Appendix

EC.1. Proof of Lemma 2

Proof of Lemma 2 Let $\pi(x,y)$ be the state joint distribution of $[X,Y]$, and let $\pi_X(x) = \sum_{y\geq 0}\pi(x,y)$ be the marginal distribution of X. For a given $x\geq 0$, we have:

$$\sum_{y\in S(x)}\pi(x,y)Q([x,y],[x+1,y]) + \sum_{y\notin S(x)}\pi(x,y)Q([x,y],[x+1,y])$$
$$= \sum_{y\in S(x+1)}\pi(x+1,y)Q([x+1,y],[x,y]) + \sum_{y\notin S(x+1)}\pi(x+1,y)Q([x+1,y],[x,y]).$$

Using Condition 2, we upper-bound the LHS and lower-bound the RHS, which results in having:

$$f(x)\mathbf{P}[X=x, Y\in S(x)] + \mathbf{P}[X=x, Y\notin S(x)] \geq g(x+1)\mathbf{P}[X=x+1, Y\in S(x+1)].$$

Let $\pi_S(x) = \mathbf{P}[X=x, Y\in S(x)] = \sum_{y\in S(x)}\pi(x,y)$. Observe that $\pi_X(x)\leq \pi_S(x) + c\delta^x \leq \pi_S(x) + c\rho^x$ from Condition 1 and for any $\rho\in[\delta,1)$. Assuming that for $x\geq\eta$, $\frac{f(x)}{g(x+1)}\leq\rho$, we get:

$$\pi_S(x+1)\leq \frac{f(x)}{g(x+1)}\pi_S(x) + \frac{\mathbf{P}[Y\notin S(x)]}{g(x+1)} \leq \rho\pi_S(x) + \frac{c\delta^x}{g(x+1)} \leq \rho\pi_S(x) + \frac{c\rho^x}{g(\eta+1)},$$

where the last inequality results from the assumption $\frac{\delta^x}{g(x+1)}\leq\frac{\rho^x}{g(\eta+1)}$. We can now prove by induction that:

$$\pi_S(\eta+i)\leq\rho^i\left(\pi_S(\eta) + i\frac{c\rho^{\eta-1}}{g(\eta+1)}\right)$$

This allows us to conclude:

$$\begin{aligned}
\mathbf{P}[X\geq\eta+k] &= \sum_{i=\eta+k}^{\infty}\pi_X(i)\\
&\leq \sum_{i=k}^{\infty}\pi_S(\eta+i) + \sum_{i=k}^{\infty}c\rho^{\eta+i}\\
&\leq \sum_{i=k}^{\infty}\rho^i\pi_S(\eta) + \frac{c\rho^{\eta-1}}{g(\eta+1)}\sum_{i=k}^{\infty}i\rho^i + \sum_{i=k}^{\infty}c\rho^{\eta+i}\\
&\leq \frac{\rho^k}{1-\rho}\left(1 + c\rho^\eta + \frac{c\rho^{\eta-1}}{g(\eta+1)}\frac{k+1}{1-\rho}\right)\\
&\leq \frac{\rho^k}{1-\rho}\left(1 + c + \frac{c(k+1)}{g(\eta+1)-f(\eta)}\right)
\end{aligned}$$

□

EC.2. Missing proofs for BilateralMatch(H) Policy

In the proofs of this section and of the next ones we will use the following facts: for any bounded, non-negative function $\xi:\mathbb{R}_+\mapsto\mathbb{R}_+$ and any constant $u>0$, in the limit where $p_H\to 0$,

Fact 1 *For* $\eta=\frac{(\ln u)+\xi(p_H)}{p_H^2}$, *we have* $(1-p_H^2)^\eta=\frac{e^{-\xi(p_H)}}{u}+O(p_H^2)$.

Fact 2 *For any constants* p_E, $r>0$ *and for* $\eta=\frac{(\ln u)+\xi(p)}{p_H^2}$, *we have* $(1-p_Ep_H)^\eta=o(p_H^r)$.

Fact 3 *For any constant* p_E, *and for* $\eta=\frac{(\ln u)+\xi(p_H)}{p_Ep_H}$, *we have* $(1-p_H^2)^\eta=1-O(p_H)$.

Fact 4 *For any constant* p_E *and for* $\eta=\frac{(\ln u)+\xi(p)}{p_Ep_H}$, *we have* $(1-p_Ep_H)^\eta=\frac{e^{-\xi(p)}}{u}+O(p_H)$.

EC.2.1. Proof of Theorem 1

Proof of Theorem 1 We first upper-bound $\mathbf{E}[H^{\mathcal{B}_H}]$. Let

$$v(\lambda_H,\lambda_E,p_E,p_H)=\begin{cases}\frac{1}{p_Ep_H}\left(\ln\left(\frac{\lambda_E}{\lambda_E-\lambda_H}\right)+c_3\sqrt{p_H}\right)+p_H^{-3/4} & \text{when } \lambda_H<\lambda_E,\\ \frac{1}{p_H^2}\left(\ln\left(\frac{2\lambda_H}{\lambda_E+\lambda_H}\right)+c_4\sqrt{p_H}\right)+p_H^{-3/4} & \text{when } \lambda_H>\lambda_E.\end{cases}$$

Where c_3 and c_4 are the constants from Proposition 4. Using the equality $\mathbf{E}[H^{\mathcal{B}_H}]=\sum_{i=1}^{\infty}\mathbf{P}[H^{\mathcal{B}_H}\ge i]$, we have:

$$\begin{aligned}\mathbf{E}[H^{\mathcal{B}_H}]&=\sum_{i=1}^{v(\lambda_H,\lambda_E,p_E,p_H)-1}\mathbf{P}[H^{\mathcal{B}_H}\ge i]+\sum_{i=v(\lambda_H,\lambda_E,p_E,p_H)}^{\infty}\mathbf{P}[H^{\mathcal{B}_H}\ge i]\\&\le v(\lambda_H,\lambda_E,p_E,p_H)+\sum_{j=p_H^{-3/4}}^{\infty}\frac{\gamma(p_H)^j}{1-\gamma(p_H)}\\&\le v(\lambda_H,\lambda_E,p_E,p_H)+\frac{\gamma''(p_H)^{p_H^{-3/4}}}{(1-\gamma''(p_H))^2}.\end{aligned}$$

Where we denote $\gamma''=\max(\gamma,\gamma')$ and we used the result from Proposition 4 : $\mathbf{P}[H^{\mathcal{B}_H}\ge v(\lambda_H,\lambda_E,p_E,p_H)+j]\le\frac{\gamma''(p_H)^j}{1-\gamma''(p_H)}$.

Applying the fact that $\gamma''(p_H)^{p_H^{-3/4}}=\left(1-\sqrt{p_H}+o(\sqrt{p_H})\right)^{p_H^{-3/4}}=o(p_H^2)$, and some algebra we get the following upper-bound on $\mathbf{E}[H^{\mathcal{B}_H}]$:

- If $\lambda_H<\lambda_E$, then $\mathbf{E}[H^{\mathcal{B}_H}]\le\frac{\ln\left(\frac{\lambda_E}{\lambda_E-\lambda_H}\right)}{p_Ep_H}+o\left(\frac{1}{p_H}\right)$.
- If $\lambda_H>\lambda_E$, then $\mathbf{E}[H^{\mathcal{B}_H}]\le\frac{\ln\left(\frac{2\lambda_H}{\lambda_H+\lambda_E}\right)}{p_H^2}+o\left(\frac{1}{p_H^2}\right)$.

Now we proceed to lower-bound $\mathbf{E}[H^{\mathcal{B}_H}]$: Applying Markov inequality to $\mathbf{E}[H^{\mathcal{B}_H}]$ and using Proposition 3, we get the following lower-bound on $\mathbf{E}[H^{\mathcal{B}_H}]$:

- If $\lambda_H<\lambda_E$, then $\mathbf{E}[H^{\mathcal{B}_H}]\ge\frac{\ln\left(\frac{\lambda_E}{\lambda_E-\lambda_H}\right)}{p_Ep_H}+o\left(\frac{1}{p_H}\right)$.
- If $\lambda_H>\lambda_E$, then $\mathbf{E}[H^{\mathcal{B}_H}]\ge\frac{\ln\left(\frac{2\lambda_H}{\lambda_H+\lambda_E}\right)}{p_H^2}+o\left(\frac{1}{p_H^2}\right)$.

This completes the proof.

□

EC.2.2. Proof of Corollaries 1 and 2

Proof of Corollary 1 Define $x = \lambda_H/\lambda_E$, and $f(x) = \frac{\ln\left(\frac{1}{1-x}\right)}{x}$. Note that the constant $\frac{\ln\left(\frac{\lambda_E}{\lambda_E-\lambda_H}\right)}{p_E\lambda_H} = \frac{f(x)}{\lambda_E p_E}$. It is easy check that $f'(x) > 0$ in $x \in (0,1)$, and therefore $f(x)$ is increasing in $x \in (0,1)$.

$\square$

Proof of Corollary 2 Define $y = \lambda_H/\lambda_E$, and $g(y) = \frac{\ln\left(\frac{2y}{1+y}\right)}{y}$. Note that the constant $\frac{\ln\left(\frac{2\lambda_H}{\lambda_H+\lambda_E}\right)}{\lambda_H} = \frac{g(y)}{\lambda_E}$. It is easy to check that $g'(y) > 0$ when $y \in (1, y^*)$, and $g'(y) < 0$ in $y > y^*$ where y^* is the solution of $g'(y) = 0$:

$$g'(y^*) = 0 \Leftrightarrow \frac{1}{y^*+1} = \ln\left(\frac{2y^*}{1+y^*}\right) \Leftrightarrow (y^*+1)\ln(2-2/(y^*+1)) = 1,$$

$\square$

EC.2.3. Proof of Proposition 4

Instead of proving Proposition 4, we prove a stronger result. This will be useful later on to prove an upper bound for E agents (see Lemma EC.1 in EC.2.4).

PROPOSITION EC.1. *Under $\mathcal{B}_H$, if $\lambda_H < \lambda_E$, for any non-negative bounded function $\xi(p_H)$, for all $k \geq 0$:*

$$\mathbf{P}\left[H^{\mathcal{B}_H} \geq \frac{1}{p_E p_H}\left(\ln\left(\frac{\lambda_E}{\lambda_E-\lambda_H}\right) + \xi(p_H)\right) + k\right] \leq \frac{\gamma(p_H)^k}{1-\gamma(p_H)},$$

where $\gamma(p_H) = \frac{\lambda_H}{\lambda_E-(\lambda_E-\lambda_H)e^{-\xi(p_H)}} + O(p_H)$.

If $\lambda_H > \lambda_E$, for any non-negative bounded function $\xi'(p_H)$, for all $k \geq 0$:

$$\mathbf{P}\left[H^{\mathcal{B}_H} \geq \frac{1}{p_H^2}\left(\ln\left(\frac{2\lambda_H}{\lambda_E+\lambda_H}\right) + \xi'(p_H)\right) + k\right] \leq \frac{\gamma'(p_H)^k}{1-\gamma'(p_H)},$$

where $\gamma'(p_H) = \frac{e^{-\xi'(p_H)}}{2-e^{-\xi'(p_H)}} + O(p_H^2)$.

