Electronic Companion:

Dynamic Mechanism Design with Budget Constrained Buyers under Limited Commitment

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A Numerical Framework

In this section, we present formally the numerical framework we develop to evaluate the performance of the mechanism obtained from the continuous time fluid formulation in the discrete model.

A.1 Step 1: Obtaining a Solution to the HJB Equations

Let $\mathcal{K} = \{k = 1, \ldots, K : x_k > 0\}$ be the subset of buyers with positive budget at a state $\mathbf{x}$. For each subset $\mathcal{K} \subseteq \{1, \ldots, K\}$ we solve the partial differential equation given in the statement of Theorem 1. We also verify that the value functions are well-behaved, i.e., satisfy the conditions in Definition 1. We next lay out the finite differences approach taken to numerically solve the system of PDEs.

A.1.1 Solving the PDEs

We denote by $\mathbf{e}_k \in \mathbb{R}^K$ the unit vector with a one in the k^{th} entry and zero otherwise. Let $h : \mathbb{R}_+^K \times 2^{\{1,\ldots,K\}} \to \mathbb{R}$ be given by

$$h(\boldsymbol{\eta}, \mathcal{K}) = \mathbb{E}_{\mathbf{v}} \left[\left(\max_{k \in \mathcal{K}} \left\{ \eta_k \phi_k(v_k) - c \right\} \right)^+ \right] ,$$

and $g_k : \mathbb{R}_+^K \times 2^{\{1,\ldots,K\}} \to \mathbb{R}$ be given by

$$g_k(\boldsymbol{\eta}, \mathcal{K}) = \mathbb{E}_{\mathbf{v}} \left[\mathbf{1} \left\{ \eta_k \phi_k(v_k) > \max \left\{ c, \max_{i \in \mathcal{K} \setminus k} \eta_i \phi_i(v_i) \right\} \right\} \frac{\bar{F}_k(v_k)}{f_k(v_k)} \right] .$$

We first discuss how to compute these expectations efficiently using single-dimension quadratures. Let

$$Q(y \,|\, \boldsymbol{\eta}, \mathcal{K}) := \mathbb{P} \left\{ \max_{k \in \mathcal{K}} \left\{ \eta_k \phi_k(v_k) \right\} \le y \right\} = \prod_{k \in \mathcal{K}} \mathbb{P} \left\{ \eta_k \phi_k(v_k) \le y \right\} = \prod_{k \in \mathcal{K}} F_k \left(\phi_k^{-1}(y/\eta_k) \right) , \quad \text{(A-1)}$$

where the second equality follows because values are independent and the third because the virtual value function ϕ_k is invertible. Then using integration by parts we obtain

$$h(\boldsymbol{\eta}, \mathcal{K}) = \int_c^{\bar{y}} (y - c) \mathrm{d}Q(y \,|\, \boldsymbol{\eta}, \mathcal{K}) = \int_c^{\bar{y}} \left(1 - Q(y \,|\, \boldsymbol{\eta}, \mathcal{K})\right) \mathrm{d}y \,,$$

where can set $\bar{y} = \infty$, or $\bar{y} = \max_{k\in\mathcal{K}}(\eta_k \bar{v}_k)$ because $Q(y \,|\, \boldsymbol{\eta}, \mathcal{K}) = 1$ for $y > \bar{y}$. Similarly, using that values are independent we obtain

$$
\begin{aligned}
g_k(\boldsymbol{\eta}, \mathcal{K}) &= \mathbb{E}_{v_k}\left[\mathbf{1}\left\{v_k > \phi_k^{-1}(c/\eta_k)\right\} Q\left(\eta_k \phi_k(v_k) \,|\, \boldsymbol{\eta}, \mathcal{K} \setminus k\right) \frac{\bar{F}_k(v_k)}{f_k(v_k)}\right] \\
&= \int_{\phi_k^{-1}(c/\eta_k)}^{\bar{v}_k} Q\left(\eta_k \phi_k(v) \,|\, \boldsymbol{\eta}, \mathcal{K} \setminus k\right) \bar{F}_k(v) \mathrm{d}v \,.
\end{aligned}
$$

With these, we lay out the precise finite differences algorithm we use to solve the PDEs below.

Algorithm 1 Solving the PDEs

```
Set a grid on the space of budgets $\mathcal{B} := \prod_{k=1}^{K} [0, B_k]$ and time $[0, T]$.
Let $\Delta t$ be the step size in time and $\Delta x_k$ be the step size in budgets.
Set $\bar{\Pi}(\mathbf{x}, t) := 0$ and $\bar{U}_k(\mathbf{x}, t) := 0$.
for $t = \Delta t, 2\Delta t, \ldots, T$ do
  for $x_1 = 0, \Delta x_1, 2\Delta x_1, \ldots, B_1$ do
    ⋮
    for $x_K = 0, \Delta x_K, 2\Delta x_K, \ldots, B_K$ do
      Let $\mathcal{K} := \{k = 1, \ldots, K : x_k > 0\}$ be the agents with positive budget.
      Compute derivatives using central differences whenever possible (we use backwards differences when $x_k = B_k$):
          $\nabla_{\mathbf{x}} \bar{\Pi}|_{\mathcal{K}} := \left(\frac{\bar{\Pi}(\mathbf{x} + \mathbf{e}_k \Delta x_k, t - \Delta t) - \bar{\Pi}(\mathbf{x} - \mathbf{e}_k \Delta x_k, t - \Delta t)}{2\Delta x_k}\right)_{k\in\mathcal{K}}$
          $D\bar{U}|_{\mathcal{K}} := \left(\frac{\bar{U}_k(\mathbf{x} + \mathbf{e}_i \Delta x_i, t - \Delta t) - \bar{U}_k(\mathbf{x} - \mathbf{e}_i \Delta x_i, t - \Delta t)}{2\Delta x_i}\right)_{k\in\mathcal{K}, i\in\mathcal{K}}$
      Check for regularity:
          (non-singularity)    $\det(\mathbf{I} + D\bar{\mathbf{U}}|_{\mathcal{K}}) \neq 0\,,$
          (non-negativity)     $\left(\mathbf{1} - \nabla_{\mathbf{x}} \bar{\Pi}|_{\mathcal{K}}\right) \left(\mathbf{I} + D\bar{\mathbf{U}}|_{\mathcal{K}}\right)^{-1} \geq 0\,.$
      Let $\boldsymbol{\eta}|_{\mathcal{K}} := \left(\mathbf{1} - \nabla_{\mathbf{x}} \bar{\Pi}|_{\mathcal{K}}\right) \left(\mathbf{I} + D\bar{\mathbf{U}}|_{\mathcal{K}}\right)^{-1}$ and $\eta_k = 0$ for $k \notin \mathcal{K}$, where $\mathbf{1} \in \mathbb{R}^{|\mathcal{K}|}$ and $\mathbf{I} \in \mathbb{R}^{|\mathcal{K}| \times |\mathcal{K}|}$.
      Let $\bar{\Pi}(\mathbf{x}, t) := \bar{\Pi}(\mathbf{x}, t - \Delta t) + \Delta t \cdot h(\boldsymbol{\eta}, \mathcal{K})$.
      Let $\bar{U}_k(\mathbf{x}, t) := \bar{U}_k(\mathbf{x}, t - \Delta t) + \Delta t \cdot g_k(\boldsymbol{\eta}, \mathcal{K})$ for $k \in \mathcal{K}$.
    end for
    ⋮
  end for
end for
```

A.2 Step 2: Performance Evaluation in the Discrete Model

Let $\hat{\mathbf{M}}[\mathbf{x},t] = (\hat{\mathbf{P}}[\mathbf{x},t], \hat{\mathbf{Z}}[\mathbf{x},t])$ be the *adjusted mechanism* given by for all $v \in \mathbb{R}^K$

$$\hat{P}_k[\mathbf{x},t](\mathbf{v}) = \mathbf{1}\{v_k > y_k(\mathbf{v}_{-k})\}\,, \qquad k = 1,\ldots,K,$$
$$\hat{Z}_k[\mathbf{x},t](\mathbf{v}) = \min\left\{\frac{x_k}{\delta}, \sum_{i\in\mathcal{K}} a_{ki} y_i(\mathbf{v}_{-i})\hat{P}_i[\mathbf{x},t](\mathbf{v})\right\}\,, \qquad k = 1,\ldots,K,$$

where we denote by $y_k(\mathbf{v}_{-k}) = \inf\big\{v : \eta_k\phi_k(v) > \max_{i\in\mathcal{K}\setminus k}\eta_i\phi_i(v_i), \eta_k\phi_k(v) > c\big\}$ the smallest value for the k^{th} buyer that wins against reports $\mathbf{v}_{-k}$ from the competitors, and $\mathbf{A} = (a_{ki}) = (\mathbf{I} + D\bar{\mathbf{U}}|_{\mathcal{K}})^{-1}$. We remark that the allocation and payment factors are time dependent (we drop the dependence on time to simplify the notation). Additionally, if $v_k = \perp$ for some buyer k then

$$\hat{P}_k[\mathbf{x},t](\mathbf{v}) = 0\,, \qquad k = 1,\ldots,K,$$
$$\hat{Z}_k[\mathbf{x},t](\mathbf{v}) = 0\,, \qquad k = 1,\ldots,K.$$

Given an initial state $(\mathbf{x},n) \in \mathcal{S}^\delta$; when all buyers report their values truthfully, budgets evolve according to the stochastic process $\{\mathbf{x}_i\}_{i=1}^n$ with dynamics $\mathbf{x}_{i-1} = \mathbf{x}_i - \delta\hat{\mathbf{Z}}[\mathbf{x}_i,i](\mathbf{v}_i)$ and initial state $\mathbf{x}_n = \mathbf{x}$, where $\hat{\mathbf{Z}}[\mathbf{x}_i,i](\mathbf{v}_i)$ is the vector of payments in market state $(\mathbf{x}_i,i)$ and value realizations $\mathbf{v}_i = (v_{i,k})_{k=1}^K$. (Recall that the mechanism for the discrete stochastic model is given by setting $\hat{\mathbf{M}}[x,i] \triangleq \hat{\mathbf{M}}[x,t_i]$ where $t_i = \delta i$.) We denote by $\mathbb{E}^{\hat{\mathbf{M}}}_{\mathbf{x},n}[-]$ the expectation with respect to this process and with respect to present and future buyers' valuations. Here we assume that the sequence of values $\{\mathbf{v}_i\}_{i=1}^m$ are independent across time periods.

We denote the total seller's profit from an initial state $(\mathbf{x},n)$ until the end of the horizon when all buyers report truthfully under mechanism $\hat{\mathbf{M}}$ by

$$\Pi^{\hat{\mathbf{M}}}(\mathbf{x},n) = \mathbb{E}^{\hat{\mathbf{M}}}_{\mathbf{x},n}\left[\delta\sum_{i=1}^n\sum_{k=1}^K\Big(\hat{Z}_k[\mathbf{x}_i,i](\mathbf{v}_i) - c\hat{P}_k[\mathbf{x}_i,i](\mathbf{v}_i)\Big)\right]\,,$$

where the expectation is taken w.r.t. the evolution of budgets when buyers bid truthfully and the dynamic mechanism $\hat{\mathbf{M}}$ is implemented. Similarly we denote the total utility of buyer k from an initial state $(\mathbf{x},n)$ when all buyers report truthfully by

$$U_k^{\hat{\mathbf{M}}}(\mathbf{x},n) = \mathbb{E}^{\hat{\mathbf{M}}}_{\mathbf{x},n}\left[\delta\sum_{i=1}^n\Big(v_{i,k}\hat{P}_k[\mathbf{x}_i,i](\mathbf{v}_i) - \hat{Z}_k[\mathbf{x}_i,i](\mathbf{v}_i)\Big)\right]\,.$$

Monte Carlo simulation:

Step 1. Set a grid on the space of budgets $\mathcal{B} := \prod_{k=1}^K[0,B_k]$.

Step 2. For each point $\mathbf{x} \in \mathcal{B}$, estimate $\Pi^{\hat{\mathbf{M}}}(\mathbf{x},N)$ and $U_k^{\hat{\mathbf{M}}}(\mathbf{x},N)$ for each k using Monte Carlo simulation. In order to reduce the variance, we use the same sample paths for each initial point $\mathbf{x}$.

A.3 Step 3: Approximate Incentive Compatibility

A stage mechanism $\mathbf{m} \in \mathcal{M}$ is $\epsilon^{\text{IC}}-$incentive compatible for the buyer at state $(\mathbf{x},n)$ with respect to dynamic mechanism $\hat{\mathbf{M}}$ if at state $(\mathbf{x},n)$ the buyer's incentive to misreport *after* learning her value is at most $\delta\epsilon^{\mathrm{IC}}$ when the seller offers the stage mechanism $\mathbf{m}$ at the current state, and the seller offers dynamic mechanism $\hat{\mathbf{M}}$ and the buyer reports truthfully onwards.

Fix a state $(\mathbf{x}, n)$. We define $\hat{U}_k(v, w)$ as the interim utility-to-go of buyer i under dynamic mechanism $\hat{\mathbf{M}}$ when she reports w and her value is v and competitors report truthfully, which is given by

$$\hat{U}_k(v,w) = \mathbb{E}_{\mathbf{v}_{-k}}\Big[\delta\left(v\hat{p}_k(w,\mathbf{v}_{-k}) - \hat{z}_k(w,\mathbf{v}_{-k})\right) + U_k^{\hat{\mathbf{M}}}\big(\mathbf{x} - \delta\hat{\mathbf{z}}(w,\mathbf{v}_{-k}), n-1\big)\Big]\,,$$

where $\hat{\mathbf{m}} = \hat{\mathbf{M}}[\mathbf{x}, n]$. The continuation values $U_k^{\hat{\mathbf{M}}}$ are computed using linear interpolation of the value functions obtained from Monte Carlo simulation. The IC error at state $(\mathbf{x}, n)$ is

$$E_\infty(\mathbf{x},n) = \frac{1}{\delta}\max_k \max_v \max_{w\in\mathbb{R}_\perp}\left\{\hat{U}_k(v,w) - \hat{U}_k(v,v)\right\}\,.$$

Computation of approximate incentive compatibility:

Step 1. Set a grid on the space of budgets $\mathcal{B} := \prod_{k=1}^{K}[0, B_k]$.

Step 2. For each point $\mathbf{x} \in \mathcal{B}$, compute the error $E_\infty(\mathbf{x}, N)$ where we let $N = T/\delta$.

Step 3. Take the average over all states $\mathbf{x}$.

A.4 Step 4: Approximate Sequential Rationally

For a given state $(\mathbf{x}, n)$, we compare the seller's profit when all buyers report truthfully under our mechanism, denoted by $\Pi^{\hat{\mathbf{M}}}(\mathbf{x}, n)$, to the profit the seller could obtain from a one-shot deviation, denoted by $\Pi^*(\mathbf{x}, n)$. Here, $\Pi^*(\mathbf{x}, n)$ is the profit obtained from offering an optimal mechanism for the current period and then offering mechanism $\hat{\mathbf{M}}$ in the following periods and is given by

$$\begin{aligned}
\Pi^*(\mathbf{x},n) = \max_{\mathbf{m}\in\mathcal{M}}\ & \mathbb{E}_{\mathbf{v}}\left[\delta\sum_{k=1}^{K}\left(z_k(\mathbf{v}) - cp_k(\mathbf{v})\right) + \Pi^{\hat{\mathbf{M}}}\big(\mathbf{x} - \delta\mathbf{z}(\mathbf{v}), n-1\big)\right] && \text{(A-2a)}\\
\text{s.t. }\ & \delta z_k(\mathbf{w}) \le x_k\,, \quad \forall k, \mathbf{w} \in \mathbb{R}_\perp^K\,, && \text{(A-2b)}\\
& \mathbb{E}_{\mathbf{v}_{-k}}\Big[\delta\left(vp_k(\mathbf{v}) - z_k(\mathbf{v})\right) + U_k^{\hat{\mathbf{M}}}\big(\mathbf{x} - \delta\mathbf{z}(\mathbf{v}), n-1\big)\Big] + \delta E_\infty(\mathbf{x},n) && \text{(A-2c)}\\
& \quad \ge \mathbb{E}_{\mathbf{v}_{-k}}\Big[\delta\left(vp_k(w,\mathbf{v}_{-k}) - z_k(w,\mathbf{v}_{-k})\right) + U_k^{\hat{\mathbf{M}}}\big(\mathbf{x} - \delta\mathbf{z}(w,\mathbf{v}_{-k}), n-1\big)\Big]\ \forall v, w \in \mathbb{R}_\perp, k\,.
\end{aligned}$$

Note that we allow for mechanisms that are $E_\infty(\mathbf{x}, n)$-incentive compatible so as to guarantee that the candidate stage mechanism $\hat{\mathbf{M}}[\mathbf{x}, n]$ is feasible for the seller's problem.

Computation of approximate sequential rationality:

Step 1. Set a grid on the space of budgets $\mathcal{B} := \prod_{k=1}^{K}[0, B_k]$.

Step 2. For each point $\mathbf{x} \in \mathcal{B}$, compute the difference $(\Pi^*(\mathbf{x}, N) - \Pi^{\mathbf{M}}(\mathbf{x}, N))/\delta$ where we let $N = T/\delta$.

Step 3. Take the average over all states $\mathbf{x}$.

We next describe how to numerically compute $\Pi^*(\mathbf{x}, n)$.

A.4.1 Solving for the Seller's Optimal Deviation

We drop the superscript $\mathbf{M}$ from value functions to simplify the notation. We consider the discrete version of the problem with a uniform mesh of ℓ_k points for buyer k. Let $\{\mathcal{V}_{k,i}\}_{i=1}^{\ell_k}$ be the uniform disjoint partition of the interval $[0, \bar{v}_k]$ given by $\mathcal{V}_{k,i} = \left(\frac{i-1}{\ell_k}\bar{v}_k, \frac{i}{\ell_k}\bar{v}_k\right]$ for $i > 1$ and $\mathcal{V}_{k,1} = \left[0, \frac{\bar{v}_k}{\ell_k}\right]$. Let the discrete probabilities $\{f_{k,i}\}_{i=1}^{\ell_k}$ be given by $f_{k,i} = \mathbb{P}\{v_k \in \mathcal{V}_{k,i}\}$ and $v_{k,i} = \inf \mathcal{V}_{k,i} = \frac{i-1}{\ell_k}\bar{v}_k$. Let $\mathcal{I}_k = \{1, \ldots, \ell_k\}$ be the discrete set of value indices for buyer k and $\mathcal{I} = \prod_{k=1}^K \mathcal{I}_k$ be the set of value all indices for all buyers. For a vector of value indices $\mathbf{i} \in \mathcal{I}$ given by $\mathbf{i} = (i_1, \ldots, i_K)$ we denote $\mathbf{v}_\mathbf{i} = (v_{k,i_k})_{k=1}^K \in \prod_{k=1}^K [0, \bar{v}_k]$ the values of the buyers and $f_\mathbf{i} = \prod_{k=1}^K f_{k,i_k}$ the probability that the vector of value indices is drawn. With some abuse of notation we denote by $\mathbf{i}_{-k} \in \mathcal{I}_{-k}$ the value indices of all buyers other than k with $\mathcal{I}_{-k} = \prod_{k' \neq k} \mathcal{I}_{k'}$. Similarly, we have $\mathbf{v}_{\mathbf{i}_{-k}} = (v_{k',i_{k'}})_{k' \neq k}$ and $f_{\mathbf{i}_{-k}} = \prod_{k' \neq k} f_{k',i_{k'}}$. Finally, we denote $\mathcal{I}^\perp = \prod_{k=1}^K \left(\mathcal{I}_k^\perp\right)$ and $\mathcal{I}_k^\perp = \mathcal{I}_k \cup \{\perp\}$.

