Electronic Companion to

"Process Flexibility for Multi-Period Production Systems"

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EC.1. Stability Condition

Here, we characterize the conditions on the flexibility structure $\mathcal{A}$, the capacity vector $\boldsymbol{c}$ and the demand distribution $\mathbf{D}$ under which there exists a policy that guarantees the finiteness of the long-run average backlogging cost. The requirement of finite backlogging cost is also known as the *stability* condition, and this consideration motivates us to define the stability of a policy.

Definition EC.1 (Stable Policy). Given a flexibility structure $\mathcal{A}$, plant capacity $\mathbf{c}$, and product demand distribution $\mathbf{D}$, a production policy π is said to be *stable* if the expected long-run average backlogging costs $\Gamma(\pi) < \infty$, and is *unstable* otherwise.

The following proposition gives a necessary and sufficient condition on $\boldsymbol{c}$ and $\mathbf{D}$ for the existence of a stable policy under $\mathcal{A}$. The proof is standard, and is included here for completeness.

Proposition EC.1. *Let the flexibility structure $\mathcal{A}$ be given. Then, a necessary and sufficient condition for the existence of a stable policy is*

$$\sum_{\mathcal{S}_i \in N(\Omega)} c_i > \sum_{\mathcal{T}_j \in \Omega} \lambda_j, \textit{for all } \Omega \subseteq \{\mathcal{T}_1, \ldots, \mathcal{T}_n\},\ \Omega \neq \emptyset. \tag{EC.1}$$

Proof of Proposition EC.1. The necessity of (EC.1) can be derived as follows. For any non-empty subset $\Omega \subseteq \{\mathcal{T}_1, \ldots, \mathcal{T}_n\}$, the corresponding aggregate demand in time period t is $\sum_{\mathcal{T}_j \in \Omega} D_j(t)$, with $\mathbb{E}\left[\sum_{\mathcal{T}_j \in \Omega} D_j(t)\right] = \sum_{\mathcal{T}_j \in \Omega} \lambda_j$, and the maximum production capacity that can be devoted to this demand is $\sum_{\mathcal{S}_i \in N(\Omega)} c_i$. Consider a single-plant production system with capacity $\sum_{\mathcal{S}_i \in N(\Omega)} c_i$ and demand given by $\sum_{\mathcal{T}_j \in \Omega} D_j(t)$ in time period t, $t = 1, 2, \cdots$. Let $Q(t)$ be the backlog of the single-plant system at time t, and suppose that this single-plant system starts empty, i.e., $Q(0) = 0$. Then, by a standard coupling argument, it can be shown that $\sum_{\mathcal{T}_j \in \Omega} B_j(t) \geq Q(t)$ for each t. Thus, for the original system to be stable, we need $\limsup \frac{1}{T} \sum_{t=1}^{T} \sum_{\mathcal{T}_j \in \Omega} B_j(t) < \infty$, and so it is necessary that $\limsup \frac{1}{T} \sum_{t=1}^{T} Q(t) < \infty$. To guarantee $\limsup \frac{1}{T} \sum_{t=1}^{T} Q(t) < \infty$, it is required that $\sum_{\mathcal{S}_i \in N(\Omega)} c_i > \sum_{\mathcal{T}_j \in \Omega} \lambda_j$. This establishes necessity.

The sufficiency of (EC.1) can be proved as follows. Suppose that for each non-empty subset $\Omega \subseteq \{\mathcal{T}_1, \cdots, \mathcal{T}_n\}$, $\sum_{\mathcal{S}_i \in N(\Omega)} c_i > \sum_{\mathcal{T}_j \in \Omega} \lambda_j$. Scale the vector $\boldsymbol{\lambda}$ by a factor α so that for every non-empty subset $\tilde{\Omega} \subseteq \{\mathcal{T}_1, \cdots, \mathcal{T}_n\}$, $\sum_{\mathcal{S}_i \in N(\tilde{\Omega})} c_i \geq \alpha \sum_{\mathcal{T}_j \in \tilde{\Omega}} \lambda_j$, and there exists a non-empty subset $\Omega \subseteq \{\mathcal{T}_1, \cdots, \mathcal{T}_n\}$ such that $\sum_{\mathcal{S}_i \in N(\Omega)} c_i = \alpha \sum_{\mathcal{T}_j \in \Omega} \lambda_j$. Necessarily, $\alpha > 1$. By the max-flow min-cut theorem, $\alpha\boldsymbol{\lambda} \in R(\mathcal{A})$. Let π be the production policy that uses the constant schedule $\alpha\boldsymbol{\lambda}$. Then, under the policy π, the system is stable. This concludes the sufficiency part. $\square$

An immediate consequence of Proposition EC.1 is the following corollary.

Corollary EC.1. *A necessary and sufficient condition for the existence of a stable policy under the full flexibility structure is $\Lambda < C$.*

Corollary EC.1 essentially spells out that the weakest condition for the existence of a stable policy under any given flexibility structure is $\Lambda < C$, justifying Equation (3) in Assumption 1.

EC.2. Proofs of Results in §3.2

EC.2.1. Proof of Proposition 1

A key fact that is used in the proof of Proposition 1 is the following lemma.

Lemma EC.1. *Let the demand rate vector $\boldsymbol{\lambda}$ and capacity vector $\boldsymbol{c}$ be given with $\Lambda < C$, and $\boldsymbol{\lambda}'$ be the projection of $\boldsymbol{\lambda}$ onto the plane defined by $\left\{\mathbf{g} \mid \sum_{j=1}^{n} g_j = C\right\}$. Let $\mathcal{A}$ be a flexibility structure that has positive GCG ($\eta > 0$). Then, for any $\mathbf{x} \in \mathbb{R}^n$ with $\sum_j x_j = 0$ and $\|\mathbf{x}\| \leq \eta/\sqrt{n}$, $\boldsymbol{\lambda}' + \mathbf{x}$ lies on the face defined by $\left\{\mathbf{g} \mid \sum_{j=1}^{n} g_j = C\right\}$, and $\boldsymbol{\lambda}' + \mathbf{x} \in R(\mathcal{A})$.*

Proof of Lemma EC.1. Let $\mathbf{x} \in \mathbb{R}^n$ be such that $\sum_j x_j = 0$ and $\|\mathbf{x}\| \leq \eta/\sqrt{n}$. Then, $\sum_j |x_j| \leq \eta$ by the Cauchy-Schwarz inequality. Second, since $\sum_j x_j = 0$, $\sum_j (\lambda'_j + x_j) = \sum_j \lambda'_j = C$, and $\boldsymbol{\lambda}' + \mathbf{x}$ lies on the face defined by $\left\{\mathbf{g} \mid \sum_{j=1}^{n} g_j = C\right\}$. Finally, for any $\Omega \subsetneq \{\mathcal{T}_1, \ldots, \mathcal{T}_n\}$,

$$\sum_{\mathcal{T}_j \in \Omega} (\lambda'_j + x_j) = \sum_{\mathcal{T}_j \in \Omega} \lambda'_j + \sum_{\mathcal{T}_j \in \Omega} x_j \leq \sum_{\mathcal{T}_j \in \Omega} \lambda'_j + \sum_j |x_j| \leq \left(\sum_{\mathcal{S}_i \in N(\Omega)} c_i - \eta \right) + \eta = \sum_{\mathcal{S}_i \in N(\Omega)} c_i.$$

We also note that for each j, $\lambda'_j + x_j \geq 0$. To see this, for each j, let $\Omega^{-j} = \{\mathcal{T}_1, \ldots \mathcal{T}_{j-1}, \mathcal{T}_{j+1}, \ldots, \mathcal{T}_n\}$, and we have

$$\eta \leq \sum_{\mathcal{S}_i \in N(\Omega^{-j})} c_i - \sum_{\mathcal{T}_j \in \Omega^{-j}} \lambda'_j \leq \sum_{i=1}^{m} c_i - \sum_{\mathcal{T}_j \in \Omega^{-j}} \lambda'_j = \lambda'_j.$$

Thus, we have that $\lambda'_j \geq \eta \geq |x_j|$, implying $\lambda'_j + x_j \geq 0$. As a result, we can conclude that $\boldsymbol{\lambda}' + \mathbf{x} \in R(\mathcal{A})$. $\square$

Lemma EC.1 states if $\mathcal{A}$ has a positive GCG η, the Euclidean ball defined on the hyperplane $\left\{\mathbf{g} \mid \sum_{j=1}^{n} g_j = C\right\}$ with center $\boldsymbol{\lambda}'$ and radius $\eta/\sqrt{n}$, lies within the production polytope $R(\mathcal{A})$. The lemma therefore allows us to connect GCG to the result in Eryilmaz and Srikant (2012), which is used for the proof of Proposition 1.

We first note that the system is stable under the Max-Weight policy. The proof of this fact is quite standard, by considering the conditional expected drift of the quadratic Lyapunov function $\sum_{j=1}^{n} B_j^2$, and invoking the so-called Foster's lemma. Similar proofs have appeared in e.g., McKeown et al. (1999), Tassiulas and Ephremides (1992), Dai and Lin (2005). We skip details.

Since the Max-Weight policy is stable, there exists a unique steady-state distribution. Let $\mathbf{B}(\infty)$ be the unique random backlog vector in steady state. Furthermore, for any backlog vector $\mathbf{B} = (B_1, B_2, \cdots, B_n)$, define $\bar{B} = (B_1 + B_2 + \cdots + B_n)/n$ to be the average of the backlogs, and let $\Delta\mathbf{B} = \mathbf{B} - \bar{B}\mathbf{1}$, where $\mathbf{1} = (1, 1, \ldots, 1)$. Note that $\Delta\mathbf{B}$ is the vector of deviations of the backlogs from their average. In addition, we define $D_{\max}$ as the maximum possible value for the aggregate demand. Under Assumption 1, we have $D_{\max} \leq nu$.

To prove Proposition 1, we first establish the following state space collapse result. Informally, state space collapse implies that under the Max-Weight policy, all backlogs B_i stays close to the mean $\bar{B}$, so that the vector $\Delta\mathbf{B}$ remains small.

THEOREM EC.1 **(State space collapse)**. *Let $\mathbf{B}(\infty)$ have the steady-state distribution of the backlog vector under Max-Weight policy, and flexibility structure $\mathcal{A}$ with the GCG η. Then, for any $\ell \in \mathbb{Z}_+$,*

$$\mathbb{P}\left(\|\Delta\mathbf{B}(\infty)\|_2 > K' + 2\xi\ell\right) \leq \left(\frac{\xi}{\xi+\gamma}\right)^{\ell+1}, \tag{EC.2}$$

where

$$K' = \frac{\left(\Sigma^2 + \sum_j \lambda_j^2 + 2C^2\right)\sqrt{n}}{\eta}; \quad \gamma = \frac{\eta}{2\sqrt{n}}; \quad \xi = \sqrt{n}(D_{\max} + C). \tag{EC.3}$$

Proof of Theorem EC.1. The proof of Theorem EC.1 invokes the following theorem in Bertsimas et al. (2001) (where a weaker version was also given in Hajek (1982)).