Proof of Proposition EC.1 We wish to apply Lemma 2 with $[X(t), Y(t)] = [H^{\mathcal{B}_H}(t), E^{\mathcal{B}_H}(t)]$, in the special case where $S(h) = \mathbb{N}$ for all h, and $c = \delta = 0$. This implies that we need to find functions $f(\cdot)$ and $g(\cdot)$ such that for all $e \in \mathbb{N}$, $f(h) \geq Q^{\mathcal{B}_H}([h,e],[h+1,e])$, and $g(h) \leq Q^{\mathcal{B}_H}([h,e],[h-1,e])$.

Let $f(h) = \lambda_H(1-p_H^2)^h$ and $g(h) = \lambda_H(1-(1-p_H^2)^h) + \lambda_E(1-(1-p_E p_H)^h)$. We have

$$\frac{f(h)}{g(h+1)} = \frac{\lambda_H(1-p_H^2)^h}{\lambda_H(1-(1-p_H^2)^{h+1}) + \lambda_E(1-(1-p_E p_H)^{h+1})}.$$

Case $\lambda_H < \lambda_E$: Take $\eta = \frac{1}{p_E p_H}\left(\ln\left(\frac{\lambda_E}{\lambda_E-\lambda_H}\right) + \xi(p_H)\right)$. Facts 3 and 4 imply respectively that $(1-p_H^2)^\eta = 1 + O(p_H)$ and $(1-p_E p_H)^\eta = \frac{\lambda_E-\lambda_H}{\lambda_E}e^{-\xi(p_H)} + O(p_H)$. This yields for all $h \geq \eta$:

$$\frac{f(h)}{g(h+1)} \leq \frac{\lambda_H}{\lambda_E-(\lambda_E-\lambda_H)e^{-\xi(p_H)}} + O(p_H) := \gamma(p_H).$$

Case $\lambda_H > \lambda_E$: Taking $\eta = \frac{1}{p_H^2}\left(\ln\left(\frac{2\lambda_H}{\lambda_E+\lambda_H}\right) + \xi'(p_H)\right)$. Facts 1 and 2 imply respectively $(1-p_H^2)^\eta = \frac{\lambda_E+\lambda_H}{2\lambda_H} e^{-\xi'(p_H)} + O(p_H^2)$ and $(1-p_E p_H)^\eta = o(p_H^2)$. This yields for all $h \geq \eta$:

$$\frac{f(h)}{g(h+1)} \leq \frac{\frac{1}{2}e^{-\xi'(p_H)}}{1-\frac{1}{2}e^{-\xi'(p_H)}} + O(p_H^2) := \gamma'(p_H).$$

In both cases, we conclude by applying Lemma 2 with $\rho = \gamma(p_H)$ or $\rho = \gamma'(p_H)$.

□

Proof of Proposition 4 The proof of Proposition 4 is a consequence of Proposition EC.1:

- In the case where $\lambda_H < \lambda_E$, take $\xi(p_H) = \sqrt{p_H}\frac{\lambda_H}{\lambda_E-\lambda_H}$. This implies that $\gamma(p_H) = \frac{\lambda_H}{\lambda_E-(\lambda_E-\lambda_H)e^{-\xi(p_H)}} + O(p_H) = 1-\sqrt{p_H} + O(\sqrt{p_H})$.

- In the case where $\lambda_H > \lambda_E$, take $\xi'(p_H) = 2\sqrt{p_H}$. This implies that $\gamma'(p_H) = \frac{e^{-\xi'(p_H)}}{2-e^{-\xi'(p_H)}} + O(p_H^2) = 1-\sqrt{p_H} + O(\sqrt{p_H})$.

□

EC.2.4. Proof of Proposition 3

The proof of Proposition 3 requires a concentration bound on the number of E agents, which we state in Lemma EC.1.

Lemma EC.1. *Under $\mathcal{B}_H$, and assuming $p_H \leq p_E$, there exist constants C and $\zeta < 1$ (which only depend on λ_H, λ_E, and p_E) such that for any $k \geq 0$, there exists p such that for any $p_H < p$, we have:*

$$\mathbf{P}\left[E^{\mathcal{B}_H} \geq \frac{1}{\sqrt{p_H}} + k\right] \leq C\zeta^k.$$

Proof of Lemma EC.1 The proof is based on a bound on the right-tail distribution of E agents in the market. To do this, we will apply Lemma 2 with $[X(t), Y(t)] = [E^{\mathcal{B}_H}(t), H^{\mathcal{B}_H}(t)]$. Therefore, we find an upper-bound $f(e) = \lambda_E(1-p_E^2)^e \geq Q^{\mathcal{B}_H}([h,e],[h,e+1])$ on the upward transition (3c). Similarly, we would like to find a lower-bound $g(e)$ on the downward transition (3d), but we cannot find one for any $h \in \mathbb{N}$. Therefore we need to restrict our attention to some subset $S(e) \subset \mathbb{N}$. Recall that $Q^{\mathcal{B}_H}([h,e],[h,e-1]) = \lambda_H(1-p_H^2)^h(1-(1-p_E p_H)^e) + \lambda_E(1-p_E p_H)^h(1-(1-p_E^2)^e))$.

Case $\lambda_H < \lambda_E$:

$$S(e) = \left\{h \in \mathbb{N} \mid h \leq \frac{1}{p_E p_H}\left(\ln\left(\frac{\lambda_E}{\lambda_E-\lambda_H}\right) + \ln 2\right) + e\right\}.$$

Applying Proposition EC.1 with $\xi(p_H) = \ln 2$, we get that $\mathbf{P}[H^{\mathcal{B}_H} \notin S(e)] \leq \frac{\gamma(p_H)^e}{1-\gamma(p_H)} = c\delta^e$, where $c = \frac{\lambda_H+\lambda_E}{\lambda_E-\lambda_H}$, $\delta = \frac{2\lambda_H}{\lambda_E+\lambda_H}$. Fact 4 implies that for all $h \in S(e)$, $(1-p_E p_H)^h \geq \frac{\lambda_E-\lambda_H}{2\lambda_E}(1-p_E p_H)^e + O(p_H)$. Keeping only the second term in $Q^{\mathcal{B}_H}([h,e],[h,e-1])$, we get the lower-bound:

$$Q^{\mathcal{B}_H}([h,e],[h,e-1]) \geq \frac{\lambda_E - \lambda_H}{2}(1-p_E p_H)^e (1-(1-p_E^2)^e) + O(p_H) := g(e)$$

This yields:

$$\frac{f(e)}{g(e+1)} = \frac{2\lambda_E}{(\lambda_E - \lambda_H)(1-(1-p_E^2)^{e+1})}\left(\frac{1-p_E^2}{1-p_E p_H}\right)^{e+1} + O(p_H).$$

We get for all $e \geq \frac{1}{\sqrt{p_H}}$, and p_H small enough, $\frac{f(e)}{g(e+1)} \leq \delta$. Furthermore, for $\rho = \frac{1+\delta}{2}$ and p_H small enough,

$$\frac{\delta^e}{g(e+1)} \leq \delta^e \frac{(1-p_E p_H)^{\frac{1}{\sqrt{p_H}}-e-1}}{g(\frac{1}{\sqrt{p_H}}+1)} \leq \left(\frac{\delta}{1-p_E p_H}\right)^e \frac{1}{g(\frac{1}{\sqrt{p_H}}+1)} \leq \frac{\rho^e}{g(\frac{1}{\sqrt{p_H}}+1)}$$

We can apply Lemma 2 which yields the desired bound:

$$\begin{aligned}
\mathbf{P}\left[E^{\mathcal{B}_H} \geq \frac{1}{\sqrt{p_H}} + k\right] &\leq \frac{\rho^k}{1-\rho}\left(1 + c + \frac{c(k+1)}{g(\frac{1}{\sqrt{p_H}}+1) - f(\frac{1}{\sqrt{p_H}})}\right) \\
&\leq \frac{\rho^k}{1-\rho}\left(1 + c + \frac{c(k+1)(\lambda_H+\lambda_E)}{2\lambda_H g(1/\sqrt{p_H}+1)}\right). \\
&\leq \frac{\rho^k}{1-\rho}\left(1 + c + \frac{c(k+1)(\lambda_H+\lambda_E)}{\lambda_H(\lambda_E-\lambda_H)}\right).
\end{aligned}$$

Where we used first the fact that $\frac{f(1/\sqrt{p_H})}{g(1/\sqrt{p_H}+1)} \leq \delta = \frac{2\lambda_H}{\lambda_E+\lambda_H}$ and therefore $g(1/\sqrt{p_H}+1) - f(1/\sqrt{p_H}) \geq \frac{\lambda_E-\lambda_H}{\lambda_E+\lambda_H}$ and second the fact that $g(1/\sqrt{p_H}+1) = \frac{\lambda_E-\lambda_H}{2} + O(p_H)$.

Case $\lambda_H > \lambda_E$:

$$S(e) = \left\{ h \in \mathbb{N} \mid h \leq \frac{1}{p_H^2}\left(\ln\left(\frac{2\lambda_H}{\lambda_E+\lambda_H}\right) + \ln 2\right) + e \right\}.$$

Applying Proposition EC.1 with $\xi(p_H) = \ln(2)$, we have $\mathbf{P}[H^{\mathcal{B}_E} \notin S(e)] \leq \frac{\gamma(p_H)^e}{1-\gamma(p_H)} = c\delta^e$, with $c = 3/2$ and $\delta = 1/3$. Fact 1 implies that for all $h \in S(e)$, $(1-p_H^2)^h \geq \frac{\lambda_H+\lambda_E}{4\lambda_H}(1-p_H^2)^e + O(p_H^2)$. Keeping only the first term in $Q^{\mathcal{B}_H}([h,e],[h,e-1])$, we get the lower-bound:

$$Q^{\mathcal{B}_H}([h,e],[h,e-1]) \geq \frac{\lambda_H+\lambda_E}{4\lambda_H}(1-p_H^2)^e(1-(1-p_E p_H)^e) + O(p_H^2) = g(e).$$

This yields:

$$\frac{f(e)}{g(e+1)} \leq \frac{4\lambda_H\lambda_E}{(\lambda_H+\lambda_E)(1-(1-p_E p_H)^e)}\left(\frac{1-p_E^2}{1-p_H^2}\right)^e + O(p_H^2).$$

Furthermore, with $\rho = 1/2$ we get that for p_H small enough, $\frac{\delta}{1-p_H^2} \leq \rho$. This leads to

$$\frac{\delta^e}{g(e+1)} \leq \delta^e \frac{(1-p_H^2)^{p_H^{-1/2}-e}}{g\left(p_H^{-1/2}+1\right)} \leq \left(\frac{\delta}{1-p_H^2}\right)^e \frac{1}{g\left(p_H^{-1/2}+1\right)} \leq \frac{\rho^e}{g\left(p_H^{-1/2}+1\right)}.$$

Therefore we can apply Lemma 2:

$$\mathbf{P}\left[E^{\mathcal{B}_H} \geq n_E + k\right] \leq \frac{\rho^k}{1-\rho}\left(1+c+\frac{c(k+1)}{g\left(p_H^{-1/2}+1\right)-f(p_H^{-1/2})}\right)$$
$$\leq \frac{\rho^k}{1-\rho}\left(1+c+\frac{c(k+1)3}{2g(p_H^{-1/2})}\right)$$
$$\leq \frac{\rho^k}{1-\rho}\left(1+c+\frac{6c(k+1)}{\lambda_H+\lambda_E}\right)$$

Where we used the fact that for all $e \geq \frac{1}{\sqrt{p_H}}$, $\frac{f(e)}{g(e+1)} = o(p_H)$. Therefore for p_H small enough, $\frac{f(e)}{g(e+1)} \leq \frac{1}{3}$ and therefore $g(e+1)-f(e) \geq \frac{2}{3}g(e+1)$. Which concludes the proof.
□

We can now prove Proposition 3. The main idea is to apply Lemma 1, in two cases separately, where in one case we have few ($\leq \frac{2}{\sqrt{p_H}}$) E agents, and in the other case where we have many. Using Lemma EC.1, we can exponentially bound the second case.

