.
- $\hat{\alpha}_{11} = \frac{\sum_{i\in\mathcal{N}_1}\sum_{i'\in\mathcal{N}_1} \alpha_{ii'}}{|\mathcal{N}_1|}$, $\hat{\alpha}_{12} = \frac{\sum_{i\in\mathcal{N}_1}\sum_{j\in\mathcal{N}_2} \alpha_{ij}}{|\mathcal{N}_1|}$, $\hat{\alpha}_{21} = \frac{\sum_{j\in\mathcal{N}_2}\sum_{i\in\mathcal{N}_1} \alpha_{ji}}{|\mathcal{N}_2|}$, and $\hat{\alpha}_{22} = \frac{\sum_{j\in\mathcal{N}_2}\sum_{j'\in\mathcal{N}_2} \alpha_{jj'}}{|\mathcal{N}_2|}$.

That is, location 1 aggregates the locations in $\mathcal{N}_1$ and location 2 aggregates those in $\mathcal{N}_2$. The demand at each location corresponds to the total demand of the locations in each of the two sets and the transition probabilities represent the average probability that a ride originating from one of the sets has as its destination a location in the same/different set.

We can now relate the optimal solution to Problem (6) in the original network with that in the two-location network as follows. Let $\{p_i^\star, \delta_i^\star, y_{ij}^\star\}_{i,j=1}^n$ be the optimal solution to Problem (6) in the original network, and let $x^\star$ denote the associated vector of supply. Since the objective function in Problem (6) is concave, we can establish that there is a symmetric solution in the original network, i.e., a solution that features the same price for all locations belonging to the same subset, i.e., $p_i^\star = p_{i'}^\star$ for all $i, i' \in \mathcal{N}_1$ and $p_j^\star = p_{j'}^\star$ for all $j, j' \in \mathcal{N}_2$. To see why, assume by way of contradiction that this is not the case, i.e., there does not exist an optimal solution that is symmetric. Then, if $\{p_i^\star, \delta_i^\star, y_{ij}^\star\}_{i,j=1}^n$ is an optimal solution to (6), consider tuple $\{p_i', \delta_i', y_{ij}'\}_{i,j=1}^n$ such that:

- $p_i' = \frac{1}{|\mathcal{N}_1|}\sum_{k\in\mathcal{N}_1} p_k^\star$ for all $i \in \mathcal{N}_1$ and $p_j' = \frac{1}{|\mathcal{N}_2|}\sum_{k\in\mathcal{N}_2} p_k^\star$ for all $j \in \mathcal{N}_2$,
- $\delta_i' = \frac{1}{|\mathcal{N}_1|}\sum_{k\in\mathcal{N}_1} \delta_k^\star$ and $\delta_j' = \frac{1}{|\mathcal{N}_2|}\sum_{k\in\mathcal{N}_2} \delta_k^\star$, and
- $y_{ij}' = \frac{1}{|\mathcal{N}_1||\mathcal{N}_2|}\sum_{k\in\mathcal{N}_1}\sum_{\ell\in\mathcal{N}_2} y_{k\ell}^\star$ and $y_{ji}' = \frac{1}{|\mathcal{N}_1||\mathcal{N}_2|}\sum_{\ell\in\mathcal{N}_2}\sum_{k\in\mathcal{N}_1} y_{\ell k}^\star$ for all $i \in \mathcal{N}_1$ and $j \in \mathcal{N}_2$. Also, $y_{ii'}' = \frac{1}{|\mathcal{N}_1|^2}\sum_{k\in\mathcal{N}_1}\sum_{k'\in\mathcal{N}_1} y_{kk'}^\star$ and $y_{jj'}' = \frac{1}{|\mathcal{N}_2|^2}\sum_{k\in\mathcal{N}_2}\sum_{k'\in\mathcal{N}_2} y_{kk'}^\star$ for all $i, i' \in \mathcal{N}_1$ and $j, j' \in \mathcal{N}_2$.

Note that the concavity of the objective function implies that the profits corresponding to tuple $\{p_i', \delta_i', y_{ij}'\}_{i,j=1}^n$ are at least as high as those corresponding to $\{p_i^\star, \delta_i^\star, y_{ij}^\star\}_{i,j=1}^n$ assuming that there is enough supply to satisfy the entire induced demand $\sum_{i=1}^n (1 - p_i')\theta_i$. To see that the latter is true, consider locations that belong to $\mathcal{N}_1$ (a similar argument holds for locations in $\mathcal{N}_2$). Note that by construction the supply of drivers and the induced demand under $\{p_i', \delta_i', y_{ij}'\}_{i,j=1}^n$ is the same at each of the locations in $\mathcal{N}_1$. Thus, it suffices to establish that $\sum_{i\in\mathcal{N}_1} x'_i \geq \sum_{i\in\mathcal{N}_1}(1-p'_i)\theta_i$. To this end, we have

$$\sum_{i\in\mathcal{N}_1} x'_i = \beta \sum_{i\in\mathcal{N}_1}\Big[\sum_{k\in\mathcal{N}_1}(\alpha_{ki}(1-p'_k)\theta_k + y_{ki}) + \sum_{k\in\mathcal{N}_2}(\alpha_{ki}(1-p'_k)\theta_k + y_{ki})\Big] + \sum_{i\in\mathcal{N}_1}\delta_i$$
$$= \sum_{i\in\mathcal{N}_1} x^\star_i \geq \sum_{i\in\mathcal{N}_1}(1-p^\star_i)\theta_i = \sum_{i\in\mathcal{N}_1}(1-p'_i)\theta_i,$$

where the equalities follow from the definition of two-type demand patterns and the construction of tuple $\{p'_i, \delta'_i, y'_{ij}\}_{i,j=1}^n$. The inequality $\sum_{i\in\mathcal{N}_1} x'_i \geq \sum_{i\in\mathcal{N}_1}(1-p'_i)\theta_i$ follows directly from the fact that in an optimal solution the available supply of drivers has to be greater than the induced demand. Thus, it follows that there exists a symmetric optimal solution, i.e., we can assume that $\{p^\star_i, \delta^\star_i, y^\star_{ij}\}_{i,j=1}^n$ is symmetric.