THEOREM EC.2. *Let $X(\cdot)$ be an irreducible, aperiodic and positively recurrent discrete-time Markov chain with a countable state space $\mathscr{X}$. Suppose that there exists a Lyapunov function $\Phi : \mathscr{X} \to \mathbb{R}_+$ with the following properties.*

(a) **Bounded increment.** *There exists a positive constant ξ such that $|\Phi(X(t+1)) - \Phi(X(t))| \leq \xi$ for all t a.s.*

(b) **Negative drift.** *There exist positive constants K' and γ such that whenever $\Phi(X(t)) > K'$,*

$$\mathbb{E}\left[\Phi(X(t+1)) - \Phi(X(t)) \mid X(t)\right] \leq -\gamma. \tag{EC.4}$$

Then, under the steady-state distribution of $X(\cdot)$, for any $\ell \in \mathbb{Z}_+$,

$$\mathbb{P}\left(\Phi(X) > K' + 2\xi\ell\right) \leq \left(\frac{\xi}{\xi+\gamma}\right)^{\ell+1}. \tag{EC.5}$$

The proof of Theorem EC.1 then relies on establishing conditions (a) and (b) of Theorem EC.2 for an appropriately chosen Lyapunov function $\Phi(\cdot)$. It is straightforward to check that $\mathbf{B}(\cdot)$ is aperiodic and positively recurrent. We also claim that if the product demand instances and plant capacities are integral (rational), then the state space of $\mathbf{B}(\cdot)$ is countable. To see this, note that the Max-Weight policy solves a network flow problem with integral (rational) input at each time period, implying that if $\mathbf{b}(t-1)$ and $\mathbf{d}(t)$ are integral (rational), then $\mathbf{b}(t)$ is integral (rational). Instead of focusing on the Markov chain $\mathbf{B}(\cdot)$, we will consider the closely related chain $\mathbf{B}'(\cdot)$, defined to be $\mathbf{B}'(t) = \mathbf{B}(t) + \mathbf{D}(t)$ for all t.

PROPOSITION EC.2. *The following inequality holds for the Lyapunov function* $\Phi(\mathbf{B}') = \|\Delta\mathbf{B}'\|_2$.

$$\mathbb{E}\left[\|\Delta\mathbf{B}'(t+1)\|_2 - \|\Delta\mathbf{B}'(t)\|_2 \mid \mathbf{B}'(t)\right] \leq -\frac{\eta}{\sqrt{n}} + \frac{\Sigma^2 + \sum_j \lambda_j^2 + 2C^2}{2\|\Delta\mathbf{B}'(t)\|_2}. \tag{EC.6}$$

In particular, whenever $\|\Delta\mathbf{B}'(t)\|_2 > \frac{\sqrt{n}\left(\Sigma^2 + \sum_j \lambda_j^2 + 2C^2\right)}{\eta}$,

$$\mathbb{E}\left[\|\Delta\mathbf{B}'(t+1)\|_2 - \|\Delta\mathbf{B}'(t)\|_2 \mid \mathbf{B}'(t)\right] \leq -\frac{\eta}{2\sqrt{n}}. \tag{EC.7}$$

Proof of Proposition EC.2. The proof of Proposition EC.2 mainly consists of establishing the following expressions.

(a) Show that

$$\|\Delta\mathbf{B}'(t+1)\|_2 - \|\Delta\mathbf{B}'(t)\|_2 \leq \frac{(\|\mathbf{B}'(t+1)\|_2^2 - \|\mathbf{B}'(t)\|_2^2) - n\left(\overline{B'(t+1)}^2 - \overline{B'(t)}^2\right)}{2\|\Delta\mathbf{B}'(t)\|_2}; \tag{EC.8}$$

(b) show that

$$\mathbb{E}\left[\|\mathbf{B}'(t+1)\|_2^2 - \|\mathbf{B}'(t)\|_2^2 \mid \mathbf{B}'(t)\right] \leq -2n\zeta\overline{B'(t)} - \frac{2\eta}{\sqrt{n}}\|\Delta\mathbf{B}'(t)\|_2 + \left(\Sigma^2 + C^2 + \sum_j \lambda_j^2\right); \tag{EC.9}$$

and

(c) show that

$$\mathbb{E}\left[n\left(\overline{B'(t+1)}^2 - \overline{B'(t)}^2\right) \mid \mathbf{B}'(t)\right] \geq -2n\zeta\overline{B'(t)} + \frac{1}{n}\left(\Sigma^2 + n^2\zeta^2 - 3C^2\right). \tag{EC.10}$$

(a) To establish (EC.8), we use the following general inequality: if $x > 0$, then $y - x \leq \frac{y^2 - x^2}{2x}$. Substituting $\|\Delta\mathbf{B}'(t)\|_2$ in place of x and $\|\Delta\mathbf{B}'(t+1)\|_2$ in place of y, we get

$$\|\Delta\mathbf{B}'(t+1)\|_2 - \|\Delta\mathbf{B}'(t)\|_2 \leq \frac{\|\Delta\mathbf{B}'(t+1)\|_2^2 - \|\Delta\mathbf{B}'(t)\|_2^2}{2\|\Delta\mathbf{B}'(t)\|_2}. \tag{EC.11}$$

It is easy to verify that in general, $\langle \Delta \mathbf{B}, \bar{B}\mathbf{1}\rangle = 0$ and $\Delta\mathbf{B} + \bar{B}\mathbf{1} = \mathbf{B}$. Thus, by the Pythagorean theorem,

$$\|\Delta\mathbf{B}'(t)\|_2^2 = \|\mathbf{B}'(t)\|_2^2 - \|\overline{B'(t)}\mathbf{1}\|_2^2 = \|\mathbf{B}'(t)\|_2^2 - n\overline{B'(t)}^2.$$

A similar identity holds for $\|\Delta\mathbf{B}'(t+1)\|_2^2$. Therefore, substituting these identities into (EC.11), we establish (EC.8).

(b) To establish (EC.9), we first write $\mathbf{B}'(t+1) = \mathbf{B}'(t) - \mathbf{G}(\mathbf{B}'(t)) + \mathbf{U}(\mathbf{B}'(t)) + \mathbf{D}(t+1)$. Here $\mathbf{G}(\mathbf{B}'(t))$ is the production schedule used at time t, which depends on the vector $\mathbf{B}'(t)$. We use the following convention here: we suppose that $\mathbf{G}(\mathbf{B}'(t))$ is obtained from some $(f_{ij})_{i,j}$ such that $\sum_j f_{ij} = c_i$ for all i; i.e., all plants use their production capability fully in each time period. It may happen that for some j, $B_j'(t) < G_j(\mathbf{B}'(t))$, in which case we let $U_j(t) = G_j(\mathbf{B}'(t)) - B_j'(t)$ be the unused capacity for product j. Otherwise, let $U_j(t) = 0$. Then, we denote the vector $(U_j(t))_j$ by $\mathbf{U}(\mathbf{B}'(t))$. An immediate consequence is that $\langle \mathbf{B}(t+1), \mathbf{U}(t)\rangle = \langle \mathbf{B}'(t) - \mathbf{G}(\mathbf{B}'(t)) + \mathbf{U}(t), \mathbf{U}(t)\rangle = 0$.

For notational convenience, we drop the time index, and then

$$\mathbb{E}\left[\|\mathbf{B}'(t+1)\|_2^2 - \|\mathbf{B}'(t)\|_2^2 \mid \mathbf{B}'(t)\right] = \mathbb{E}\left[\|\mathbf{B}' - \mathbf{G} + \mathbf{U} + \mathbf{D}\|_2^2 - \|\mathbf{B}'\|_2^2 \mid \mathbf{B}'\right].$$

We now focus on the term $\mathbb{E}\left[\|\mathbf{B}' - \mathbf{G} + \mathbf{U} + \mathbf{D}\|_2^2 - \|\mathbf{B}'\|_2^2 \mid \mathbf{B}'\right]$. We have

$$\begin{aligned}
&\mathbb{E}\left[\|\mathbf{B}' - \mathbf{G} + \mathbf{U} + \mathbf{D}\|_2^2 - \|\mathbf{B}'\|_2^2 \mid \mathbf{B}'\right] \\
&= \mathbb{E}\left[\|\mathbf{B}' - \mathbf{G} + \mathbf{D}\|_2^2 \mid \mathbf{B}'\right] + \mathbb{E}\left[\|\mathbf{U}\|_2^2 \mid \mathbf{B}'\right] + 2\mathbb{E}\left[\langle \mathbf{D}, \mathbf{U}\rangle \mid \mathbf{B}'\right] - \mathbb{E}\left[2\|\mathbf{U}\|_2^2 \mid \mathbf{B}'\right] - \mathbb{E}\left[\|\mathbf{B}'\|_2^2 \mid \mathbf{B}'\right] \\
&= \mathbb{E}\left[2\langle \mathbf{D} - \mathbf{G}, \mathbf{B}'\rangle \mid \mathbf{B}'\right] + \mathbb{E}\left[\|\mathbf{D} - \mathbf{G}\|_2^2 \mid \mathbf{B}'\right] - \mathbb{E}\left[\|\mathbf{U}\|_2^2 \mid \mathbf{B}'\right] + 2\mathbb{E}\left[\langle \mathbf{D}, \mathbf{U}\rangle \mid \mathbf{B}'\right] \\
&\le \mathbb{E}\left[2\langle \mathbf{D} - \mathbf{G}, \mathbf{B}'\rangle \mid \mathbf{B}'\right] + \mathbb{E}\left[\|\mathbf{D} - \mathbf{G}\|_2^2 \mid \mathbf{B}'\right] + 2\mathbb{E}\left[\langle \mathbf{D}, \mathbf{U}\rangle \mid \mathbf{B}'\right],
\end{aligned} \tag{EC.12}$$

where the first equality holds because $\mathbb{E}\left[\langle \mathbf{B}' - \mathbf{G}, \mathbf{U}\rangle \mid \mathbf{B}'\right] = -\mathbb{E}\left[\|\mathbf{U}\|_2^2 \mid \mathbf{B}'\right]$. Next, let us start with $\mathbb{E}\left[\langle \mathbf{D} - \mathbf{G}, \mathbf{B}'\rangle \mid \mathbf{B}'\right]$ and observe that

$$\begin{aligned}
\mathbb{E}\left[\langle \mathbf{D} - \mathbf{G}, \mathbf{B}'\rangle \mid \mathbf{B}'\right] &= \langle \boldsymbol{\lambda} - \mathbf{G}, \mathbf{B}'\rangle \\
&= \langle \boldsymbol{\lambda}' - \mathbf{G}, \mathbf{B}'\rangle - \langle \boldsymbol{\lambda}' - \boldsymbol{\lambda}, \mathbf{B}'\rangle \\
&= \langle \boldsymbol{\lambda}' - \mathbf{G}, \mathbf{B}'\rangle - n\zeta\overline{B'}.
\end{aligned}$$

By the Max-Weight policy, $\mathbf{G}$ is chosen from $R(\mathcal{A})$ to maximize the inner product $\langle \mathbf{G}, \mathbf{B}'\rangle$. By Lemma EC.1, $\boldsymbol{\lambda}' + \frac{\eta}{\sqrt{n}} \cdot \frac{\Delta\mathbf{B}'}{\|\Delta\mathbf{B}'\|} \in R(\mathcal{A})$. Therefore, we have

$$\begin{aligned}
\mathbb{E}\left[\langle \mathbf{D} - \mathbf{G}, \mathbf{B}'\rangle \mid \mathbf{B}'\right] &= \langle \boldsymbol{\lambda}' - \mathbf{G}, \mathbf{B}'\rangle - n\zeta\overline{B'} \\
&\le -\frac{\eta}{\sqrt{n}\cdot\|\Delta\mathbf{B}'\|}\langle \Delta\mathbf{B}', \mathbf{B}'\rangle - n\zeta\overline{B'}
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{\eta}{\sqrt{n}\cdot\|\Delta\mathbf{B}'\|}\langle\Delta\mathbf{B}',\Delta\mathbf{B}'\rangle - n\zeta\overline{B'} \\
&= -\frac{\eta}{\sqrt{n}}\|\Delta\mathbf{B}'\| - n\zeta\overline{B'}. \qquad \text{(EC.13)}
\end{aligned}
$$