Proof of Proposition 3 We will apply Lemma 1. Consider $S = \left\{e \in \mathbb{R}, e \leq p_H^{-1/2} + p_H^{-1/2}\right\}$. Using Lemma EC.1 with $k = p_H^{-1/2}$, we get that $\mathbf{P}[E^{\mathcal{B}_H} \notin S] = o(p_H^4)$.

For $e \in S$, we have $(1-p_Ep_H)^e \geq 1 - p_Ep_H\left(p_H^{-1/2}+p_H^{-1/2}+o(\sqrt{p_H})\right) = 1-2p_E\sqrt{p_H}+o(\sqrt{p_H})$. Taking $f(h) = \lambda_H(1-p_H^2)^h\left(1-2p_E\sqrt{p_H}\right)+o(\sqrt{p_H})$, we have for $e \in S$, $Q^{\mathcal{B}_H}([h,e],[h+1,e]) \geq f(h)$. Let $g(h) = Q^{\mathcal{B}_H}([h,e],[h-1,e]) = \lambda_H(1-(1-p_H^2)^h)+\lambda_E(1-(1-p_Ep_H)^h)$, and note that $f(h)$ is non-increasing and $g(h)$ is non-decreasing. We have:

$$\frac{g(h+1)}{f(h)} = \frac{\lambda_H+\lambda_E-\lambda_H(1-p_H^2)^{h+1}-\lambda_E(1-p_Ep_H)^{h+1}}{\lambda_H(1-p_H^2)^h\left(1-2p_E\sqrt{p_H}+o(\sqrt{p_H})\right)}$$

In the case $\lambda_H < \lambda_E$, Take $\eta = \frac{1}{p_Ep_H}\left(\ln\left(\frac{\lambda_E}{\lambda_E-\lambda_H}\right)-c_1'\sqrt{p_H}\right)$. Using Facts 3 and 4, we have $(1-p_H^2)^\eta = 1+O(p_H)$ and $(1-p_Ep_H)^\eta = \frac{\lambda_E-\lambda_H}{\lambda_E}e^{c_1'\sqrt{p_H}}+O(p_H) = \frac{\lambda_E-\lambda_H}{\lambda_E}(1+c_1'\sqrt{p_H}+o(\sqrt{p_H}))$, therefore:

$$\frac{g(\eta+1)}{f(\eta)} = \frac{\lambda_H+\lambda_E-\lambda_H-(\lambda_E-\lambda_H)(1+c_1'\sqrt{p_H})+o(\sqrt{p_H})}{\lambda_H\left(1-2p_E\sqrt{p_H}\right)+O(p_H)}$$
$$= 1-\left(\frac{\lambda_E-\lambda_H}{\lambda_H}c_1'-2p_E\right)\sqrt{p_H}+o(\sqrt{p_H}).$$
$$= 1-\sqrt{p_H}+o(\sqrt{p_H}).$$

Where we fixed $c_1' = \frac{\lambda_H(1+2p_E)}{\lambda_E-\lambda_H}$. Using Lemma 1 with $[X(t),Y(t)] = [H^{\mathcal{B}_H}(t), E^{\mathcal{B}_H}(t)]$, $k = p_H^{-3/4}$ and $\rho = \frac{g(\eta+1)}{f(\eta)}$, we get $\rho^k = o(p_H^2)$ and:

$$\mathbf{P}[H^{\mathcal{B}_H} \leq \eta - k] \leq \eta \cdot o(p_H^4)\left(1+\frac{1}{f(\eta)-g(\eta+1)}\right)+\frac{\rho^k}{1-\rho}$$
$$\leq \eta \cdot o(p_H^4)\left(1+\frac{1}{f(\eta)\sqrt{p_H}+o(\sqrt{p_H})}\right)+\frac{o(p_H^2)}{\sqrt{p_H}+o(\sqrt{p_H})}$$
$$= o(p_H).$$

Taking $c_1 = c_1' + p_E$, this enables us to conclude that $\mathbf{P}[H^{\mathcal{B}_H} \le \frac{1}{p_E p_H}\left(\ln\left(\frac{\lambda_E}{\lambda_E-\lambda_H}\right) - c_1 p_H^{1/4}\right)] \le o(p_H)$.

In the case $\lambda_H > \lambda_E$, Let $\eta = \frac{1}{p_H^2}\left(\ln\left(\frac{2\lambda_H}{\lambda_H+\lambda_E}\right) - c_2' p_H^{1/4}\right)$. Using Facts 1 and 2, we have $(1-p_H^2)^\eta = \frac{\lambda_H+\lambda_E}{2\lambda_H} e^{c_2' p_H^{1/4}} + O(p_H^2)$ and $(1-p_E p_H)^\eta = O(p_H^2)$. This implies that:

$$\begin{aligned}\rho = \frac{g(\eta+1)}{f(\eta)} &= \frac{(\lambda_H+\lambda_E) - \frac{(\lambda_H+\lambda_E)}{2} e^{c_2'\sqrt{p_H}} + O(p_H^2)}{\frac{(\lambda_H+\lambda_E)}{2} e^{c_2'\sqrt{p_H}}\left(1-2p_E\sqrt{p_H}\right) + O(p_H^2)} \\ &= \frac{2-(1+c_2'\sqrt{p_H})}{(1+c_2'\sqrt{p_H})(1-2p_E\sqrt{p_H})} + o(\sqrt{p_H}) \\ &= 1 - (2c_2' - 2p_E)\sqrt{p_H} + o(\sqrt{p_H}) \\ &= 1 - \sqrt{p_H} + o(\sqrt{p_H}).\end{aligned}$$

Where we have chosen $c_2' = \frac{1+2p_E}{2}$. Taking $k = p_H^{-3/4}$, we have $\rho^k = o(p_H^2)$. Applying Lemma 1, we get that for all :

$$\begin{aligned}\mathbf{P}[H^{\mathcal{B}_H} \le \eta - k] &\le \eta \cdot o(p_H^4)\left(1 + \frac{1}{f(\eta)-g(\eta+1)}\right) + \frac{\rho^k}{1-\rho} \\ &= \eta \cdot o(p_H^3) + \frac{o(p_H^2)}{f(\eta)\sqrt{p_H} + o(\sqrt{p_H})} \\ &= o(p_H)\end{aligned}$$

Taking $c_2 = c_2' + 1$, this enables us to conclude that $\mathbf{P}[H^{\mathcal{B}_H} \le \frac{1}{p_H^2}\left(\ln\left(\frac{2\lambda_H}{\lambda_H+\lambda_E}\right) - c_2\sqrt{p_H}\right)] \le o(p_H)$.
□

EC.3. Missing proofs for BilateralMatch(E) Policy

The proof of Theorem 2 relies on a concentration result on the left tail of the distribution of $H^{\mathcal{B}_E}$ (Proposition EC.2), and a coupling argument to upper-bound $\mathbf{E}[H^{\mathcal{B}_E}]$ (Proposition EC.3).

Proposition EC.2. [Lower Bound] *Under $\mathcal{B}_E$ in steady-state,*

- *If $\lambda_H < \lambda_E$, there exist a function $\gamma(p_H) = 1 - \sqrt{p_H} + o(\sqrt{p_H})$ and a constant $c_5 \ge 0$ such that for all $k \ge 0$:*

$$\mathbf{P}\left[H^{\mathcal{B}_E} \le \frac{1}{p_E p_H}\left(\ln\left(\frac{\lambda_E}{\lambda_E-\lambda_H}\right) - c_5 p_H^{1/4}\right) - k\right] \le \frac{\gamma(p_H)^k}{1-\gamma(p_H)}.$$

- *If $\lambda_H > \lambda_E$, there exist a function $\gamma'(p_H) = 1 - \sqrt{p_H} + o(\sqrt{p_H})$ and a constant $c_6 \ge 0$ such that for all $k \ge 0$:*

$$\mathbf{P}\left[H^{\mathcal{B}_E} \le \frac{1}{p_H^2}\left(\ln\left(\frac{2\lambda_H}{\lambda_H+\lambda_E}\right) - c_6\sqrt{p_H}\right) - k\right] \le \frac{\gamma'(p_H)^k}{1-\gamma'(p_H)}.$$

Proposition EC.3. [Upper-bound] *Under $\mathcal{B}_E$ and in steady-state,*

- *If $\lambda_H < \lambda_E$, then $\mathbf{E}[H^{\mathcal{B}_E}] \le \frac{\ln\left(\frac{2\lambda_E}{\lambda_E-\lambda_H}\right)}{p_E p_H} + o\left(\frac{1}{p_H}\right)$*
- *If $\lambda_H > \lambda_E$, then $\mathbf{E}[H^{\mathcal{B}_E}] \le \frac{1}{p_H^2}\ln\left(\frac{2\lambda_H}{\lambda_H+\lambda_E}\right) + o\left(\frac{1}{p_H^2}\right)$.*

Proof of Theorem 2 We will first compute a lower bound for $\mathbf{E}[H^{\mathcal{B}_E}]$: Applying Markov inequality to $\mathbf{E}[H^{\mathcal{B}_E}]$ and using Proposition EC.2 with $k = p_H^{-3/4}$, we get the following lower-bound on $\mathbf{E}[H^{\mathcal{B}_E}]$:

- If $\lambda_H < \lambda_E$, then $\mathbf{E}[H^{\mathcal{B}_E}] \geq \frac{\ln\left(\frac{\lambda_E}{\lambda_E - \lambda_H}\right)}{p_E p_H} + o\left(\frac{1}{p_H}\right)$.
- If $\lambda_H > \lambda_E$, then $\mathbf{E}[H^{\mathcal{B}_E}] \geq \frac{\ln\left(\frac{2\lambda_H}{\lambda_H + \lambda_E}\right)}{p_H^2} + o\left(\frac{1}{p_H^2}\right)$.