A discrete stage mechanism is given by $\mathbf{m} = (\mathbf{p}, \mathbf{z})$ where $\mathbf{p} : \mathcal{I}_\perp \to \Delta$ is a probability allocation function and $\mathbf{z} : \mathcal{I}_\perp \to \mathbb{R}_+$ is a payment function. The discrete problem is given by:

$$\max_{\mathbf{m}} \sum_{\mathbf{i} \in \mathcal{I}} f_\mathbf{i} \left[\delta \sum_{k=1}^K (z_k(\mathbf{i}) - c p_k(\mathbf{i})) + \mathit{\Pi}\big(\mathbf{x} - \delta \mathbf{z}(\mathbf{i}), n-1\big)\right] \tag{A-3a}$$

$$\text{s.t. } \delta z_k(\mathbf{i}) \leq x_k\,, \quad \forall k, \mathbf{i} \in \mathcal{I}_\perp\,, \tag{A-3b}$$

$$\tilde{U}_k(\mathbf{m}, i, i) \geq \tilde{U}_k(\mathbf{m}, i, i') - \delta E_\infty(\mathbf{x}, n)\,, \quad \forall k, i \in \mathcal{I}_k, i' \in \mathcal{I}_k\,, \tag{A-3c}$$

$$\tilde{U}_k(\mathbf{m}, i, i) \geq \tilde{U}_k(\mathbf{m}, i, \perp) - \delta E_\infty(\mathbf{x}, n)\,, \quad \forall k, i \in \mathcal{I}_k\,, \tag{A-3d}$$

where we denote the interim utility-to-go of buyer i under mechanism $\mathbf{m}$ when she reports $i' \in \mathcal{I}_k$ and her value is $i \in \mathcal{I}_k$ as

$$\tilde{U}_k(\mathbf{m}, i, i') = \sum_{\mathbf{i}_{-k} \in \mathcal{I}_{-k}} f_{\mathbf{i}_{-k}} \Big[\delta \left(v_{k,i} p_k(i', \mathbf{i}_{-k}) - z_k(i', \mathbf{i}_{-k})\right) + U_k\big(\mathbf{x} - \delta \mathbf{z}(i', \mathbf{i}_{-k}), n-1\big)\Big]\,.$$

Note that $\tilde{U}_k(\mathbf{m}, i, i')$ is a non-linear function of the payments $\mathbf{z}$. We tackle this problem by allowing for randomized mechanisms over a grid a possible payments $\mathcal{Y} = \prod_{k=1}^K \mathcal{Y}_k$ with $\mathcal{Y}_k \subset \mathbb{R}$. For each vector of value indices $\mathbf{i} \in \mathcal{I}_\perp$, we now have that $\mathbf{z}(\mathbf{i})$ is a multi-variate probability distribution over $\mathcal{Y}$. We denote the probability that the payments are $\mathbf{y} \in \mathcal{Y}$ when buyers report $\mathbf{i}$ by $\zeta(\mathbf{i}, \mathbf{y}) = \mathbb{P}\{\mathbf{z}(i) = \mathbf{y}\}$. A mechanism is now given by a pair $(\mathbf{p}, \boldsymbol{\zeta})$ with $\boldsymbol{\zeta} : \mathcal{I}_\perp \times \mathcal{Y} \to \mathbb{R}$ and the probability distribution over payments satisfying

$$\zeta(\mathbf{i}, \mathbf{y}) \geq 0, \sum_{\mathbf{y} \in \mathcal{Y}} \zeta(\mathbf{i}, \mathbf{y}) = 1\,. \tag{A-4}$$

Using these variables, we can write the non-linear term in the interim-utility-to-go as

$$\begin{aligned}\sum_{\mathbf{i}_{-k} \in \mathcal{I}_{-k}} f_{\mathbf{i}_{-k}} U_k\big(\mathbf{x} - \delta \mathbf{z}(i, \mathbf{i}_{-k}), n-1\big) &= \sum_{\mathbf{i}_{-k} \in \mathcal{I}_{-k}} f_{\mathbf{i}_{-k}} \sum_{\mathbf{y} \in \mathcal{Y}} \zeta((i, \mathbf{i}_{-k}), \mathbf{y}) U_k\big(\mathbf{x} - \delta \mathbf{y}, n-1\big) \\ &= \sum_{\mathbf{y} \in \mathcal{Y}} U_k\big(\mathbf{x} - \delta \mathbf{y}, n-1\big) \sum_{\mathbf{i}_{-k} \in \mathcal{I}_{-k}} f_{\mathbf{i}_{-k}} \zeta((i, \mathbf{i}_{-k}), \mathbf{y})\,.\end{aligned}$$

A similar expression holds for the non-linear term in the seller's objective.

The number of variables is $K \prod_{k=1}^K (\ell_k + 1)$ for the allocation and $|\mathcal{Y}| \prod_{k=1}^K (\ell_k + 1)$ for the probability distribution over payments. In the case when $\ell_k = \ell$ and $|\mathcal{Y}_k| = \ell$, we obtain that the number of variables are roughly $K\ell^K$ and ℓ^{2K}, respectively.

Interim representation. We now discuss an alternative formulation based on interim payment probabilities. Note that the non-linear term in the interim-utility-to-go can be further simplified to

$$\sum_{\mathbf{i}_{-k}\in\mathcal{I}_{-k}} f_{\mathbf{i}_{-k}} U_k\big(\mathbf{x}-\delta\mathbf{z}(i,\mathbf{i}_{-k}), n-1\big) = \sum_{\mathbf{y}\in\mathcal{Y}} U_k\big(\mathbf{x}-\delta\mathbf{y}, n-1\big)\hat{\zeta}_k(i,\mathbf{y})\,,$$

where we denote by

$$\hat{\zeta}_k(i,\mathbf{y}) := \sum_{\mathbf{i}_{-k}\in\mathcal{I}_{-k}} f_{\mathbf{i}_{-k}}\zeta((i,\mathbf{i}_{-k}),\mathbf{y})\,, \tag{A-5}$$

the interim probability that the mechanism charges payment vector $\mathbf{y}$ to the buyers when buyer k reports $i \in \mathcal{I}_k^{\perp}$. Using these decision variables, we can reduce the number of variables for the payments to $|\mathcal{Y}|\sum_{k=1}^{K}(\ell_k+1)$. In the case when $\ell_k=\ell$ and $|\mathcal{Y}_k|=\ell$, we obtain that the number of variables for the allocation and the interim probabilities for payments are roughly $K\ell^K$ and $K\ell^{K+1}$, respectively.

Because we are projecting to a lower dimensional space we need to add some Border-like inequalities. We call the interim payment probabilities $\hat{\zeta}_k(i,\mathbf{y})$ *achievable* if there exists payment probabilities $\zeta(\mathbf{i},\mathbf{y})$ satisfying (A-4) such that (A-5) holds. Let $\bigsqcup_{k=1}^{K}\mathcal{I}_k^{\perp}$ be the disjoint union of value indices. The disjoint union differs from the standard union in that the elements are indexed by the set they come from. Formally, $\bigsqcup_{k=1}^{K}\mathcal{I}_k^{\perp} = \bigcup_{k=1}^{K}\left\{(k,i) | i\in\mathcal{I}_k^{\perp}\right\}$. Following Che et al. (2013), the interim payment probabilities $\hat{\zeta}_k(i,\mathbf{y})$ *achievable* if and only if

$$\sum_{(k,i)\in S} f_{k,i}\sum_{\mathbf{y}\in\mathcal{Y}}\hat{\zeta}_k(i,\mathbf{y}) \le \sum_{\mathbf{i}\in I(S)} f_{\mathbf{i}}\,,\quad \forall S\subseteq \bigsqcup_{k=1}^{K}\mathcal{I}_k^{\perp}\,, \tag{A-6}$$

where $I(S)=\{\mathbf{i}\in\mathcal{I} : \exists(k,i)\in S \text{ with } i_k = i\}$.

A.5 Results for $K=3$

For three buyers, as noted in Section 5.2, the discrete time model can only be handled with significantly coarser grids. Here, we discretize the budgets by setting a uniform grid with 25 points (as opposed to 50 with one or two buyers, respectively) and the possible payments setting a uniform grid over the support of values $[0,\bar{v}]$ with 5 points (as opposed to 50 and 40 with one and two buyers, respectively).

Table 1a reports the utility a buyer can gain from misreporting her value in the first period of the horizon, averaged over all possible initial budgets in the grid, for $c=0.5$ and $K=3$ buyers. As in the case of 1 and 2 buyers, the results show that the optimal fluid mechanism is $\epsilon^{\mathrm{IC}}-$incentive compatible in the discrete time model, where ϵ^{IC} approaches zero as the number of auctions N becomes large. Table 1b reports the profit the seller can gain from offering an optimal mechanism in the first time period and then offering our mechanism in the following time periods with $K=3$ buyers. When the number of players is $K=3$, we do not observe convergence to zero. We conjecture that this is due to numerical errors introduced by the coarse grids. This being said, for $K=3$ we computed an appropriately normalized average error (normalized with the value function) yielding a 1.7% approximate error, so it is arguably small even in this case.

K N	3
10	0.114707
10^2	0.007642
10^3	0.000273

(a) Approximate Incentive Compatibility (Step 3)

K N	3
10	0.132173
10^2	0.023206
10^3	0.019850

(b) Approximate Sequential Rationality (Step 4)

Table 1: Table 1a reports the maximum IC error $E_\infty(\mathbf{x}, N)$ across buyers, averaged over all budgets $\mathbf{x}$, evaluated at the first period (Step 3). Table 1b reports the difference $(\Pi^*(\mathbf{x}, N) - \Pi^{\hat{\mathbf{M}}}(\mathbf{x}, N))/\delta$ averaged over all budgets $\mathbf{x}$, evaluated at the first period (Step 4). Results are for $K = 3$.

B Informal Derivation of HJB Equations

Let $\bar{\Pi} : \mathcal{S} \to \mathbb{R}$ be the cumulative profit-to-go of the seller and let $\bar{U}_k : \mathcal{S} \to \mathbb{R}$ be the expected cumulative utility-to-go for buyer k in the fluid model. Suppose that the value functions $\bar{\Pi}(\mathbf{x}, t)$ and $\bar{U}_k(\mathbf{x}, t)$ are differentiable everywhere. Given a state $(\mathbf{x}, n) \in \mathcal{S}^\delta$ of the discrete stochastic model, the corresponding state in the fluid model is $(\mathbf{x}, t_{n^\delta})$ with $t_{n^\delta} = \delta n$. With some abuse of notation we denote by $\Pi^\delta(\mathbf{x}, n) = \bar{\Pi}(\mathbf{x}, \delta n)$ and $U_k^\delta(\mathbf{x}, n) = \bar{U}_k(\mathbf{x}, \delta n)$ the evaluation of the fluid model value functions in a discrete state.

Consider a sequence of states $(\mathbf{x}, t_{n^\delta}) \in \mathcal{S}$ converging to $(\mathbf{x}, t) \in \mathcal{S}$ as $\delta \to 0$. Fix the optimal mechanism $\mathbf{M}^*[\mathbf{x}, t]$ of the fluid model. We first informally argue that the HJB equations (2) and (3) converge to (7) and (8), respectively. Performing a first-order expansion of the seller's value function around the current state we obtain that for $0 \le \mathbf{z} \le \delta\mathbf{x}$

$$\begin{aligned} \Pi^\delta\big(\mathbf{x} - \delta\mathbf{z}, n^\delta - 1\big) &= \bar{\Pi}\big(\mathbf{x} - \delta\mathbf{z}, t_{n^\delta} - \delta\big) \\ &= \bar{\Pi}\big(\mathbf{x}, t_{n^\delta}\big) - \delta\nabla_{\mathbf{x}}\bar{\Pi}\big(\mathbf{x}, t_{n^\delta}\big) \cdot \mathbf{z} - \delta\frac{\partial\bar{\Pi}}{\partial t}\big(\mathbf{x}, t_{n^\delta}\big) + o(\delta)\,, \end{aligned} \tag{B-7}$$

where $o(\delta)$ is a bound on the error term of the first-order expansion in little-o notation, that is, $\lim_{\delta\to 0} o(\delta)/\delta = 0$. Substituting this first-order expansion in the seller's HJB equation in the discrete model given in (2), we obtain after cancelling the term $\Pi\big(\mathbf{x}, t_{n^\delta}\big)$ and reordering

$$\delta\frac{\partial\bar{\Pi}}{\partial t}\big(\mathbf{x}, t_{n^\delta}\big) = \delta\pi(\mathbf{M}^*[\mathbf{x}, t_{n^\delta}]) - \delta\nabla_{\mathbf{x}}\bar{\Pi}\big(\mathbf{x}, t_{n^\delta}\big) \cdot \mathbb{E}_{\mathbf{v}}\Big[\mathbf{Z}^*[\mathbf{x}, t_{n^\delta}](\mathbf{v})\Big] + o(\delta)\,.$$

Dividing by $\delta > 0$ and taking limits $\delta \to 0$ we obtain under proper continuity assumptions that

$$\frac{\partial\bar{\Pi}}{\partial t}(\mathbf{x}, t) = \pi(\mathbf{M}^*[\mathbf{x}, t]) - \nabla_{\mathbf{x}}\bar{\Pi}(\mathbf{x}, t) \cdot \mathbf{e}(\mathbf{M}^*[\mathbf{x}, t])\,,$$

which coincides with the seller's HJB equation in the fluid model given in (7). A similar argument follows for the buyer's HJB equation by leveraging the first-order expansion

$$U_k^\delta\big(\mathbf{x} - \delta\mathbf{z}, n^\delta - 1\big) = \bar{U}_k\big(\mathbf{x}, t_{n^\delta}\big) - \delta\nabla_{\mathbf{x}}\bar{U}_k\big(\mathbf{x}, t_{n^\delta}\big) \cdot \mathbf{z} - \delta\frac{\partial\bar{U}_k}{\partial t}\big(\mathbf{x}, t_{n^\delta}\big) + o(\delta)\,, \tag{B-8}$$

for each buyer k.

We next argue that the mechanism design problem for the stage game in the fluid model (6) follows from the discrete counterpart (1). The objective (6a) follows from (1a) via the first-order expansion for the seller's value function in (B-7). Indeed, using the latter, the objective in (1a) can be rewritten as

$$\begin{aligned}
& \delta\pi(\mathbf{m}) + \mathbb{E}_{\mathbf{v}}\Big[\mathit{\Pi}\big(\mathbf{x} - \delta\mathbf{z}(\mathbf{v}), n^{\delta} - 1\big)\Big] \\
= \quad & \delta\pi(\mathbf{m}) + \mathbb{E}_{\mathbf{v}}\Big[\bar{\mathit{\Pi}}\big(\mathbf{x}, t_{n^{\delta}}\big) - \delta\nabla_{\mathbf{x}}\bar{\mathit{\Pi}}\big(\mathbf{x}, t_{n^{\delta}}\big) \cdot \mathbf{z}(\mathbf{v}) - \delta\frac{\partial\bar{\mathit{\Pi}}}{\partial t}\big(\mathbf{x}, t_{n^{\delta}}\big) + o(\delta)\Big] \\
= \quad & \delta\left[\pi(\mathbf{m}) - \nabla_{\mathbf{x}}\bar{\mathit{\Pi}}(\mathbf{x}, t) \cdot \mathbf{e}(\mathbf{m})\right] + \bar{\mathit{\Pi}}\big(\mathbf{x}, t_{n^{\delta}}\big) - \delta\frac{\partial\bar{\mathit{\Pi}}}{\partial t}\big(\mathbf{x}, t_{n^{\delta}}\big) + o(\delta).
\end{aligned}$$

Hence we have

$$\arg\max_{\mathbf{m}\in\mathcal{M}} \; \delta\pi(\mathbf{m}) + \mathbb{E}_{\mathbf{v}}\Big[\mathit{\Pi}\big(\mathbf{x} - \delta\mathbf{z}(\mathbf{v}), n^{\delta} - 1\big)\Big] \;=\; \arg\max_{\mathbf{m}\in\mathcal{M}} \; \pi(\mathbf{m}) - \nabla_{\mathbf{x}}\bar{\mathit{\Pi}}(\mathbf{x}, t) \cdot \mathbf{e}(\mathbf{m}) + o(1).$$

The dynamic incentive compatibility constraints (6c) follows from (1c) via the first-order expansion for the buyers' value functions in (B-8). Indeed, we have

$$\begin{aligned}
& \delta u_k(\mathbf{m}, v, w) + \mathbb{E}_{\mathbf{v}_{-k}}\Big[U_k\big(\mathbf{x} - \delta\mathbf{z}(w, \mathbf{v}_{-k}), n^{\delta} - 1\big)\Big] \\
= \quad & \delta u_k(\mathbf{m}, v, w) + \mathbb{E}_{\mathbf{v}_{-k}}\Big[\bar{U}_k\big(\mathbf{x}, t_{n^{\delta}}\big) - \delta\nabla_{\mathbf{x}}\bar{U}_k\big(\mathbf{x}, t_{n^{\delta}}\big) \cdot \mathbf{z}(w, \mathbf{v}_{-k}) - \delta\frac{\partial\bar{U}_k}{\partial t}\big(\mathbf{x}, t_{n^{\delta}}\big) + o(\delta)\Big] \\
= \quad & \delta\left[u_k(\mathbf{m}, v, w) - \nabla_{\mathbf{x}}\bar{U}_k\big(\mathbf{x}, t_{n^{\delta}}\big) \cdot E_{\mathbf{v}_{-k}}\left[\mathbf{z}(w, \mathbf{v}_{-k})\right]\right] + \bar{U}_k\big(\mathbf{x}, t_{n^{\delta}}\big) - \delta\frac{\partial\bar{U}_k}{\partial t}\big(\mathbf{x}, t_{n^{\delta}}\big) + o(\delta)
\end{aligned}$$

Hence we have

$$\begin{aligned}
& \arg\max_{w\in\mathbb{R}_{\perp}} \; \delta u_k(\mathbf{m}, v, w) + \mathbb{E}_{\mathbf{v}_{-k}}\Big[U_k\big(\mathbf{x} - \delta\mathbf{z}(w, \mathbf{v}_{-k}), n^{\delta} - 1\big)\Big] \\
= \quad & \arg\max_{w\in\mathbb{R}_{\perp}} \; u_k(\mathbf{m}, v, w) - \nabla_{\mathbf{x}}\bar{U}_k(\mathbf{x}, t) \cdot \mathbb{E}_{\mathbf{v}_{-k}}\left[\mathbf{z}(w, \mathbf{v}_{-k})\right] + o(1)
\end{aligned}$$

The budget feasibility constraint (6b) follows from (1b) by dividing by $\delta > 0$ and taking limits as $\delta \to 0$. Here we use that when $x_k > 0$ for some buyer k the constraint gives $z_k(\mathbf{w}) \le x_k/\delta$, which implies that the buyer's payment are not constrained in the limit since the right-hand side goes to infinite.

C Proofs for Section 4

C.1 Proof of Theorem 1

We first show that, when the value functions satisfy the conditions in the statement, the mechanism in the statement is optimal at any given state, that is, it is an optimal solution of problem (6). We then show that evaluating HJB equations (7) and (8) at this optimal mechanism yield the equations (9a)-(9b) in the statement. Thus the value functions and mechanism in the statement satisfy HJB equations (6)-(8). While this result shows that every solution of the equations (9a)-(9b) in the statement of Theorem 1 solve HJB equations (6)-(8), it does not exclude the existence of a solution to (6)-(8) that does not satisfy the conditions in the statement of the theorem.