For the remaining terms, we have

$$
\begin{aligned}
\mathbb{E}\left[\|\mathbf{D}-\mathbf{G}\|_2^2 \mid \mathbf{B}'\right] + 2\mathbb{E}\left[\langle\mathbf{D},\mathbf{U}\rangle \mid \mathbf{B}'\right] &= \mathbb{E}\left[\sum_j (D_j-G_j)^2 \mid \mathbf{B}'\right] + 2\mathbb{E}\left[\sum_j D_jU_j \mid \mathbf{B}'\right] \\
&= \mathbb{E}\left[\sum_j D_j^2 + \sum_j G_j^2 \mid \mathbf{B}'\right] - 2\mathbb{E}\left[\sum_j D_jG_j \mid \mathbf{B}'\right] + 2\mathbb{E}\left[\sum_j D_jU_j \mid \mathbf{B}'\right] \\
&\le \mathbb{E}\left[\sum_j D_j^2 + \sum_j G_j^2 \mid \mathbf{B}'\right] \le \Sigma^2 + \sum_j \lambda_j^2 + C^2. \qquad \text{(EC.14)}
\end{aligned}
$$

Combining (EC.12), (EC.13) and (EC.14), we have established (EC.9).

(c) To establish (EC.10), we have

$$
\begin{aligned}
&\mathbb{E}\left[n\left(\overline{B'(t+1)}^2 - \overline{B'(t)}^2\right) \mid \mathbf{B}'(t)\right] \\
&= \mathbb{E}\left[\frac{1}{n}\left(\sum_j B_j'(t+1)\right)^2 - \frac{1}{n}\left(\sum_j B_j'(t)\right)^2 \,\middle|\, \mathbf{B}'(t)\right] \\
&= \frac{1}{n}\mathbb{E}\left[\left(\sum_j (B_j' - G_j + U_j + D_j)\right)^2 - \left(\sum_j B_j'\right)^2 \,\middle|\, \mathbf{B}'\right] \\
&= \frac{2}{n}\left(\sum_j B_j'\right)\mathbb{E}\left[\sum_j (D_j - G_j)\,\middle|\, \mathbf{B}'\right] + \frac{1}{n}\mathbb{E}\left[\left(\sum_j (D_j-G_j)\right)^2 \,\middle|\, \mathbf{B}'\right] \\
&\quad + \frac{2}{n}\mathbb{E}\left[\left(\sum_j (B_j' + D_j - G_j)\right)\left(\sum_j U_j\right) \,\middle|\, \mathbf{B}'\right] \\
&= -2n\zeta\overline{B'} + \frac{1}{n}\left(\Sigma^2 + (\Lambda - C)^2\right) + \frac{2}{n}\mathbb{E}\left[\left(\sum_j (B_j' + D_j)\right)\left(\sum_j U_j\right) \,\middle|\, \mathbf{B}'\right] - \frac{2C}{n}\mathbb{E}\left[\sum_j U_j \,\middle|\, \mathbf{B}'\right] \\
&\ge -2n\zeta\overline{B'} + \frac{1}{n}\left(\Sigma^2 + (\Lambda - C)^2\right) - \frac{2C^2}{n} = -2n\zeta\overline{B'} + \frac{1}{n}\left(\Sigma^2 + n^2\zeta^2 - 3C^2\right).
\end{aligned}
$$

Combining (EC.9) and (EC.10), we have

$$
\begin{aligned}
&\mathbb{E}\left[\left(\|\mathbf{B}'(t+1)\|_2^2 - \|\mathbf{B}'(t)\|_2^2\right) - n\left(\overline{B'(t+1)}^2 - \overline{B'(t)}^2\right) \,\middle|\, \mathbf{B}'(t)\right] \\
&\le -2n\zeta\overline{B'(t)} - \frac{2\eta}{\sqrt{n}}\|\Delta\mathbf{B}'(t)\|_2 + \left(\Sigma^2 + C^2 + \sum_j \lambda_j^2\right) + 2n\zeta\overline{B'(t)} - \frac{1}{n}\left(\Sigma^2 + n^2\zeta^2 - 3C^2\right) \\
&\le -\frac{2\eta}{\sqrt{n}}\|\Delta\mathbf{B}'(t)\|_2 + \left(\Sigma^2 + 2C^2 + \sum_j \lambda_j^2\right).
\end{aligned}
$$

Thus, by (EC.8),

$$\mathbb{E}\left[\|\Delta\mathbf{B}'(t+1)\|_2 - \|\Delta\mathbf{B}'(t)\|_2 \mid \mathbf{B}'(t)\right] \le \frac{1}{2\|\Delta\mathbf{B}'(t)\|_2}\left(-\frac{2\eta}{\sqrt{n}}\|\Delta\mathbf{B}'(t)\|_2 + \left(\Sigma^2 + 2C^2 + \sum_j \lambda_j^2\right)\right)$$
$$\le -\frac{\eta}{\sqrt{n}} + \frac{\Sigma^2 + \sum_j \lambda_j^2 + 2C^2}{2\|\Delta\mathbf{B}'(t)\|_2}.$$

This concludes the proof of Proposition EC.2. □

With Proposition EC.2, we can now complete the proof of Theorem EC.1.

Proof of Theorem EC.1. We have already established the negative drift condition (b) (of Theorem EC.2) in Proposition EC.2. To establish condition (a), first note that

$$\Big|\|\Delta\mathbf{B}'(t+1)\|_2 - \|\Delta\mathbf{B}'(t)\|_2\Big| \le \Big|\|\mathbf{B}'(t+1)\|_2 - \|\mathbf{B}'(t)\|_2\Big|.$$

Second, the maximum decrease in each $B'_j(t)$ is C, and the maximum increase in each $B'_j(t)$ is $D_{\max}$. Therefore, almost surely, for each j,

$$|B'_j(t+1) - B'_j(t)| \le C + D_{\max}.$$

This implies that almost surely, for every t,

$$\Big|\|\mathbf{B}'(t+1)\|_2 - \|\mathbf{B}'(t)\|_2\Big| \le \|(C + D_{\max})\mathbf{1}\|_2 = \sqrt{n}(C + D_{\max}).$$

By setting $K' = \sqrt{n}\left(\sum_j \lambda_j^2 + 2C^2\right)/\eta$, $\gamma = \eta/(2\sqrt{n})$, and $\xi = \sqrt{n}(C + D_{\max})$, and invoking Theorem EC.2, we can establish Theorem EC.1. □

An immediate consequence of Theorem EC.1 is that all moments of $\|\Delta\mathbf{B}(\infty)\|_2$ are finite, and that $\mathbb{E}[\|\Delta\mathbf{B}(\infty)\|_2^2]$ is of order $O(1/\eta^2)$. There is also an immediate corollary to the state space collapse result in Theorem EC.1.

Corollary EC.2. *Let the setup be the same as in Theorem EC.1. Then,*

$$\mathbb{E}[\|\Delta\mathbf{B}\|_2^2] \le \left(\frac{\sqrt{n}\left(\Sigma^2 + \sum_j \lambda_j^2\right) + 14n^{3/2}C^2 + 12n^{3/2}D_{\max}^2}{\eta} + \sqrt{n}(D_{\max} + C)\right)^2. \tag{EC.15}$$

We now proceed to the formal proof of Proposition 1.

Proof of Proposition 1. Similar to the proof of Theorem EC.1, let $\mathbf{B}(\infty)$ be a random vector that has the stationary distribution of the Markov chain $\mathbf{B}(\cdot)$ under the Max-Weight policy. We are interested in the steady-state expected total backlog $\mathbb{E}\left[\sum_j B_j(\infty)\right]$. Suppose that at time -1, the initial backlog vector has the distribution of $\mathbf{B}(\infty)$. We will focus instead on the in-period backlog vector $\mathbf{B}'(\infty) \triangleq \mathbf{B}(\infty) + \mathbf{D}(0)$, where $\mathbf{D}(0)$ is the random demand vector in period 0, realized after $\mathbf{B}(\infty)$. Now, consider the backlog vector $\mathbf{B}^+(\infty)$, and the in-period backlog vector $\mathbf{B}'^+(\infty)$, both in time period 1. Then, by stationarity, $\mathbf{B}^+(\infty)$ and $\mathbf{B}'(\infty)$ have the same distribution, and we can write

$$\mathbf{B}^+(\infty) = \mathbf{B}'(\infty) - \mathbf{G}(\mathbf{B}'(\infty)) + \mathbf{U}(\mathbf{B}'(\infty)), \quad \text{and} \quad \mathbf{B}'^+(\infty) = \mathbf{B}^+(\infty) + \mathbf{D}(1),$$

where $\mathbf{G}(\mathbf{B}'(\infty))$ is the production allocation under the Max-Weight policy, based on the updated backlog vector $\mathbf{B}'(\infty)$, $\mathbf{U}(\mathbf{B}'(\infty))$ is the vector of unused capacities, and $\mathbf{D}(1)$ the random demand vector in time period 1, realized immediately after $\mathbf{B}^+(\infty)$. With a slight abuse of notation, and to simplify notation, we write $\mathbf{D}$ for $\mathbf{D}(1)$ for the rest of this section. It is useful to note that because $\mathbf{D}$ is the demand vector in time period 1, but not time period 0, $\mathbf{D}$ is independent from $\mathbf{B}'(\infty)$, $\mathbf{B}^+(\infty)$, and $\mathbf{U}(\mathbf{B}'(\infty))$.