Using Proposition EC.3, we can get the desired upper-bounds for $\mathbf{E}[H^{\mathcal{B}_E}]$. We can conclude using Little's law: $w_H = \frac{\mathbf{E}[H^{\mathcal{B}_E}]}{\lambda_H}$.

□

EC.3.1. Proof of Proposition EC.2.

In order to prove Proposition EC.2, we first need a concentration result on $E^{\mathcal{B}_E}$ agents. This is stated in Lemma EC.2.

LEMMA EC.2. *Under $\mathcal{B}_E$ in steady-state, for any $\alpha \in [0,1]$, let $n_E(\alpha) = \frac{\ln(\alpha/(1+\alpha(1-p_E^2))}{\ln(1-p_E^2))}$. For any $k \geq 0$, we have:*

$$\mathbf{P}\left[E^{\mathcal{B}_E} \geq n_E(\alpha) + k\right] \leq \frac{\alpha^k}{1-\alpha}.$$

Also, $(1-p_E^2)^{n_E(\alpha)} = \frac{\alpha}{1+\alpha(1-p_E^2)}$.

Proof of Lemma EC.2 In the same way that upper-bounding H agents when they get priority is easy because waiting E agents can be ignored, upper-bounding the number of E agents is easy when they get the priority because H agents can be ignored. We can get the following bounds on the transition probabilities: $Q^{\mathcal{B}_E}([h,e],[h,e+1]) \leq \lambda_E(1-p_E^2)^e = f(e)$, and $Q^{\mathcal{B}_E}([h,e],[h,e-1]) \geq \lambda_E(1-(1-p_E^2)^e) = g(e)$ which leads to:

$$\frac{f(e)}{g(e+1)} = \frac{(1-p_E^2)^e}{1-(1-p_E^2)^{e+1}}.$$

Setting $\eta = n_E(\alpha)$ to be the solution to $\frac{f(n)}{g(n+1)} = \alpha$, and applying Lemma 2 with $[X(t), Y(t)] = [E(t), H(t)]$, and $S(e) = \mathbb{N}$ for all h, $c = 0$ and $\delta = 0$, we get the desired result.

□

We can now prove Proposition EC.2.

Proof of Proposition EC.2 Notice that $Q^{\mathcal{B}_E}([h,e],[h+1,e]) = Q^{\mathcal{B}_H}([h,e],[h+1,e])$, therefore the function $f(h) = \lambda_H(1-p_H^2)^h\left(1-2p_E\sqrt{p_H}\right) + o(\sqrt{p_H})$ used in the proof of Proposition 3 remains a lower bound for $e \in S = \left\{e \leq 2p_H^{-1/2}\right\}$. Furthermore, $Q^{\mathcal{B}_E}([h,e],[h-1,e]) \leq Q^{\mathcal{B}_H}([h,e],[h-1,e])$ therefore the function $g(h) = \lambda_H(1-(1-p_H^2)^h) + \lambda_E(1-(1-p_E p_H)^h)$ remains an upper bound. Therefore, the proof is exactly the same as the proof for the lower bounds for Proposition 3, except that instead of Lemma EC.1 we use Lemma EC.2 in the special case $n_E(\alpha) = p_H^{-1/2}$ with $S = \left\{e \in \mathbb{R}, e \leq 2p_H^{-1/2}\right\}$, and $k = p_H^{-1/2}$. We still have, $\mathbf{P}[E^{\mathcal{B}_H} \notin S] = o(p_H^4)$.

□

EC.3.2. Proof of Proposition EC.3

This proof is based on a coupling argument: instead of analysing the CTMC resulted from $\mathcal{B}_E$, we analyse the CTMC underlying another process that we call $\widetilde{\mathcal{B}_E}$. $\widetilde{\mathcal{B}_E}$ works similarly to $\mathcal{B}_E$ with one crucial difference: each time an E agent arrives which cannot be matched immediately, it changes type and becomes an H agent and then joins the market. First note that under $\widetilde{\mathcal{B}_E}$, no E agent joins the market, making its underlying CTMC a 1-dimensional Markov chain. Proving that the stochastic process underlying $\widetilde{\mathcal{B}_E}$ is a positive recurrent CTMC, and therefore it reaches steady-state is similar to the proof of positive recurrence of $\mathcal{B}_E$ (See Claim EC.3) and therefore omitted.

Using a coupling argument we show that in steady-state number of H agents under $\mathcal{B}_E$ can be upper-bounded by the number of H agents under $\widetilde{\mathcal{B}_E}$. Our motivation behind defining $\widetilde{\mathcal{B}_E}$ is that in $\mathcal{B}_E$, unmatched E agents in the market are competing against H agents over whom they have priority. The idea is that by turning unmatched E agents into H ones, we expect to have more H agents waiting. Let $H^{\widetilde{\mathcal{B}_E}}$ be the random number of H agents under $\widetilde{\mathcal{B}_E}$ in steady-state. First in the next lemma we show that $\mathbf{E}[H^{\mathcal{B}_E}] \leq \mathbf{E}[H^{\widetilde{\mathcal{B}_E}}]+1$, then we compute an upper-bound on $\mathbf{E}[H^{\widetilde{\mathcal{B}_E}}]$.

Lemma EC.3. $\mathbf{E}[H^{\mathcal{B}_E}] \leq \mathbf{E}[H^{\widetilde{\mathcal{B}_E}}]+1$.

Proof of Lemma EC.3 Consider two copies of the arrival process, one under the setting of $\mathcal{B}_E$ and one under $\widetilde{\mathcal{B}_E}$. Let $[H_k^{\mathcal{B}_E}, E_k^{\mathcal{B}_E}]$ and $H_k^{\widetilde{\mathcal{B}_E}}$ denote the embedded (discrete-time) Markov chain resulting from observing the two dynamic systems at arrival epochs. We will prove the following (stronger) result: there exists a coupling such that at any step k, $H_k^{\mathcal{B}_E}+E_k^{\mathcal{B}_E} \leq H_k^{\widetilde{\mathcal{B}_E}}+1$.

We start by coupling all arrivals: with probability $\frac{\lambda_H}{\lambda_E+\lambda_H}$ ($\frac{\lambda_E}{\lambda_E+\lambda_H}$), arrivals to $\mathcal{B}_E$ and $\widetilde{\mathcal{B}_E}$ at k are both H (E) agents. Three cases can happen:

1. $H_k^{\mathcal{B}_E}+E_k^{\mathcal{B}_E} < H_k^{\widetilde{\mathcal{B}_E}}$; in this case, we let the two chains evolve independently.
2. $H_k^{\widetilde{\mathcal{B}_E}} \leq H_k^{\mathcal{B}_E}+E_k^{\mathcal{B}_E} \leq H_k^{\widetilde{\mathcal{B}_E}}+1$. Let $h = H_k^{\widetilde{\mathcal{B}_E}}$. We consider two sub-cases:
   (a) The arrival at $k+1$ is an H agent. We couple the events that $\widetilde{\mathcal{B}_E}$ and $\mathcal{B}_E$ cannot find a match as follows: we draw two independent Bernoulli random variables $\mathbb{B}_1$, $\mathbb{B}_2$ with respective parameters of $(1-p_H^2)^h$ and $(1-p_H^2)^{H_k^{\mathcal{B}_E}}(1-p_Hp_E)^{E_k^{\mathcal{B}_E}}(1-p_H^2)^{-h}$. $\mathbb{B}_1=1$ corresponds to the event that $\widetilde{\mathcal{B}_E}$ cannot find a match; similarly $\mathbb{B}_1\mathbb{B}_2=1$ corresponds to the event that $\mathcal{B}_E$ cannot find a match.
   (b) The arrival at $k+1$ is an E agent. Similarly we couple the events that $\widetilde{\mathcal{B}_E}$ and $\mathcal{B}_E$ cannot find a match as follows: we draw two independent Bernoulli random variables $\mathbb{B}_3$, $\mathbb{B}_4$ with respective parameters of $(1-p_Ep_H)^h$ and $(1-p_Ep_H)^{H_k^{\mathcal{B}_E}}(1-p_E^2)^{E_k^{\mathcal{B}_E}}(1-p_Ep_H)^{-h}$. $\mathbb{B}_3=1$ corresponds to the event that $\widetilde{\mathcal{B}_E}$ cannot find a match. Similarly $\mathbb{B}_3\mathbb{B}_4=1$ corresponds to the event that $\mathcal{B}_E$ cannot find a match.

3. $H_k^{\mathcal{B}_E} + E_k^{\mathcal{B}_E} > H_k^{\widetilde{\mathcal{B}_E}} + 1$; in this case, we let the two chains evolve independently.

Having the above coupling, we finish the proof by induction. The base case $k=0$ is trivial: $H_0^{\mathcal{B}_E} + E_0^{\mathcal{B}_E} = H_0^{\widetilde{\mathcal{B}_E}} = 0$. Suppose $H_k^{\mathcal{B}_E} + E_k^{\mathcal{B}_E} \le H_k^{\widetilde{\mathcal{B}_E}} + 1$, we show that $H_{k+1}^{\mathcal{B}_E} + E_{k+1}^{\mathcal{B}_E} \le H_{k+1}^{\widetilde{\mathcal{B}_E}} + 1$: In Case (1), note that the number of agents waiting can increase or decrease by at most 1. For both cases, we have $H_{k+1}^{\mathcal{B}_E} + E_{k+1}^{\mathcal{B}_E} \le H_{k+1}^{\widetilde{\mathcal{B}_E}} + 1$. In Case (2a), we have:

- If $\mathbb{B}_1 = 0$, then $H_{k+1}^{\widetilde{\mathcal{B}_E}} = H_k^{\widetilde{\mathcal{B}_E}} - 1$. Further if $\mathbb{B}_1 = 0$ then $\mathbb{B}_1\mathbb{B}_2 = 0$, and $H_{k+1}^{\mathcal{B}_E} + E_{k+1}^{\mathcal{B}_E} = H_k^{\mathcal{B}_E} + E_k^{\mathcal{B}_E} - 1$. Therefore $H_{k+1}^{\mathcal{B}_E} + E_{k+1}^{\mathcal{B}_E} \le H_{k+1}^{\widetilde{\mathcal{B}_E}} + 1$

- If $\mathbb{B}_1 = 1$, then $H_{k+1}^{\widetilde{\mathcal{B}_E}} = H_k^{\widetilde{\mathcal{B}_E}} + 1$. Note that number of agents under $\mathcal{B}_E$ can increase by at most one, therefore $H_{k+1}^{\mathcal{B}_E} + E_{k+1}^{\mathcal{B}_E} \le H_{k+1}^{\widetilde{\mathcal{B}_E}} + 1$.