Next, we define a solution $\{\hat{\mathbf{p}}, \hat{\boldsymbol{\delta}}, \hat{\mathbf{Y}}\}$ for the two-location network that generates the same profits as $\{p^\star_i, \delta^\star_i, y^\star_{ij}\}_{i,j=1}^n$ in the original network as follows:

- $\hat{p}_1 = p^\star_i$ for $i \in \mathcal{N}_1$ and $\hat{p}_2 = p^\star_j$ for $j \in \mathcal{N}_2$.
- $\hat{\delta}_1 = \sum_{k\in\mathcal{N}_1}\delta^\star_k = |\mathcal{N}_1|\delta^\star_i$ for $i \in \mathcal{N}_1$, and $\hat{\delta}_2 = \sum_{k\in\mathcal{N}_2}\delta^\star_k = |\mathcal{N}_2|\delta^\star_j$ for $j \in \mathcal{N}_2$.
- $\hat{y}_{12} = \sum_{i\in\mathcal{N}_1}\sum_{j\in\mathcal{N}_2} y^\star_{ij}$ and $\hat{y}_{21} = \sum_{j\in\mathcal{N}_2}\sum_{i\in\mathcal{N}_1} y^\star_{ji}$.

It is straightforward that $\{\hat{\mathbf{p}}, \hat{\boldsymbol{\delta}}, \hat{\mathbf{Y}}\}$ is in fact an optimal solution to Problem (6) in the two-location case. By way of contradiction, suppose that there exists another solution $\{\tilde{\mathbf{p}}, \tilde{\boldsymbol{\delta}}, \tilde{\mathbf{Y}}\}$ in the two-location network that generates higher profits for the platform. Then, the following is a solution to Problem (6) that generates higher profits for the platform than $\{p^\star_i, \delta^\star_i, y^\star_{ij}\}_{i,j=1}^n$ in the original network, which leads to a contradiction:

- $p''_i = \tilde{p}_1$ for all $i \in \mathcal{N}_1$ and $p''_j = \tilde{p}_2$ for all $i \in \mathcal{N}_2$,
- $\delta''_i = \frac{1}{|\mathcal{N}_1|}\tilde{\delta}_1$ for all $i \in \mathcal{N}_1$ and $\delta''_j = \frac{1}{|\mathcal{N}_2|}\tilde{\delta}_2$ for all $i \in \mathcal{N}_2$, and
- $y''_{ij} = \frac{1}{|\mathcal{N}_1||\mathcal{N}_2|}\tilde{y}_{12}$ and $y''_{ji} = \frac{1}{|\mathcal{N}_1||\mathcal{N}_2|}\tilde{y}_{21}$ for all $i \in \mathcal{N}_1$ and $j \in \mathcal{N}_2$. Also, $y''_{ii'} = 0$ and $y''_{jj'} = 0$ for all $i, i' \in \mathcal{N}_1$ and $j, j' \in \mathcal{N}_2$.

The optimality of $\{\hat{\mathbf{p}}, \hat{\boldsymbol{\delta}}, \hat{\mathbf{Y}}\}$ in the two-location network and Lemma EC.2 imply that there exists a $\gamma^\star$ such that $\{\hat{x}_i, \hat{\delta}_i, \hat{y}_{ij}\}_{i=1}^n$ is an equilibrium under a fixed commission rate $\gamma^\star$ and prices $\{\hat{p}_i\}_{i=1}^n$ in the two-location network. To complete the proof, it suffices to argue that $\{x_i^\star, \delta_i^\star, y_{ij}^\star\}_{i,j=1}^n$ is an equilibrium under a fixed commission rate with $\{p_i^\star\}_{i=1}^n$ and $\gamma = \gamma^\star$ that achieves the optimal profit. However, this follows from the mapping between the solutions in the original and two-location networks—a driver entering the platform to provide service makes exactly the same profit in both cases. $\square$

Proof of Proposition 8

Let $\psi_i (i = 1, \cdots, n)$, $\phi_i (i = 1, \cdots, n)$, $\omega_{ij} (i, j = 1, \cdots, n)$, $\nu_i (i = 1, \cdots, n)$ denote the dual variables corresponding to the inequality constraints $-p_i \leq 0, -\delta_i \leq 0, -y_{ij} \leq 0, p_i - 1 \leq 0$, respectively, and $\lambda_i (i = 1, \cdots, n)$ denote those corresponding to the equality constraints in optimization problem (15). Then, the corresponding KKT conditions imply the following for the optimal solution to (15) under Assumption 2 (note that the platform's optimization problem is a convex program with affine constraints and, therefore, Slater's condition holds):

(1) Taking the derivative of the objective function and constraints with respect to δ_i yields $-1 = -\lambda_i^\star - \phi_i^\star$. Given that $\phi_i^\star \geq 0$ we have that $\lambda_i^\star \leq 1$. In addition, from complementary slackness we obtain that $\lambda_i^\star = 1$ when $\delta_i^\star > 0$.