Fix a state $(\mathbf{x}, t) \in \mathcal{S}$. We prove the result when all buyers have positive budget, that is, $\mathcal{K} = \{1, \ldots, K\}$. The result when $\mathcal{K}$ is a proper subset follows mutatis mutandis. We prove the result in four steps.

Step 1 (Envelope Condition). In this step, we leverage the continuous time fluid formulation to adapt the Myerson approach (Myerson, 1981) to solve for payments as a linear functional of the allocation and then reformulate the seller's optimization problem in the next step. Let $\tilde{u}_k(v,w)$ be the interim utility-to-go of buyer k when her true value is v and she reports w to the mechanism. This is given by

$$\tilde{u}_k(v,w) = \mathbb{E}_{\mathbf{v}_{-k}}\big[\, v\, p_k\left(w,\mathbf{v}_{-k}\right) - z_k\left(w,\mathbf{v}_{-k}\right)\big] - \sum_{j=1}^{K} \frac{\partial \bar{U}_k}{\partial x_j} \mathbb{E}_{\mathbf{v}_{-k}}\left[z_j(w,\mathbf{v}_{-k})\right] ,$$

where we ignored the dependence on $(\mathbf{x},t)$ to simplify notation.

The DIC condition (6c) implies that for any feasible mechanism: $\tilde{u}_k(v,v) = \max_{w\geq 0} u_k(v,w)$. Applying the Envelope Theorem we obtain that:

$$\frac{\mathrm{d}\tilde{u}_k(v,v)}{\mathrm{d}v} = \mathbb{E}_{\mathbf{v}_{-k}}\big[p_k\left(v,\mathbf{v}_{-k}\right)\big] .$$

This implies that the interim allocation $\mathbb{E}_{\mathbf{v}_{-k}}\big[p_k\left(v,\mathbf{v}_{-k}\right)\big]$ is non-decreasing in the value v because $\tilde{u}_k(v,v)$ is convex in the value v (since it is the maximum of linear functions). Convexity of $\tilde{u}_k(v,v)$ implies absolute continuity and integrating we obtain that:

$$\tilde{u}_k(v,v) = \tilde{u}_k(0,0) + \int_0^v \mathbb{E}_{\mathbf{v}_{-k}}\big[p_k\left(y,\mathbf{v}_{-k}\right)\big]\ \mathrm{d}y\,. \tag{C-9}$$

Using the definition of $\tilde{u}_k(v,v)$ we get that the envelope condition corresponding to the DIC constraints is given by:

$$\mathbb{E}_{\mathbf{v}_{-k}}\big[vp_k(v,\mathbf{v}_{-k}) - z_k(v,\mathbf{v}_{-k}) - \nabla_{\mathbf{x}}\bar{U}_k\cdot\mathbf{z}(v,\mathbf{v}_{-k})\big] = \tilde{u}_k(0,0) + \int_0^v \mathbb{E}_{\mathbf{v}_{-k}}\big[p_k\left(y,\mathbf{v}_{k}\right)\big]\ \mathrm{d}y\,. \tag{C-10}$$

where $\tilde{u}_k(0,0) = -\mathbb{E}_{\mathbf{v}_{-k}}\big[z_k(0,\mathbf{v}_{-k}) + \nabla_{\mathbf{x}}\bar{U}_k\cdot\mathbf{z}(0,\mathbf{v}_{-k})\big]$ is the utility-to-go of the lowest type. Further, the monotonicity of the expected interim allocation together with the envelope condition are necessary and sufficient conditions for incentive compatibility (see, e.g., Theorem 4.3 in Milgrom (2004)).

The IR constraint ensures that buyer k is better off participating in each auction, or equivalently $\tilde{u}_k(v,v)\geq\tilde{u}_k(v,\perp)$. Because the allocation is non-negative, it follows from (C-9) that the interim utility-to-go $\tilde{u}_k(v,v)$ is non-decreasing in v and it thus suffices to impose the IR constraint at the lowest type. Under our assumption that transfers are not allowed if one buyer does not participate in an auction, the IR constraint implies that

$$\tilde{u}_k(0,0)\geq\tilde{u}_k(0,\perp) = -\mathbb{E}_{\mathbf{v}_{-k}}\big[z_k(\perp,\mathbf{v}_{-k}) + \nabla_{\mathbf{x}}\bar{U}_k\cdot\mathbf{z}(\perp,\mathbf{v}_{-k})\big] = 0\,, \tag{C-11}$$

because $\mathbf{z}(\mathbf{w}) = 0$ if $w_j = \perp$ for some buyer j, by assumption.

Step 2 (Reformulating the Seller's Problem). Taking expectations of the k^{th} buyer DIC constraint (C-10) w.r.t. her value we obtain that

$$\begin{aligned}\mathbb{E}_{\mathbf{v}}\big[z_k(\mathbf{v}) + \nabla_{\mathbf{x}}\bar{U}_k\cdot\mathbf{z}(\mathbf{v})\big] &= -\tilde{u}_k(0,0) + \mathbb{E}_{\mathbf{v}}\Big[v_kp_k(\mathbf{v}) - \int_0^{v_k} p_k\left(y,\mathbf{v}_{-k}\right)\ \mathrm{d}y\Big] \\ &= -\tilde{u}_k(0,0) + \mathbb{E}_{\mathbf{v}}\big[\phi_k(v_k)p_k(\mathbf{v})\big]\,,\end{aligned}$$

where the last equation follows from integration by parts. Denoting by $\circ$ the Hadamard (or entry-wise product) the latter system of equations can be written in matrix form as

$$(\mathbf{I}+D\bar{\mathbf{U}})\mathbb{E}_{\mathbf{v}}\big[\mathbf{z}(\mathbf{v})\big]=-\tilde{\mathbf{u}}(0,0)+\mathbb{E}_{\mathbf{v}}\big[\boldsymbol{\phi}(\mathbf{v})\circ\mathbf{p}(\mathbf{v})\big]\,,$$

where $\mathbf{I}\in\mathbb{R}^{K\times K}$ is the identity matrix, and $D\bar{\mathbf{U}}:\mathbb{R}^K\to\mathbb{R}^{K\times K}$ denotes the Jacobian of the vector function $\bar{\mathbf{U}}$. Because the matrix $(\mathbf{I}+D\bar{\mathbf{U}})$ is invertible we obtain that $\mathbb{E}_{\mathbf{v}}\big[\mathbf{z}(\mathbf{v})\big]=(\mathbf{I}+D\bar{\mathbf{U}})^{-1}\left(-\tilde{\mathbf{u}}(0,0)+\mathbb{E}_{\mathbf{v}}\big[\boldsymbol{\phi}(\mathbf{v})\circ\mathbf{p}(\mathbf{v})\big]\right)$.

Replacing the expression above for expected payments, the seller's objective in the right-hand side of the HJB equation (6a) is given by

$$\begin{aligned}\mathbb{E}_{\mathbf{v}}\Big[\big(\mathbf{1}-\nabla_{\mathbf{x}}\bar{\Pi}\big)\cdot\mathbf{z}(\mathbf{v})-c\mathbf{1}\mathbf{p}(\mathbf{v})\Big]&=-\boldsymbol{\eta}\cdot\tilde{\mathbf{u}}(0,0)+\mathbb{E}_{\mathbf{v}}\Big[\big(\boldsymbol{\eta}\circ\boldsymbol{\phi}(\mathbf{v})-c\mathbf{1}\big)\cdot\mathbf{p}(\mathbf{v})\Big]\\&=-\boldsymbol{\eta}\cdot\tilde{\mathbf{u}}(0,0)+\sum_{k=1}^{K}\mathbb{E}_{\mathbf{v}}\Big[\big(\eta_k\phi_k(v_k)-c\big)p_k(\mathbf{v})\Big]\,,\end{aligned}\tag{C-12}$$

where we denoted $\boldsymbol{\eta}\in\mathbb{R}^K$ as $\boldsymbol{\eta}=\big(\mathbf{1}-\nabla_{\mathbf{x}}\bar{\Pi}\big)\,(\mathbf{I}+D\bar{\mathbf{U}})^{-1}$ and used that $\boldsymbol{\eta}\cdot(\boldsymbol{\phi}(\mathbf{v})\circ\mathbf{p}(\mathbf{v}))=(\boldsymbol{\eta}\circ\boldsymbol{\phi}(\mathbf{v}))\cdot\mathbf{p}(\mathbf{v})$. Because value functions are well-behaved we have that $\boldsymbol{\eta}\geq 0$.

Step 3 (Allocation and Payments). Relaxing momentarily the constraint that the interim allocation is non-decreasing and optimizing point-wise we obtain that the optimal allocation rule is $p_k(\mathbf{v})=\mathbf{1}\{v_k>y_k(\mathbf{v}_{-k})\}$ where $y_k(\mathbf{v}_{-k})=\inf\big\{v:\eta_k\phi_k(v)>\max_{i\neq k}\eta_i\phi_i(v_i),\eta_k\phi_k(v)>c\big\}$. Observe that because $\eta_k\geq 0$ and $\phi_k(\cdot)$ is increasing we obtain that the interim allocation is monotonic, as required. Additionally, because $\eta_k\geq 0$ and $\tilde{u}_k(0,0)\geq 0$ from the IR constraint (C-11), we obtain that the auctioneer would like to set $\tilde{u}_k(0,0)=0$.

Payments are chosen so that they satisfy the interim DIC point-wise for every report $\mathbf{v}$. That is, for each buyer k they should satisfy that

$$\begin{aligned}z_k(\mathbf{v})+\nabla_{\mathbf{x}}\bar{U}_k\cdot\mathbf{z}(\mathbf{v})&=v_kp_k(\mathbf{v})-\int_0^{v_k}p_k\left(x,\mathbf{v}_{-k}\right)\,\mathrm{d}x\\&=y_k(\mathbf{v}_{-k})\mathbf{1}\{v_k>y_k(\mathbf{v}_{-k})\}\,,\end{aligned}\tag{C-13}$$

where the last equation follows from integrating the allocation. The latter is stronger than the requirement imposed by equation (C-10). The previous system can be written in matrix form as $(\mathbf{I}+D\bar{\mathbf{U}})\cdot\mathbf{z}(\mathbf{v})=\mathbf{y}(\mathbf{v})\circ\mathbf{p}(\mathbf{v})$, and solving for the payments we obtain that $\mathbf{z}(\mathbf{v})=(\mathbf{I}+D\bar{\mathbf{U}})^{-1}\left(\mathbf{y}(\mathbf{v})\circ\mathbf{p}(\mathbf{v})\right)$. Letting $\mathbf{A}=(a_{ki})=(\mathbf{I}+D\bar{\mathbf{U}})^{-1}$, the latter can be written as $z_k(\mathbf{v})=\sum_{i=1}^{K}a_{ki}y_i(\mathbf{v}_{-i})\mathbf{1}\{v_i>y_i(\mathbf{v}_{-i})\}$. Additionally, by equation (C-13), $z_k(0,\mathbf{v}_{-k})+\nabla_{\mathbf{x}}\bar{U}_k\cdot\mathbf{z}(0,\mathbf{v}_{-k})=0$ and $\tilde{u}_k(0,0)=0$ as required. Thus the IR constraint is satisfied.

Step 4 (HJB Equation). Putting everything together we obtain that the seller's HJB equation can be written as

$$\frac{\partial\bar{\Pi}}{\partial t}=\mathbb{E}_{\mathbf{v}}\left[\Big(\max_k\{\eta_k\phi_k(v_k)-c\}\Big)^+\right]\,,$$

and the equation for buyer k is

$$\begin{aligned}\frac{\partial \bar{U}_k}{\partial t} &= \mathbb{E}_{\mathbf{v}}\left[v_k p_k(\mathbf{v}) - z_k(\mathbf{v}) - \nabla_{\mathbf{x}} \bar{U}_k \cdot \mathbf{z}(\mathbf{v})\right] \\ &= \mathbb{E}_{\mathbf{v}}\left[\int_0^{v_k} p_k\left(x, \mathbf{v}_{-k}\right)\, \mathrm{d}x\right] = \mathbb{E}_{\mathbf{v}}\left[\mathbf{1}\left\{v_k > y_k(\mathbf{v}_{-k})\right\} \frac{\bar{F}_k(v_k)}{f_k(v_k)}\right],\end{aligned}$$

where the second equation follows from the DIC constraint and the last from exchanging the order of integration.

C.2 Proof of Theorem 2

We will next show that there exists a solution to the system of coupled partial differential equations by transforming it to a system of conservation laws. Let $\gamma(x,t) = \frac{\partial \bar{\Pi}}{\partial x}(x,t)$, $\mu(x,t) = \frac{\partial \bar{U}}{\partial x}(x,t)$,

$$\begin{aligned} h(\eta) &= \mathbb{E}_v\left[(\eta\phi(v) - c)^+\right], \text{ and} \\ g(\eta) &= \mathbb{E}_v\left[\mathbf{1}\left\{\eta\phi(v) \geq c\right\} \bar{F}(v)/f(v)\right].\end{aligned}$$

We assume that the solution is differentiable everywhere except at the origin. We proceed by differentiating the PDEs and exchanging derivatives to obtain the new system of conservation laws:

$$\frac{\partial \gamma}{\partial t}(x,t) = \frac{\partial}{\partial x} h\left(\frac{1-\gamma(x,t)}{1+\mu(x,t)}\right), \tag{C-14}$$

$$\frac{\partial \mu}{\partial t}(x,t) = \frac{\partial}{\partial x} g\left(\frac{1-\gamma(x,t)}{1+\mu(x,t)}\right), \tag{C-15}$$

with the boundary conditions that $\gamma(x,0) = \mu(x,0) = 0$ for all $x \geq 0$ and $(1-\gamma(0,t))/(1+\mu(0,t)) \leq c/\bar{v}$. The second boundary condition follows from letting $x \downarrow 0$ in the original PDE and using that $\bar{\Pi}(0,t) = 0$ (respectively, $\bar{U}(0,t) = 0$) and $h(\eta) = 0$ (respectively, $g(\eta) = 0$) if $\eta \leq c/\bar{v}$.

Consider the vector function $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ given by $\mathbf{F}(\gamma,\mu) = -[h(\eta), g(\eta)]$ where we let $\eta = (1-\gamma)/(1+\mu)$. We seek solutions having the structure $\gamma(x,t) = v_1(w(x,t))$, and $\mu(x,t) = v_2(w(x,t))$, where $\mathbf{v} : \mathbb{R} \to \mathbb{R}^2$ and $w : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ are to be found. These solutions are referred as simple waves (see Evans (2010)). Substituting these functions in the PDEs we obtain

$$\dot{\mathbf{v}}(w)\frac{\partial w}{\partial t} + D\mathbf{F}(\mathbf{v}(w))\dot{\mathbf{v}}(w)\frac{\partial w}{\partial x} = 0, \tag{C-16}$$

where $D\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^{2\times 2}$ denotes the Jacobian of the vector function $\mathbf{F}$. Let $\lambda_1(\gamma,\mu)$ and $\lambda_2(\gamma,\mu)$ be the eigenvalues and $\mathbf{r}_1(\gamma,\mu)$ and $\mathbf{r}_2(\gamma,\mu)$ be the corresponding eigenvectors of the Jacobian matrix $D\mathbf{F}(\gamma,\mu)$, that is, $D\mathbf{F}(\gamma,\mu)\mathbf{r}_i(\gamma,\mu) = \lambda_i(\gamma,\mu)\mathbf{r}_i(\gamma,\mu)$ for $i = 1,2$. For $i = 1,2$ we postulate that $\mathbf{v}$ solves the ODE

$$\dot{\mathbf{v}}(s) = \mathbf{r}_i(\mathbf{v}(s)), \tag{C-17}$$

with $\mathbf{v}(0) = 0$, while w solves the PDE

$$\frac{\partial w}{\partial t} + \lambda_i(\mathbf{v}(w))\frac{\partial w}{\partial x} = 0, \tag{C-18}$$

with boundary condition $w(x,0) = 0$ for all $x \geq 0$. It is not hard to verify that any solution satisfying (C-17) and (C-18) should satisfy (C-16). We shall momentarily drop the second boundary condition on γ and μ and then prove that the candidate solution satisfies it.

We next study the differential equations associated with the different eigenvalue/eigenvector pairs. Note that the Jacobian matrix is given by

$$D\mathbf{F} = \frac{1}{1+\mu}\begin{pmatrix} h'(\eta) & h'(\eta)\eta \\ g'(\eta) & g'(\eta)\eta \end{pmatrix},$$

the eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = (h'(\eta) + \eta g'(\eta))/(1+\mu)$, and the eigenvectors are $\mathbf{r}_1 = [-\eta, 1]$ and $\mathbf{r}_2 = [h'(\eta), g'(\eta)]$. The solution associated with the first eigenvalue/eigenvector pair is $w(x,t) = 0$, which implies that $\gamma(x,t) = \mu(x,t) = 0$. This solution is not valid because it does not satisfy the second boundary condition.

In the remainder of this proof we show that the solution associated to the second eigenvalue/eigenvector pair exists and satisfies the original PDE. The system of ODEs in (C-17) is given by

$$\dot{v}_1(s) = h'\left(\frac{1-v_1(s)}{1+v_2(s)}\right), \qquad \dot{v}_2(s) = g'\left(\frac{1-v_1(s)}{1+v_2(s)}\right), \tag{C-19}$$

with the initial condition $\mathbf{v}(0) = 0$. Plots of the right-hand side functions and the solution to the ODE are provided in Figure 1 and Figure 2, respectively, when values are $U[0,1]$ and the seller's cost is $c = 0.25$.

The steps are as follows:

Step 1. Properties on the primitives. (i) The functions $h(\eta)$ and $g(\eta)$ are continuous for $\eta \geq 0$. (ii) The derivatives $h'(\eta)$ and $g'(\eta)$ are Lipschitz continuous for $\eta \geq 0$. (iii) The derivatives satisfy that $h'(\eta) > 0$ and $g'(\eta) > 0$ for $\eta > c/\bar{v}$ and are zero otherwise. (iv) The second derivatives $h''(\eta)$ and $g''(\eta)$ are Lipschitz continuous for $\eta > c/\bar{v}$.

Step 2. Any solution of the system of ODEs (C-17) satisfies that $\eta(s) = \frac{1-v_1(s)}{1+v_2(s)} > c/\bar{v} \triangleq \eta_\infty$ for all $s \geq 0$. Additionally we have that $\lim_{s\to\infty} \eta(s) = \eta_\infty$.

Step 3. The system of ODEs (C-17) admits a unique solution.

Step 4. The function $\lambda_2(s) \triangleq \lambda_2(\mathbf{v}(s))$ is decreasing in $s \geq 0$. Additionally, $\lambda_2(0) = h'(1) + g'(1) < \infty$ and $\lim_{s\to\infty} \lambda_2(s) = 0$.

Step 5. The PDE (C-18) admits a unique entropy solution given by $w(x,t) = \lambda_2^{-1}(x/t)$ when $x/t < \lambda_2(0)$ and $w(x,t) = 0$ otherwise.

Step 6. The solution $\mathbf{v}(w(x,t))$ satisfies the boundary conditions of the conservation law (C-14)-(C-15).

Step 7. A solution of the original system of PDEs (10a)-(10b) is given by $\bar{\Pi}(x,0) = \bar{U}(x,0) = 0$ for all $x \geq 0$, and for all $t > 0$ and $x \geq 0$ by

$$\bar{\Pi}(x,t) = \int_0^x v_1(w(y,t))\,\mathrm{d}y = tj(x/t),$$
$$\bar{U}(x,t) = \int_0^x v_2(w(y,t))\,\mathrm{d}y = tu(x/t),$$

where we denote by $j(y) = \int_0^{\min\{y,\lambda_2(0)\}} v_1\left(\lambda_2^{-1}(z)\right)\,\mathrm{d}z$, and $u(y) = \int_0^{\min\{y,\lambda_2(0)\}} v_2\left(\lambda_2^{-1}(z)\right)\,\mathrm{d}z$. The solution is continuously differentiable for all $(x,t) \neq 0$ and satisfies the boundary conditions.