Write $\mathbf{G}(\infty) = \mathbf{G}(\mathbf{B}'(\infty))$ and $\mathbf{U}(\infty) = \mathbf{U}(\mathbf{B}'(\infty))$. Then, by stationarity,

$$\mathbb{E}\left[\left(\sum_j B_j'^+(\infty)\right)^2\right] = \mathbb{E}\left[\left(\sum_j B_j'(\infty)\right)^2\right],$$

and

$$\begin{aligned}
0 &= \mathbb{E}\left[\left(\sum_j B_j'^+(\infty)\right)^2 - \left(\sum_j B_j'(\infty)\right)^2\right] \\
&= \mathbb{E}\left[\left(\sum_j B'(\infty) - \sum_j G_j(\infty) + \sum_j U_j(\infty) + \sum_j D_j\right)^2 - \left(\sum_j B_j'(\infty)\right)^2\right] \\
&= 2\mathbb{E}\left[\left(\sum_j D_j - \sum_j G_j(\infty)\right)\left(\sum_j B_j'(\infty)\right)\right] + \mathbb{E}\left[\left(\sum_j D_j - \sum_j G_j(\infty)\right)^2\right] \\
&\quad + \mathbb{E}\left[\left(\sum_j U_j(\infty)\right)^2\right] + 2\mathbb{E}\left[\left(\sum_j B_j'(\infty) - \sum_j G_j(\infty) + \sum_j D_j\right)\left(\sum_j U_j(\infty)\right)\right] \\
&= 2\mathbb{E}\left[\left(\sum_j D_j - \sum_j G_j(\infty)\right)\left(\sum_j B_j'(\infty)\right)\right] + \mathbb{E}\left[\left(\sum_j D_j - \sum_j G_j(\infty)\right)^2\right] \\
&\quad + \mathbb{E}\left[\left(\sum_j U_j(\infty)\right)^2\right] + 2\mathbb{E}\left[\left(\sum_j B_j'^+(\infty) - \sum_j U_j(\infty)\right)\left(\sum_j U_j(\infty)\right)\right] \\
&= 2\mathbb{E}\left[\left(\sum_j D_j - \sum_j G_j(\infty)\right)\left(\sum_j B_j'(\infty)\right)\right] + \mathbb{E}\left[\left(\sum_j D_j - \sum_j G_j(\infty)\right)^2\right]
\end{aligned}$$

$$+2\mathbb{E}\left[\left(\sum_j B_j'^+(\infty)\right)\left(\sum_j U_j(\infty)\right)\right]-\mathbb{E}\left[\left(\sum_j U_j(\infty)\right)^2\right].$$

Write $\tilde{D}=\sum_j D_j$, and note that $\sum_j G_j(\infty)=C$. Then,

$$\begin{aligned}\mathbb{E}\left[\left(\sum_j D_j-\sum_j G_j(\infty)\right)\left(\sum_j B_j'(\infty)\right)\right] &= \mathbb{E}\left[(\tilde{D}-C)\sum_j B_j'(\infty)\right]\\ &= (\Lambda-C)\mathbb{E}\left[\sum_j B_j'(\infty)\right]=-n\zeta\mathbb{E}\left[\sum_j B_j'(\infty)\right],\end{aligned}$$

where the second inequality follows from the independence between $\mathbf{D}$ and $\mathbf{B}'(\infty)$. Therefore,

$$\begin{aligned}2n\zeta\mathbb{E}\left[\sum_j B_j'(\infty)\right] &= \mathbb{E}\left[\left(\tilde{D}-C\right)^2\right]+2\mathbb{E}\left[\left(\sum_j B_j'^+(\infty)\right)\left(\sum_j U_j(\infty)\right)\right]-\mathbb{E}\left[\left(\sum_j U_j(\infty)\right)^2\right]\\ &\le \mathbb{E}\left[\left(\tilde{D}-C\right)^2\right]+2\mathbb{E}\left[\left(\sum_j B_j'^+(\infty)\right)\left(\sum_j U_j(\infty)\right)\right]-\mathbb{E}\left[\left(\sum_j U_j(\infty)\right)\right]^2\\ &\le \mathbb{E}\left[\left(\tilde{D}-C\right)^2\right]+2\mathbb{E}\left[\left(\sum_j B_j'^+(\infty)\right)\left(\sum_j U_j(\infty)\right)\right]-n^2\zeta^2. \end{aligned} \tag{EC.16}$$

We now analyze the first two terms on the right-hand side separately.

(a) $\mathbb{E}\left[\left(\tilde{D}-C\right)^2\right]$. Noting that $\mathbb{E}[\tilde{D}]=\Lambda$ and $Var[\tilde{D}]=\sum_j\sigma_j^2=\Sigma^2$, we have

$$\mathbb{E}\left[\left(\tilde{D}-C\right)^2\right]=\Sigma^2+(C-\Lambda)^2=\Sigma^2+n^2\zeta^2. \tag{EC.17}$$

(b) $\mathbb{E}\left[\left(\sum_j B_j'^+(\infty)\right)\left(\sum_j U_j(\infty)\right)\right]$. First, we have

$$\mathbb{E}\left[\left(\sum_j B_j'^+(\infty)\right)\left(\sum_j U_j(\infty)\right)\right]=\mathbb{E}\left[\left(\sum_j B_j^+(\infty)+\tilde{D}\right)\left(\sum_j U_j(\infty)\right)\right].$$

Second, $\tilde{D}$ and $\mathbf{U}(\infty)$ are independent, so

$$\mathbb{E}\left[\tilde{D}\sum_j U_j(\infty)\right]=\Lambda\mathbb{E}\left[\sum_j U_j(\infty)\right]=\Lambda(C-\Lambda)=\Lambda n\zeta.$$

We now consider the term $\mathbb{E}\left[\left(\sum_j B_j^+(\infty)\right)\left(\sum_j U_j(\infty)\right)\right]$. Let us first note that for each j, $B_j^+(\infty)U_j(\infty)=0$, since if there were any unused capacity for product j (i.e., $U_j(\infty)>0$), there would be no backlog after production (i.e., $B_j^+(\infty)=0$). Thus,

$$\begin{aligned}\frac{1}{n}\mathbb{E}\left[\left(\sum_j B_j^+(\infty)\right)\left(\sum_j U_j(\infty)\right)\right] &= \mathbb{E}\left[\overline{B^+(\infty)}\sum_j U_j(\infty)\right]\\ &= \mathbb{E}\left[\sum_j \overline{B^+(\infty)}U_j(\infty)-\sum_j B_j^+(\infty)U_j(\infty)\right]\\ &= \mathbb{E}\left[\sum_j (\overline{B^+(\infty)}-B_j^+(\infty))U_j(\infty)\right]\\ &= \mathbb{E}\left[\sum_j \big(-\Delta B_j^+(\infty)\big)U_j(\infty)\right]\\ &\le \sqrt{\mathbb{E}\left[\|\Delta\mathbf{B}^+(\infty)\|_2^2\right]}\cdot\sqrt{\mathbb{E}\left[\|\mathbf{U}(\infty)\|_2^2\right]}.\end{aligned}$$

Now each individual $U_j(\infty)$ cannot exceed the total capacity, so $\|\mathbf{U}(\infty)\|_2^2=\sum_j U_j^2(\infty)\le C\sum_j U_j(\infty)$. Furthermore, by stationarity,

$$\mathbb{E}\left[\sum_j U_j(\infty)\right]=C-\mathbb{E}[\tilde{D}]=C-\Lambda=n\zeta.$$

Thus,

$$\mathbb{E}\left[\|\mathbf{U}(\infty)\|_2^2\right]\le\mathbb{E}\left[C\sum_j U_j(\infty)\right]=Cn\zeta.$$

By Corollary EC.2,

$$\sqrt{\mathbb{E}\left[\|\Delta\mathbf{B}^+(\infty)\|_2^2\right]}\le\frac{\sqrt{n}\sum_j\lambda_j^2+14n^{3/2}C^2+12n^{3/2}D_{\max}^2}{\eta}+\sqrt{n}(D_{\max}+C).$$

Thus,

$$\begin{aligned}\frac{1}{n}\mathbb{E}\left[\left(\sum_j B_j^+(\infty)\right)\left(\sum_j U_j(\infty)\right)\right] &\le \sqrt{\mathbb{E}\left[\|\Delta\mathbf{B}^+(\infty)\|_2^2\right]}\cdot\sqrt{\mathbb{E}\left[\|\mathbf{U}(\infty)\|_2^2\right]}\\ &\le\left(\frac{\sqrt{n}\sum_j\lambda_j^2+14n^{3/2}C^2+12n^{3/2}D_{\max}^2}{\eta}+\sqrt{n}(D_{\max}+C)\right)\cdot\sqrt{Cn\zeta}\\ &=\sqrt{C\zeta}\cdot\left(\frac{n\sum_j\lambda_j^2+14n^2C^2+12n^2D_{\max}^2}{\eta}+n(D_{\max}+C)\right),\end{aligned}$$

and

$$\mathbb{E}\left[\left(\sum_j B_j^+(\infty)\right)\left(\sum_j U_j(\infty)\right)\right]\le\sqrt{C\zeta}\cdot\left(\frac{n^2\sum_j\lambda_j^2+14n^3C^2+12n^3D_{\max}^2}{\eta}+n^2(D_{\max}+C)\right). \tag{EC.18}$$

To complete the proof of Proposition 1, we plug the bounds in (EC.17) and (EC.18) into (EC.16), and get

$$\mathbb{E}\left[\sum_j B'_j(\infty)\right] \leq \frac{\Sigma^2}{2n\zeta} + \frac{K'_1 + \eta K'_2}{\eta\sqrt{\zeta}}, \tag{EC.19}$$

where

$$K'_1 = \sqrt{C}(n\sum_j \lambda_j^2 + 14n^2C^2 + 12n^2D_{\max}^2), K'_2 = \sqrt{C}n(D_{\max} + C).$$

Note that by Assumption 1, K'_1 (and K'_2) can be upper-bounded by some constants $K_1(l,u)$ (and $K_2(l,u)$) respectively, and this concludes the proof. □

EC.2.2. Proofs of Proposition 2, Corollary 1 and Corollary 2

Proof of Proposition 2. Consider a discrete-time make-to-order system with a single plant of capacity C that produces only one type of product. In each time period t, demand $\sum_{j=1}^n D_j(t)$ arrives, where $D_j(t)$ is the amount of demand for product j in the original system. Let $\tilde{D}(t) = \sum_{j=1}^n D_j(t)$. Then, $\mathbb{E}[\tilde{D}(t)] = \Lambda < C$ and $Var[\tilde{D}(t)] = \sum_{j=1}^n \sigma_j^2 = \Sigma^2$. Let $\tilde{B}(t)$ be the backlog at the end of period t, then for all t,

$$\tilde{B}(t) = (\tilde{B}(t-1) + \tilde{D}(t) - C)^+.$$

Alternatively, we can write

$$\tilde{B}(t) = \tilde{B}(t-1) + \tilde{D}(t) - C + \tilde{U}(t), \tag{EC.20}$$

where $\tilde{U}(t)$ is the unused capacity in period t. For now let us note that $\tilde{U}(t)\tilde{B}(t) = 0$, since if there is positive unused capacity, i.e., $\tilde{U}(t) > 0$, then the backlog in the next time period must have been cleared, i.e., $\tilde{B}(t) = 0$. Consequently, $\tilde{U}(t)(\tilde{B}(t-1) + \tilde{D}(t) - C + \tilde{U}(t)) = 0$, or $\tilde{U}(t)(\tilde{B}(t-1) + \tilde{D}(t) - C) = -\tilde{U}^2(t)$ for all t.