(2) Furthermore, the derivative with respect to y_{ij} yields $0 = \lambda_i^\star - \beta^{\zeta_{ij}} \lambda_j^\star - \omega_{ij}^\star$. Note that since $\omega_{ij}^\star \geq 0$ we have $\lambda_i^\star \geq \beta^{\zeta_{ij}} \lambda_j^\star$ and from complementary slackness we have that if $y_{ij}^\star > 0$, i.e., if it is optimal to relocate excess supply from location i to j, then $\lambda_i^\star = \beta^{\zeta_{ij}} \lambda_j^\star$.

Next, we establish that the compensations defined by Equation (16) can support $\{\delta_i^\star, x_i^\star, y_{ij}^\star\}_{i,j=1}^n$, i.e., the optimal solution to Problem (15), as an equilibrium under price vector $\{p_i^\star\}_{i=1}$. First, note that the compensations defined in (16) are the solution to Equation (8) below, i.e., the equation that describes the drivers' expected earnings, when we set $V_i = \lambda_i^\star$ for all i:

$$V_i = \frac{1 - p_i^\star}{x_i^\star} \sum_j \alpha_{ij} \left(c_i^\star \zeta_{ij} + \beta^{\zeta_{ij}} V_j \right) + \left(1 - \frac{1 - p_i^\star}{x_i^\star} \right) \max_k \beta^{\zeta_{ik}} V_k. \tag{8}$$

To see this, first consider the case where $x_i^\star > (1 - p_i^\star)$, which, in turn, implies that $y_{ij}^\star > 0$ for some

j. Then, we can rewrite (8) by setting $V_j = \lambda_j^\star$ in the right hand side as follows:

$$\begin{aligned} V_i &= \frac{1-p_i^\star}{x_i^\star}\sum_j \alpha_{ij}\left(c_i^\star\zeta_{ij} + \beta^{\zeta_{ij}}\lambda_j^\star\right) + \left(1 - \frac{1-p_i^\star}{x_i^\star}\right)\max_k \beta^{\zeta_{ik}}\lambda_k^\star \\ &= \frac{1-p_i^\star}{x_i^\star}c_i^\star\sum_j \alpha_{ij}\zeta_{ij} + \frac{1-p_i^\star}{x_i^\star}\sum_j \alpha_{ij}\beta^{\zeta_{ij}}\lambda_j^\star + \left(1 - \frac{1-p_i^\star}{x_i^\star}\right)\max_k \beta^{\zeta_{ik}}\lambda_k^\star \\ &= \frac{1-p_i^\star}{x_i^\star}\lambda_i^\star + \left(1 - \frac{1-p_i^\star}{x_i^\star}\right)\lambda_i^\star = \lambda_i^\star, \end{aligned}$$

where the equality in the last line follows directly from the definition of compensation $c_i^\star$ (Equation (16)) and the fact that $\lambda_i^\star = \max_k \beta^{\zeta_{ik}}\lambda_k^\star$ from item (2) above. The claim for the case where $x_i^\star = (1-p_i^\star)$ follows immediately from (8) and the definition of $c_i^\star$.

Therefore, the V_i's as defined here satisfy Equation (8) (which is equivalent to Equation (2) when the compensations and the terms involving β are appropriately scaled with the ζ_{ij}'s). In addition, the drivers' incentive-compatibility constraints, i.e., Equation (3), are satisfied as well, since $\lambda_i^\star = V_i = 1 = w$, when $\delta_i^\star > 0$ and $\lambda_i^\star \le 1$ for all i. Finally, condition (ii) in the equilibrium definition, i.e., Equation (1) (appropriately scaled to incorporate the ζ_{ij}'s), is satisfied trivially as $\{\delta_i^\star, x_i^\star, y_{ij}^\star\}_{i,j=1}^n$ is feasible for Problem (16) and $x_i^\star \ge (1-p_i^\star)$.

Thus, we conclude that the compensations defined by Equation (16) can support $\{\delta_i^\star, x_i^\star, y_{ij}^\star\}_{i,j=1}^n$ as an equilibrium under price vector $\{p_i^\star\}_{i=1}$ and expected future earnings for a driver at location i given by $V_i = \lambda_i^\star$.[1] □

EC.3 Proofs from Appendix A

Proof of Proposition A.1

Optimization problem (17) is a quadratic maximization problem with a concave objective function and affine constraints. Thus, Slater's condition is satisfied for (17) and, consequently, strong duality also holds. Substituting $w = 1$, we obtain that the Lagrangian of Problem (17) is given by:

$$L(\mathbf{p}, \mathbf{Y}, \boldsymbol{\delta}, \boldsymbol{\lambda}) = \mathbf{p}^T\boldsymbol{\Theta}(1-\mathbf{p}) - \mathbf{1}^T\boldsymbol{\delta} + \boldsymbol{\lambda}^T(\boldsymbol{\delta} + \beta\mathbf{Y}^T\mathbf{1} + \beta\mathbf{A}^T\boldsymbol{\Theta}(1-\mathbf{p})) - \boldsymbol{\lambda}^T(\mathbf{Y}\mathbf{1} + \boldsymbol{\Theta}(1-\mathbf{p})). \quad (9)$$

[1]Recall that in the case of unequal distances, the expressions involved in the equilibrium definition are scaled appropriately according to the ζ_{ij}'s as we discuss in Subsection 5.1 (footnote 32).