Step 8. The solution satisfies $0 \leq \bar{\Pi}_x(x,t) \leq 1$ and $\bar{U}_x(x,t) \geq 0$ for all $(x,t) \neq 0$.

Step 1. Recall that the primitives are given by $h(\eta) = \int_0^{\bar{v}} \max\{\eta\phi(x) - c, 0\} f(x)\,\mathrm{d}x$ and $g(\eta) = \int_0^{\bar{v}} \mathbf{1}\{\eta\phi(x) \geq c\}\bar{F}(x)\,\mathrm{d}x$.

We begin with the seller's function $h(\cdot)$. Continuity is trivial. The integrand is differentiable almost everywhere w.r.t. η with derivative $\phi(x)\mathbf{1}\{\eta\phi(x) \geq c\}f(x)$. In turn, the derivative is bounded by the integrable function $\phi(x)^+ f(x) \leq x f(x)$ and we conclude by Leibniz rule that $h'(\eta) = \int_0^{\bar{v}} \phi(x)\mathbf{1}\{\eta\phi(x) \geq c\}f(x)\,\mathrm{d}x$. Let $\psi(x) = \phi^{-1}(x)$. Using the fact $\phi(\cdot)$ is invertible and $\phi(\bar{v}) = \bar{v}$ we conclude that $h'(\eta) = \int_{\psi(c/\eta)}^{\bar{v}} \phi(x) f(x)\,\mathrm{d}x$ for $\eta > c/\bar{v}$ and zero otherwise. The last expression can be alternatively written for $\eta > c/\bar{v}$ as

$$h'(\eta) = \int_{\psi(c/\eta)}^{\bar{v}} \phi(x) f(x)\,\mathrm{d}x = \int_{\psi(c/\eta)}^{\bar{v}} x f(x) - \bar{F}(x)\,\mathrm{d}x = -x\bar{F}(x)\Big|_{\psi(c/\eta)}^{\bar{v}} = \psi(\frac{c}{\eta})\bar{F}(\psi(\frac{c}{\eta}))\,, \quad \text{(C-20)}$$

where the third equation follows from integrating by parts the first term in the integrand and canceling integrals, and the last because $\bar{F}(\bar{v}) = 0$. Since $\phi(x)$ is continuous, and $\phi(0) = -1/f(0) < 0$ and $\phi(\bar{v}) = \bar{v} > 0$ we obtain that $\psi(0) > 0$, which implies that $\psi(x) > 0$ for all $x > 0$ since $\psi(x)$ is increasing. Positivity of $h'(\cdot)$ follows because $\psi(x) > 0$ for all $x > 0$ and $\bar{F}(c/\eta) > 0$ for $\eta > c/\bar{v}$. Lipschitz continuity follows from the fact that composition and product of functions preserves Lipschitz continuity, and $\bar{F}(\cdot)$ and $\psi(\cdot)$ are continuously differentiable (and thus Lipschitz). From Assumption 1 we obtain that $\psi(x)$ is differentiable and we conclude by the Fundamental Theorem of Calculus that

$$h''(\eta) = \frac{c^2}{\eta^3} f(\psi(\frac{c}{\eta}))\psi'(\frac{c}{\eta})$$

for $\eta > c/\bar{v}$, which is Lipschitz continuous by assumption because $f(\cdot)$ is continuously differentiable (and thus Lipschitz).

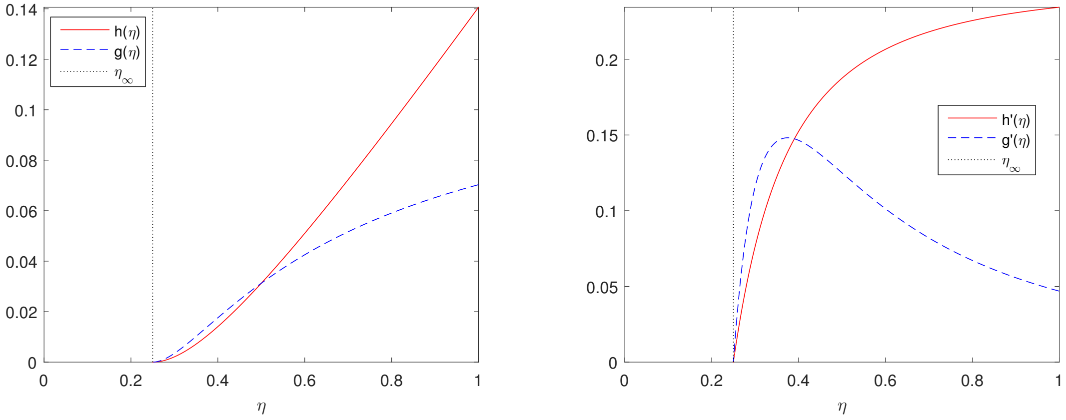
We now proceed to the buyer's function $g(\cdot)$, which can be written as $g(\eta) = \int_{\psi(c/\eta)}^{\bar{v}} \bar{F}(x)\,\mathrm{d}x$ for $\eta > c/\bar{v}$ and zero otherwise. Continuity is again trivial because $\psi(\cdot)$ is continuous. The Fundamental Theorem of Calculus yields that

$$g'(\eta) = \frac{c}{\eta^2}\bar{F}(\psi(\frac{c}{\eta}))\psi'(\frac{c}{\eta})$$

for $\eta > c/\bar{v}$. Note that $\lim_{\eta \to (c/\bar{v})^+} g'(\eta) = 0$ because $\psi(\bar{v}) = \bar{v}$, and continuity of $g'(\eta)$. Because $\phi(\cdot)$ is increasing, we conclude that $\psi'(\cdot) > 0$ and as a result we have that $g'(\eta) > 0$ for $\eta > c/\bar{v}$. Lipschitz continuity follows from the fact that composition and product of functions preserves Lipschitz continuity, and $\bar{F}(\cdot)$, $\psi(\cdot)$ and $\psi'(\cdot)$ are continuously differentiable (and thus Lipschitz). Taking derivatives again we conclude that for $\eta > c/\bar{v}$ it is the case that

$$g''(\eta) = -2\frac{c}{\eta^3}\bar{F}(\psi(\frac{c}{\eta}))\psi'(\frac{c}{\eta}) + \frac{c^2}{\eta^4} f(\psi(\frac{c}{\eta}))\psi'(\frac{c}{\eta})^2 - \frac{c^2}{\eta^4}\bar{F}(\psi(\frac{c}{\eta}))\psi''(\frac{c}{\eta})\,,$$

which is Lipschitz continuous by assumption.



(a) $h(\eta)$ and $g(\eta)$ **(b)** $h'(\eta)$ and $g'(\eta)$

Figure 1: Plots of the functions $h(\eta)$ and $g(\eta)$, and the derivatives $h'(\eta)$ and $g'(\eta)$ as a function of the allocation factor η. Values are $U[0,1]$ and the seller's cost is $c = 0.25$.

Step 2. Because the function $g'(\cdot)$ is non-negative and $v_2(0) = 0$ we obtain that from (C-19) that $v_2(s) \geq 0$. Let $\eta(s) = \frac{1-v_1(s)}{1+v_2(s)}$ and $\eta_\infty = c/\bar{v}$. Taking derivatives and using (C-19) we obtain that the function $\eta(s)$ satisfies the ODE

$$\dot{\eta}(s) = -\frac{1}{1+v_2(s)} \left(h'(\eta(s)) + \eta(s) g'(\eta(s)) \right) . \tag{C-21}$$

Now using that $v_2(s)$ is non-negative we find that $\dot{\eta}(s) \geq -A(\eta(s))$ with $A(\eta) = h'(\eta) + \eta g'(\eta)$ (where we used that $A(\eta) \geq 0$ by step 1 under the assumption that $\eta \geq 0$, which we shall prove as correct). Note that $A(\eta) = 0$ for $\eta \leq \eta_\infty$ and $A(\eta)$ is Lipschitz continuous. Therefore we have that $A(\eta) \leq L \max\{\eta - \eta_\infty, 0\}$ for some $L > 0$. By the Comparison Theorem we get that $\eta(s) \geq a(s)$ with $a(s)$ the solution of the ODE $\dot{a}(s) = -L \max\{\eta - \eta_\infty, 0\}$ and $a(0) = 1$. The previous ODE admits the closed-form solution $a(s) = \eta_\infty + e^{-Ls}(1-\eta_\infty)$, and the result follows because $a(s) > \eta_\infty$.

For the second point we begin by noting that $\eta(s) > \eta_\infty$ and $v_1(s) \geq 0$ implies that $v_2(s) \leq \eta_\infty^{-1} - 1 \in (0,1)$. As a result we obtain that $\dot{\eta}(s) \leq -A(\eta(s))/\eta_\infty$. Another application of the Comparison Theorem yields that $\eta(s) \leq b(s)$ with $b(s)$ the solution of the ODE $\dot{b}(s) = -L/\eta_\infty \max\{\eta - \eta_\infty, 0\}$ and $b(0) = 1$. The previous ODE admits the closed-form solution $b(s) = \eta_\infty + e^{-L/\eta_\infty s}(1-\eta_\infty)$, and the result follows because $\lim_{s\to\infty} b(s) = \eta_\infty$.

Step 3. Because $h'(\cdot)$ and $g'(\cdot)$ are non-negative we have that $\mathbf{v}(s) \geq 0$. The latter together with step 2 implies that the image of the function $\mathbf{v}(s)$ lies in the compact set $\mathcal{V} \triangleq [0, 1-\eta_\infty] \times [0, \eta_\infty^{-1} - 1]$, which is non-empty because $0 < \eta_\infty < 1$. Note that the function $H(v_1, v_2) = \left[h'(\frac{1-v_1}{1+v_2}), g'(\frac{1-v_1}{1+v_2}) \right]$ is Lipschitz continuous in $\mathcal{V}$ because $h'(\eta)$, $g'(\eta)$, and $\frac{1-v_1}{1+v_2}$ are Lipschitz continuous and the composition of Lipschitz continuous functions is Lipschitz continuous. By Picard–Lindelöf theorem we have that there exits a unique solution to the system of ODEs.

Step 4. Letting $\lambda_2(s) \triangleq \lambda_2(\mathbf{v}(s))$ we have that $\lambda_2(s) = (h'(\eta(s)) + \eta(s)g'(\eta(s)))/(1 + v_2(s))$. The total derivative is

$$\dot{\lambda}_2(s) = \frac{\partial \lambda_2}{\partial \gamma}\Big|_{\mathbf{v}(s)} \dot{v}_1(s) + \frac{\partial \lambda_2}{\partial \mu}\Big|_{\mathbf{v}(s)} \dot{v}_2(s)\,.$$

Using the fact that $\eta(s) \geq \eta_\infty$, we obtain that the primitives are differentiable and the partial derivatives of the eigenvalue are

$$\begin{aligned}\frac{\partial \lambda_2}{\partial \gamma} &= -\frac{1}{(1+\mu)^2}\left(h''(\eta) + \eta g''(\eta) + g'(\eta)\right)\,,\\ \frac{\partial \lambda_2}{\partial \mu} &= -\frac{1}{(1+\mu)^2}\left(\eta h''(\eta) + \eta^2 g''(\eta) + 2\eta g'(\eta) + h'(\eta)\right)\,.\end{aligned}$$

Substituting in our previous expression and using the ODE for the total derivatives of $\mathbf{v}(s)$ we obtain that

$$\dot{\lambda}_2(s) = -\frac{1}{(1+v_2(s))^2}\left(h'(\eta(s)) + \eta(s)g'(\eta(s))\right)\left(h''(\eta(s)) + \eta(s)g''(\eta(s)) + 2g'(\eta(s))\right)\,. \tag{C-22}$$

From step 1 we know that the first term in parenthesis is strictly positive because $\eta(s) > \eta_\infty$. For the second term notice that by letting $\kappa(y) \triangleq \psi'(y)(1+\psi'(y)) + (y - \psi(y))\psi''(y)$ we obtain

$$\begin{aligned}h''(\eta) + \eta g''(\eta) + 2g'(\eta) &= \frac{c^2}{\eta^3}\left(f(\psi(\frac{c}{\eta}))\psi'(\frac{c}{\eta})\left(1 + \psi'(\frac{c}{\eta})\right) - \bar{F}(\psi(\frac{c}{\eta}))\psi''(\frac{c}{\eta})\right)\\ &= \frac{c^2}{\eta^3} f(\psi(\frac{c}{\eta}))\kappa(\frac{c}{\eta}) > 0\,, \qquad\qquad \text{(C-23)}\end{aligned}$$

where the second equality follows from noting that $y - \psi(y) = -\bar{F}(\psi(y))/f(\psi(y))$ from the definition of the virtual value function, and the last inequality $f(\cdot) > 0$ and $\kappa(\frac{c}{\eta}) > 0$ from Assumption 1. Thus we conclude that $\dot{\lambda}_2(s) < 0$ and the first point follows.

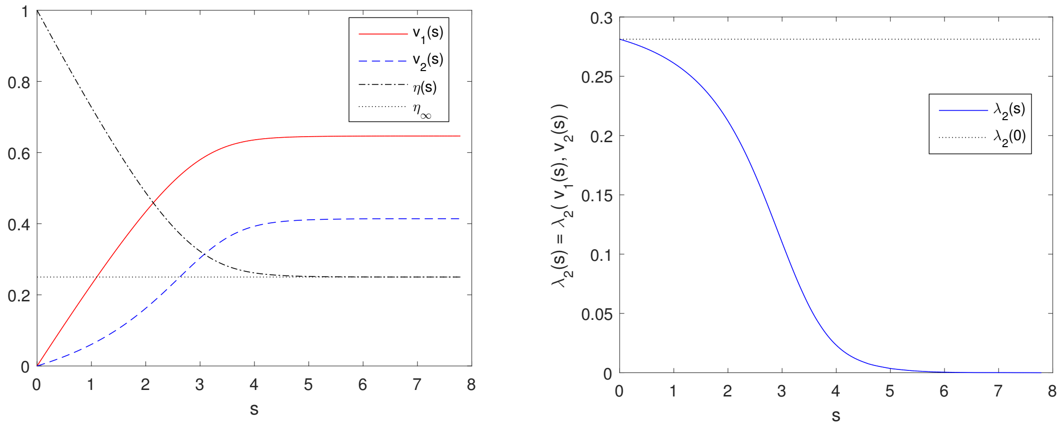
The second point is trivial. For the last point note that from step 2 we know that $\lim_{s\to\infty}\eta(s) = \eta_\infty$. Recall from step 1 that the primitives $h'(\eta)$ and $g'(\eta)$ are continuous and $h'(\eta_\infty) = g'(\eta_\infty) = 0$, implying that $\lim_{s\to\infty}\lambda_2(s) = 0$.

Step 5. In view of step 4 we can apply Theorem 3 in Section 3.4 of Evans (2010) to conclude the existence of a unique entropy solution. We reproduce the construction of the solution for completeness. We postulate a solution of the form $w(x,t) = G(x/t)$. Substituting in the PDE (C-18) we obtain that

$$\frac{1}{t}G'\left(\frac{x}{t}\right)\left(\lambda_2\left(G\left(\frac{x}{t}\right)\right) - \frac{x}{t}\right) = 0\,,$$

which implies that $G(\cdot) = \lambda_2^{-1}(\cdot)$ whenever $G(\cdot)$ is not constant. Observe that the image of $\lambda_2(s)$ is $[0, \lambda_2(0)] \subset \mathbb{R}_+$. The result follows from pasting the solutions $w(x,t) = \lambda_2^{-1}(x/t)$ when $x/t < \lambda_2(0)$ and $w(x,t) = 0$ otherwise, to obtain continuity.

Step 6. It suffices to show that the solution $\gamma(x,t) = v_1(w(x,t))$ and $\mu(x,t) = v_2(w(x,t))$ satisfies the boundary conditions that $\gamma(x,0) = \mu(x,0) = 0$ for all $x > 0$ and $(1-\gamma(0,t))/(1+\mu(0,t)) \leq \eta_\infty$ for all $t > 0$. For $x > 0$ and $t \to 0$ we have that $w(x,t) = 0$ and $\mathbf{v}(0) = \mathbf{0}$ as expected. For $t > 0$ and $x \to 0$ we have that $\lim_{x\to 0} w(x,t) = \lim_{x\to 0}\lambda_2^{-1}(x) = \infty$. Because $\lim_{s\to\infty}\eta(s) = \eta_\infty$ we conclude that $(1-\gamma(0,t))/(1+\mu(0,t)) = \eta_\infty$ as required.



(a) $v_1(s)$, $v_2(s)$ and $\eta(s) = \frac{1-v_1(s)}{1+v_2(s)}$ **(b)** $\lambda_2(s) \triangleq \lambda_2(\mathbf{v}(s))$

Figure 2: The first figure plots the solution $\mathbf{v}(s) = \big(v_1(s), v_2(s)\big)$ to the ODE (C-19) when values are $U[0,1]$ and the seller's cost is $c = 0.25$. The second figure plots $\lambda_2(s)$, which is equal to the second eigenvalue of $D\mathbf{F}(\gamma, \mu)$ evaluated at the solution of the ODE.

Step 7. We prove the result for the seller's value function (a similar argument holds for the buyer). Using the fact that $v_1(0) = 0$ and $w(x,t) = 0$ when $t > 0$ and $x \geq t\lambda_2(0)$ we obtain that

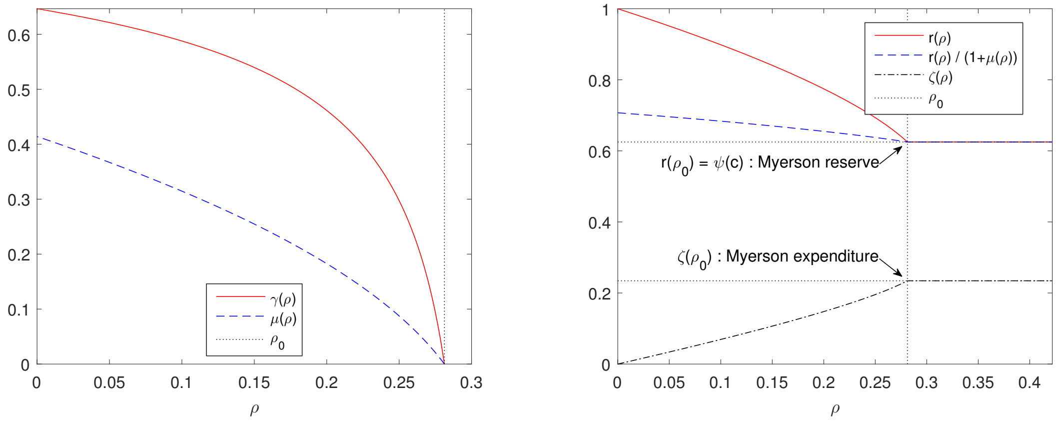
$$\begin{aligned}\bar{\Pi}(x,t) &= \int_0^x v_1(w(y,t))\,\mathrm{d}y = \int_0^{\min\{x,t\lambda_2(0)\}} v_1\left(\lambda_2^{-1}(y/t)\right)\,\mathrm{d}y \\ &= t\int_0^{\min\{x/t,\lambda_2(0)\}} v_1\left(\lambda_2^{-1}(z)\right)\,\mathrm{d}z = tj(x/t)\,,\end{aligned}$$

where the next to last equation follows from the change of variables $z = y/t$ and the last from our definition of $j(y) = \int_0^{\min\{y,\lambda_2(0)\}} v_1\left(\lambda_2^{-1}(z)\right)\,\mathrm{d}z$. Observe that the boundary conditions trivially hold for $\bar{\Pi}(x,t)$, and by construction the PDE holds in the interior of the domain.