Suppose that both the original and the single-plant system start empty. Let π be a greedy production policy that produces as much as possible each time. Then, using the recursions (10) and (EC.20), it can be shown that for each t, $\sum_{j=1}^n B_j^\pi(t) = \tilde{B}(t)$. Therefore,

$$\Gamma(\pi) = \limsup_{T\to\infty} \frac{1}{T}\sum_{t=1}^T \sum_{j=1}^n B_j^\pi(t) = \limsup_{T\to\infty} \frac{1}{T}\sum_{t=1}^T \tilde{B}(t).$$

We next bound $\limsup_{T\to\infty} \frac{1}{T}\sum_{t=1}^T \tilde{B}(t)$. First, note that $\Lambda < C$, so $\limsup_{T\to\infty} \frac{1}{T}\sum_{t=1}^T \tilde{B}(t)$ is finite. Furthermore, there exists a unique stationary distribution for the process $\tilde{B}(\cdot)$, and if we let $\tilde{B}(\infty)$ be a random variable with this stationary distribution, then all moments of $\tilde{B}(\infty)$ are finite, and

$$\limsup_{T\to\infty} \frac{1}{T}\sum_{t=1}^T \tilde{B}(t) = \mathbb{E}[\tilde{B}(\infty)].$$

We now show that $\frac{\Sigma^2}{2n\zeta}+\frac{C-\Lambda}{2}\geq \mathbb{E}[\tilde{B}(\infty)]\geq \frac{\Sigma^2+n^2\zeta^2}{2n\zeta}-\frac{1}{2}C$. To this end, consider the conditional drift term $\mathbb{E}[\tilde{B}^2(t+1)-\tilde{B}^2(t)\mid \tilde{B}(t)]$. We have

$$\begin{aligned}\mathbb{E}[\tilde{B}^2(t+1)-\tilde{B}^2(t)\mid \tilde{B}(t)] &= \mathbb{E}[(\tilde{B}(t)+\tilde{D}(t+1)-C+\tilde{U}(t+1))^2-\tilde{B}^2(t)\mid \tilde{B}(t)]\\ &= \mathbb{E}[(\tilde{D}(t+1)-C)^2\mid \tilde{B}(t)]+2\mathbb{E}[\tilde{D}(t+1)-C\mid \tilde{B}(t)]\tilde{B}(t)\\ &\quad -\mathbb{E}[\tilde{U}^2(t+1)\mid \tilde{B}(t)]\\ &= \Sigma^2+(C-\Lambda)^2-2(C-\Lambda)\tilde{B}(t)-\mathbb{E}[\tilde{U}^2(t+1)\mid \tilde{B}(t)].\end{aligned}$$

Now take expectation on both sides, over the stationary distribution of $\tilde{B}(\cdot)$. Then, by stationarity, the left-hand side is zero, and

$$0=\Sigma^2+n^2\zeta^2-2n\zeta\mathbb{E}[\tilde{B}(\infty)]-\mathbb{E}[\tilde{U}^2(\infty)]\leq \Sigma^2+n^2\zeta^2-2n\zeta\mathbb{E}[\tilde{B}(\infty)].$$

Thus, we have $\frac{\Sigma^2}{2n\zeta}+\frac{C-\Lambda}{2}\geq \mathbb{E}[\tilde{B}(\infty)]$. Next, using $\tilde{B}(t)=\tilde{B}(t-1)+\tilde{D}(t)-C+\tilde{U}(t)$, we know that $\mathbb{E}[\tilde{U}(\infty)]=C-\Lambda=n\zeta$. Furthermore, unused capacity can never exceed total capacity, so $\tilde{U}(\infty)\leq C$. Thus, $\mathbb{E}[\tilde{U}^2(\infty)]\leq \mathbb{E}[C\tilde{U}(\infty)]=Cn\zeta$. This implies that

$$\begin{aligned}\mathbb{E}[\tilde{B}(\infty)] &= \frac{1}{2n\zeta}\left(\Sigma^2+n^2\zeta^2-\mathbb{E}[\tilde{U}^2(\infty)]\right)\\ &\geq \frac{\Sigma^2}{2n\zeta}-\frac{C-n\zeta}{2}.\end{aligned}$$

This completes the proof of Proposition 2. □

Proof of Corollary 1. Let $\mathcal{A}$ be a flexibility structure, and let π be a production policy that respects the structure $\mathcal{A}$. It is easy to see that $BL(\mathcal{A})\geq BL(\mathcal{F})\geq \frac{\Sigma^2}{2n\zeta}-\frac{C-n\zeta}{2}$. This proves the corollary. □

Proof of Corollary 2. Define $c=c_1$. By Remark 2, η of $\mathcal{LC}$ is equal to c. Combining this with Proposition 1 and Corollary 1, we have

$$\begin{aligned}\frac{BL(\mathcal{LC})}{BL(\mathcal{F})} &\leq \Big(\frac{\Sigma^2}{2n\zeta}-\frac{C-n\zeta}{2}\Big)^{-1}\cdot\Big(\frac{\Sigma^2}{2n\zeta}+\frac{K_1+cK_2}{c\sqrt{\zeta}}\Big)\\ &=1+\Big(\frac{C-n\zeta}{2}+\frac{K_1+cK_2}{c\sqrt{\zeta}}\Big)\cdot\Big(\frac{\Sigma^2}{2n\zeta}-\frac{C-n\zeta}{2}\Big)^{-1}.\end{aligned}$$

Now, pick a small $K^*>0$ such that for all $\zeta\leq K^*$, $\frac{\Sigma^2}{2n\zeta}-\frac{C-n\zeta}{2}\geq\frac{\Sigma^2}{4n\zeta}$. We have then for all $\zeta\leq K^*$,

$$\begin{aligned}\frac{BL(\mathcal{LC})}{BL(\mathcal{F})} &\leq 1+\Big(\frac{C-n\zeta}{2}+\frac{K_1+cK_2}{c\sqrt{\zeta}}\Big)\cdot\Big(\frac{\Sigma^2}{2n\zeta}-\frac{C-n\zeta}{2}\Big)^{-1}\\ &\leq 1+\Big(\frac{C-n\zeta}{2}+\frac{K_1+cK_2}{c\sqrt{\zeta}}\Big)\cdot\Big(\frac{4n\zeta}{\Sigma^2}\Big)\end{aligned}$$

$$\leq 1+\Big(\frac{4n\sqrt{\zeta}}{\Sigma^2}\Big)\Big(\frac{\sqrt{\zeta}C}{2}+\frac{K_1+cK_2}{c}\Big),$$

where K_1 and K_2 are values specified in Proposition 1. Now, by Assumption 1, ζ, C, K_1 and K_2 are upper-bounded by a positive constant, while c, and Σ^2 are lower-bounded by a positive constant. Therefore, we have that there exists $K=K(l,u)>0$ such that

$$\frac{BL(\mathcal{LC})}{BL(\mathcal{F})}\leq 1+K\sqrt{\zeta}. \quad \square$$

EC.3. Relationship between Max-Weight and Max-Flow Policies

In this section, we discuss the relationship between the Max-Weight and Max-Flow policies. First, we formally introduce the definition of Max-Flow policies.

EC.3.1. Max-Flow Policies

In a nutshell, Max-Flow policies consist of all production policies that solve for a production schedule to *greedily* minimize the total backlog at the end of each time period. More precisely, under a Max-Flow policy, in each time period t, the production output $\mathbf{g}(t)$ at time t is an optimal solution of the optimization problem Opt-M defined below:

$$\min_{\mathbf{g}(t)\in R(\mathcal{A})}\sum_{j=1}^{n} b_j(t),\ \text{where } \mathbf{b}(t)=\big(\mathbf{b}(t-1)+\mathbf{d}(t)-\mathbf{g}(t)\big)^{+}. \tag{Opt-M}$$

Problem Opt-M is the same optimal policy of the one-period MTO model studied by Jordan and Graves (1995), with product demand $\mathbf{b}(t-1)+\mathbf{d}(t)$. As suggested in Jordan and Graves (1995), Opt-M can be solved as a max-flow problem (hence the name Max-Flow policy). To see that this is the case, recall the definition of constraints (6)–(8) of the production polytope $R(\mathcal{A})$, and we obtain the following equivalent optimization problem:

$$\begin{aligned}
&\min \sum_{j=1}^{n} b_j(t) && \text{(Flow-M)}\\
\text{s.t.}\quad &\sum_{i=1}^{m} f_{i,j}+b_j(t)=b_j(t-1)+d_j(t),\ \forall 1\leq j\leq n,\\
&\sum_{j=1}^{n} f_{i,j}\leq c_i,\ \forall 1\leq i\leq m,\\
&f_{i,j}=0,\forall(\mathcal{S}_i,\mathcal{T}_j)\notin\mathcal{A},\\
&\mathbf{b}(t)\in\mathbb{R}^n_+,\mathbf{f}\in\mathbb{R}^{mn}_+.
\end{aligned}$$

Note that if we write variables $b_j(t)$ in the objective of Flow-M as $b_j(t)=b_j(t-1)+d_j(t)-\sum_{i=1}^{m} f_{i,j}$, it is easy to see that Flow-M is equivalent to an optimization problem of maximizing the objective $\sum_{j=1}^{n}\sum_{i=1}^{m} f_{i,j}$ under the appropriate constraints. This optimization problem is equivalent to a bipartite max-flow problem with graph $\mathcal{A}$, supply nodes $\{\mathcal{S}_1,...,\mathcal{S}_m\}$, and demand nodes $\{\mathcal{T}_1,...,\mathcal{T}_n\}$, where supply node $\mathcal{S}_i$ supplies up to c_i units of flow and demand node $\mathcal{T}_j$ receives up to $b_j(t-1)+d_j(t)$ units of flow. If $f^*_{i,j}$ is a max-flow solution, then for each $j\in\{1,2,\ldots,n\}$, by letting $b^*_j(t)=b_j(t-1)+d_j(t)-\sum_{i=1}^{m} f^*_{i,j}$, we have that $(\mathbf{b}^*(t),\mathbf{f}^*)$ is an optimal solution for Flow-M. An example of a bipartite max-flow problem equivalent to Flow-M is illustrated in Figure EC.1.

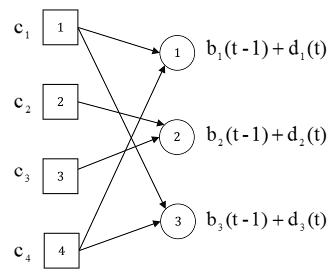

Figure EC.1 Max-flow Diagram for Flow-M with 4 plants and 3 products.

EC.3.2. Max-Weight Policies

Recall that a policy is a Max-Weight policy if it solves the optimization problem defined in Opt-MW at every time period t. Similar to the optimization problem Opt-M, we can expand $R(\mathcal{A})$ in Opt-MW to obtain a (weighted) max-flow formulation. In particular, Opt-MW is equivalent to:

$$\begin{aligned}
&\max \sum_{i=1}^{m}\sum_{j=1}^{n} f_{i,j}(b_j(t-1)+d_j(t)) && \text{(Flow-MW)}\\
\text{s.t.}\quad &\sum_{i=1}^{m} f_{i,j} \le b_j(t-1)+d_j(t),\\
&\sum_{j=1}^{n} f_{i,j} \le c_i,\ \forall 1\le i\le m\\
&f_{i,j}=0, \forall(\mathcal{S}_i,\mathcal{T}_j)\notin\mathcal{A},\ \mathbf{f}\in\mathbb{R}^{mn}_{+}.
\end{aligned}$$

EC.3.3. Generalized Max-Flow Policies

Next, we present a proposition to show that a very general class of network flow optimization problems will lead to a Max-Flow policy. The proposition proves that the class of Max-Weight policies is contained in the class of Max-Flow policies as a special case.

Proposition EC.3. *Consider the optimization problem*

$$\max \sum_{j=1}^{n} \Theta_j\left(\sum_{i=1}^{m} f_{i,j}\right) \qquad \text{(Flow-Monotone)}$$

$$
\begin{aligned}
s.t. \quad & \sum_{i=1}^{m} f_{i,j} \leq b_j(t-1) + d_j(t), \\
& \sum_{j=1}^{n} f_{i,j} \leq c_i, \forall 1 \leq i \leq m \\
& f_{i,j} = 0, \forall (\mathcal{S}_i, \mathcal{T}_j) \notin \mathcal{A} \\
& \mathbf{f} \in \mathbb{R}_+^{mn}.
\end{aligned}
$$

Suppose the Θ_j is a strictly increasing function for each $j = 1, \ldots, n$. Then, any optimal solution of Flow-Monotone is also an optimal solution of Flow-M.