By strong duality we obtain

$$\max_{\mathbf{p},\mathbf{Y},\boldsymbol{\delta}\geq 0}\min_{\boldsymbol{\lambda}} L(\mathbf{p},\mathbf{Y},\boldsymbol{\delta},\boldsymbol{\lambda}) = \min_{\boldsymbol{\lambda}}\max_{\mathbf{p},\mathbf{Y},\boldsymbol{\delta}\geq 0} L(\mathbf{p},\mathbf{Y},\boldsymbol{\delta},\boldsymbol{\lambda}).$$

Let $g(\boldsymbol{\lambda}) := \max_{\mathbf{p},\mathbf{Y},\boldsymbol{\delta}\geq 0} L(\mathbf{p},\mathbf{Y},\boldsymbol{\delta},\boldsymbol{\lambda})$. The dual problem, which has the same optimal objective value as the primal one, is given by $\min_{\boldsymbol{\lambda}} g(\boldsymbol{\lambda})$. Furthermore, the feasibility of the primal problem implies that both the primal and the dual optimal objectives are bounded and the corresponding optimal solutions exist.

Next, we consider expression $\max_{\mathbf{p},\mathbf{Y},\boldsymbol{\delta}\geq 0} L(\mathbf{p},\mathbf{Y},\boldsymbol{\delta},\boldsymbol{\lambda})$ for some fixed $\boldsymbol{\lambda}$. First, observe that

$$\frac{\partial L}{\partial y_{ij}} = \beta\lambda_j - \lambda_i.$$

Given that the Lagrangian is linear in $\mathbf{Y}$, it follows that $g(\boldsymbol{\lambda}) = \infty$, if $\lambda_i < \beta\lambda_j$. Moreover, if in the optimal solution $y_{ij} > 0$, then $\lambda_i = \beta\lambda_j$. Similarly,

$$\frac{\partial L}{\partial \delta_i} = -1 + \lambda_i,$$

and hence $g(\boldsymbol{\lambda}) = \infty$, if $\lambda_i > 1$ as the Lagrangian is linear in δ_i. Moreover, if in the optimal solution $\delta_i > 0$, then $\lambda_i = 1$. These observations imply that $g(\boldsymbol{\lambda}) < \infty$ only when $\lambda_i \leq 1$ for all i and $\lambda_i \geq \beta\lambda_j$ for all i, j. Thus, we can rewrite the dual problem as follows:

$$\begin{aligned} \min_{\boldsymbol{\lambda}} g(\boldsymbol{\lambda}) = \min_{\boldsymbol{\lambda}} \quad & \max_{\mathbf{p}\geq 0} \mathbf{p}^T\boldsymbol{\Theta}(1-\mathbf{p}) + \boldsymbol{\lambda}^T\Big[\beta\mathbf{A}^T\boldsymbol{\Theta}(1-\mathbf{p}) - \boldsymbol{\Theta}(1-\mathbf{p})\Big] \\ s.t. \quad & \lambda_i \geq \beta\lambda_j, \text{ for all } i,j, \\ & \lambda_i \leq 1, \text{ for all } i, \end{aligned} \tag{10}$$

where in the objective function we replace $\max_{\mathbf{p},\mathbf{Y},\boldsymbol{\delta}\geq 0} L(\mathbf{p},\mathbf{Y},\boldsymbol{\delta},\boldsymbol{\lambda})$ with:

$$\max_{\mathbf{p}\geq 0} \mathbf{p}^T\boldsymbol{\Theta}(1-\mathbf{p}) + \boldsymbol{\lambda}^T\Big[\beta\mathbf{A}^T\boldsymbol{\Theta}(1-\mathbf{p}) - \boldsymbol{\Theta}(1-\mathbf{p})\Big], \tag{11}$$

since, as mentioned above, in the optimal solution, $\delta_i > 0$ implies $\lambda_i = 1$ and $y_{ij} > 0$ implies $\lambda_i = \beta\lambda_j$; thus, we can remove the terms that involve $\boldsymbol{\delta}$ and $\mathbf{Y}$.

Ignoring the non-negativity constraint on the vector of prices for a moment, the first order optimality conditions of the optimization problem in the right hand side of (10) suggest that:

$$2\mathbf{p} - \mathbf{1} + \beta \mathbf{A}\boldsymbol{\lambda} - \boldsymbol{\lambda} = \mathbf{0},$$

or equivalently $\mathbf{p} = \frac{\mathbf{1}+\boldsymbol{\lambda}-\beta\mathbf{A}\boldsymbol{\lambda}}{2}$. Using the fact that matrix $\mathbf{A}$ is row-stochastic and $\lambda_i \geq \beta\lambda_j$ for all i, j yields:

$$\boldsymbol{\lambda} - \beta\mathbf{A}\boldsymbol{\lambda} \geq \boldsymbol{\lambda} - \Big(\beta \max_k \lambda_k\Big)\mathbf{A}\mathbf{1} \geq \boldsymbol{\lambda} - \Big(\beta \max_k \lambda_k\Big)\mathbf{1} \geq \mathbf{0}.$$