Similarly in Case (2b), we have:

- If $\mathbb{B}_3 = 0$, then $H_{k+1}^{\widetilde{\mathcal{B}_E}} = H_k^{\widetilde{\mathcal{B}_E}} - 1$. Further if $\mathbb{B}_3 = 0$ then $\mathbb{B}_3\mathbb{B}_4 = 0$, and $H_{k+1}^{\mathcal{B}_E} + E_{k+1}^{\mathcal{B}_E} = H_k^{\mathcal{B}_E} + E_k^{\mathcal{B}_E} - 1$; therfore $H_{k+1}^{\mathcal{B}_E} + E_{k+1}^{\mathcal{B}_E} \le H_{k+1}^{\widetilde{\mathcal{B}_E}} + 1$

- If $\mathbb{B}_3 = 1$, then $H_{k+1}^{\widetilde{\mathcal{B}_E}} = H_k^{\widetilde{\mathcal{B}_E}} + 1$. Note that number of agents under $\mathcal{B}_E$ can increase by at most one, therefore and $H_{k+1}^{\mathcal{B}_E} + E_{k+1}^{\mathcal{B}_E} \le H_{k+1}^{\widetilde{\mathcal{B}_E}} + 1$.

Thus, in all possible cases $H_{k+1}^{\mathcal{B}_E} + E_{k+1}^{\mathcal{B}_E} \le H_{k+1}^{\widetilde{\mathcal{B}_E}} + 1$ finishing the proof. Note that the above induction also implies that Case (3) never occurs.

□

Proof of Proposition EC.3 Observe that the CTMC $H_t^{\widetilde{\mathcal{B}_E}}$ has the following rate matrix:

$$Q^{\widetilde{\mathcal{B}_E}}(h, h+1) = \lambda_H (1 - p_H^2)^h + \lambda_E (1 - p_E p_H)^h$$
$$Q^{\widetilde{\mathcal{B}_E}}(h, h-1) = \lambda_H (1 - (1 - p_H^2)^h) + \lambda_E (1 - (1 - p_E p_H)^h)$$

And let us define:

$$\rho(h) = \frac{Q^{\widetilde{\mathcal{B}_E}}(h, h+1)}{Q^{\widetilde{\mathcal{B}_E}}(h+1, h)} = \frac{\lambda_H (1 - p_H^2)^h + \lambda_E (1 - p_E p_H)^h}{\lambda_H (1 - (1 - p_H^2)^{h+1}) + \lambda_E (1 - (1 - p_E p_H)^{h+1})}.$$

Note that $\rho(h)$ is a decreasing function, and suppose that there exists $\eta \ge 0$ such that $\rho(\eta) < 1 - \sqrt{p_H}$ and let $\pi = \pi(h)_{h \ge 0}$ be the stationnary distribution of $H^{\widetilde{\mathcal{B}_E}}$. Then we have for all $i \ge 0$, $\pi(\eta + i) \le \rho(\eta)^i \pi(\eta) \le \rho(\eta)^i$ and for all $k \ge 0$, $\mathbf{P}[H^{\widetilde{\mathcal{B}_E}} \ge \eta + k] \le \pi(\eta) \sum_{i \ge k} \rho(\eta)^i \le \frac{\rho(\eta)^k}{1 - \rho(\eta)}$. We then have

$$\mathbf{E}[H^{\widetilde{\mathcal{B}_E}}] = \sum_{h \ge 1} \mathbf{P}[H^{\widetilde{\mathcal{B}_E}} \ge h]$$
$$\mathbf{E}[H^{\widetilde{\mathcal{B}_E}}] \le (\eta + k) + \sum_{h \ge \eta + k} \mathbf{P}[H^{\widetilde{\mathcal{B}_E}} \ge h]$$
$$\mathbf{E}[H^{\widetilde{\mathcal{B}_E}}] \le (\eta + k) + \frac{\rho(\eta)^k}{(1 - \rho(\eta))^2}$$

Let us consider the case when $\lambda_H < \lambda_E$. Let $\eta = \frac{\ln(1/u) + c\sqrt{p_H}}{p_E p_H}$. For $h \ge \eta$, we have $(1 - p_E p_H)^h \le u\left(1 - c\sqrt{p_H} + o(\sqrt{p_H})\right)$ and $(1 - p_H^2)^h \le 1 - \ln(1/u) p_H^2 + o(p_H^2)$. We have $\rho(\eta) = \frac{\lambda_H + \lambda_E u(1 - c\sqrt{p_H})}{\lambda_E (1 - u(1 - c\sqrt{p_H}))} +$

$o(\sqrt{p_H})$. Taking $u = \frac{\lambda_E - \lambda_H}{2\lambda_E}$ and $c = \frac{\lambda_E+\lambda_H}{2(\lambda_E-\lambda_H)}$, we get $\rho(\eta) = 1 - \sqrt{p_H} + o(\sqrt{p_H})$. Taking $k = p_H^{-3/4}$ we get:

$$\mathbf{E}[H^{\widetilde{\mathcal{B}_E}}] \le \eta + o(1/p_H) = \frac{\ln(\frac{2\lambda_E}{\lambda_E-\lambda_H})}{p_E p_H} + o(1/p_H), \quad \lambda_H < \lambda_E. \tag{EC.1}$$

In the case when $\lambda_H > \lambda_E$, let $\eta = \frac{\ln(1/u)+c\sqrt{p_H}}{p_H^2}$. For $h \ge \eta$, we have $(1-p_E p_H)^h = o(p_H)$ and $(1-p_H^2)^h \le u(1-c\sqrt{p_H}) + o(\sqrt{p_H})$. Taking $u = \frac{\lambda_H+\lambda_E}{2\lambda_H}$ and $c = 1/2$, we get $\rho(\eta) = 1 - \sqrt{p_H} + o(\sqrt{p_H})$. Taking $k = p_H^{-3/4}$ we get:

$$\mathbf{E}[H^{\widetilde{\mathcal{B}_E}}] \le \eta + o(1/p_H^2) = \frac{\ln(\frac{2\lambda_H}{\lambda_E+\lambda_H})}{p_H^2} + o(1/p_H^2), \quad \lambda_H > \lambda_E. \tag{EC.2}$$

Putting Lemma EC.3 and upper-bounds given in (EC.1) and (EC.2) together finishes the proof.
□

EC.4. Missing proofs for ChainMatch(d) Policy

Proof of Proposition 1 As pointed at the end of Subsection 5.4, when $p_E = 1$, an arriving E agent is matched immediately by a bridge agent, implying that $E_t^{\mathcal{C}(d)} = 0$ and $H_t^{\mathcal{C}(d)} = H_t^{\hat{\mathcal{C}}(d)}$; consequently Proposition 5 implies the limit stated in the Proposition 1.

Further note that fixing λ_H, it is straightforward to check that function $f(\lambda_E) = \frac{\ln\left(\frac{\lambda_H}{\lambda_E}+1\right)}{\lambda_H}$ is decreasing for $\lambda_E > 0$. Similarly fixing λ_E, it is straightforward to check that function $g(\lambda_H) = \frac{\ln\left(\frac{\lambda_H}{\lambda_E}+1\right)}{\lambda_H}$ is decreasing for $\lambda_H > 0$.
□

Proof of Corollary 3 First we observe that $\ln\left(\frac{\lambda_H}{\lambda_E(1-(1-p_E)^d)}+1\right)$ is decreasing in d. Therefore the worst upper-bound on $w_H^{\mathcal{C}(d)}$ is for $d = 1$. Next we show that:

$$\begin{aligned}\ln\left(\frac{\lambda_H}{\lambda_E p_E}+1\right)^{p_E} &\le \ln\left(\frac{\lambda_H}{\lambda_E}+1\right)\\ &\le \ln\left(\frac{\lambda_H}{\lambda_E-\lambda_H}+1\right),\end{aligned}$$

where the first inequality holds because function $f(x) := (\frac{\lambda_H/\lambda_E}{x}+1)^x$ is increasing in $x \in (0,1]$.
□

Proof of Proposition 2 In steady-state, for any function $f(\cdot,\cdot)$, $0 = \mathbb{E}[f(H_k,E_k) - f(H_{k+1},E_{k+1})]$, and in particular for $f(h,e) = h+e$, we get

$$\begin{aligned}0 &= (\lambda_H(1-p_H)^d + \lambda_E(1-p_E)^d)\mathbb{E}[H_k+E_k-H_{k+1}-E_{k+1} | H_k+E_k < H_{k+1}+E_{k+1}]\\ &\quad + (\lambda_H(1-(1-p_H)^d) + \lambda_E(1-(1-p_E)^d))\mathbb{E}[H_k+E_k-H_{k+1}-E_{k+1} | H_k+E_k \ge H_{k+1}+E_{k+1}]\\ &\quad = -(\lambda_H(1-p_H)^d + \lambda_E(1-p_E)^d) + (\lambda_H(1-(1-p_H)^d) + \lambda_E(1-(1-p_E)^d))\mathbf{E}[L-1 \mid L \ge 1].\end{aligned} \tag{EC.3}$$

This gives us:

$$\mathbb{E}[L \mid L \geq 1] = \frac{\lambda_H(1-p_H)^d + \lambda_E(1-p_E)^d}{(\lambda_H(1-(1-p_H)^d) + \lambda_E(1-(1-p_E)^d))} + 1 = \frac{\lambda_H + \lambda_E(1-p_E)^d}{\lambda_E(1-(1-p_E)^d)} + 1 + o(1).$$

□

EC.5. Heuristic argument for ChainMatch(d) Policy

In what follows we provide a heuristic analysis of the CTMC underlying the *ChainMatch(d)*. This heuristic adds intuition for the behavior of the policy and further establishes what is supposedly the right constant in the case in which our theory only generates an upper bound ($p_E < 1$).