Proof of Proposition EC.3. Let $\mathbf{f}^*$ be an optimal solution of Flow-Monotone. Suppose that there exists some augmenting path $P = (\mathcal{S}_{i_1}, \mathcal{T}_{i_2}, \mathcal{S}_{i_3}, \ldots \mathcal{S}_{i_{2k-1}}, \mathcal{T}_{i_{2k}})$ of $\mathbf{f}^*$, i.e., there exists $\epsilon > 0$ for which $\mathbf{f}^* + \epsilon(\sum_{l=1}^{k} \mathbf{e}^{i_{2l-1}, i_{2l}} - \sum_{l=1}^{k-1} \mathbf{e}^{i_{2l}, i_{2l+1}})$ is feasible, where for any pair $(i,j) \in \{(i_{2l-1}, i_{2l}) | l = 1, \ldots k\}$, $(i,j) \in \{(i_{2l}, i_{2l+1}) | l = 1, \ldots k-1\}$, $e_{i,j}^{i,j} = 1$ and $e_{i',j'}^{i,j} = 0, \forall (i', j') \neq (i,j)$.

Let $\mathbf{g} = \epsilon(\sum_{l=1}^{k} \mathbf{e}^{i_{2l-1}, i_{2l}} - \sum_{l=1}^{k-1} \mathbf{e}^{i_{2l}, i_{2l+1}})$. By definition, $\mathbf{f}^* + \mathbf{g}$ is feasible. Moreover, note that by the construction of $\mathbf{g}$,

$$
\begin{aligned}
& \sum_{i=1}^{m} f_{i,j} + g_{i,j} = \sum_{i=1}^{m} f_{i,j}, \forall j \neq i_2, \ldots, i_{2k} \\
& \sum_{i=1}^{m} f_{i,j} + g_{i,j} = \sum_{i=1}^{m} f_{i,j} + \epsilon - \epsilon = \sum_{i=1}^{m} f_{i,j}, \forall j \in \{i_2, \ldots, i_{2k-2}\} \\
& \sum_{i=1}^{m} f_{i,j} + g_{i,j} = \sum_{i=1}^{m} f_{i,j} + \epsilon, \text{ if } j = i_{2k}.
\end{aligned}
$$

Because Θ_j is a strictly increasing function for each $j = 1, \ldots, n$, we must have that

$$
\sum_{j=1}^{n} \Theta_j(\sum_{i=1}^{m} (f_{i,j} + g_{i,j})) > \sum_{j=1}^{n} \Theta_j(\sum_{i=1}^{m} (f_{i,j})),
$$

which contradicts the fact that $\mathbf{f}^*$ is optimal.

Therefore, there cannot exist any augmenting path of $\mathbf{f}^*$, that starts at plant node $\mathcal{S}_{i_1}$ and ends at product node $\mathcal{T}_{i_{2k}}$. Note that any augmenting path must start at a plant (supply) node and end at a product (demand) node and by the classical Ford-Fulkerson algorithm, $\mathbf{f}^*$ is an optimal solution of Flow-M. $\square$

To see why Flow-MW returns a max-flow solution, consider the optimization problem Flow-MW', which has the same constraints as Flow-MW, and objective function $\sum_{i=1}^{m} \sum_{j=1}^{n} f_{i,j} w_j$ where

$$
\begin{aligned}
& w_j = b_j(t-1) + d_j(t), && \text{if } b_j(t-1) + d_j(t) > 0 \\
\text{and} \quad & w_j = 1, && \text{if } b_j(t-1) + d_j(t) = 0.
\end{aligned}
$$

Then, Flow-MW' is equivalent to Flow-MW, because if $b_j(t-1)+d_j(t)=0$, we have that $\sum_{j=1}^n f_{i,j} \le b_j(t-1)+d_j(t)=0$.

Because Flow-MW' is a special instance of Flow-Monotone with $\Theta_j\left(\sum_{i=1}^m f_{i,j}\right)=w_j\sum_{i=1}^m f_{i,j}$, by Proposition EC.3, we immediately have that any optimal solution of Flow-MW' (and Flow-MW) is an optimal solution of Flow-M.

Next, we further analyze the optimization problems in the class of Flow-Monotone, with linear objectives. In particular, we present a result which states that if the objective function is in the form of $\sum_{i=1}^m\sum_{j=1}^n f_{i,j}w_j$, then the optimal solution is determined only by the ordering of the $w_1,\dots w_n$, and is independent of their absolute differences.

Proposition EC.4. *Consider the optimization problem*

$$\begin{aligned}
\max \quad & \sum_{j=1}^n\sum_{i=1}^m w_j\cdot f_{i,j} && (\text{Flow}(\boldsymbol{w}))\\
s.t. \quad & \sum_{i=1}^m f_{i,j}\le b_j(t-1)+d_j(t),\\
& \sum_{j=1}^n f_{i,j}\le c_i,\ \forall 1\le i\le m\\
& f_{i,j}=0,\forall(\mathcal{S}_i,\mathcal{T}_j)\notin\mathcal{A}\\
& \mathbf{f}\in\mathbb{R}_+^{mn},
\end{aligned}$$

where w_j is the linear weight for all of the flows that enter demand node $\mathcal{T}_j$. Let $\boldsymbol{w}^1$, $\boldsymbol{w}^2\in\mathbb{R}_+^n$ be two strictly positive vectors where the entries have the same order, i.e.,

$$w_i^1\le w_j^1 \text{ if and only if } w_i^2\le w_j^2,\ \forall 1\le i,j\le n.$$

Then, the set of optimal solutions for Flow($\boldsymbol{w}^1$) coincides the optimal solutions of Flow($\boldsymbol{w}^2$).

Proof of Proposition EC.4. Let $\mathbf{f}^1$ be an optimal solution of Flow($\boldsymbol{w}^1$). Suppose that $\mathbf{f}^1$ is not optimal for Flow($\boldsymbol{w}^2$), then there must exist some vector $\mathbf{g}$ such that $\mathbf{f}^1+\mathbf{g}$ is feasible for Flow($\boldsymbol{w}^2$), and

$$\sum_{j=1}^n\sum_{i=1}^m w_j^2\cdot g_{i,j}>0.$$

By the Flow Decomposition Theorem (see Theorem 3.5 in Ahuja et al. (1993) for details), we can always decompose $\mathbf{g}$ into a flow on path and cycles.

Suppose that $\mathbf{g}^C$ is a *cycle flow* on some cycle C with flow value ϵ. Note that for any product node $\mathcal{T}_j$ in C, we must have exactly two plant nodes, say $\mathcal{S}_{i_1}$ and $\mathcal{S}_{i_2}$, such that $(\mathcal{S}_{i_1},\mathcal{T}_j,\mathcal{S}_{i_2})$ is directed path in C. This implies that $\epsilon = g_{i_1,j} = -g_{i_2,j}$, which in turn implies that $w_j^2 \cdot \sum_{i=1}^m \cdot g_{i,j} = 0$. Therefore, for any $\mathbf{g}^C$, we must have that

$$\sum_{j=1}^n \sum_{i=1}^m w_j^2 \cdot g_{i,j}^C = 0.$$

Therefore, we must have some vector $\mathbf{g}^P$ that is a *path flow* on some path P such that

$$\sum_{j=1}^n \sum_{i=1}^m w_j^2 \cdot g_{i,j}^P > 0. \tag{EC.21}$$

Suppose that $\mathbf{g}^P$ is one such path flow with flow value ϵ. Note that for any product node $\mathcal{T}_j$ in P, there exists integers i_1, i_2 such that we either have $(\mathcal{T}_j, \mathcal{S}_{i_2})$ to be the first arc in path P, or $(\mathcal{S}_{i_1}, \mathcal{T}_j)$ to be the last arc in path P, or $(\mathcal{S}_{i_1}, \mathcal{T}_j, \mathcal{S}_{i_2})$ to be a directed path in P.

If $(\mathcal{T}_j, \mathcal{S}_{i_2})$ is the first arc in P, then

$$\sum_{i=1}^m w_j^2 \cdot g_{i,j}^P = w_j^2 g_{i_2,j}^P = -w_j^2 \epsilon.$$

If $(\mathcal{S}_{i_1}, \mathcal{T}_j)$ is the last arc in P, then

$$\sum_{i=1}^m w_j^2 \cdot g_{i,j}^P = w_j^2 g_{i_2,j}^P = w_j^2 \epsilon.$$

And finally, if $(\mathcal{S}_{i_1}, \mathcal{T}_j, \mathcal{S}_{i_2})$ is a directed path in P, then

$$\sum_{i=1}^m w_j^2 \cdot g_{i,j}^P = 0.$$

By (EC.21), and the equations above, we must have some product node $\mathcal{T}_{j_2}$ such that it is the last node in P. If the first node in P is a plant node, note that $\mathbf{f}^1 + \mathbf{g}^P$ is feasible and by equations above, we have that

$$\sum_{j=1}^n \sum_{i=1}^m w_j^1 \cdot g_{i,j}^P = w_{j_2}^1 \epsilon > 0,$$

which is a contradiction to the optimality of $\mathbf{f}^1$ of Flow($\boldsymbol{w}^1$).

Thus, the first node in P must be a product node. Let the product node be $\mathcal{T}_{j_1}$, then we have that

$$0 < \sum_{j=1}^n \sum_{i=1}^m w_j^2 \cdot g_{i,j}^P = (w_{j_2}^2 - w_{j_1}^2)\epsilon \implies w_{j_2}^2 > w_{j_1}^2.$$

But this implies that $w^1_{j_2} > w^1_{j_1}$. And because

$$\sum_{j=1}^{n}\sum_{i=1}^{m} w^1_j \cdot g^P_{i,j} = (w^1_{j_2} - w^1_{j_1})\epsilon,$$

we have that $\mathbf{f}^1 + \mathbf{g}^P$ is a strictly better feasible solution for Flow($\boldsymbol{w}^1$), which results in a contradiction. $\square$

Since Flow($\boldsymbol{w}$) belongs to the class of Flow-Monotone, it is a max-flow solution by Proposition EC.3. Therefore, Proposition EC.4 suggests that any optimization of the form Flow($\boldsymbol{w}$) is essentially an optimal solution of Flow-M that prioritizes products with higher linear weights.

Another interesting implication of Proposition EC.4 concerns the generality of the Max-Weight policy. Suppose that $f(\cdot)$ is a strictly increasing function, and the factors $b_j(t-1) + d_j(t)$ in the objective of Flow-MW are replaced by $f(b_j(t-1) + d_j(t))$. By Proposition EC.4, this new optimization problem has the same set of optimal solutions as Flow-MW. In particular, this implies that in our model, the well-studied Max-Weight-α policies (where $f(x) = x^\alpha$, $\alpha > 0$, see Keslassy and McKeown (2001), Shah and Wischik (2012) for more details) all coincide with the Max-Weight policy, a fact that is not necessarily true in other queueing models.