Thus, it follows that $\mathbf{0} \leq \frac{\mathbf{1}+\boldsymbol{\lambda}-\beta\mathbf{A}\boldsymbol{\lambda}}{2}$ and the non-negativity constraint in the right hand side of (10) can be relaxed without affecting the optimal solution, i.e., the optimal solution is interior. By strong duality, it follows that the primal optimal solution $(\mathbf{p}^\star, \mathbf{Y}^\star, \boldsymbol{\delta}^\star)$ satisfies

$$(\mathbf{p}^\star, \mathbf{Y}^\star, \boldsymbol{\delta}^\star) \in \arg \max_{(\mathbf{p},\mathbf{Y},\boldsymbol{\delta})} L(\mathbf{p}, \mathbf{Y}, \boldsymbol{\delta}, \boldsymbol{\lambda}^\star),$$

for the optimal dual solution $\boldsymbol{\lambda}^\star$. Thus, the vector $\mathbf{p}^\star$ that solves (11) for $\boldsymbol{\lambda} = \boldsymbol{\lambda}^\star$ is also equal to the vector of optimal prices in (17). That is,

$$\mathbf{p}^\star = \frac{\mathbf{1} + \boldsymbol{\lambda}^\star - \beta\mathbf{A}\boldsymbol{\lambda}^\star}{2},$$

as stated in the proposition. Using the characterization for the vector of optimal prices $\mathbf{p} = (\mathbf{1} + \boldsymbol{\lambda} - \beta\mathbf{A}\boldsymbol{\lambda})/2$ derived above (for $\boldsymbol{\lambda}$ such that $\lambda_i \geq \beta\lambda_j$ for all i, j), we then conclude that the dual problem can be rewritten as

$$\begin{aligned} \min_{\boldsymbol{\lambda}\in\mathbb{R}^n} \quad & \frac{1}{4}\Big(\mathbf{1} - (\mathbf{I} - \beta\mathbf{A})\boldsymbol{\lambda}\Big)^T \boldsymbol{\Theta}\Big(\mathbf{1} - (\mathbf{I} - \beta\mathbf{A})\boldsymbol{\lambda}\Big) \\ s.t. \quad & \lambda_i \geq \beta\lambda_j, \text{ for all } i, j, \\ & \lambda_i \leq 1, \text{ for all } i. \end{aligned}$$

Finally, strong duality and Expression (19) directly imply that the platform's optimal profits, i.e., the value of the objective function of optimization problem (17) at the optimal vector of prices $\mathbf{p}^\star$, are given as

$$\frac{1}{4}\Big(\mathbf{1} - (\mathbf{I} - \beta\mathbf{A})\boldsymbol{\lambda}^\star\Big)^T \boldsymbol{\Theta}\Big(\mathbf{1} - (\mathbf{I} - \beta\mathbf{A})\boldsymbol{\lambda}^\star\Big) = (\mathbf{1} - \mathbf{p}^\star)^T \boldsymbol{\Theta}(\mathbf{1} - \mathbf{p}^\star).$$

□

Proof of Lemma A.1

Given that $w = 1$, in any optimal solution for (17), there exists location k such that $\delta_k^\star > 0$, since otherwise serving some demand at k would lead to a solution with a higher value for the objective function. To see this, note that setting $p_k = 1 - \epsilon$, $\delta_k = \epsilon(1 - \beta^2)$, and $y_{ik} = \beta\alpha_{ki}(1 - p_k)\theta_k$ for all i and for some $\epsilon \ll 1$ is a feasible solution for (17) and generates positive profits for the platform (by contrast, setting $\delta_i = 0$ for all i generates zero profits).

For part (b), recall from the proof of Proposition A.1 that the primal optimal solution $(\mathbf{p}^\star, \mathbf{Y}^\star, \boldsymbol{\delta}^\star)$ satisfies

$$(\mathbf{p}^\star, \mathbf{Y}^\star, \boldsymbol{\delta}^\star) \in \arg\max_{(\mathbf{p}, \mathbf{Y}, \boldsymbol{\delta})} L(\mathbf{p}, \mathbf{Y}, \boldsymbol{\delta}, \boldsymbol{\lambda}^\star),$$

for the optimal dual solution $\boldsymbol{\lambda}^\star$. In addition, again from the proof of Proposition A.1, we have $\lambda_i^\star = 1$ when $\delta_i^\star > 0$, which establishes part (b)(ii) of the lemma. Also, by part (a) we have that there exists location i such that $\delta_i^\star > 0$, which together with $\lambda_i^\star \geq \beta\lambda_j^\star$ from the feasibility constraints of the dual, establishes part (b)(i) of the lemma. Finally, noting that $\lambda_i^\star = \beta\lambda_j^\star$ when $y_{ij}^\star > 0$ in combination with part (b)(i), establishes part (b)(iii) of the lemma. □

Proof of Lemma A.2

From Equation (19) the optimal vector of prices $\mathbf{p}^\star$ and the corresponding optimal dual vector $\boldsymbol{\lambda}^\star$ satisfy $(I - \beta\mathbf{A})\boldsymbol{\lambda}^\star = (2\mathbf{p}^\star - 1)$. Restricting attention to the i-th row of the vectors in this equation, we obtain

$$\lambda_i^\star - \beta\sum_j \alpha_{ij}\lambda_j^\star = 2p_i^\star - 1. \tag{12}$$