We introduce an auxiliary 3-dimensional CTMC, in which a chain-segment is not formed instantaneously; instead a chain-segment can only advance at certain "tokens" (or times) that arrive according to a Poisson process with rate μ. Note that this process is independent from the Poisson processes guiding the arrivals of H and E agents. We denote this auxiliary CTMC by $\widetilde{\mathcal{C}}(d,\mu)$ and its states by $[H_t^{\widetilde{\mathcal{C}}(d,\mu)}, E_t^{\widetilde{\mathcal{C}}(d,\mu)}, U_t] \in \mathbb{N}^2 \times \{0,1\}$. The first two dimensions represent the number of H and E agents. The third dimension U_t indicates whether a chain-segment is being conducted. Initially $U_0 = 0$. Suppose at time t_1 the first agent i arrives that is matched by an altruistic agent. At this time U_{t_1} becomes 1 indicating that a chain-segment formation is in process. The policy waits x unit of time for a token to arrive to find an agent who can be matched by i (note that x is exponentially distributed with rate μ) Note that it is possible that there is a new arrivals of H and/or E agents in the interval $(t_1, t_1 + x]$. In this heuristic, we assume that the arriving H or E agent is deleted. The intuition is that when μ is large, this event happens rarely. If such an agent does not exist, then chain-segment is stopped; and U_{t_1+x} becomes 0. Otherwise U_{t_1+x} remains 1 and the same process repeats. The transition rates of $\widetilde{\mathcal{C}}(d,\mu)$ are:

$$Q^{\widetilde{\mathcal{C}}(d,\mu)}([h,e,0],[h+1,e,0]) = \lambda_H(1-p_H)^d \tag{EC.4a}$$

$$Q^{\widetilde{\mathcal{C}}(d,\mu)}([h,e,0],[h,e+1,0]) = \lambda_E(1-p_E)^d \tag{EC.4b}$$

$$Q^{\widetilde{\mathcal{C}}(d,\mu)}([h,e,0],[h,e,1]) = \lambda_H(1-(1-p_H)^d) + \lambda_E(1-(1-p_E)^d) \tag{EC.4c}$$

$$Q^{\widetilde{\mathcal{C}}(d,\mu)}([h,e,1],[h,e,0]) = \mu(1-p_H)^h(1-p_E)^e \tag{EC.4d}$$

$$Q^{\widetilde{\mathcal{C}}(d,\mu)}([h,e,1],[h-1,e,1]) = \mu\left(1-(1-p_H)^h\right) \tag{EC.4e}$$

$$Q^{\widetilde{\mathcal{C}}(d,\mu)}([h,e,1],[h,e-1,1]) = \mu(1-p_H)^h\left(1-(1-p_E)^e\right) \tag{EC.4f}$$

Figure EC.1 illustrates the transitions of $\widetilde{\mathcal{C}}(d,\mu)$ in $H_t^{\widetilde{\mathcal{C}}(d,\mu)}$ and U_t dimensions along with the transition rates. The rate (EC.4a) ((EC.4b)) corresponds to the event that an H (E) agent arrives

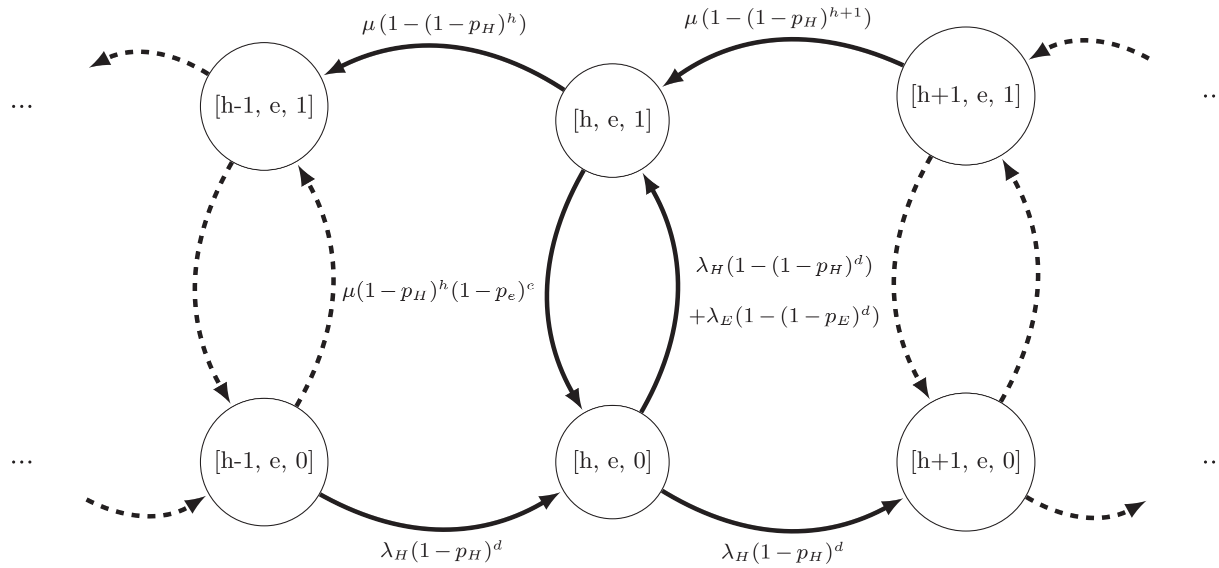

Figure EC.1 Transitions for the CTMC underlying $\widetilde{\mathcal{C}}(d,\mu)$ in the first and third dimensions, i.e, transitions to states with a different number of E agents are not shown. Transition rates are given only for solid arrows.

but cannot be matched by any bridge agent which happens with probability $(1-p_H)^d$ ($(1-p_E)^d$). The rate (EC.4c) corresponds to the case an H arrives and starts a chain-segment (occurs with probability $(1-(1-p_H)^d)$) or an E agent arrives and starts a chain-segment (occurs with probability $(1-(1-p_E)^d)$; similarly (EC.4d) represents the case where the chain-segment cannot advance any further (happens with probability $(1-p_H)^h(1-p_E)^e$). The last two rates correspond to adding one more H and E agent to the chain-segment respectively (with probability $(1-(1-p_H)^h)$ and $(1-p_H)^h(1-(1-p_E)^e)$ respectively).

Similar to the heuristic argument for *BilateralMatch(H)* and *BilateralMatch(E)*, we can try to solve the following system of nonlinear equations (that result from setting expected drifts in all three dimensions to be zero):

$$\mathbf{E}\left[\left(\lambda_H(1-(1-p_H)^d)+\lambda_E(1-(1-p_E)^d)\right)\mathbb{I}(U=0)-\mu(1-p_H)^{H^{\widetilde{\mathcal{C}}(d,\mu)}}(1-p_E)^{E^{\widetilde{\mathcal{C}}(d,\mu)}}\mathbb{I}(U=1)\right]=0$$
$$\mathbf{E}\left[\lambda_E(1-p_E)^d\mathbb{I}(U=0)-\mu(1-p_H)^{H^{\widetilde{\mathcal{C}}(d,\mu)}}(1-(1-p_E)^{E^{\widetilde{\mathcal{C}}(d,\mu)}})\mathbb{I}(U=1)\right]=0$$
$$\mathbf{E}\left[\lambda_H(1-p_H)^d\mathbb{I}(U=0)-\mu(1-(1-p_H)^{H^{\widetilde{\mathcal{C}}(d,\mu)}})\mathbb{I}(U=1)\right]=0,$$

where $\mathbb{I}(\cdot)$ is the indicator function. We find that $\ln\left(\frac{\lambda_H+\lambda_E}{\lambda_H(1-(1-p_H)^d)+\lambda_E}\right)/p_H$ is an approximate solution for $\mathbf{E}[H^{\widetilde{\mathcal{C}}(d,\mu)}]$. Further, note that in $\widetilde{\mathcal{C}}(d,\mu)$ if an H/E agent arrives while a chain-segment is being formed then the agent will not join the market. However, probability of such an event vanishes as $\mu\to\infty$ (i.e, forming chain-segments becomes instantaneous). Therefore, it is reasonable to approximate $\mathbf{E}[H^{\mathcal{C}(d)}]$ with $\lim_{\mu\to\infty}\mathbf{E}[H^{\widetilde{\mathcal{C}}(d,\mu)}]$, and thus with $\ln\left(\frac{\lambda_H+\lambda_E}{\lambda_H(1-(1-p_H)^d)+\lambda_E}\right)/p_H$. Finally note that:

$$\lim_{p_H\to 0}\ln\left(\frac{\lambda_H+\lambda_E}{\lambda_H(1-(1-p_H)^d)+\lambda_E}\right)=\ln\left(\frac{\lambda_H+\lambda_E}{\lambda_E}\right),$$

which is the constant in Proposition 1.

We emphasize that the analysis above is only a heuristic for guessing the right constant when $p_E<1$. We refer the reader to Section 4.5 for numerical simulations that show the tightness of this constant.

EC.6. Intuition and Another Heuristic for Limit of $w_H^{\mathcal{C}(d)}$

In this section, we provide another heuristic argument that explains why p_E and d do not impact waiting time in the limit.

Let us look at the formation of a chain-segment starting with an E agent; it will have the following structure: $E-H-\ldots-H-E-H-\ldots-H-E-H-\ldots-H$; in other words, the chain-segment can be divided into sub-segments, each starting with an E agent, and following by a number of H agents until the sub-segment gets "stuck", meaning that there is no H agent that can continue the sub-segment (Recall that priority is given to H agents; therefore we will not search among E agents as long as there is an H agent that can continue the sub-segment). At that stage, the policy will look for an E agent to start a new sub-segment. Let Σ_E be the number of H agents in each sub-segment (between two E agents).

Next, let us look at the formation of a chain-segment starting with an H agent; the chain-segment has the following structure: $H-H-\ldots-H-E-H-\ldots-H-E-H-\ldots-H$, i.e, the first sub-segment starts with an H, but all the subsequent ones start with an E. Let us denote the expected number of H agents in the very first sub-segment by Σ_H.

With the above, let us consider the rate at which H agents join the market and the rate at which they depart. The former is simply $\lambda_H(1-p_H)^d$. The latter is $\lambda_E\mathbf{E}[\Sigma_E]+\lambda_H[1-(1-p_H)^d](\mathbf{E}[\Sigma_H]-1)$. Thus by conservation of H agents, we have:

$$(1-p_H)^d\lambda_H=\lambda_E\mathbf{E}[\Sigma_E]+\lambda_H[1-(1-p_H)^d](\mathbf{E}[\Sigma_H]-1).$$

Note that Proposition 2 - which shows that the expected length of a chain-segment is a constant - implies that $\mathbf{E}[\Sigma_H]$ is a constant, because $\mathbf{E}[\Sigma_H]\le\mathbf{E}[L|L\ge 1]$, and Proposition 2 states that $\mathbf{E}[L|L\ge 1]$ remains a constant as $p_H\to 0$. Therefore, we can conclude that:

$$\mathbf{E}[\Sigma_E]=\lambda_H/\lambda_E+o(1),$$

i.e., in the limit $p_H\to 0$, almost all H agents match through sub-segments initiated by an E, and the length of such sub-segment only depends on the arrival rates. In other words, if we view the number of H agents that an E agent can help match as the "usefulness" of the E, this already suggests that (in the limit) the usefulness of an E does not depend on p_E nor on d.

However, we note that the time at which an E agent initiates a sub-segment changes with p_E:

• When $p_E = 1$, an arriving E agent initiates the first (and only) sub-segment of the chain-segment.

• When $p_E < 1$:

—With probability $(1-(1-p_E)^d)$, the E agent is matched by the bridge agent, and initiates the 1-st sub-segment.

—With probability $(1-p_E)^d$, the E agent joins the market, and initiates a sub-segment (that is not the first) at a later time.