EC.4. Proofs in §5

Proof of Lemma 4. We prove the lemma using backward induction. $t = T + 1$ is trivially true. Suppose the statement is true for $t = K$. For any b_1, b_2 where $b_1 \geq 1$, let $p^K_1(\mathbf{b},\mathbf{d})$, $p^K_2(\mathbf{b},\mathbf{d})$ be the optimal production of product 1 and 2 at time t given that $\mathbf{b}$ and $\mathbf{d}$ are the backlog and demand at time $t = K$, respectively. Note that by induction hypothesis, we must have

$$p^K_1(\mathbf{b},\mathbf{d}) = \min\{c_1, b_1 + d_1\} \tag{EC.22}$$

$$p^K_2(\mathbf{b},\mathbf{d}) = \min\{c_2 + (c_1 - b_1 - d_1)^+, b_2 + d_2\}. \tag{EC.23}$$

By Equations (EC.22) and (EC.23), we can define $b^{K+1}_1(\mathbf{b},\mathbf{d})$, $b^{K+1}_2(\mathbf{b},\mathbf{d})$, the backlogs in time period $K + 1$, as follows.

$$b^{K+1}_1(\mathbf{b},\mathbf{d}) = \max\{b_1 + d_1 - c_1, 0\} \tag{EC.24}$$

$$b^{K+1}_2(\mathbf{b},\mathbf{d}) = \max\{b_2 + d_2 - c_2 - (c_1 - b_1 - d_1)^+, 0\}. \tag{EC.25}$$

Let $\mathbf{b}' = [b_1 - 1, b_2 + 1]$. First, it is simple to check that for any $\mathbf{d}$, if $b^{K+1}_1(\mathbf{b},\mathbf{d}) \geq 1$, then we must have that

$$b^{K+1}_1(\mathbf{b},\mathbf{d}) = b^{K+1}_1(\mathbf{b}',\mathbf{d}) - 1,$$

$$\text{and} \qquad b_2^{K+1}(\mathbf{b},\mathbf{d}) \geq b_2^{K+1}(\mathbf{b}',\mathbf{d}) + 1.$$

Therefore, if $b_1^{K+1}(\mathbf{b},\mathbf{d}) \geq 1$, we must have that for any $\mathbf{d}$,

$$b_1^{K+1}(\mathbf{b},\mathbf{d}) + b_2^{K+1}(\mathbf{b},\mathbf{d}) \geq b_1^{K+1}(\mathbf{b}',\mathbf{d}) + b_2^{K+1}(\mathbf{b}',\mathbf{d}),$$
$$\text{and} \qquad J^{K+1}(b_1^{K+1}(\mathbf{b},\mathbf{d}), b_2^{K+1}(\mathbf{b},\mathbf{d})) \geq J^{K+1}(b_1^{K+1}(\mathbf{b}',\mathbf{d}), b_2^{K+1}(\mathbf{b}',\mathbf{d})).$$

This implies that

$$J^K(b_1,b_2) \geq J^K(b_1-1,b_2+1).$$

Next, if $b_1^{K+1}(\mathbf{b},\mathbf{d}) = 0$, and $b_2^{K+1}(\mathbf{b},\mathbf{d}) \geq 1$, then we must have that

$$b_1^{K+1}(\mathbf{b},\mathbf{d}) = b_1^{K+1}(\mathbf{b}',\mathbf{d}) = 0,$$
$$\text{and} \qquad b_2^{K+1}(\mathbf{b},\mathbf{d}) = b_2^{K+1}(\mathbf{b}',\mathbf{d}),$$

which implies that

$$J^K(b_1,b_2) = J^K(b_1-1,b_2+1).$$

Therefore, in either case, we must have that

$$J^K(b_1,b_2) \geq J^K(b_1-1,b_2+1),$$

and the proof is done by induction. □

Proof of Corollary 4. Applying Proposition 6, we have

$$\begin{aligned}
\frac{BL(\mathcal{A})}{BL(\mathcal{F})} &\geq \left(\frac{\Sigma_\Omega^2}{2(\eta+|\Omega|\zeta)} + \frac{\Sigma_{\Omega^c}^2}{2n\zeta} - \frac{\Lambda + \sum_{\mathcal{T}_j\in\Omega}\lambda_j}{2}\right)\cdot\left(\frac{\Sigma^2}{2n\zeta} + \frac{C-\Lambda}{2}\right)^{-1} \\
&= 1 + \left(\frac{\Sigma_\Omega^2}{2(\eta+|\Omega|\zeta)} - \frac{\Sigma_\Omega^2}{2n\zeta} - \frac{\Lambda + \sum_{\mathcal{T}_j\in\Omega}\lambda_j}{2} - \frac{C-\Lambda}{2}\right)\cdot\left(\frac{\Sigma^2}{2n\zeta} + \frac{C-\Lambda}{2}\right)^{-1} \\
&\geq 1 + \left(\frac{\Sigma_\Omega^2}{2(\eta+|\Omega|\zeta)} - \frac{\Sigma_\Omega^2}{2n\zeta} - C\right)\cdot\left(\frac{\Sigma^2}{2n\zeta} + \frac{n\zeta}{2}\right)^{-1} \\
&\geq 1 + \left(\frac{\Sigma_\Omega^2}{2(n-\alpha)\zeta} - \frac{\Sigma_\Omega^2}{2n\zeta} - C\right)\cdot\left(\frac{\Sigma^2}{2n\zeta} + \frac{n\zeta}{2}\right)^{-1} \\
&\geq 1 + \left(\frac{\Sigma_\Omega^2}{2(n-\alpha)\zeta} - \frac{\Sigma_\Omega^2}{2n\zeta} - C\right)\cdot\frac{n\zeta}{\Sigma^2} \\
&= 1 + \left(\frac{\alpha\Sigma_\Omega^2}{2n(n-\alpha)\zeta} - C\right)\cdot\frac{n\zeta}{\Sigma^2} \\
&\geq 1 + \frac{\alpha\Sigma_\Omega^2}{4(n-\alpha)\Sigma^2} \\
&\geq 1 + \frac{\alpha l}{4n(n-\alpha)u^2}.
\end{aligned}$$

The third inequality above follows from the facts that $\eta < (1-\alpha)\zeta$, $|\Omega| \leq n-1$; the fourth inequality follows from $\zeta \leq \frac{l}{n} \leq \frac{\sqrt{\Sigma^2}}{n}$ thus implying that $\frac{n\zeta}{2} \leq \frac{\Sigma^2}{2n\zeta}$; and the fifth inequality above follows from the fact that $\zeta \leq \frac{\alpha l^2}{4mn(n-\alpha)u} \leq \frac{\alpha \Sigma_\Omega^2}{4Cn(n-\alpha)}$. $\square$

EC.5. Numerical Results for §6

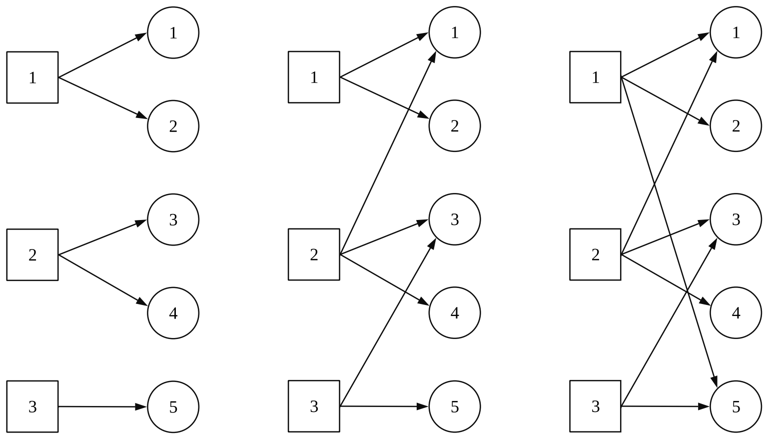

Figure EC.2 Structures for §6.1.2: the 3 by 5 systems from left to right: Dedicated, $\mathcal{C}^-$, $\mathcal{C}$.

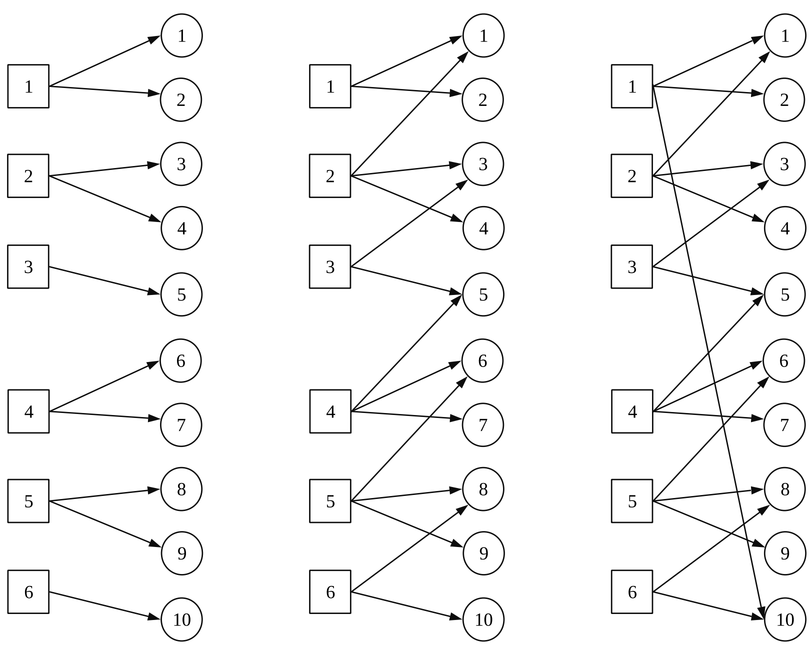

Figure EC.3 Structures for §6.1.2: the 6 by 10 systems from left to right: Dedicated, $\mathcal{C}^-$, $\mathcal{C}$.

ρ	cv	$R(\mathcal{LC})$	$\Delta(\mathcal{LC})$	$B(\mathcal{LC})$	SE%	$R(\mathcal{LC})$	$\Delta(\mathcal{LC})$	$B(\mathcal{LC})$	SE%
		$n=5$				$n=10$			
0.8	0.3	100.0%	100.0%	0.7	1.2%	112.5%	100.0%	0.1	3.0%
0.8	0.4	100.9%	99.9%	3.1	0.8%	145.2%	99.6%	1.3	1.3%
0.8	0.5	103.2%	99.8%	7.2	0.7%	182.0%	98.8%	5.1	0.9%
0.9	0.3	100.3%	100.0%	12.0	0.6%	109.1%	99.7%	7.6	0.7%
0.9	0.4	101.7%	99.8%	28.6	0.6%	131.6%	98.7%	25.1	0.6%
0.9	0.5	104.5%	99.4%	50.5	0.7%	159.7%	97.1%	57.3	0.6%
0.95	0.3	100.4%	100.0%	50.0	0.7%	108.1%	99.6%	42.9	0.6%
0.95	0.4	101.6%	99.8%	99.6	0.8%	124.9%	98.5%	104.5	0.7%
0.95	0.5	103.4%	99.5%	157.5	0.9%	142.8%	97.0%	196.6	0.8%
0.975	0.3	100.3%	100.0%	137.2	0.9%	106.5%	99.6%	131.4	0.8%
0.975	0.4	101.1%	99.8%	254.4	1.0%	117.8%	98.5%	272.9	0.9%
0.975	0.5	102.1%	99.6%	380.7	1.0%	128.1%	97.0%	462.7	0.9%
0.9875	0.3	100.2%	100.0%	298.1	1.0%	104.5%	99.5%	311.6	1.0%
0.9875	0.4	100.7%	99.8%	496.1	0.9%	111.2%	98.3%	590.3	1.0%
0.9875	0.5	101.3%	99.5%	705.6	0.9%	117.0%	96.9%	907.6	1.0%