Note that since $\lambda_j^\star \in [\beta, 1]$ (Lemma A.1) and $\mathbf{A}$ is a row stochastic matrix, we get

$$\beta^2 = \beta\sum_j(\alpha_{ij}\beta) \leq \beta\sum_j \alpha_{ij}\lambda_j^\star \leq \beta\sum_j \alpha_{ij} = \beta. \tag{13}$$

Using (12) and the inequalities in (13), we obtain

$$\lambda_i^\star - \beta \leq 2p_i^\star - 1 \leq \lambda_i^\star - \beta^2, \tag{14}$$

which, by rearranging terms, concludes the proof of the lemma. □

EC.4 Supporting Material

EC.4.1 Nonuniqueness of Equilibria

In this subsection, we illustrate that the equilibrium need not be unique. As a simple example, consider a network consisting of a single location with $\alpha_{11} = 1$, and let $c_1 = (1-\beta)w$ and $p_1 = 1/2$. Suppose that the riders' willingness to pay is uniformly distributed in $[0, 1]$. Then, we can construct a continuum of equilibria. In particular, any tuple $\{\delta_1, x_1, y_{11}\}$ with

$$\delta_1 \leq \theta_1(1 - F(1/2))(1-\beta),$$

and $x_1 = \delta_1/(1-\beta)$ and $y_{11} = 0$, constitutes an equilibrium under $(p_1, c_1) = (1/2, (1-\beta)w)$. This is straightforward to see, since for any such δ_1 drivers are indifferent between entering or not and upon entering they always get assigned to a ride. Moreover, the profits for the platform corresponding to these equilibria are not the same. In particular, the flow rate of profits for the platform is given by

$$p_1 x_1 - w\delta_1 = \left(\frac{1}{2(1-\beta)} - w\right)\delta_1.$$

□

EC.4.2 Derivations for the Example in Figure 3

Example. Consider the network depicted in Figure 3 and assume that $w = 1$. Then, restrict attention to location 3 and let γ take some fixed value in $[0, 1]$. For a driver to find it optimal to enter and provide service at location 3, i.e., for any demand to be served at location 3, it has to be the case that the price p_3 set by the platform at location 3 satisfies:

$$w = V_3 = \min\{1, (1-p_3)/x_3\} \cdot \gamma p_3 + \beta V_3 \leq \frac{\gamma p_3}{1-\beta}, \tag{15}$$

where x_3 denotes the supply of drivers who provide service at location 3. The right hand side of the inequality is equal to the expected lifetime earnings of a driver when the probability of getting assigned to a rider at location 3 is equal to one (and, therefore, it is an upper bound to the earnings that a driver can make by providing service at 3 when the compensation per ride is equal to γp_3). Expression (15) further implies that, when w is normalized to one, we must have $p_3 \geq (1-\beta)/\gamma$ for any demand to be served at location 3. Thus, in such solutions, the platform's optimal profits at location 3 are equal to the solution to the following problem:

$$\max_{p_3 \in [0,1]} (1-\gamma)\theta_3 p_3 (1-p_3), \text{ subject to } p_3 \geq \frac{1-\beta}{\gamma},$$

which implies that for fixed γ the optimal price $p_3^\star(\gamma)$ is equal to $\max\{1/2, (1-\beta)/\gamma\}$ whereas the platform's optimal profits from location 3 as a function of γ, which we denote by $\Pi_3^\star(\gamma)$, take the following form:

$$\Pi_3^\star(\gamma) = \begin{cases} 0 & \text{if } \gamma \leq 1-\beta, \\ (1-\gamma)\theta_3 \frac{1-\beta}{\gamma}\left(1 - \frac{1-\beta}{\gamma}\right) & \text{if } \gamma \in (1-\beta, \min\{2(1-\beta), 1\}), \\ 1/4(1-\gamma)\theta_3 & \text{otherwise.} \end{cases}$$

To complete the description of the equilibrium outcome for location 3 for a fixed γ, we have

$$p_3^\star(\gamma) = \begin{cases} 1 & \text{if } \gamma \leq 1-\beta, \\ (1-\beta)/\gamma & \text{if } \gamma \in (1-\beta, \min\{2(1-\beta), 1\}), \\ 1/2 & \text{otherwise} \end{cases}$$

where with some abuse of notation $p_3^\star(\gamma)$ denotes the optimal price for the platform at location 3 as a function of γ. Similarly,

$$\delta_3^\star(\gamma) = \begin{cases} 0 & \text{if } \gamma \leq 1-\beta, \\ (1-\beta)(1-p_3^\star(\gamma))\theta_3 & \text{if } \gamma \in (1-\beta, \min\{2(1-\beta), 1\}), \\ 1/4 \cdot \gamma\theta_3 & \text{otherwise} \end{cases}$$

and $x_3^\star(\gamma) = \delta_3^\star(\gamma)/(1-\beta)$, whereas $y_{33}^\star(\gamma) = x_3^\star(\gamma) - (1-p_3^\star(\gamma))\theta_3$ and $y_{i3}^\star(\gamma) = y_{3i}^\star(\gamma) = 0$ for $i \neq 3$.