The function $j(y)$ is absolutely continuous and thus almost everywhere differentiable with derivative $j'(y) = v_1\left(\lambda_2^{-1}(y)\right)$ for $y < \lambda_2(0)$ and $j'(y) = 0$ for $y > \lambda_2(0)$. Continuity of the derivative follows because $v_1(\cdot)$ and $\lambda_2^{-1}(\cdot)$ are continuous, and $\lim_{y\to\lambda_2(0)^-} j'(y) = 0$ since $v_1(0) = 0$. As a result $j(y)$ is continuously differentiable.

The previous observation yields that $\bar{\Pi}(x,t)$ is continuously differentiable for $t > 0$ with partial derivatives given by $\frac{\partial\bar{\Pi}}{\partial x} = j'\left(\frac{x}{t}\right)$ and $\frac{\partial\bar{\Pi}}{\partial t} = j\left(\frac{x}{t}\right) - \frac{x}{t}j'\left(\frac{x}{t}\right)$. Continuity of $\bar{\Pi}(x,t)$ at $t = 0$ for $x \geq 0$ follows from the fact that $j(\cdot)$ is bounded by $j(\lambda_2(0)) < \infty$, and as result $\lim_{t\to 0}\bar{\Pi}(x,t) = \lim_{t\to 0} tj(x/t) = 0$. Because $j'(y) = 0$ for $y > \lambda_2(0)$ we conclude that the partial derivatives are continuous for all $(x,t) \neq 0$.

Step 8. We need to show that $0 \leq \bar{\Pi}_x(x,t) \leq 1$ and $\bar{U}_x(x,t) \geq 0$. We have that $\bar{\Pi}_x(x,t) = \gamma(x,t) = v_1(w(x,t))$, and $\bar{U}_x(x,t) = \mu(x,t) = v_2(w(x,t))$. In step 3 we showed that the image of the function $\mathbf{v}(s)$ lies in the compact set $\mathcal{V} = [0, 1-\eta_\infty] \times [0, \eta_\infty^{-1} - 1]$. The result follows because $\eta_\infty \in (0,1)$.



(a) $\gamma(\rho)$ and $\mu(\rho)$

(b) $r(\rho)$, $r(\rho)/(1+\mu(\rho))$ and $\zeta(\rho)$

Figure 3: The first figure plots shadow prices $\mu(\rho)$ and $\gamma(\rho)$ as a function of the budget-to-time ratio ρ when values are $U[0,1]$ and the seller's cost is $c = 0.25$. The second figure plots the threshold value $r(\rho)$, the payment $r(\rho)/(1+\mu(\rho))$, and expenditure rate $\zeta(\rho)$ as function of the budget-to-time ratio ρ. Note that when $\rho \geq \rho_0 \triangleq \lambda_2(0)$, the shadow prices are zero and the threshold value and payment coincide with the Myerson optimal reserve $\psi(c)$.

C.3 Structural Properties for Single-Buyer Problem

Before proving the result we show that it suffices to consider the ratio of budget to time remaining $\rho = x/t$ to study the evolution of budget. This result heavily leverages the structure of the solution of the PDE in the existence result.

Lemma C.1. *Consider the function* $\zeta : \mathbb{R}_+ \to \mathbb{R}_+$

$$\zeta(\rho) = \frac{r(\rho)}{1+\mu(\rho)} \bar{F}\big(r(\rho)\big)$$

where the threshold value is $r(\rho) \triangleq \psi\left(c\frac{1+\mu(\rho)}{1-\gamma(\rho)}\right)$*, and the shadow prices are* $\gamma(\rho) \triangleq v_1\left(\lambda_2^{-1}(\rho)\right)$ *and* $\mu(\rho) \triangleq v_2\left(\lambda_2^{-1}(\rho)\right)$ *whenever* $\rho < \lambda_2(0)$ *and* $\gamma(\rho) = \mu(\rho) = 0$ *otherwise.*

The evolution of budgets is given by the ODE

$$\dot{\bar{x}}(t) = \zeta\left(\frac{\bar{x}(t)}{t}\right).$$

Proof. The existence result shows that there exists a solution of the PDE where the shadow prices are given by $\gamma(x,t) = v_1\left(w(x,t)\right))$ and $\mu(x,t) = v_2\left(w(x,t)\right)$. Because $w(x,t) = \lambda_2^{-1}(x/t)$ if $x/t < \lambda_2(0)$ and $w(x,t) = 0$ otherwise, it suffices to consider the ratio of budget to time remaining $\rho = x/t$. Let $\gamma(\rho) = v_1\left(\lambda_2^{-1}(\rho)\right)$, $\mu(\rho) = v_2\left(\lambda_2^{-1}(\rho)\right)$. We obtain that the budget along the equilibrium path evolves according to

$$\dot{\bar{x}}(t) = e\left(\bar{x}(t),t\right) = \frac{r(\bar{x}(t),t)}{1+\mu(\bar{x}(t),t)}\bar{F}\big(r(\bar{x}(t),t)\big) = \frac{r(\bar{x}(t)/t)}{1+\mu(\bar{x}(t)/t)}\bar{F}\big(r(\bar{x}(t)/t)\big) = \zeta\left(\frac{\bar{x}(t)}{t}\right). \qquad \square$$

Plots of the shadow prices, the threshold value, the payment and the expenditure rate are provided in Figure 3 when values are $U[0,1]$ and the seller's cost is $c = 0.25$. We next prove some properties of the functions $\zeta(\rho)$, $r(\rho)$, $\gamma(\rho)$, and $\mu(\rho)$.

Lemma C.2. *The functions ζ, r, γ, μ satisfy the following properties:*

1. *$\gamma(\rho)$ and $\mu(\rho)$ are non-negative and non-increasing in ρ.*
2. *$r(\rho)$ is non-increasing in ρ.*
3. *$\zeta(\rho) < \rho$ for all $\rho > 0$.*
4. *$\frac{r(\rho)}{1+\mu(\rho)}$ is non-increasing in ρ.*
5. *$r(\rho) \geq \frac{r(\rho)}{1+\mu(\rho)} \geq \psi(c)$.*
6. *$\rho - \zeta(\rho) \geq \alpha\rho$ for some $\alpha \in (0,1)$.*
7. *$\zeta(\rho)$, $\mu(\rho)$ and $\gamma(\rho)$ are Lipschitz continuous in ρ.*

Proof. We prove each statement at a time.

Item 1. We first prove the result for the seller's shadow price. When $\rho > \lambda_2(0)$ the result is trivial since $\gamma(\rho) = 0$. Recall that when $\rho < \lambda_2(0)$ we have that $\gamma(\rho) = v_1\left(\lambda_2^{-1}(\rho)\right))$. Step 4 from the proof of Theorem 2 implies that $\lambda_2^{-1}(\rho)$ is decreasing. From step 1 of the same proof we have that $h'(\eta) \geq 0$ for all $\eta \geq 0$, which implies that the $\dot{v}_1(s) \geq 0$ from ODE (C-19). As a result we have that $v_1(s)$ is non-decreasing and the result follows. Non-negativity follows because from the initial conditions of the ODE (C-19) we have that $v_1(0) = 0$, implying that $v_1(s) \geq 0$ because the function is non-decreasing.

For the buyer's shadow price we have that when $\rho < \lambda_2(0)$ it is the case that $\mu(\rho) = v_2\left(\lambda_2^{-1}(\rho)\right))$. A similar argument applies by exploiting that $v_2(0) = 0$, and $g'(\eta) \geq 0$ from step 1 of the proof of Theorem 2.

Item 2. First notice that the previous point implies that the ratio $\eta(\rho) = \frac{1-\gamma(\rho)}{1+\mu(\rho)}$ is non-decreasing in ρ. The result follows because $r(\rho) = \psi\left(c/\eta(\rho)\right)$, and $\psi(\cdot)$ is non-decreasing.

Item 3. It suffices to prove the claim for all $0 < \rho \leq \lambda_2(0)$ since $\zeta(\rho)$ is constant otherwise. We have by step 4 of the proof of Theorem 2 that $\lambda_2(s)$ is decreasing and continuous, and $\lim_{s\to\infty} \lambda_2(s) = 0$. Thus, for all $0 < \rho \leq \lambda_2(0)$ there exists $s = \lambda_2^{-1}(\rho)$, and it suffices to show that $\zeta(\lambda_2(s)) < \lambda_2(s)$. Equation (C-20) implies that

$$\zeta(\rho) = \frac{h'(\eta(\rho))}{1+\mu(\rho)},$$

therefore it suffices to show that

$$\frac{h'(\eta(s))}{1+v_2(s)} < \lambda_2(s),$$

where used that $\eta(s) = \frac{1-v_1(s)}{1+v_2(s)}$. Using the formula for the second eigenvalue we can write the previous inequality as

$$\lambda_2(s) = \frac{h'(\eta(s)) + \eta(s)g'(\eta(s))}{1+v_2(s)} > \frac{h'(\eta(s))}{1+v_2(s)},$$

where the inequality follows because $v_2(s) \geq 0$, $\eta(s) > \eta_\infty > 0$ because $\rho > 0$, and $g'(\eta) > 0$ from step 1 of the proof of Theorem 2. The result follows.

Item 4. It suffices to prove the claim for all $\rho < \lambda_2(0)$ since the payment $r(\rho)/(1+\mu(\rho))$ is constant otherwise. Letting $s = \lambda_2^{-1}(\rho)$ as in the proof of item 3, we shall show that

$$q(s) \triangleq \frac{1}{1+v_2(s)}\psi\left(\frac{c}{\eta(s)}\right)$$

is non-decreasing because $\lambda_2^{-1}(\rho)$ is decreasing. Taking derivatives with respect to s and dropping the dependence on s to simplify the notation we obtain that

$$\begin{aligned}\dot{q} &= -\frac{\dot{v}_2}{(1+v_2)^2}\psi\left(\frac{c}{\eta}\right) - \frac{c\dot{\eta}}{\eta^2(1+v_2)}\psi'\left(\frac{c}{\eta}\right) \\ &= -\frac{g'(\eta)h'(\eta) + (1+v_2)g'(\eta)\dot{\eta}}{(1+v_2)^2\bar{F}(\psi(c/\eta))} \\ &= \left(\frac{g'(\eta)}{1+v_2}\right)^2 \frac{\eta}{\bar{F}(\psi(c/\eta))},\end{aligned}$$

where the second equality follows because $\dot{v}_2 = g'(\eta)$ from ODE (C-19), together with the formulas for $g'(\eta)$ and $h'(\eta)$ in step 1 of the proof of Theorem 2; and the last equality follows because $\dot{\eta} = -\frac{1}{1+v_2}\left(h'(\eta) + \eta g'(\eta)\right)$ from step 2 of the proof of Theorem 2. The result follows because $\dot{q}(s) \geq 0$ since $\eta(s) \geq 0$ for all $s \geq 0$.

Item 5. The first inequality is trivial because $\mu(\rho) \geq 0$. For the second note that when $\rho \geq \lambda_2(0)$ we have from Lemma C.1 that $\lambda(\rho) = \mu(\rho) = 0$, which implies that $\frac{r(\rho)}{1+\mu(\rho)} = \psi(c)$. When $\rho < \lambda_2(0)$ the result follows because the ratio $\frac{r(\rho)}{1+\mu(\rho)}$ is non-increasing in ρ from item 4 and thus

$$\frac{r(\rho)}{1+\mu(\rho)} \geq \frac{r(\lambda_2(0))}{1+\mu(\lambda_2(0))} = \psi(c)\,.$$

Item 6. Let $q(\rho) \triangleq \rho - \zeta(\rho)$ and $\rho_0 = \lambda_2(0)$. It suffices to prove the claim for all $\rho \leq \rho_0$ since for $\rho > \rho_0$ we have that

$$q(\rho) = \rho - \zeta(\rho) = \rho - \zeta(\rho_0) = \rho - \rho_0 + \rho_0 - \zeta(\rho_0) \geq \alpha(\rho - \rho_0) + \alpha\rho_0 = \alpha\rho\,,$$

where the first equality follows because $\zeta(\rho)$ is constant for $\rho > \rho_0$, and the last because $\alpha \in (0,1)$ and using the claim for $\rho = \rho_0$. We next prove the result for $\rho \leq \rho_0$.

We have by step 4 of the proof of Theorem 2 that $\lambda_2(s)$ is decreasing and continuous, and $\lim_{s\to\infty} \lambda_2(s) = 0$. Thus, for all $0 < \rho \leq \lambda_2(0)$ there exists $s = \lambda_2^{-1}(\rho)$, and using the expressions in the proof of item 3 we obtain that

$$\begin{aligned}q(\lambda_2(s)) = \lambda_2(s) - \zeta(\lambda_2(s)) = \lambda_2(s) - \frac{1}{1+v_2(s)}h'(\eta(s)) = \frac{\eta(s)g'(\eta(s))}{1+v_2(s)} \\ = c\frac{1}{1-v_1(s)}\bar{F}(\psi(\frac{c}{\eta(s)}))\psi'(\frac{c}{\eta(s)})\,.\end{aligned}$$

where used that $\eta(s) = \frac{1-v_1(s)}{1+v_2(s)}$ and the formulae for $\lambda_2(s)$, $h'(\eta)$ and $g'(\eta)$ in step 1 from the proof of Theorem 2. We next lower bound each term at a time since all terms are positive by steps 1, 2 and 3 from the proof of Theorem 2. From step 3 we have $v_1(s) \geq 0$ and thus $(1 - v_1(s))^{-1} \geq 1$. From Assumption 1 we have $\psi'(\cdot) \geq \underline{\psi}'$ for some $\underline{\psi}' > 0$ (because the domain is bounded and the function is continuous). Letting $\eta(\rho) = \eta(\lambda_2^{-1}(\rho))$ we obtain that

$$q(\rho) \geq c\underline{\psi}'\bar{F}(\psi(\frac{c}{\eta(\rho)})) \triangleq \underline{q}(\rho)\,.$$

We conclude the proof by showing that $\underline{q}(\rho) \geq \alpha\rho$ for some $\alpha \in (0,1)$.

By the Mean Value Theorem we obtain that for some $\xi \in [0,\rho]$

$$\underline{q}(\rho) = \underline{q}(0) + \underline{q}'(\xi)\rho\,.$$

First note that $\underline{q}(0) = 0$ because $\eta(0) = \eta(\lambda_2^{-1}(0)) = \lim_{s\to\infty}\eta(s) = \eta_\infty$, $\psi(\bar{v}) = \bar{v}$ and $\bar{F}(\bar{v}) = 0$. Additionally,

$$\underline{q}'(\xi) = c\underline{\psi}'f(\psi(\frac{c}{\eta(\xi)}))\psi'(\frac{c}{\eta(\xi)})\frac{c}{\eta(\xi)^2}\eta'(\xi) \geq c(\underline{\psi}')^2\underline{f}\frac{c}{\eta(\xi)^2}\eta'(\xi)\,,$$

where the inequality follows because $f(\cdot) \geq \underline{f}$ for some $\underline{f} > 0$ from Assumption 1 (because the domain is bounded and the function is continuous). Finally using equations (C-21) and (C-22), and dropping the dependence of ρ we obtain by the Implicit Function Theorem (because $\dot{\lambda}_2(s) > 0$) that

$$\begin{aligned}
\eta'(\rho) = \frac{\mathrm{d}\eta}{\mathrm{d}s}\left(\frac{\mathrm{d}\lambda_2}{\mathrm{d}s}\right)^{-1} &= (1+v_2(s))\left(h''(\eta) + \eta g''(\eta) + 2g'(\eta)\right)^{-1} \\
&= (1+v_2(s))\frac{\eta^3}{c^2}\left(f(\psi(\frac{c}{\eta}))\psi'(\frac{c}{\eta})\left(1+\psi'(\frac{c}{\eta})\right) - \bar{F}(\psi(\frac{c}{\eta}))\psi''(\frac{c}{\eta})\right)^{-1} \\
&\geq \frac{\eta^2}{c}\left(\bar{f}\bar{\psi}'\left(1+\bar{\psi}'\right) + \bar{\psi}''\right)^{-1}\,,
\end{aligned}$$

where the second equality follows from (C-23), and the inequality because $\eta \geq \eta_\infty = c/\bar{v}$ and $f(\cdot) \leq \bar{f}$ for some $\bar{f} < \infty$ and $\left|\psi''(\cdot)\right| \leq \bar{\psi}''$ for some $\bar{\psi}'' < \infty$ from Assumption 1 (because the domain is bounded and the functions are continuous).

Putting everything together we obtain that

$$q(\rho) \geq c\underline{f}(\underline{\psi}')^2\left(\bar{f}\bar{\psi}'\left(1+\bar{\psi}'\right) + \bar{\psi}''\right)^{-1}\rho = \alpha\rho\,,$$

for some suitable α and the result follows because $c > 0$. Observe that $\zeta(\rho) > 0$ implies that $q(\rho) < \rho$ and thus we have that $\alpha \in (0,1)$.

Item 7. We prove the result for $\mu(\rho)$. Lipschitz continuity of $\gamma(\rho)$ follows similarly. Lipschitz continuity of $\zeta(\rho) = h'(\eta(\rho))/(1+\mu(\rho))$ follows because $h'(\cdot)$ is Lipschitz continuous from step 1 of the proof of Theorem 2, $\mu(\rho) \geq 0$ and $\eta(\rho) = \frac{1-\gamma(\rho)}{1+\mu(\rho)}$ is Lipschitz continuous.

We show that $\mu(\rho)$ is Lipschitz continuous by upper bounding the derivative $|\mu'(\rho)|$ for $\rho \leq \lambda_2(0)$ since $\mu(\rho) = 0$ otherwise. Using equation (C-22) and dropping the dependence of ρ we obtain by the Implicit Function Theorem (because $\dot{\lambda}_2(s) > 0$) that

$$\mu'(\rho) = \frac{\mathrm{d}v_2}{\mathrm{d}s}\left(\frac{\mathrm{d}\lambda_2}{\mathrm{d}s}\right)^{-1} = -\underbrace{(1+v_2)^2}_{(I)}\underbrace{\frac{g'(\eta)}{h'(\eta)+\eta g'(\eta)}}_{(II)}\underbrace{\frac{1}{h''(\eta)+\eta g''(\eta)+2g'(\eta)}}_{(III)}\,,$$

The first term is bounded by $|(I)| \leq \eta_\infty^{-2}$ because $0 \leq v_2(s) \leq \eta_\infty^{-1} - 1$ from step 3 of the proof of Theorem 2. The second term can be bounded using the expressions in step 1 of the proof of

Theorem 2 as

$$|(II)| = \frac{\frac{c}{\eta^2}\bar{F}(\psi(\frac{c}{\eta}))\psi'(\frac{c}{\eta})}{\bar{F}(\psi(\frac{c}{\eta}))\psi(\frac{c}{\eta}) + \frac{c}{\eta}\bar{F}(\psi(\frac{c}{\eta}))\psi'(\frac{c}{\eta})} = \frac{c\psi'(\frac{c}{\eta})}{\eta^2\psi(\frac{c}{\eta}) + c\eta\psi'(\frac{c}{\eta})} \leq \frac{\bar{\psi}'}{\eta_\infty \underline{\psi}'}\,,$$

because $\psi(\cdot) \geq 0$, and $\underline{\psi}' \leq \psi'(\cdot) \leq \bar{\psi}'$ from Assumption 1. For the third term we (C-23) to obtain

$$|(III)|^{-1} = \frac{c^2}{\eta^3} f(\psi(\frac{c}{\eta}))\kappa(\frac{c}{\eta}) \geq c^2 \underline{f}\underline{\kappa}$$

where the equality follows because $\kappa(y) \triangleq \psi'(y)(1+\psi'(y)) + (y-\psi(y))\psi''(y)$ and the inequality follows because $\eta \leq 1$ and $\kappa(\cdot) \geq \underline{\kappa}$ for some $\underline{\kappa} > 0$ from Assumption 1 (because the domain is bounded and the $\kappa(\cdot)$ continuous and positive). $\square$

The next result shows that ODE (11) admits an unique solution.