Table EC.1 Performance of the long chain with $n=5$ and 10

ρ	cv	$R(\mathcal{LC})$	$\Delta(\mathcal{LC})$	$B(\mathcal{LC})$	SE%	$R(\mathcal{LC})$	$\Delta(\mathcal{LC}))$	$B(\mathcal{LC})$	SE%
		$n=15$				$n=20$			
0.8	0.3	335.6%	100.0%	0.0	8.0%	1570.3%	99.9%	0.1	18.7%
0.8	0.4	453.7%	99.4%	1.3	2.3%	1678.0%	99.3%	1.6	4.0%
0.8	0.5	522.5%	98.2%	6.5	1.4%	1500.4%	98.0%	8.5	2.0%
0.9	0.3	152.9%	99.4%	6.5	0.8%	260.9%	99.1%	7.3	0.9%
0.9	0.4	222.5%	97.6%	30.7	0.7%	378.4%	96.9%	39.6	0.7%
0.9	0.5	282.5%	95.4%	78.8	0.6%	472.1%	94.3%	104.3	0.6%
0.95	0.3	134.6%	99.0%	45.2	0.7%	184.1%	98.4%	52.6	0.6%
0.95	0.4	181.8%	97.0%	134.7	0.7%	262.8%	95.9%	173.2	0.6%
0.95	0.5	222.0%	94.8%	269.7	0.7%	318.1%	93.5%	356.9	0.7%
0.975	0.3	127.1%	98.8%	144.0	0.8%	161.6%	98.1%	169.6	0.7%
0.975	0.4	158.1%	96.7%	345.0	0.9%	212.5%	95.4%	443.4	0.8%
0.975	0.5	179.6%	94.6%	610.5	0.9%	244.6%	93.0%	788.1	0.8%
0.9875	0.3	119.2%	98.6%	341.3	1.0%	143.7%	97.6%	401.0	0.9%
0.9875	0.4	137.6%	96.3%	722.5	1.0%	173.2%	94.6%	900.4	1.0%
0.9875	0.5	148.2%	94.1%	1171.2	1.0%	190.3%	92.1%	1450.7	1.0%

Table EC.2 Performance of the long chain with $n=15$ and 20

ρ	cv	$R(\mathcal{LC}^-)$	$\Delta(\mathcal{LC}^-)$	$R(\mathcal{LC}^-)$	$\Delta(\mathcal{LC}^-)$	$R(\mathcal{LC}^-)$	$\Delta(\mathcal{LC}^-)$	$R(\mathcal{LC}^-)$	$\Delta(\mathcal{LC}^-)$
		$n=5$		$n=10$		$n=15$		$n=20$	
0.8	0.3	809%	80%	5704%	88%	40662%	92%	169102%	94%
0.8	0.4	647%	70%	2412%	82%	7681%	87%	23141%	90%
0.8	0.5	586%	64%	1739%	76%	4094%	83%	9367%	86%
0.9	0.3	417%	70%	1295%	66%	2271%	75%	3556%	80%
0.9	0.4	361%	69%	964%	65%	1852%	66%	2571%	72%
0.9	0.5	333%	69%	814%	65%	1512%	64%	2154%	69%
0.95	0.3	298%	81%	683%	73%	1175%	70%	1810%	68%
0.95	0.4	273%	78%	587%	71%	961%	68%	1422%	67%
0.95	0.5	261%	76%	532%	70%	866%	67%	1232%	66%
0.975	0.3	265%	80%	523%	72%	817%	69%	1136%	68%
0.975	0.4	235%	77%	458%	69%	703%	66%	961%	65%
0.975	0.5	218%	74%	412%	67%	630%	64%	871%	63%
0.9875	0.3	238%	74%	420%	65%	615%	62%	819%	61%
0.9875	0.4	202%	73%	345%	63%	503%	60%	664%	58%
0.9875	0.5	185%	72%	308%	62%	442%	58%	591%	57%

Table EC.3 Performance of long chain less arc $(n,1)$

ρ	cv	$R(\mathcal{C})$	$\Delta(\mathcal{C})$	$B(\mathcal{C})$	$R(\mathcal{C}^-)$	$\Delta(\mathcal{C}^-)$	SE%
0.8	0.3	100.3%	100.0%	0.53	569.8%	60.2%	1.1%
0.8	0.4	101.6%	99.8%	2.24	393.7%	57.7%	0.7%
0.8	0.5	104.2%	99.2%	5.17	337.7%	55.8%	0.7%
0.9	0.3	101.1%	99.8%	8.25	231.8%	71.5%	0.6%
0.9	0.4	104.0%	99.0%	19.92	223.6%	67.4%	0.6%
0.9	0.5	107.9%	97.7%	34.99	222.1%	64.6%	0.7%
0.95	0.3	102.1%	99.7%	34.13	179.2%	89.7%	0.8%
0.95	0.4	105.2%	99.1%	68.75	181.9%	85.1%	0.8%
0.95	0.5	108.4%	98.1%	111.22	177.7%	82.5%	0.8%
0.975	0.3	102.4%	99.7%	92.04	153.5%	93.7%	1.0%
0.975	0.4	104.6%	99.1%	171.09	161.0%	88.0%	1.0%
0.975	0.5	106.4%	98.2%	262.83	158.2%	84.1%	1.0%
0.9875	0.3	102.1%	99.6%	199.74	137.1%	93.1%	1.0%
0.9875	0.4	103.4%	99.0%	337.47	141.4%	87.6%	0.9%
0.9875	0.5	104.6%	98.2%	464.54	143.4%	82.9%	0.9%

Table EC.4 Performance of $\mathcal{C}^-$ and $\mathcal{C}$ in the 3 by 5 system

ρ	cv	$R(\mathcal{C})$	$\Delta(\mathcal{C})$	$B(\mathcal{C})$	$R(\mathcal{C}^-)$	$\Delta(\mathcal{C}^-)$	SE%
0.8	0.3	118.6%	99.9%	0.11	3520.2%	77.1%	2.6%
0.8	0.4	151.3%	99.0%	1.04	1421.9%	73.4%	1.2%
0.8	0.5	176.3%	97.4%	3.67	973.3%	70.2%	0.9%
0.9	0.3	113.5%	99.3%	5.50	559.5%	74.8%	0.7%
0.9	0.4	134.9%	97.3%	17.85	527.4%	67.1%	0.6%
0.9	0.5	158.6%	94.7%	38.08	505.4%	63.5%	0.6%
0.95	0.3	113.3%	99.4%	30.44	425.1%	84.4%	0.7%
0.95	0.4	131.2%	97.8%	72.22	382.6%	80.5%	0.7%
0.95	0.5	147.5%	95.8%	132.60	354.1%	77.3%	0.8%
0.975	0.3	112.9%	99.3%	91.74	381.6%	85.8%	0.8%
0.975	0.4	125.3%	97.9%	191.95	330.4%	80.9%	0.9%
0.975	0.5	135.8%	96.0%	317.60	304.3%	77.0%	0.9%
0.9875	0.3	110.8%	99.1%	212.96	377.7%	76.9%	1.0%
0.9875	0.4	118.2%	97.5%	405.35	286.7%	74.3%	1.0%
0.9875	0.5	123.7%	95.5%	615.38	250.2%	71.8%	1.0%

Table EC.5 Performance of $\mathcal{C}^-$ and $\mathcal{C}$ in the 6 by 10 system

ρ	cv	$R(\mathcal{C})$	$\Delta(\mathcal{C})$	$B(\mathcal{C})$	$R(\mathcal{C}^-)$	$\Delta(\mathcal{C}^-)$	SE%
0.8	0.3	386.3%	99.8%	0.06	19390.1%	84.6%	5.9%
0.8	0.4	442.7%	98.4%	1.07	4215.1%	80.9%	2.1%
0.8	0.5	471.9%	96.2%	4.57	2278.7%	77.5%	1.2%
0.9	0.3	165.0%	98.5%	5.16	966.2%	80.2%	0.8%
0.9	0.4	220.6%	95.6%	21.36	834.0%	73.3%	0.7%
0.9	0.5	267.9%	92.3%	51.27	814.2%	67.2%	0.6%
0.95	0.3	144.1%	98.8%	32.70	727.3%	83.2%	0.7%
0.95	0.4	184.0%	96.6%	91.68	609.7%	79.3%	0.7%
0.95	0.5	218.1%	93.9%	178.46	560.0%	76.3%	0.7%
0.975	0.3	135.4%	98.9%	101.74	601.0%	84.7%	0.8%
0.975	0.4	162.9%	96.8%	236.50	504.8%	79.3%	0.9%
0.975	0.5	182.4%	94.2%	415.39	455.6%	74.8%	0.9%
0.9875	0.3	127.5%	98.6%	239.23	570.6%	75.2%	1.0%
0.9875	0.4	144.2%	96.1%	493.41	421.0%	71.7%	1.0%
0.9875	0.5	154.6%	93.5%	774.19	359.8%	69.2%	1.0%

Table EC.6 Performance of $\mathcal{C}^-$ and $\mathcal{C}$ in the 9 by 15 system

n	ρ	$R(\mathcal{LC})$	$\Delta(\mathcal{LC})$	$R(\mathcal{LC}^-)$	$\Delta(\mathcal{LC}^-)$	SE%
5	0.8	179.2%	87.6%	356.8%	59.8%	1.1%
5	0.9	137.0%	93.4%	274.9%	68.9%	1.5%
5	0.95	119.1%	97.3%	228.3%	81.8%	2.1%
5	0.975	108.8%	98.6%	187.2%	85.9%	2.7%
5	0.9875	104.5%	98.9%	159.7%	84.9%	3.0%
10	0.8	456.9%	80.9%	738.1%	65.8%	0.9%
10	0.9	271.3%	87.0%	493.7%	70.2%	1.3%
10	0.95	189.5%	93.2%	390.3%	78.1%	1.9%
10	0.975	146.1%	95.8%	310.2%	80.8%	2.6%
10	0.9875	124.0%	96.7%	250.4%	79.2%	2.9%
15	0.8	890.1%	78.7%	1313.7%	67.3%	0.9%
15	0.9	447.3%	84.2%	739.8%	70.9%	1.1%
15	0.95	290.5%	90.8%	563.8%	77.5%	1.6%
15	0.975	202.0%	93.9%	444.8%	79.2%	2.6%
15	0.9875	154.9%	95.0%	356.9%	76.7%	2.7%

Table EC.7 Performance of $\mathcal{LC}$ and $\mathcal{LC}^-$ in a continuous parallel system

n	wip	$R(\mathcal{LC})$	$\Delta(\mathcal{LC})$	$R(\mathcal{LC}^-)$	$\Delta(\mathcal{LC}^-)$	SE%
5	5.00	82.6%	61.0%	64.8%	20.9%	0.03%
5	10.00	95.5%	84.3%	77.4%	21.5%	0.02%
5	25.00	100.0%	99.7%	88.4%	18.2%	0.02%
10	10.00	75.4%	48.2%	65.2%	26.7%	0.03%
10	20.00	91.1%	71.3%	79.4%	33.9%	0.03%
10	50.00	99.3%	95.2%	88.5%	25.7%	0.03%
15	15.00	72.8%	43.8%	65.7%	29.0%	0.03%
15	30.00	89.4%	66.8%	80.6%	39.3%	0.03%
15	75.00	98.6%	91.5%	89.7%	35.5%	0.03%

Table EC.8 Performance of $\mathcal{LC}$ and $\mathcal{LC}^-$ in a serial production line