Furthermore, for the subgraph consisting of locations 1 and 2 a similar analysis when $\theta_2 = \epsilon \to 0$ yields the following for the optimal profits in the subgraph as a function of γ (which we denote by

$\Pi^\star_{1,2}(\gamma)$):

$$\Pi^\star_{1,2}(\gamma) = \begin{cases} 0 & \text{if } \gamma \le 1-\beta^2, \\ (1-\gamma)\theta_1 \frac{1-\beta^2}{\gamma}\left(1 - \frac{1-\beta^2}{\gamma}\right) & \text{if } \gamma \in (1-\beta^2, \min\{2(1-\beta^2), 1\}), \\ 1/4(1-\gamma)\theta_1 & \text{otherwise.} \end{cases}$$

To complete the description of the equilibrium outcome for the subgraph for a fixed γ, we have

$$p^\star_1(\gamma) = \begin{cases} 1 & \text{if } \gamma \le 1-\beta^2, \\ (1-\beta^2)/\gamma & \text{if } \gamma \in (1-\beta^2, \min\{2(1-\beta^2), 1\}), \\ 1/2 & \text{otherwise} \end{cases}$$

and $p^\star_2(\gamma) = 0$. Also,

$$\delta^\star_1(\gamma) = \begin{cases} 0 & \text{if } \gamma \le 1-\beta^2, \\ (1-\beta^2)(1-p^\star_1(\gamma))\theta_1 & \text{if } \gamma \in (1-\beta^2, \min\{2(1-\beta^2), 1\}), \\ 1/4 \cdot \gamma\theta_1 & \text{otherwise} \end{cases}$$

and $\delta^\star_2(\gamma) = 0$. Finally, $x^\star_1(\gamma) = \delta^\star_1(\gamma)/(1-\beta^2), x^\star_2(\gamma) = \beta(1-p^\star_1(\gamma))$, $y^\star_{21}(\gamma) = \beta(1-p^\star_1(\gamma))$, and $y^\star_{11}(\gamma) = x^\star_1(\gamma) - (1-p^\star_1(\gamma))\theta_1$, which completes the description of the equilibrium outcome for locations 1,2 for a fixed γ.

To see this, note that if any demand is served at the subgraph consisting of locations 1 and 2, it has to be the case that $\delta_1 > 0 = \delta_2$, i.e., drivers enter at location 1. Moreover, a driver who gets assigned to a ride at 1, completes it at location 2, and then returns to 1 without earning additional compensation. Thus, the supply of new drivers who enter the platform at location 1 in each time period is equal to $(1-\beta^2)x^\star_1(\gamma)$ given that β^2-fraction of the drivers who provide service at 1 return back to this location and continue providing service. Thus, the platform's problem at the subgraph consisting of locations 1 and 2 is essentially equivalent to the problem at location 3 when the fraction of drivers continuing to provide service changes from β to β^2. Finally, the platform's optimal choice of γ is the value that maximizes $\Pi^\star_{1,2}(\gamma) + \Pi^\star_3(\gamma)$. Let Π_{opt} denote the corresponding optimal objective of (6) for the same network. In Figure 3, we illustrate the profit gap between Π_{opt} and $\max_\gamma \Pi^\star_{1,2}(\gamma) + \Pi^\star_3(\gamma)$, i.e., $1 - (\max_\gamma(\Pi^\star_{1,2}(\gamma) + \Pi^\star_3(\gamma))/\Pi_{opt}$ for different values of β.

EC.4.3 Heterogeneity in the Drivers' Outside Option

Throughout the paper, we assume that the drivers have the same outside option, which amounts to lifetime earnings equal to w. This allows us to focus on the questions we are mostly interested in; i.e., how imbalances in the demand and its destination preferences across a network of locations may affect a platform's pricing policy and profits.

That said, we describe below how we could accommodate heterogeneity in the outside option among the platform's potential drivers. Suppose that the mass of potential drivers that could join and start providing service for the platform at each time period is Δ and their outside option (reservation wage) is distributed with CDF $G(\cdot)$; i.e., $\Delta \cdot G(w)$ is the total mass of drivers that would be willing to join the platform when their expected lifetime earnings by participating are equal to w. In the remainder of this appendix, we consider optimizing over prices and compensations $\{p_i, c_i\}_{i=1}^n$, where p_i denotes the price that a rider has to pay for a ride that originates from location i, and c_i is the corresponding compensation for the driver (as in the main body of the paper). Then, the platform's optimization problem would be written as (this is an extension of optimization problem (6) incorporating heterogeneity in the reservation wages—note that the relaxation implied by Lemma 1 applies here as well):

$$\begin{aligned} \max_{w,\{p_i,\delta_i,y_{ij}\}_{i,j=1}^n} \quad & \sum_i p_i(1-p_i)\theta_i - w \cdot \Delta \cdot G(w) \\ s.t. \quad & \sum_j y_{ij} = \beta\left[\sum_j \alpha_{ji}(1-p_j)\theta_j + \sum_j y_{ji}\right] + \delta_i - (1-p_i)\theta_i, \text{ for all } i, \\ & p_i, \delta_i, y_{ij} \geq 0, \text{ for all } i, j, \\ & p_i \leq 1, \text{ for all } i, \\ & \sum_i \delta_i \leq \Delta \cdot G(w), \end{aligned} \tag{16}$$

where in the objective function we substitute $w \sum_i \delta_i = w \cdot \Delta \cdot G(w)$, which is a consequence of the fact that the platform will set w so that the total mass of drivers $\sum_i \delta_i$ that would enter each time period is precisely equal to the drivers the platform needs.