Lemma C.3. *Fix an initial state $(B, T) \in \mathcal{S}$, and let $\bar{x}(t) : [0, T] \to [0, B]$ be the evolution of budget along the equilibrium path. Then ODE* (11) *has an unique solution.*

Proof. Existence of a solution follows by Carathéodory Theorem because, according to Lemma C.1, the right-hand side of ODE (11) is bounded, continuous in x and measurable in t. We next prove uniqueness. By Proposition 1 we have $t_0 > 0$ where t_0 is the first time that the shadow prices are zero (note that this result does not rely on the uniqueness of the solution). For $t \in [0, t_0)$ the budget depletes at a constant rate and the ODE trivially has a unique solution. For $t \in [t_0, T]$ the right-hand side $\zeta(x/t)$ is continuous in t and Lipschitz continuous in x by item 7 of Lemma C.2. Invoking Picard–Lindelöf Theorem we have that there exits a unique solution to the ODE (11). To wit, Lipschitz continuity follows because for $x, y \geq 0$ and fixed t

$$|\zeta(x/t) - \zeta(y/t)| \leq L|x-y|/t \leq L|x-y|/t_0\,,$$

where the first inequality follows from Lipschitz continuity of ζ and the second because $t \geq t_0 > 0$. $\square$

The following result characterizes the payments and threshold values along the equilibrium path. The result shows monotonicity of the shadow prices, the threshold value, and the payment along the equilibrium path. It also shows that both the threshold value and the payment are never lower than the Myerson optimal reserve price $\psi(c)$ throughout the time horizon.

Proposition C.1. *Fix an initial state $(B, T) \in \mathcal{S}$, and let $x(t) : [0, T] \to [0, B]$ be the evolution of budget along the equilibrium path.*

1. *The seller's shadow price $\gamma(t) \triangleq \gamma(\bar{x}(t), t)$, the buyer's shadow price $\mu(t) \triangleq \mu(\bar{x}(t), t)$, the threshold value $r(t) \triangleq r(\bar{x}(t), t)$, and the payment $r(t)/(1+\mu(t))$ along the equilibrium path are non-decreasing in t, that is, they are non-increasing as time progresses.*

2. *The threshold value and payment along the equilibrium path satisfy $r(t) \geq \frac{r(t)}{1+\mu(t)} \geq \psi(c)$.*

Proof. We prove each item at a time.

Item 1. The shadow price of the seller along the equilibrium path is given by $\gamma(t) = \gamma(\bar{x}(t), t) = \gamma(\rho(t))$. From item 1 of Lemma C.2 we have that $\gamma(\rho)$ is non-increasing in ρ, and $\rho(t)$ is non-decreasing with time t from item 4 of Proposition 1. As a result we obtain that $\gamma(t)$ is non-increasing with time t and the claim follows. A similar result applies to the buyer's shadow price $\mu(t)$ and the threshold value $r(t)$.

Item 2. This follows directly from item 5 of Lemma C.2. $\square$

C.4 Proof of Proposition 1

We first show that $\rho_0 = \lambda_2(0)$ and then prove each item at a time. Step 4 of proof of Theorem 2 implies that $\lambda_2(0)$ is given by

$$\lambda_2(0) = h'(1) + g'(1) = \psi(c)\bar{F}(\psi(c)) + c\psi'(c)\bar{F}(\psi(c)) = \big(\psi(c) + c\psi'(c)\big)\bar{F}(\psi(c)) \triangleq \rho_0$$

where the second equation follows from equation (C-20) and the formula for $g'(\eta)$.

Item 1. Because $\rho(t)$ is non-decreasing as time t decreases (from item 4), we obtain that $\rho(t) \geq \rho(T) = B/T \geq \rho_0$. Recall that from Lemma C.1 we have that $\gamma(\rho) = \mu(\rho) = 0$ whenever $\rho \geq \rho_0$. Therefore $\gamma(t) = \mu(t) = 0$ for all $t \in [0, T]$ and the result follows.

Item 2. From item 4 we have that the budget-to-time ratio $\rho(t) = \bar{x}(t)/t$ satisfies the following ODE

$$\dot{\rho}(t) = -\frac{1}{t}\left(\rho(t) - \zeta(\rho(t))\right) \leq -\alpha\frac{\rho(t)}{t},$$

where the last inequality follows from item 6 of Lemma C.2. By the Comparison Theorem we obtain that (note that signs are inverted because time goes backwards)

$$\rho(t) \geq \frac{B}{T}\left(\frac{T}{t}\right)^{\alpha}. \tag{C-24}$$

Because $\rho(t)$ is continuous, $\rho(T) < \rho_0$, and $\lim_{t\to 0}\rho(t) = \infty$ (from bound C-24); the Intermediate Value Theorem implies that there exists $t_0 \in (0, T)$ with $\rho(t_0) = \rho_0$. Because $\rho(t)$ is increasing as time decreases, $\mu(t) = \gamma(t) = 0$ only if $t \leq t_0$. The previous bound given in (C-24) evaluated at $\rho(t_0) = \rho_0$ leads to $\rho_0 \geq B/T\,(T/t_0)^{\alpha}$, and the result follows from solving for t_0.

Item 3. Because the budget is non-increasing with time, it suffices to show that $\bar{x}(0) > 0$. From items 1 and 2, we have that there exists a time $t_0 \in (0, T]$ during which the Myerson auction is implemented. Let $e_0 \triangleq \psi(c)\bar{F}(\psi(c))$ be the expenditure rate under the Myerson auction. We have that

$$\bar{x}(0) = \bar{x}(t_0) - e_0 t_0 \geq (\rho_0 - e_0)t_0 > 0,$$

where the first inequality follows because $\bar{x}(t_0) \geq \rho_0 t_0$, and the second because $t_0 > 0$ and $\rho_0 > e_0$ by Assumption 1.

Item 4. Using Lemma C.1 we obtain that the ratio of budget to time remaining $\rho(t) = \bar{x}(t)/t$ satisfies the following ODE

$$\dot{\rho}(t) = \frac{\dot{\bar{x}}(t)}{t} - \frac{\bar{x}(t)}{t^2} = \frac{1}{t}\left(\zeta(\rho(t)) - \rho(t)\right).$$

From item 3 of Lemma C.2 we obtain that $\dot{\rho}(t) < 0$ because $\zeta(\rho) < \rho$, and as a result $\rho(t)$ is increasing as time t decreases.

D Proofs for Section 5

In the following section we denote by $C > 0$ a generic constant that is independent of the initial state and the scaling δ. Additionally, we denote $f(\delta) = O(g(\delta))$ if and only if there exists $C > 0$ and a real number δ_0 such that $|f(\delta)| \le C|g(\delta)|$ for all $0 \le \delta \le \delta_0$.

D.1 Proof of Theorem 3

We need to show that the stage mechanism $\hat{M}[x,n]$ is $O(\sqrt{\delta})-$incentive compatible for the buyer at state $(x,n) \in \mathcal{S}^\delta$ with respect to dynamic mechanism $\hat{M}$. We prove the result under the assumption that (x,n) is such that $(x,t_n) \in \hat{\mathcal{S}}^\delta$. This guarantees that when the initial state is $(x-\delta\bar{v}, t_n)$ the budget remaining at time zero in the fluid model is greater or equal than $\delta\bar{v}$. This implies that $x \ge \delta\bar{v}$ and as a result $\hat{M}[x,n] = M^*[x,n]$. Letting $p^* = P^*[x,n]$ and $z^* = Z^*[x,n]$ this is equivalent to

$$\delta\big(vp^*(w) - z^*(w)\big) + U^{\hat{M}}(x - \delta z^*(w), n-1) \le \delta\big(vp^*(v) - z^*(v)\big) + U^{\hat{M}}(x - \delta z^*(v), n-1) + O(\delta^{3/2}),$$

for every value $v \in \mathbb{R}_+$ and report $w \in \mathbb{R}_\perp$.

The DIC constraint of the fluid model (6c) implies that the optimal mechanism satisfies

$$vp^*(w) - \big(1+\mu(x,t_n)\big)z^*(w) \le vp^*(v) - \big(1+\mu(x,t_n)\big)z^*(v), \tag{D-25}$$

for every value $v \in \mathbb{R}_+$ and report $w \in \mathbb{R}_\perp$. We need to compare this condition to the one in the statement of the result. Proposition 2 is critical to this end. Proposition 2 shows that the marginal utility of an additional unit of budget for the buyer under dynamic mechanism $\hat{M}$ and truthful reporting in the discrete model is close to $\mu(x,t) = \bar{U}_x(x,t)$, that is, the marginal utility of budget for the buyer in the fluid model. Note that $(x,t_n) \in \hat{\mathcal{S}}^\delta$ implies that $(x,t_{n-1}) \in \hat{\mathcal{S}}^\delta$ because $\bar{x}(0;y,t)$ increases as t decreases. Because $x > \delta\bar{v}$ and $z^*(w) \le \bar{v}$ from Proposition 2 there exists some constant $C > 0$ such that

$$\begin{aligned}
U^{\hat{M}}(x - \delta z^*(w), n-1) &\le U^{\hat{M}}(x, n-1) - \mu(x, t_n - \delta)\delta z^*(w) + C\delta^{\frac{3}{2}} \\
&\le U^{\hat{M}}(x, n-1) - \mu(x, t_n)\delta z^*(w) + C\delta^{\frac{3}{2}} + L/t_n\bar{v}\delta^2
\end{aligned}$$

where the second inequality follows from $\big(\mu(x,t_n) - \mu(x,t_n - \delta)\big)\delta z^*(w) \le \delta^2\bar{v}L/t_n$ because payments are bounded by $z^*(w) \le \bar{v}$ and $\mu(x,t) = \bar{U}_x(x,t)$ is $L/\max(t,s)$-Lipschitz continuous between states (x,t) and (y,s) from Lemma D.3. Similarly we have

$$\begin{aligned}
U^{\hat{M}}(x - \delta z^*(v), n-1) &\ge U^{\hat{M}}(x, n-1) - \mu(x, t_n - \delta)\delta z^*(v) - C\delta^{\frac{3}{2}} \\
&\ge U^{\hat{M}}(x, n-1) - \mu(x, t_n)\delta z^*(v) - C\delta^{\frac{3}{2}} - L/t_n\bar{v}\delta^2 .
\end{aligned}$$

Multiplying condition (D-25) by δ, using the previous bounds to control the terms $\mu(x,t_n)\delta z^*(w)$ and $\mu(x,t_n)\delta z^*(v)$ respectively, and canceling the term $U^{\hat{M}}(x, n-1)$ we obtain that the feasible deviation condition can be written as

$$\delta\big(vp^*(w) - z^*(w)\big) + U^{\hat{M}}(x - \delta z^*(w), n-1) \le \delta\big(vp^*(v) - z^*(v)\big) + U^{\hat{M}}(x - \delta z^*(v), n-1) + O(\delta^{3/2}),$$

for every value $v \in \mathbb{R}_+$ and report $w \in \mathbb{R}_\perp$ as required.

D.2 Proof of Theorem 4

We need to show that the stage mechanism $\hat{M}[x,n]$ is $O(\delta^{1/3})-$sequentially rational for the seller at state $(x,n) \in \mathcal{S}^\delta$. We prove the result under the assumption that (x,n) is such that $(x,t_n) \in \hat{\mathcal{S}}^\delta$. This guarantees that when the initial state is $(x - \delta\bar{v}, t_n)$ the budget remaining at time zero in the fluid model is greater or equal than $\delta\bar{v}$. This implies that $x \geq \delta\bar{v}$ and as a result $\hat{M}[x,n] = M^*[x,n]$. Letting $p^*(v) = P^*[x,n]$, $z^*(v) = Z^*[x,n]$ and $m^* = M^*[x,n]$ this is equivalent to

$$\delta\pi(m) + \mathbb{E}\left[\mathit{\Pi}^{\hat{M}}(x - \delta z(v), n-1)\right] \leq \delta\pi(m^*) + \mathbb{E}\left[\mathit{\Pi}^{\hat{M}}(x - \delta z^*(v), n-1)\right] + \delta O(\delta^{1/3}),$$

for every mechanism $m \in \mathcal{M}$ satisfying budget feasibility; approximate incentive compatible; and bounded payments, and increasingness of the allocation and payments.

We prove the result in three steps. Because mechanism $m \in \mathcal{M}$ is only approximate incentive compatible, it is not guaranteed to be feasible for the fluid mechanism design problem in (6). We first use that from Proposition 2 the marginal utility of budget in the discrete and fluid model are close, to show that m is $O(\sqrt{\delta})$-incentive compatible in (6). We then use the novel sensitive analysis result for single-auction mechanism design from Balseiro et al. (2017) that shows that ϵ-incentive compatibility and ϵ-individual rationality increases the seller's profit by at most $O(\epsilon + \epsilon^{2/3})$. Using the optimality of mechanism M^* in the fluid model, we bound the expected profit of the stage mechanism in terms of the profit of the optimal fluid mechanism. We conclude by translating approximate optimality in the fluid model to the condition on the statement of the result (in terms of performance in the stochastic model).

Step 1. The approximate incentive compatibility condition from $m \in \mathcal{M}$ reads:

$$\delta v p(w) - \delta z(w) + U^{\hat{M}}(x - \delta z(w), n-1) \leq \delta v p(v) - \delta z(v) + U^{\hat{M}}(x - \delta z(v), n-1) + O(\delta\sqrt{\delta}),$$

for every value $v \in \mathbb{R}_+$ and report $w \in \mathbb{R}_\perp$. Note that $(x,t_n) \in \hat{\mathcal{S}}^\delta$ implies that $(x,t_{n-1}) \in \hat{\mathcal{S}}^\delta$ because $\bar{x}(0;y,t)$ increases as t decreases. Because $x > \delta\bar{v}$, $z(v) \leq \bar{v}$ from Proposition 2 there exists some constant $C > 0$ such that

$$\begin{aligned} U^{\hat{M}}(x - \delta z(v), n-1) &\leq U^{\hat{M}}(x, n-1) - \mu(x, t_n - \delta)\delta z(v) + C\delta^{\frac{3}{2}} \\ &\leq U^{\hat{M}}(x, n-1) - \mu(x, t_n)\delta z(v) + C\delta^{\frac{3}{2}} + L/t_n\bar{v}\delta^2 \end{aligned}$$

where the second inequality follows from $\big(\mu(x,t_n) - \mu(x,t_n-\delta)\big)\delta z(v) \leq \delta^2\bar{v}L/t_n$ because payments are bounded by $z(v) \leq \bar{v}$ by assumption and $\mu(x,t) = \bar{U}_x(x,t)$ is Lipschitz continuous from item 4 of Lemma D.3. Similarly we have

$$\begin{aligned} U^{\hat{M}}(x - \delta z(w), n-1) &\geq U^{M^*}(x, n-1) - \mu(x, t_n - \delta)\delta z(w) - C\delta^{\frac{3}{2}} \\ &\geq U^{\hat{M}}(x, n-1) - \mu(x, t_n)\delta z(w) - C\delta^{\frac{3}{2}} - L/t_n\bar{v}\delta^2 \,. \end{aligned}$$

Combining these bounds, canceling the term $U^{\hat{M}}(x, n-1)$ and dividing by $\delta > 0$ we obtain that the feasible deviation condition can be written as

$$vp(v) - \big(1 + \mu(x, t_n)\big) z(v) \geq vp(w) - \big(1 + \mu(x, t_n)\big) z(w) - \big(\underbrace{C\sqrt{\delta} + L/t_n \bar{v}\delta}_{\epsilon}\big) . \tag{D-26}$$

for every value $v \in \mathbb{R}_+$ and report $w \in \mathbb{R}_\perp$. This last condition coincides with the DIC constraint of the fluid model (6c) modulo the error term ϵ.

Step 2. For $\epsilon > 0$ and $\eta \in [0, 1]$ we define $h_\epsilon(\eta)$ as the optimal value of the following mechanism design problem with $\epsilon-$incentive compatibility and $\epsilon-$individually rationally constraints:

$$\begin{aligned} h_\epsilon(\eta) \triangleq \max_{(p,\tilde{z}) \in \mathcal{M}} \; & \mathbb{E}_v \Big[\eta \tilde{z}(v) - cp(v)\Big] \\ \text{s.t. } & vp(v) - \tilde{z}(v) \geq -\epsilon \, , \forall v \in \mathbb{R}_+ \, , \\ & vp(v) - \tilde{z}(v) \geq vp(w) - \tilde{z}(w) - \epsilon \, , \forall v, w \in \mathbb{R}_+ \, . \end{aligned}$$

When $\epsilon = 0$ the problem is the one-shot mechanism design problem of Corollary 1 and we have that $h_0(\eta) = h(\eta) = \mathbb{E}_v \left[(\eta\phi(v) - c)^+\right]$. Clearly the value function $h_\epsilon(\eta)$ is non-decreasing with ϵ since the feasible set enlarges as ϵ grows. The following result bounds the growth of the value function $h_\epsilon(\eta)$ as a function of ϵ.

Lemma D.1 (Balseiro et al. (2017)). *For every $\epsilon > 0$ and $\eta \in [0, 1]$ we have that $h_\epsilon(\eta) \leq h(\eta) + O(\epsilon + \epsilon^{2/3})$.*

We can bound the expected profit at state (x, n) of stage mechanism $m \in \mathcal{M}$ as

$$\begin{aligned} \pi(m) &= \mathbb{E}_v \Big[\big(1 - \gamma(x, t_n)\big) z(v) - cp(v)\Big] + \gamma(x, t_n) e(m) \\ &= \mathbb{E}_v \left[\frac{1 - \gamma(x, t_n)}{1 + \mu(x, t_n)} \tilde{z}(v) - cp(v)\right] + \gamma(x, t_n) e(m) \\ &\leq h_\epsilon(\eta(x, t_n)) + \gamma(x, t_n) e(m) \\ &\leq h(\eta(x, t_n)) + \gamma(x, t_n) e(m) + O(\epsilon^{2/3}) \\ &= \pi(m^*) - \gamma(x, t_n) e(m^*) + \gamma(x, t_n) e(m) + O(\epsilon^{2/3}) \, , \end{aligned}$$

where the first equality holds because $\pi(m) = \mathbb{E}_v \left[z(v) - cp(v)\right]$ and $e(m) = \mathbb{E}_v \left[z(v)\right]$; second equality follows from performing the change of variables $\tilde{z}(v) = \big(1+\mu(x, t_n)\big) z(v)$; the first inequality because $(\tilde{z}, p)$ is feasible for the optimization problem of $h_\epsilon(\eta(x, t_n))$ with $\eta(x, t_n) = \frac{1-\gamma(x,t_n)}{1+\mu(x,t_n)}$ because of condition (D-26), $\tilde{z}(v)$ is non-decreasing and $\tilde{z}(v) \geq 0$ since $1 + \mu(x, t_n) \geq 0$; the second inequality follows from Lemma D.1 because $\eta(x, t_n) \in (0, 1]$ from step 2 from the proof of Theorem 2; and the last equation follows from the HJB equation (6) because the fluid optimal mechanism $m^* = M^*[x, t]$ is a maximizer of the mechanism design problem $h_0(\eta(x, t_n))$. This implies that

$$\pi(m) - \gamma(x, t_n) e(m) \leq \pi(m^*) - \gamma(x, t_n) e(m^*) + O(\epsilon^{2/3}) \, . \tag{D-27}$$

Step 3. Repeating the steps of the proof of Theorem 3 and using Proposition 3 together with $\epsilon = O(\sqrt{\delta})$ we can write condition (D-27) as

$$\delta\pi(m) + \mathbb{E}\left[\mathit{\Pi}^{\hat{M}}(x - \delta z(v), n-1)\right] \leq \delta\pi(m^*) + \mathbb{E}\left[\mathit{\Pi}^{\hat{M}}(x - \delta z^*(v), n-1)\right] + \delta O(\delta^{1/3}),$$

and the result follows.