Next, we analyze $\mathbf{E}[\Sigma_E]$ and relate it to the expected number of H agents. Recall that we denote the random number of H agents in steady-state by $H^{\mathcal{C}(d)}$. Consider the first sub-segment formed upon arrival of an E agent that receives from the bridge agent. We denote this sub-segment by Σ_1. Observe that

$$\mathbf{E}[\Sigma_1 | H^{\mathcal{C}(d)} = h] = \sum_{i=1}^{h} \prod_{j=0}^{i-1} (1-(1-p_H)^{h-j}) := g(H^{\mathcal{C}(d)} = h). \tag{EC.6}$$

Note that this equation is the direct consequence of giving priority to H agents. Therefore, $\mathbf{E}[\Sigma_1] = \mathbf{E}[\mathbf{E}[\Sigma_1 | H^{\mathcal{C}(d)} = h]] = \mathbf{E}[g(H^{\mathcal{C}(d)})]$. Note that $g(H^{\mathcal{C}(d)})$ does not depend on p_E or d. We make the following observations:

1. Function $g(\cdot)$ is a monotone increasing function, therefore it has a well-defined inverse function. Further, we have:

   Claim EC.1. *For $y > x$, we have $g(y) \geq g(x) + \ln\left(\frac{1}{1-p_H}\right)(y-x)$.*

   The above claim implies that if $g(y) - g(x) = o(1)$, then $y - x = o(1/p_H)$. The proof of the claim follows the same induction steps as that of Lemma EC.4; note that function $g(\cdot)$ has the same form as function $f(\cdot)$ in that lemma. Therefore, we omit the proof of the above claim.
2. For p_H small enough, $\mathbf{E}[g(H^{\mathcal{C}(d)})] = g(\mathbf{E}[H^{\mathcal{C}(d)}]) + o(1)$. Even though we cannot prove this concentration result, in Figure EC.2, we provide numerical simulation verifying it for different p_E's.

Finally we claim that for p_H small enough, $\mathbf{E}[\Sigma_E] \approx \mathbf{E}[\Sigma_1]$; this follows from two observations:

1. $\mathbf{E}[\Sigma_1]$ is a constant; to see this, we use Proposition 2 which shows that the expected length of a chain-segment is a constant. Given that $\mathbf{E}[\Sigma_1] \leq \mathbf{E}[L | L \geq 1]$, we can conclude that $\mathbf{E}[\Sigma_1]$ is also a constant (L denotes the chain-segment length as it is defined above Proposition 2). Therefore when the second sub-segment is being formed, the number of H agents in the market

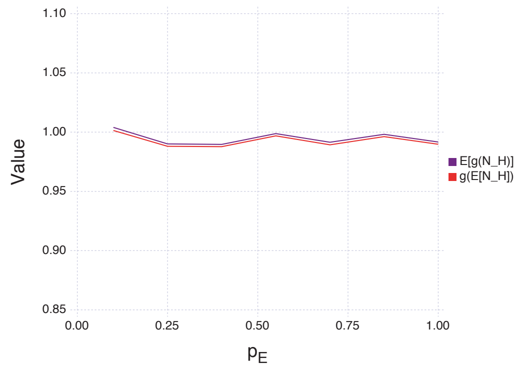

Figure EC.2 Values of $g(\mathbf{E}[H^{\mathcal{C}(d)}])$ and $\mathbf{E}[g(H^{\mathcal{C}(d)})]$ as a function of p_E, for $p_H = 0.002$, $\lambda_H = 2$, $\lambda_E = 1$, $d = 1$ and $T = 10^6$.

will be $H^{\mathcal{C}(d)} - \Theta(1) \approx H^{\mathcal{C}(d)}$ (remember that $H^{\mathcal{C}(d)} = \Theta(1/p_H)$ following the upper-bound of Theorem 3 and the lower-bound of Appendix EC.9).

2. The number of sub-segments is a constant: this follows from the observation that number of E agents in the market is an upper-bound on the number of sub-segments. In Lemma EC.4, we show that the expected number of E agents is a constant (independent of p_H).

Combining the above observations, for sufficiently small p_H, we have that:

$$g(\mathbf{E}[H^{\mathcal{C}(d)}(p_E = 1)]) + o(1) = \lambda_H/\lambda_E + o(1) = g(\mathbf{E}[H^{\mathcal{C}(d)}(p_E < 1)]) + o(1), \tag{EC.7}$$

where with a slight abuse of notation we denoted the number of H agents in the market with parameter p_E by $H^{\mathcal{C}(d)}(p_E)$. Since $g(\cdot)$ has an inverse and using Claim EC.1, (EC.7) implies that:

$$H^{\mathcal{C}(d)}(p_E = 1) = H^{\mathcal{C}(d)}(p_E < 1) + o(1/p_H). \tag{EC.8}$$

We believe that the approximation becomes exact in the limit $p_H \to 0$. Even though the above argument is not rigorous, we hope that it sheds light on why p_E and d do not impact waiting time in the limit.

EC.7. Positive recurrence proofs

To prove existence of a stationary distribution, we use a special case of a result from Meyn and Tweedie (1993), as stated in Prieto-Rumeau and Hernández-Lerma (2016).

THEOREM EC.1 **(Meyn and Tweedie (1993))**. *Suppose that X_t is an irreducible continuous time Markov chain, and suppose that there exist a nonnegative function V on S, a function $w \geq 1$ on S, a finite set $C \subset S$, and constants $c > 0$ and $b \in \mathbb{R}$ such that*

$$\sum_{j \in S} q_{ij} V(j) \leq -cw(i) + b \cdot \mathbb{I}_C(i) \text{ for all } i \in S,$$

where $\mathbb{I}_C$ *denotes the indicator function of the set* C. *Then the Markov chain* X *is ergodic.*

It is clear that all four markov chains are irreducible, so our proofs will focus on finding a suitable set C and function V for each case.

EC.7.1. Positive recurrence of $\mathcal{B}_H$ and $\mathcal{B}_E$.

Claim EC.2. *The CTMC* $[H_t^{\mathcal{B}_H}, E_t^{\mathcal{B}_H}]$ *defined in* (3a), (3b), (3c), (3d), *is positive recurrent.*

Proof of Claim EC.2 Let $V([h,e]) = h + e$. Observe that for a continuous time random walk, we can write for any state $i = [h,e]$:

$$\begin{aligned}
\sum_{j \in \mathbb{N}^2} q_{i,j} V(j) &= \sum_{j \neq i} q_{i,j}(V(j) - V(i)) \\
&= Q([h,e],[h+1,e]) - Q([h,e],[h-1,e]) + Q([h,e],[h,e+1]) - Q([h,e],[h,e-1]). \\
&= \lambda_H(1-p_H^2)^h(1-p_E p_H)^e - \lambda_H(1-(1-p_H^2)^h) - \lambda_E(1-(1-p_E p_H)^h) + \\
&\lambda_E(1-p_E p_H)^h(1-p_E^2)^e - \lambda_H(1-p_H^2)^h(1-(1-p_E p_H)^e) - \lambda_E(1-p_E p_H)^h(1-(1-p_E^2)^e)) \\
&= 2\lambda_H(1-p_H^2)^h(1-p_E p_H)^e - \lambda_H + 2\lambda_E(1-p_E p_H)^h(1-p_E^2)^e - \lambda_E.
\end{aligned}$$

Take $M(p_H)$ such that $(1-p_H^2)^{M(p_H)} = \frac{1}{3}$, and let $C = \{[x,y] \text{ s.t. } x + y \leq 2M(p_H)\}$, note that C is finite. For any $[h,e] \notin C$, either $h \geq M(p_H)$ or $e \geq M(p_H)$. In both cases, we have $\sum_{j \in \mathbb{N}^2} q_{i,j} V(j) \leq -\frac{\lambda_H + \lambda_E}{3}$.

□

Claim EC.3. *The CTMC* $[H_t^{\mathcal{B}_E}, E_t^{\mathcal{B}_E}]$ *defined in* (5a), (5b), (5c), (5d) *is irreducible and positive recurrent.*

The proof follows similar ideas as Claim EC.2.

EC.7.2. Positive recurrence of $\mathcal{C}(d)$ and $\hat{\mathcal{C}}(d)$

Claim EC.4. *Under* $\mathcal{C}(d)$, *the number of* H *and* E *agents* $[H_t^{\mathcal{C}(d)}, E_t^{\mathcal{C}(d)}]$ *is a positive recurrent CTMC.*

Proof of Claim EC.4 Similarly to the proof of Claim EC.2, let $V([h,e]) = h+e$, and $C = \{[x,y] \text{ s.t. } x+y \leq 2M\}$ for an appropriately chosen M. Consider a state $[h,e] \notin C$, and assume first that $h \geq M$. Denoting $\Lambda = \lambda_H(1-p_H)^d + \lambda_E(1-p_E)^d$, we have:

$$\begin{aligned}
\sum_{j\in\mathbb{N}^2} q_{i,j}V(j) &= \Lambda - (\lambda_H+\lambda_E-\Lambda) \sum_{[k_H,k_E]\leq[h,e]} (k_H+k_E)\mathbf{P}[k_H(k_E) \text{ H (E) agents get matched}] \\
&\leq \Lambda - (\lambda_H+\lambda_E-\Lambda) \sum_{[k_H]\leq[h,e]} k_H\mathbf{P}[k_H \text{ H agents get matched}] \\
&\leq \Lambda - (\lambda_H+\lambda_E-\Lambda) \sum_{k\leq h}\prod_{i=h}^{h-k}(1-(1-p_H)^i) \\
&\leq \Lambda - (\lambda_H+\lambda_E-\Lambda) \sum_{k\leq M/2}(1-(1-p_H)^{M/2})^k
\end{aligned}$$

Going from the second to the third inequality bounds the number of agents matched with the number of agents matched in the first sub-chain-segment (before matching an E agent). Because the function $M \to \sum_{k\leq M/2}(1-(1-p_H)^{M/2})^k$ tends to infinity as M grows large, we can find M such that $\sum_{j\in\mathbb{N}^2} q_{i,j}V(j) \leq -1$, which concludes the proof.

The case where $h < M$ and $e \geq M$ can be treated similarly using the following observation: every-time the chain cannot match to any H agent, it tries to match an E, and it succeeds with probability $1-(1-p_E)^e$ irrespective of h. Put together, these events constitute a set of E agents that are matched sequentially, and the total length of the chain is more than the number of E agents selected in that way.

□

Claim EC.5. $H_t^{\hat{\mathcal{C}}(d)}$ *is a positive recurrent CTMC.*

The proof follows similar arguments as that of Claim EC.4.

EC.8. Lower Bound

Definition EC.1. We call a matching policy *anonymous Markovian* if matching decisions are made at arrival epochs, and only depend on the current compatibility graph $\mathcal{G}_t = (\mathcal{V}_t, \mathcal{E}_t)$, and are anonymous among agents of the same type. In that case, the market $\mathcal{G}_t$ is a continuous-time Markov chain.

We say that an anonymous Markovian policy $\mathcal{P}$ is *stable* if the resulting Markov chain is ergodic and has a stationary distribution.

In line with our previous notation, we denote $w_H^{\mathcal{P}}$ ($w_E^{\mathcal{P}}$) the expected stationary waiting times for H and E agents under policy $\mathcal{P}$.

Remark EC.1. Observe that all the matching policies described in this paper are anonymous Markovian.