Note that although in general optimization problem (16) is non-convex for a general $G(\cdot)$, it is convex for a number of distributions (similar to our discussion in the paper for the riders' willingness to pay distribution $F(\cdot)$). For example, if the distribution of drivers' reservation wages is uniform with support in $[0, \bar{w}]$, we could rewrite (16) as:

$$
\begin{aligned}
\max_{w, \{p_i, \delta_i, y_{ij}\}_{i,j=1}^n} \quad & \sum_i p_i(1-p_i)\theta_i - \Delta/\bar{w} \cdot w^2 \\
s.t. \quad & \sum_j y_{ij} = \beta \left[\sum_j \alpha_{ji}(1-p_j)\theta_j + \sum_j y_{ji} \right] + \delta_i - (1-p_i)\theta_i, \text{ for all } i, \\
& p_i, \delta_i, y_{ij} \geq 0, \text{ for all } i, j, \\
& p_i \leq 1, \text{ for all } i, \\
& \sum_i \delta_i \leq \Delta/\bar{w} \cdot w, \\
& w \leq \bar{w}.
\end{aligned}
\tag{17}
$$

Note that imposing $w \leq \bar{w}$ above (which is necessary given our assumption on the support of the reservation wages) is without any loss of optimality as the platform would never find it optimal to set the wage w higher than $\bar{w}$ (even in the absence of the $w \leq \bar{w}$ constraint). The resulting optimization problem (17) is convex and upon solving it we obtain the equilibrium wage w the platform would find optimal to induce as a function of its demand pattern.

Moreover, note that in the context of the model of the main body, where w is fixed and there is free-entry, drivers' surplus is equal to zero (as their expected lifetime earnings at equilibrium are equal to their outside option). On the other hand, if drivers are heterogeneous in their outside option, i.e., their outside options are distributed according to $G(w)$, we obtain the following expression for their surplus when expected earnings in the platform at equilibrium are equal to w:

$$\Delta \int_0^w (w-x)g(x)dx,$$

where recall that Δ denotes the total mass of drivers who are willing to provide service at every time period. When outside options are distributed uniformly in $[0, \bar{w}]$, we can rewrite the expression for the drivers' surplus as follows

$$\Delta/\bar{w} \int_0^w (w-x)dx = \frac{\Delta}{2\bar{w}} \cdot w^2,$$

i.e., drivers' surplus is increasing with the prevailing equilibrium wage in the platform.

Although drivers' surplus is increasing with the equilibrium wage induced by the platform's pricing policy, this wage and, consequently, drivers' surplus do not satisfy a monotonicity property as a function of the demand pattern's balancedness. In particular, as the following figure illustrates, equilibrium wage/drivers' surplus may increase or decrease as the network becomes more balanced (unlike platforms' profits and consumer surplus that always increase with balancedness even under heterogeneity in the drivers' reservation wages as also illustrated in the figure).

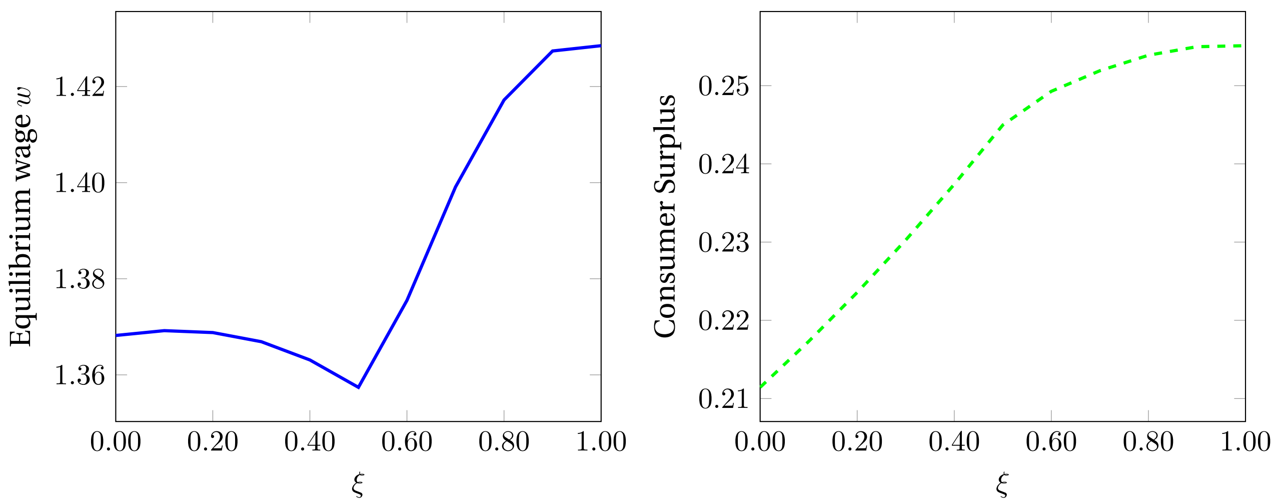

Figure 1: Induced equilibrium wage w and consumer surplus corresponding to the platform's optimal origin pricing policy for the class of star-to-complete networks with $n = 4$ locations, $\boldsymbol{\theta} = \mathbf{1}$, and $\beta = 0.9$. Here, the riders' willingness to pay is uniformly distributed in $[0, 1]$ and the drivers' reservation wage is uniformly distributed in $[0, 5]$. Finally, $\Delta = 0.5$.