D.3 Proofs of Proposition 2 and Proposition 3

D.3.1 Preliminaries: Evolution of Budgets along the Equilibrium Path

We begin by showing that the evolution of budgets in the stochastic game and its fluid counterpart are close in expectation when the buyer bids truthfully. Fix a state $(x, n) \in \mathcal{S}^\delta$ and let $(x, t_n) \in \mathcal{S}$ with $t_n = n\delta$ be the corresponding state in the fluid model. Let $\{x_i\}_{i=1}^n$ be the stochastic process

$$x_{i-1} = x_i - \delta Z^*[x_i, t_i](v_i),$$

with initial condition $x_n = x$, and let $\bar{x}(s) : [0, t_n] \to \mathbb{R}$ be the solution of the ODE

$$\dot{\bar{x}}(s) = e(\bar{x}(s), s),$$

with initial condition $\bar{x}(t_n) = x$. A challenge in this step is that the dynamic mechanism is a "closed loop" mechanism designed to stay on the equilibrium path of the fluid model. In the discrete model, the presence of stochastic fluctuations will take the state off the fluid equilibrium path and the discrete model path may slowly diverge from the original continuous time equilibrium path. In order to establish that these paths stay appropriately close we first leverage the fact that the expected expenditure under the optimal mechanism is Lipschitz continuous (close to the fluid path) to show that deviations from the fluid path accumulate linearly. We next bound deviations in an almost sure sense via Gronwall's Lemma and then conclude that the stochastic path converges to the fluid model with an application of Doob's Martingale Inequality.

Another difficulty is that the expected expenditure under the optimal mechanism is not Lipschitz continuous when the budget is small and close the origin (see item 2 of Lemma D.3). We tackle this issue by constructing a tube along the fluid path with the property that, for states inside the tube, the expected expenditure is Lipschitz continuous with the Lipschitz constant depending on the size of the tube. The proof carefully selects the width of the tube to balance the probability of not exiting the tube and the Lipschitz constant.

We first formally define a tube of radius ν around the unique solution of the expenditure ODE.

Definition D.1. *Let $\mathcal{S}_\nu = \{(y, s) \in \mathcal{S} : |\bar{x}(s) - y| \leq \nu, s \leq t_n\} \subseteq \mathcal{S}$ be a tube of radius ν around the unique solution $\bar{x}(s)$ of the ODE $\dot{\bar{x}}(s) = e(\bar{x}(s), s)$ with initial condition $\bar{x}(t_n) = x$.*

The next result characterizes the states in a regular tube, that is, when the radius satisfies $\nu < t_0(\rho_0 - e_0)$ where we define t_0, following Proposition 1, as the first time that the shadow prices are zero in the fluid model when the starting state is $(x, t_n) \in \mathcal{S}$. Because $e_0 = \zeta(\rho_0)$ is the expected expenditure under the second-price auction with reserve price $\psi(c)$, the later imposes that the radius ν is smaller than the budget remaining at the end of the horizon. First, we show that if the tube has a small enough radius then around the tube budgets are never depleted. Additionally, we explicitly characterize the smallest time around the tube with budget-to-time ratio less than ρ_0. Figure 4 provides an example of an equilibrium path under the fluid optimal mechanism and a regular tube.

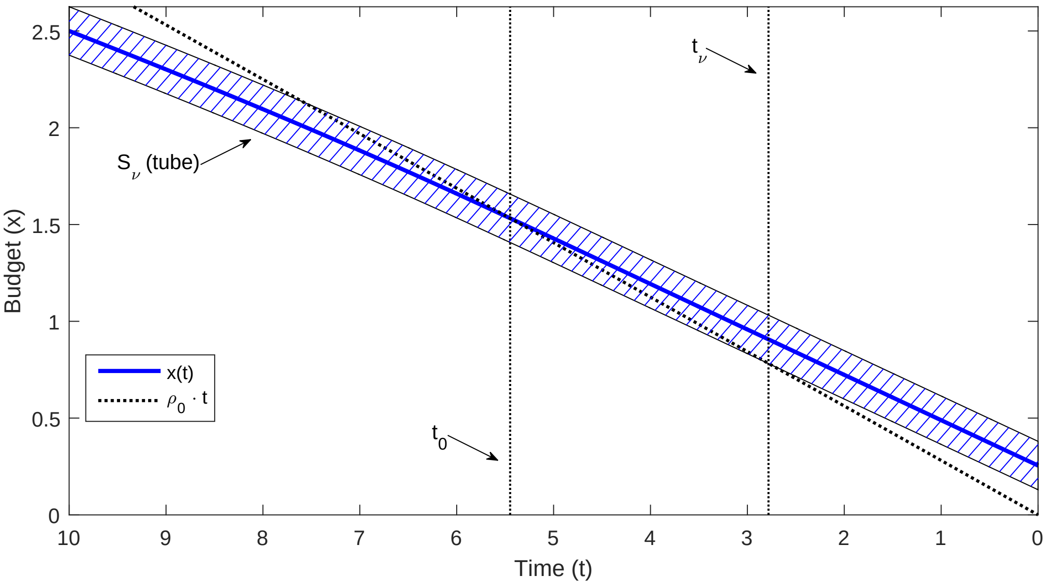


Figure 4: Equilibrium path under the fluid optimal mechanism M^* in the case of a single buyer. Values are $U[0,1]$, the seller's cost is $c = 0.25$, the initial budget is $B = 2.5$ and the horizon length is $T = 10$. The hatched area represents the regular tube S_ν of radius $\nu = 0.125$. t_0 is the first time that the budget-to-time is greater than ρ_0 along the equilibrium path and $t_\nu = t_0 - \nu/(\rho_0 - e_0)$ is the smallest time in the tube with budget-to-time ratio less than ρ_0. During time $[0, t_0]$ the budget is depleted at a rate e_0 along the equilibrium path.

Lemma D.2. *Suppose that the tube $\mathcal{S}_\nu$ is regular, that is, the radius satisfies $\nu < t_0(\rho_0 - e_0)$ where $e_0 = \zeta(\rho_0)$. Then the following holds:*

1. *Every state $(y, s) \in \mathcal{S}_\nu$ satisfies $y > 0$.*
2. *Let $t_\nu = \inf\{s \in [0, t_n] :$ there exists y such that $(y, s) \in \mathcal{S}_\nu, y < \rho_0 s\}$ be the smallest time in the tube with budget-to-time ratio less than ρ_0. Then, $t_\nu = t_0 - \nu/(\rho_0 - e_0) > 0$.*

The next result proves some useful properties of the modified mechanism $\hat{M}$ and the fluid optimal mechanism M^*. The second property shows that the expected expenditure is Lipschitz continuous about a tube $\mathcal{S}_\nu$ of radius ν under the assumption that the tube is regular. The Lipschitz constant is given by $L_\nu = L/t_\nu$ where t_ν is the smallest time around the tube with budget-to-time ratio less than ρ_0. Thus, when the tube is regular, Lemma D.2 implies that the Lipschitz constant L_ν is bounded.

Lemma D.3. *We have:*

1. *The payments are bounded by $0 \le \hat{Z}[y, s](v) \le \bar{v}$ and $0 \le Z^*[y, s](v) \le \bar{v}$ for all $(y, s) \in \mathcal{S}$.*
2. *The truthful expected expenditure $e(y, s) \triangleq e(M^*[y, s])$, truthful expected utility $u(y, s) \triangleq u(M^*[y, s])$, and truthful expected profit $\pi(y, s) \triangleq \pi(M^*[y, s])$ are $L_\nu-$Lipschitz continuous for all states (y, s) in the regular tube $\mathcal{S}_\nu$ where the Lipschitz constant is given by $L_\nu = L/t_\nu$ and the constant L is independent of the initial state (x, t_n) and ν.*
3. *The partial derivatives $\bar{U}_x(\cdot,\cdot)$, $\bar{U}_t(\cdot,\cdot)$, $\bar{\Pi}_x(\cdot,\cdot)$, $\bar{\Pi}_t(\cdot,\cdot)$ of the buyer's and seller's value function in the fluid model are $L/\max(s, s')-$Lipschitz continuous between states $(y, s) \in \mathcal{S}$ and $(y', s') \in \mathcal{S}$.*

4. *The partial derivatives of the truthful expected expenditure* $e_x(\cdot,\cdot)$ *and truthful expected utility* $u_x(\cdot,\cdot)$ *are* L/t_ν^2-*Lipschitz continuous for all states* (y,s) *in the regular tube* $\mathcal{S}_\nu$ *such that* $y/s < \rho_0$ *and* $e_x(\cdot,\cdot) = u_x(\cdot,\cdot) = 0$ *for* $y/s > \rho$.
5. *The partial derivatives of the truthful expected expenditure* $e_x(\cdot,\cdot)$ *and truthful expected utility* $u_x(\cdot,\cdot)$ *for states* $(y,s) \in \mathcal{S}$ *with* $y/s \neq \rho_0$ *are bounded by* L/s *for some constant* $L > 0$.

Consider the stopping time $\tau_\nu = \sup\{1 \le j \le n : (x_j, t_j) \notin S_\nu\}$, which represents the first time that the stochastic evolution of budget falls outside the tube $\mathcal{S}_\nu$, if ever. We are now in position to state the main result of this section.

Lemma D.4. *Let* $\mathcal{S}_\nu$ *be a regular tube of radius* ν*. Then, for* $1 \le i \le n$*:*

$$\mathbb{E}\left[\sup_{i\le j\le n} \left|x_j - \bar{x}(t_j)\right|^2 \mathbf{1}\{j \ge \tau_\nu\}\right] \le C\delta e^{2TL_\nu}\,,$$

for some constant $C > 0$ *independent of the initial state* (x, t_n)*, the tube radius* ν *and the scaling* δ*.*

As a straightforward corollary of the previous result we obtain the following bound on the probability that the budget evolution falls outside the tube S_ν.

Corollary D.1. *The probability that the stochastic budget evolution falls outside the tube* $\mathcal{S}_v$*, for* $1 \le i \le n$*, is bounded by,*

$$\mathbb{P}\left\{i \le \tau_\nu\right\} \le C\delta\nu^{-2}e^{2TL_\nu}\,,$$

for some constant $C > 0$ *independent of the initial state* (x, t_n)*, the tube radius* ν *and the scaling* δ*.*

The following result provides a refinement by leveraging the fourth moment of the martingale differences.

Corollary D.2. *Let* $\mathcal{S}_\nu$ *be a regular tube of radius* ν*. Then, for* $1 \le i \le n$*:*

$$\mathbb{E}\left[\sup_{i\le j\le n} \left|x_j - \bar{x}(t_j)\right|^4 \mathbf{1}\{j \ge \tau_\nu\}\right] \le C\delta^2 e^{4TL_\nu}\,,$$

for some constant $C > 0$ *independent of the initial state* (x, t_n)*, the tube radius* ν *and the scaling* δ*.*

The following result provides a refinement on the probability that the budget evolution falls outside the tube S_ν. The proof is straightforward and is omitted.

Corollary D.3. *The probability that the stochastic budget evolution falls outside the tube* $\mathcal{S}_v$*, for* $1 \le i \le n$*, is bounded by*

$$\mathbb{P}\left\{i \le \tau_\nu\right\} \le C\delta^2\nu^{-4}e^{4TL_\nu}\,,$$

for some constant $C > 0$ *independent of the initial state* (x, t_n)*, the tube radius* ν *and the scaling* δ*.*

D.3.2 Proof of Proposition 2

Consider two systems coupled by the realization of values $\{v_i\}_{i=1}^n$ such that in the first system the budgets evolve according to the stochastic process $\{x_i\}_{i=1}^n$ given by $x_{i-1} = x_i - \delta \hat{Z}[x_i, t_i](v_i)$ with initial condition $x_n = x$, and in the second system the budgets evolve according to the stochastic process $\{y_i\}_{i=1}^n$ given by $y_{i-1} = y_i - \delta \hat{Z}[y_i, t_i](v_i)$ with initial condition $y_n = y$.

Recall that in the fluid model budgets evolve deterministically according to the solution $\bar{x}(s; x, t_n) : [0, t_n] \to \mathbb{R}$ of the ODE $\frac{d}{ds}\bar{x}(s; x, t_n) = e(\bar{x}(s; x, t_n), s)$ with initial condition $x(t_n; x, t_n) = x$. We denote by $\frac{\partial \bar{x}}{\partial x}(s; x, t_n)$ the partial derivative of the state at time s with respect to the initial state x. In the following we drop the dependence on the initial state to simplify the notation and denote the evolution in the fluid model as $\bar{x}(s)$. We shall compare the marginal impact of budgets in the stochastic and fluid model by studying the difference between the *discrete perturbation* $q_i = x_i - y_i$ and the *fluid perturbation* $\bar{q}(s) = (x - y)\frac{\partial \bar{x}}{\partial x}(s)$.

We first characterize the fluid perturbation $\bar{q}(s)$. In order to characterize $\bar{q}(s)$ one needs to show that the expenditure under the optimal fluid mechanism is differentiable with respect to the budget. Lemma D.3 shows that the partial derivatives of the expenditure and utilities are bounded and Lipschitz whenever the state does not jump over the line

$$\mathcal{L} \triangleq \{(y, s) \in \mathcal{S} : y/s = \rho_0\}, \tag{D-28}$$

that is, when the seller switches from the two-tier auction to the static optimal mechanism. Leveraging the previous result we next show that $\bar{q}(s)$ is well-defined and satisfies the following ODE:

$$\dot{\bar{q}}(s) = e_x(\bar{x}(s), s)\bar{q}(s), \tag{D-29}$$

with initial condition $\bar{q}(t_n) = x - y$.

Lemma D.5. *The budget under the fluid model at time s, given by $\bar{x}(s)$, is differentiable with respect to the initial state x and $\bar{q}(s) = (x-y)\frac{\partial \bar{x}}{\partial x}(s)$ satisfies the ODE* (D-29). *Moreover, $\bar{q}(s) = O(|x-y|)$.*

The shadow price of an additional unit of budget in the buyer's total utility in the fluid model is given by

$$\mu(x, t_n) = \frac{\partial \bar{U}}{\partial x}(x, t_n) = \frac{\partial}{\partial x}\int_0^{t_n} u(\bar{x}(s), s)\, \mathrm{d}s = \int_0^{t_n} u_x(\bar{x}(s), s)\frac{\partial \bar{x}}{\partial x}(s)\, \mathrm{d}s,$$

where the third equality follows from Leibniz's rule because the integrand is bounded and integrable. Thus we have that

$$(x - y)\mu(x, t_n) = \int_0^{t_n} u_x(\bar{x}(s), s)\bar{q}(s)\, \mathrm{d}s = \sum_{i=1}^n \int_{t_{i-1}}^{t_i} u_x(\bar{x}(s), s)\bar{q}(s)\, \mathrm{d}s.$$

And the expression in consideration can be written as

$$U^{\hat{M}}(x, n) - U^{\hat{M}}(y, n) - (x - y)\mu(x, t_n) = \sum_{i=1}^n \mathbb{E}[\gamma_i]$$

where

$$\gamma_i = \delta\Big(\hat{u}(x_i, t_i) - \hat{u}(y_i, t_i)\Big) - \int_{t_{i-1}}^{t_i} u_x(\bar{x}(s), s)\bar{q}(s)\, \mathrm{d}s.$$

The key idea of the proof is to decompose the term γ_i in terms of the perturbation $q_i = x_i - y_i$ at time i in the discrete model and the error term $q_i - \bar{q}(t_i)$. We shall show that:

$$\gamma_i = \delta u_x(\bar{x}(t_i), t_i)\left(q_i - \bar{q}(t_i)\right) + O\Big(\delta |q_i| \cdot |x_i - \bar{x}(t_i)| + \delta |q_i|^2 + \delta^2 |x - y|\Big) ,$$

and the result follows from bounding each term in the expression.

There are three challenges in proving this result: (i) the discontinuity of the win event with respect to the threshold value, (ii) the partial derivative of the expenditure under the fluid optimal mechanism not being Lipschitz continuous everywhere, and (iii) the expenditure under the fluid optimal mechanism not being Lipschitz continuous when the state is close to zero.

The first and more fundamental challenge is the discontinuity of the win event with respect to the threshold value. Intuitively one can think of q_i as the "perturbation" of the state w.r.t. an additional amount of initial budget, and the result revolves around showing that the perturbation of the state w.r.t. the initial budget in the stochastic model q_i and the fluid model $\bar{q}(t_i)$ are close. One would be tempted to determine q_i by computing the pathwise derivative of the budget evolutions in the stochastic model and then taking expectations. Unfortunately, sample paths are not differentiable with respect to the initial state and instead we need to characterize the perturbations explicitly. To see this, recall that under states x_i and y_i, the threshold values are $r(x_i, t_i)$ and $r(y_i, t_i)$, respectively. Because $\{x_i\}_i$ and $\{y_i\}_i$ are coupled via the realization of values $\{v_i\}_i$, a perturbation in the initial amount of budget yields two effects: different payments when winning and different winning outcomes. The first effect is smooth w.r.t. budget: when the buyer wins the item in both systems ($v_i > r(x_i, t_i)$ and $v_i > r(y_i, t_i)$) the budgets are depleted by a similar amount in both systems because payments are close when x_i and y_i are close. The second effect is non-smooth w.r.t. budget: when the buyer wins the item in only one system (say $r(x_i, t_i) < v_i < r(y_i, t_i)$), the budget jumps in one system leading to large discrepancies between x_i and y_i. While the latter event is rare, it is guaranteed to occur a countable amount of times with non-vanishing probability. We handle this issue by carefully handling the impact of the latter effect.

The second challenge is that, in view of Lemma D.3, the partial derivative of the expenditure is not Lipschitz continuous when the state jumps over the line $\mathcal{L}$, which leads to large discrepancies between x_i and y_i whenever the latter event occurs. While the expected number of jumps is not guaranteed to converge to zero, we are able to bound the expected number of jumps and show that the total contribution of these jumps is vanishing. In order to bound the number of jumps we leverage that the state under the fluid optimal mechanism $\bar{x}(s)$ crosses over the line $\mathcal{L}$ at most once (since $\bar{x}(s)/s$ increases as time decreases from Proposition 1), together with the fact that x_i and y_i are close to $\bar{x}(s)$ from Lemma D.4. See Figure 4 for an example of the fluid equilibrium path crossing the line $\mathcal{L}$.