PROPOSITION EC.4. *For any $p_E \in [0,1]$, $\lambda_H > 0, \lambda_E \geq 0$, there exists a constant c such that for any $p_H > 0$, under any stable anonymous Markovian matching policy $\mathcal{P}$, $w_H^{\mathcal{P}} + w_E^{\mathcal{P}} \geq \frac{c}{p_H}$.*

The proof follows ideas used in Anderson et al. (2017). The main intuition is the following: Suppose the market size is too small, then an arriving agent has to wait a *long* time to obtain at least one incoming edge. This long waiting time contradicts the small market size (with Little's law).

Proof of Proposition EC.4 In this proof, we fix a Markovian policy $\mathcal{P}$, and we will omit the superscript notations. Observe that under $\mathcal{P}$, the market $\mathcal{G}_t$ only evolves at arrival epochs, and we can analyze the embedded discrete-time Markov chain resulting from observing the system at arrival epochs which we denote by $\mathcal{G}_i$. Let $n = \mathbf{E}[|\mathcal{V}|] = \mathbf{E}[H] + \mathbf{E}[E]$ be the expected size of the market in steady-state. Let us denote $\theta = \frac{\lambda_H}{\lambda_H + \lambda_E}$, and let $\hat{w}_H$ be the expected number of (discrete) time steps that an H agent arriving in steady-state has to wait before getting matched. Little's law for discrete Markov chains implies that $\hat{w}_H = \mathbf{E}[H]/\theta$.

Note that it is enough to prove that there exists a constant k such that $n \geq k/p_H$ (we then choose $c = \frac{k}{\max(\lambda_H, \lambda_E)}$). Let k be a constant to be defined later. Assume for contradiction that there exists p_H such that $n < k/p_H$. Let a be an H agent entering the market in steady-state. Let $\mathcal{V}_a$ be the set of agents in the market when agent a arrives. Note that we assumed $\mathbb{E}[|\mathcal{V}_a|] = n < k/p_H$. Define the event $E_1 = \{|\mathcal{V}_a| \leq 3n/\theta\}$. By Markov's inequality, $\mathbb{P}[E_1] \geq 1 - \frac{\mathbb{E}[|\mathcal{V}_a|]\theta}{3n} \geq 1 - \theta/3$.

Let $\mathcal{A}_a$ be the first $3n/\theta$ arrivals after a, and let E_2 be the event that at least one agent from $\mathcal{V}_a \cup \mathcal{A}_a$ has an outgoing edge towards a. We have

$$\mathbb{P}[E_2] = \mathbb{P}[\mathbf{Bin}(|\mathcal{V}_a| + |\mathcal{A}_a|, p_H) \geq 1].$$

Note that we abuse the notation of **Bin(u,p)** by allowing its parameters to be random variables. In this case, conditional on the event $|\mathcal{V}_a| + |\mathcal{A}_a| = u$, the random variable $\mathbf{Bin}(|\mathcal{V}_a| + |\mathcal{A}_a|, p)$ has a binomial distribution with parameters u and p. Therefore we get:

$$\mathbb{P}[E_2 | E_1] \leq \mathbb{P}[\mathbf{Bin}(6n/\theta, p_H) \geq 1] \leq \mathbb{P}[\mathrm{Bin}(6k/\theta p_H, p_H) \geq 1] \leq 6k/\theta.$$

Where the first inequality derives from the definition of E_1, the second uses the fact that $n \leq k/p_H$ and the third is Markov's inequality.

We now use the fact that if a does not have any edge from either $\mathcal{V}_a$ or $\mathcal{A}_a$, then she must wait longer than $3n/\theta$ time steps.

$$\hat{w}_H \geq \frac{3n}{\theta}\mathbb{P}[E_2^c] \geq \frac{3n}{\theta}\mathbb{P}[E_2^c|E_1]\mathbb{P}[E_1] \geq \frac{3n}{\theta}(1 - 6k/\theta)(1 - \theta/3) \geq \frac{3n}{\theta}(1 - 6k/\theta)(2/3).$$

Thus we get:

$$n \geq \mathbf{E}[H] = \hat{w}_H \theta \geq 2n(1 - \frac{6k}{\theta}).$$

Therefore for $k = \frac{\theta}{24}$, we obtain a contradiction.

□

Observe that similar to Anderson et al. (2017), the above reasoning could be generalized to the case of *periodic* Markovian policies (see Definition 2 of Anderson et al. (2017)) which includes batching polices with constant batch size.

EC.9. A Lower Bound on $w_H^{\mathcal{C}(d)}$

Here, we apply Proposition EC.4 to the *ChainMatch(d)* policy (which is a stable anonymous Markovian matching policy) to prove a lower bound on $w_H^{\mathcal{C}(d)}$:

$$w_H^{\mathcal{C}(d)} + w_E^{\mathcal{C}(d)} \geq \frac{c}{p_H}. \tag{EC.9}$$

In the following lemma, we show that $w_E^{\mathcal{C}(d)} < k_E$, where k_E is a constant independent of p_H; therefore (EC.9) implies that $w_H^{\mathcal{C}(d)} \geq \frac{c}{p_H} - k_E$, i.e., there exists a lower-bound on $w_H^{\mathcal{C}(d)}$ that scales with $1/p_H$.

Lemma EC.4. *Under ChainMatch(d) and for $0 < p_E \leq 1$, we have that $w_E^{\mathcal{C}(d)} < k_E$, where k_E is a constant independent of p_H.*

Proof of Lemma EC.4 Let $[H^{\mathcal{C}(d)}, E^{\mathcal{C}(d)}]$ denote is the random number of H and E agents in steady-state. First observe that conditional on $E^{\mathcal{C}(d)} = x$, the expected number of E agents matched in a chain-segment is given by:

$$f(x) := \sum_{k=0}^{x} \prod_{i=0}^{k} (1 - (1 - p_E)^{x-i}).$$

Note that the above holds because under *ChainMatch(d)*, we give priority to H agents; therefore, when computing the probability E agents in a chain-segment, we can ignore the H agents that are matched between two consecutive E agents.

Function $f(x)$ can be re-written recursively as $f(x) = (1 - (1 - p_E)^x)(1 + f(x-1))$. We now wish to provide a lower bound of $f(x)$. Let α be a constant to be determined later, and let us assume by induction that $f(x-1) \geq \alpha(x-1)$. Then:

$$f(x) \geq (1 - (1 - p_E)^x)(1 + \alpha(x-1)) = \alpha(x-1) + 1 - (1 - p_E)^x (1 + \alpha(x-1)).$$

Let $\xi_\alpha(y) = (1 - p_E)^y (1 + \alpha(y-1))$. Its derivative satisfies

$$\xi'_\alpha(y) = (1 - p_E)^y \left(\ln(1 - p_E)(1 + \alpha(y-1)) + \alpha \right).$$

Note that in $\xi_\alpha(y)$, y is a continuous variable. Let $\alpha = \ln(\frac{1}{1-p_E})$, for all $y \geq 1$, $\xi'_\alpha(y) \leq 0$. Further, $\xi_\alpha(1) \leq \xi_\alpha(0)$. Thus we have $\xi_\alpha(x) \leq \xi_\alpha(0)$, for $x \in \mathbb{N}$. Therefore we can complete our induction.

$$f(x) \geq \alpha(x-1) + 1 - \xi_\alpha(x) \geq \ln(\frac{1}{1-p_E})x.$$

We can now conclude our proof using conservation of E agents in the market:

$$\lambda_E(1-p_E)^d = (\lambda_E(1-(1-p_E)^d)) + \lambda_H(1-(1-p_H)^d))\mathbf{E}[f(E^{\mathcal{C}(d)})] \geq \lambda_E(1-(1-p_E)^d)\ln(\frac{1}{1-p_E})\mathbf{E}[E^{\mathcal{C}(d)}]$$

where in the last inequality we use $f(x) \geq \ln(\frac{1}{1-p_E})x$ and $(1-(1-p_H)^d) \geq 0$. We take $k_E = \frac{(1-p_E)^d}{(1-(1-p_E)^d)\ln(\frac{1}{1-p_E})}$, which does not depend on p_H.

□

EC.10. Scaling of $w_H^{\mathcal{B}_H}$ and $w_H^{\mathcal{B}_E}$ when $\lambda_H = \lambda_E$

In Figure EC.3, we present our numerical study on how $w_H^{\mathcal{B}_H}$ and $w_H^{\mathcal{B}_E}$ scale with p_H in the case that $\lambda_H = \lambda_E$; the figure plots $p_H w_H^{\mathcal{B}_H}$ and $p_H w_H^{\mathcal{B}_E}$ when p_H ranges form $10^{-3.5}$ to 10^{-7} while $p_E = 0.5$. As the plot shows, for p_H smaller than 10^{-5}, the normalized waiting time $p_H w_H^{\mathcal{B}_H}$ and $p_H w_H^{\mathcal{B}_E}$ both remain constant, which implies the asymptotic scaling is $1/p_H$ under both policies.

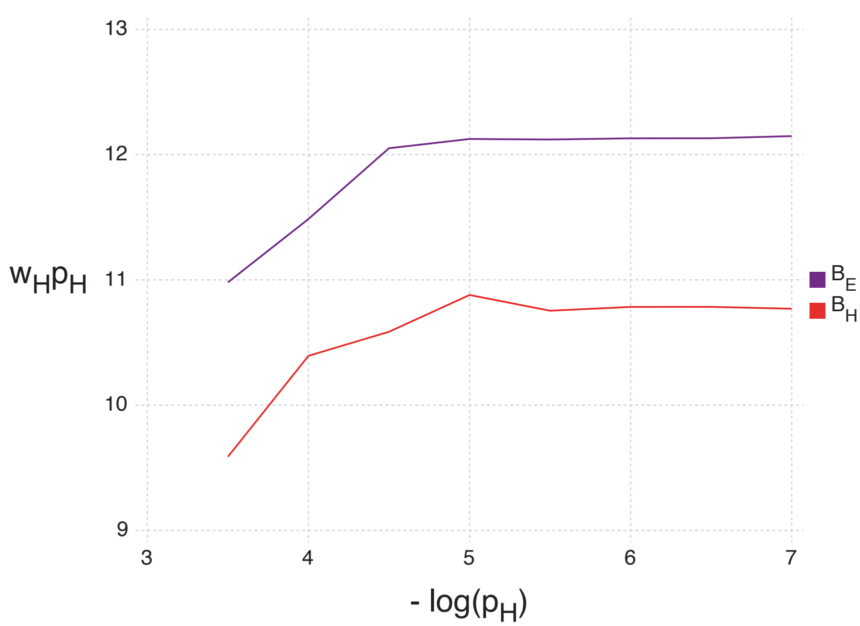

Figure EC.3 Normalized waiting time $w_H p_H$ as a function of p_H (x axis is in log scale), for both $\mathcal{B}_H$, and $\mathcal{B}_E$. $T = 10^{12}$, $\lambda_H = \lambda_E = 1$, $p_E = 0.5$.