The third challenge is that the expenditure under the fluid optimal mechanism is not Lipschitz continuous when the state is close to zero. We tackle this issue by considering the tubes $\mathcal{S}_\nu^x$ and $\mathcal{S}_\nu^y$ of radius ν along the fluid budget evolutions $\bar{x}(s)$ and $\bar{y}(s)$, respectively. We need to choose the tube radius ν such that budgets along the tubes are greater than $\delta\bar{v}$ and the expenditure is Lipschitz continuous. This is attained, for example, if $0 < \nu < \bar{x}(0; y, t_n) - \delta\bar{v}$ where $\bar{x}(0; y, t_n)$ gives the budget remaining at time 0 when the initial state is (y, t_n). Because the initial states satisfy $0 \le x - y \le \delta\bar{v}$ and $\bar{x}(0; y, t_n) > \delta\bar{v}$ since $(x, t_n) \in \hat{\mathcal{S}}^\delta$, we satisfy the previous requirements by setting the tube radius to

$$\nu \triangleq \epsilon\left(\bar{x}(0; y, t_n) - \delta\bar{v}\right) ,$$

for some $\epsilon \in (0, 1)$. As a result $\hat{M} = M^*$ along the tube. We shall study the evolutions under the event that the stochastic evolutions of budgets fall within their respective tubes. To this end we introduce the stopping times $\tau_\nu^x = \sup\{1 \leq j \leq n : (x_j, t_j) \notin S_\nu^x\}$ and $\tau_\nu^y = \sup\{1 \leq j \leq n : (y_j, t_j) \notin S_\nu^y\}$, which represents the first time that the stochastic evolution of budgets falls outside the tubes $\mathcal{S}_\nu^x$ and $\mathcal{S}_\nu^y$, respectively; and $\tau_\nu = \max(\tau_\nu^x, \tau_\nu^y)$, which represents the first time that one of the processes falls outside of their respective tube. When the state falls within the tube, item 2 of Lemma D.3 implies that the expected expenditure is Lipschitz continuous with constant $L_\nu = L/t_\nu$ where t_ν is the smallest time around the tube with budget-to-time ratio less than ρ_0 (see Lemma D.2). Under our choice of the radius ν we have that the Lipschitz constant L_ν is bounded. Thus, in the following results we shall suppress the dependence on ν and write L as a general Lipschitz constant.

The first step is to bound the L_1 norm of the budget deviation q_i in terms of the initial budget difference $|x-y|$. The proof proceeds by recursively studying the evolution of the budget deviation q_i when the systems are coupled via the same realization of values. We first leverage the Lipschitz continuity of the threshold value and the payment of the two-tier auction to show that the budget deviations accumulate linearly. We then use a discrete version of Gronwall's Lemma to bound the absolute deviations.

Lemma D.6. *Let $q_i = x_i - y_i$. Suppose that $|x-y| = O(\delta)$, then we have:*

$$\mathbb{E}|q_i \mathbf{1}\{i \geq \tau_\nu\}| = O(|x-y|) = O(\delta)\,.$$

Because the difference γ_i involves second moments of the budget difference q_i, we also need to bound the L_2 norm of q_i in terms of the initial budget difference $|x-y|$ and δ. The proof of this result is more involved than that of Lemma D.6 because, when studying the second moment of q_i, one needs to carefully deal with the discontinuity of the winning event. We next provide some rough intuition for this result. Recall that when the buyer wins the item in only one system (say $r(x_j, t_j) < v_j < r(y_j, t_j)$), the budget jumps in one system leading to large discrepancies between x_j and y_j. Suppose that the likelihood of this event in period j is proportional to δ and the budget depletes proportionally to δ if the buyer wins in only one system. We have that the difference in expenditures in period j is approximately equal to $\alpha_j = \delta \mathrm{Be}(\delta)$, where $\mathrm{Be}(\delta)$ is a Bernoulli random variable with success probability δ. Ignoring other effects and assuming that errors do not propagate, we have that the total budget deviation is $q_i \approx \delta + \sum_{j=1}^i \alpha_j$. To bound the L_1 norm of the budget deviation is suffices to study each term independently because $\|\sum_{j=1}^i \alpha_j\|_1 \leq \sum_{j=1}^i \|\alpha_j\|_1 = O(\delta)$ since $\|\alpha_j\|_1 = \delta \mathbb{E}[\mathrm{Be}(\delta)] = \delta^2$ and $\delta i = O(1)$. Unfortunately, for the L_2 norm we can not use the same argument because $\|\alpha_j\|_2 = \delta\sqrt{\mathbb{E}[\mathrm{Be}(\delta)]} = \delta\sqrt{\delta}$, which leads to the insufficient rate of $\sum_{j=1}^i \|\alpha_j\|_2 = O(\sqrt{\delta})$. To obtain the same rate of convergence we need to study all terms simultaneously. In particular, using that $\sum_{j=1}^i \alpha_j = \delta \mathrm{Bin}(i, \delta)$ where $\mathrm{Bin}(i, \delta)$ is a Binomial random variable with i successes and success probability δ, we obtain that $\|\sum_{j=1}^i \alpha_j\|_2 = \delta\sqrt{\mathbb{E}[\mathrm{Bin}(i,\delta)^2]} = O(\delta)$. The proof of Lemma D.7 makes this argument precise.

Lemma D.7. *Let $q_i = x_i - y_i$. Suppose that $|x-y| = O(\delta)$, then we have:*

$$\|q_i \mathbf{1}\{i \geq \tau_\nu\}\|_2 = O\left(|x-y| + \sqrt{\delta|x-y|}\right) = O(\delta)\,.$$

Lemma D.3 shows that the partial derivatives are not Lipschitz continuous when the state jumps over the line $\mathcal{L}$, that is, when the seller switches from the two-tier auction to the static optimal mechanism. As a result, we need to control for the number of times that the state jumps over the line $\mathcal{L}$. Let

$$
\begin{aligned}
A_j^1 &= \left\{ \frac{\bar{x}(t_j)}{t_j}, \frac{\bar{x}(t_{j-1})}{t_{j-1}} < \rho_0 \right\} \cup \left\{ \frac{\bar{x}(t_j)}{t_j}, \frac{\bar{x}(t_{j-1})}{t_{j-1}} > \rho_0 \right\} , \\
A_j^2 &= \left\{ \frac{x_j}{t_j}, \frac{\bar{x}(t_j)}{t_j} < \rho_0 \right\} \cup \left\{ \frac{x_j}{t_j}, \frac{\bar{x}(t_j)}{t_j} > \rho_0 \right\} , \\
A_j^3 &= \left\{ \frac{x_j}{t_j}, \frac{y_j}{t_j} < \rho_0 \right\} \cup \left\{ \frac{x_j}{t_j}, \frac{y_j}{t_j} > \rho_0 \right\} ,
\end{aligned}
$$

where A_j^1 is the deterministic event that the state under the fluid model does not jump over the line $\mathcal{L}$, A_j^2 is the stochastic event that the state under the fluid model and the stochastic system starting at x lie in the same side of the line $\mathcal{L}$, and A_j^3 is the stochastic event that the state under the stochastic systems starting at x and y lie in the same side of the line $\mathcal{L}$. Lemma D.3 implies that the partial derivative of expenditure is Lipschitz continuous under $A_j^1 \cap A_j^2 \cap A_j^3$ since the state of the fluid model and the discrete stochastic model do not cross the line $\mathcal{L}$. The next result controls the probability that the state jumps over the line $\mathcal{L}$ by leveraging that the evolution of budget under the stochastic model and fluid model are close.

Lemma D.8. *Let* $E_j = \mathbf{1}\{\bar{A}_j^1\} + \sqrt{\mathbb{P}\{\bar{A}_j^2, j \geq \tau_\nu\}} + \sqrt{\mathbb{P}\{\bar{A}_j^3, j \geq \tau_\nu\}}$*, where* $\bar{A}_j^\bullet$ *denotes the complement of event* $A_j^\bullet$*. Suppose that* $|x - y| = O(\delta)$*. Then for all* $1 \leq i < n$ *we have*

$$
\sum_{j=i+1}^{n} E_j = O\left(\frac{1}{\sqrt{\delta}}\right) .
$$

We next bound the deviation between the budget difference q_i and the derivative of the budget remaining w.r.t. the initial budget in the fluid model. While the expected absolute deviation $\mathbb{E}\left[\left|q_i - \bar{q}(t_i)\right|\right]$ does not converge to zero at a fast enough rate, the following result shows that the *absolute value of the expected deviation* $\left|\mathbb{E}\left[q_i - \bar{q}(t_i)\right]\right|$ does converge sufficiently fast for our results to hold.

Lemma D.9. *Let* $q_i = x_i - y_i$ *and* $\bar{q}(t_i) = (x - y)\frac{\partial \bar{x}}{\partial x}(t_i)$*. Suppose that* $|x - y| = O(\delta)$*, then we have:*

$$
\left|\mathbb{E}\left[\left(q_i - \bar{q}(t_i)\right)\mathbf{1}\{i \geq \tau_\nu\}\right]\right| = O\left(\delta^{3/2}\right) .
$$

We finally bound the absolute expected value of γ_i in terms of the deviations in budgets $q_i = x_i - y_i$, the error term $q_i - \bar{q}(t_i)$ and the probability that the state jumps over the line $\mathcal{L}$.

Lemma D.10. *Suppose that* $|x - y| = O(\delta)$*, then we have:*

$$
\left|\mathbb{E}\left[\gamma_i\right]\right| = O\left(\delta\left|\mathbb{E}\left[\left(q_i - \bar{q}(t_i)\right)\mathbf{1}\{i \geq \tau_\nu\}\right]\right| + \delta^2\sqrt{\delta} + \delta^2 E_i\right) ,
$$

where $E_i = \mathbf{1}\{\bar{A}_i^1\} + \sqrt{\mathbb{P}\{\bar{A}_i^2, i \geq \tau_\nu\}} + \sqrt{\mathbb{P}\{\bar{A}_i^3, i \geq \tau_\nu\}}$.

We are now in position to prove the main result. Putting everything together we obtain that there exists some constant $C > 0$ such that

$$\begin{aligned}\left|U^{\hat{M}}(x,n) - U^{\hat{M}}(y,n) - (x-y)\mu(x,t_n)\right| &\leq \sum_{i=1}^{n} \left|\mathbb{E}[\gamma_i]\right| \\ &\leq C\delta^{3/2} + \delta^2 C \sum_{i=1}^{n} E_i + \delta C \sum_{i=1}^{n} \left|\mathbb{E}\left[(q_i - \bar{q}(t_i))\mathbf{1}\{i \geq \tau_\nu\}\right]\right| \\ &\leq C\delta^{3/2}\,,\end{aligned}$$

where the second inequality follows from Lemma D.10, third inequality follows because $\sum_{i=1}^{n} E_i = O\left(\frac{1}{\sqrt{\delta}}\right)$ from Lemma D.8 and $\left|\mathbb{E}\left[(q_i - \bar{q}(t_i))\mathbf{1}\{i \geq \tau_\nu\}\right]\right| = O(\delta^{3/2})$ from Lemma D.9 together with the fact that $|x-y| = O(\delta)$ and $\delta(n-i) = t_n - t_i = O(1)$.

This concludes the proof of Proposition 2. The proof of Proposition 3 is similar and is thus omitted.

D.4 Proofs of Lemmas D.2 - D.10

D.4.1 Proof of Lemma D.2

We prove each item at a time.

Item 1. We show that all $(y, s) \in \mathcal{S}_\nu$ satisfy $y > 0$ when the tube is regular. Because $\bar{x}(s)$ is monotonic, the smaller budget is verified at the end of the horizon and it suffices to check that the lowest point of the tube satisfies $\bar{x}(0) - \nu > 0$. From Proposition 1 we know that there exists a time $t_0 > 0$ such that budgets are depleted at the constant Myerson rate e_0 after time $t_0 > 0$. Thus we have that

$$\bar{x}(0) - \nu = \bar{x}(t_0) - e_0 t_0 - \nu = t_0(\rho_0 - e_0) - \nu > 0\,.$$

where the first equality follows because budgets are depleted at the constant Myerson rate e_0 after time $t_0 > 0$, the second equality because $\bar{x}(t_0)/t_0 = \rho_0$, and the inequality from our assumption on ν.

Item 2. Let $\tilde{t}_\nu \triangleq t_0 - \nu/(\rho_0 - e_0)$. We first show that $t_\nu \geq \tilde{t}_\nu$, that is, all $(y, s) \in \mathcal{S}_\nu$ with $y < \rho_0 s$ satisfy that $s > \tilde{t}_\nu$. Suppose not, that is, there exists some (y, s) with $s \leq \tilde{t}_\nu$. First note that $t_0 > \tilde{t}_\nu > 0$ by definition. Because $(y, s) \in \mathcal{S}_\nu$ we have

$$y \;\geq\; \bar{x}(s) - \nu = \bar{x}(t_0) - e_0(t_0 - s) - \nu = t_0(\rho_0 - e_0) + s e_0 - \nu > s e_0\,,$$

where the first equality follows because budgets are depleted at the constant Myerson rate e_0 after time t_0 (since $s \leq \tilde{t}_\nu < t_0$), the second equality because $\bar{x}(t_0)/t_0 = \rho_0$, and the inequality from our assumption on ν. Thus, $y/s > e_0 \geq \rho_0$, a contradiction.

We conclude that $t_\nu = \tilde{t}_\nu$ by showing that for every $\epsilon > 0$ there exists a state $(y^\epsilon, s^\epsilon) \in \mathcal{S}_\nu$ with $y^\epsilon < \rho_0 s^\epsilon$ and $s^\epsilon = \tilde{t}_\nu + \epsilon$. Such state is given, for example, by $(y^\epsilon, s^\epsilon) = (\rho_0 \tilde{t}_\nu + e_0 \epsilon, \tilde{t}_\nu + \epsilon)$.

D.4.2 Proof of Lemma D.3

We prove each item at a time.

Item 1. Recall that $Z^*[y, s](v) = \frac{r(y,s)}{1+\mu(y,s)} \mathbf{1}\{v \geq r(y, s)\}$. The lower bound follows trivially. The upper bound follows because (i) $r(\rho) \leq \bar{v}$ and (ii) $\mu(\rho) \geq 0$. Point (i) holds because $r(\rho) = \psi\left(\frac{c}{\eta(\rho)}\right) \leq \psi(\bar{v}) = \bar{v}$ since $\psi(\cdot)$ is increasing and $\eta \geq \eta_\infty = c/\bar{v}$. The result for $\hat{Z}[y, s](v)$ follows similarly.

Item 2 (expenditure). Let $\mathcal{S}_\nu^0 = \{(y, s) \in \mathcal{S}_\nu : y \geq s\rho_0\}$ be the restriction of the tube to the points where the budget-to-time ratio is greater than ρ_0. By Lemma C.1 the expenditure rate $e(\cdot, \cdot)$ is constant in $\mathcal{S}_\nu^0$ and the result is trivial. We prove the result by showing that the expenditure rate $e(\cdot, \cdot)$ is L_ν−Lipschitz continuous in $\mathcal{S}_\nu \setminus \mathcal{S}_\nu^0$ and exploiting that the expenditure rate is continuous in its domain.

Consider two points $(y', s'), (y, s) \in \mathcal{S}_\nu \setminus \mathcal{S}_\nu^0$. We have by Lemma D.2 that the fluid optimal mechanism is implemented around tube and thus

$$\begin{aligned}|e(y',s')-e(y,s)| &= \left|\zeta\left(\frac{y'}{s'}\right)-\zeta\left(\frac{y}{s}\right)\right| \le L\left|\frac{y'}{s'}-\frac{y}{s}\right| \le L\left|\frac{y'}{s'}-\frac{y'}{s}\right| + L\left|\frac{y'}{s}-\frac{y}{s}\right| \\ &= L\frac{y'}{s's}\left|s-s'\right| + \frac{L}{s}\left|y'-y\right| \le \frac{L\rho_0}{t_\nu}\left|s'-s\right| + \frac{L}{t_\nu}\left|y'-y\right| ,\end{aligned}$$

where the first inequality follows from item 7 of Lemma C.2, the second from the triangle inequality, and the last because in this region $s \ge t_\nu$ from Lemma D.2 and $y' \le \rho_0 s'$ from definition.

Item 2 (utility). The utility rate under the fluid optimal mechanism can be written as

$$u(y,s) = \mathbb{E}_v\left[vP^*[y,s](v) - Z^*[y,s](v)\right] = g(\eta(y,s)) + \mu(y,s)e(y,s) .$$

Let $u(\rho) \triangleq g(\eta(\rho))+\mu(\rho)\zeta(\rho)$. Item 7 of Lemma C.2 together with the continuous differentiability of g imply that $u(\rho)$ is Lipschitz continuous in ρ; because composition, product and addition preserve Lipschitz continuity. Since $u(\rho)$ is constant for $\rho > \rho_0$, a similar argument that before yields that $u(y,s) = u(y/s)$ is $L_\nu-$Lipschitz continuous in the regular tube $\mathcal{S}_v$.

Item 2 (profit). The profit rate under the fluid optimal mechanism can be written as

$$\pi(y,s) = \mathbb{E}_v\left[Z^*[y,s](v) - cP^*[y,s](v)\right] = h(\eta(y,s)) + \gamma(y,s)e(y,s) .$$

Let $\pi(\rho) \triangleq h(\eta(\rho))+\gamma(\rho)\zeta(\rho)$. Item 7 of Lemma C.2 together with the continuous differentiability of h imply that $\pi(\rho)$ is Lipschitz continuous in ρ; because composition, product and addition preserve Lipschitz continuity. Since $\pi(\rho)$ is constant for $\rho > \rho_0$, a similar argument that before yields that $\pi(y,s) = \pi(y/s)$ is $L_\nu-$Lipschitz continuous in the regular tube $\mathcal{S}_v$.

Item 3. We prove the result for the buyer's value function (a similar argument holds for the seller). For the partial derivative w.r.t. budget, note that $\bar{U}_x(y,s) = \mu(y,s)$ and $\mu(y,s) = \mu\big(\min\{y/s,\rho_0\}\big)$. Because $\mu(\rho)$ is Lipschitz continuous from Item 7 of Lemma C.2, the result follows from Lemma D.11. For the partial derivative w.r.t. time, we have by step 7 from the proof of Theorem 2 that $\bar{U}_t(y,s) = \ell\big(\min\{y/s,\rho_0\}\big)$ with

$$\ell(\rho) = \int_0^\rho \mu(\nu)\,\mathrm{d}\nu - \rho\mu(\rho) .$$

Clearly $\ell(\rho)$ is Lipschitz continuous for $\rho \le \rho_0$ and the result follows as before.

Items 4 and 5. We prove the result for the expected expenditure $e(y,s)$, a similar results follows for the expected utility. When $y/s \ge \rho_0$ we have that the expected expenditure $e(y,s) = \zeta(\rho_0)$ is constant and the result is trivial. In the remainder of the proof we consider the case when $y/s < \rho_0$. We have that $e(y,s) = \zeta(y/s)$ and thus $e_x(y,s) = \zeta'(y/s)/s$. We claim that $\zeta'(\rho)$ is bounded and Lipschitz continuous for $\rho \in [0,\rho_0)$. A similar argument that the one in item 2 then implies that $e_x(y,s)$ is L/t_ν^2-Lipschitz continuous.

We now prove the claim that $\zeta'(\rho)$ is bounded and Lipschitz continuous for $\rho \in [0,\rho_0)$. Recall that in the proof of item 3 of Lemma C.2 we have seen that

$$\zeta(\rho) = \frac{h'(\eta(\rho))}{1+\mu(\rho)}\,,$$

and thus taking derivatives with respect to ρ and dropping the dependence on ρ to simplify the notation yields:

$$\zeta' = \frac{h''(\eta)\eta'}{1+\mu} - \frac{h'(\eta)\mu'}{(1+\mu)^2} = -\frac{1}{(1+\mu)^2}\left(h''(\eta)(\gamma' + \eta\mu') + h'(\eta)\mu'\right)$$

where the second equation follows from $\eta(\rho) = (1-\gamma(\rho))/(1+\mu(\rho))$. From Lemma C.1 we have $\gamma(\rho) = v_1(\lambda_2^{-1}(\rho))$ and $\mu(\rho) = v_2(\lambda_2^{-1}(\rho))$. By the Implicit Function Theorem we obtain $\gamma'(\rho) = \frac{\mathrm{d}v_1}{\mathrm{d}s}\left(\frac{\mathrm{d}\lambda_2}{\mathrm{d}s}\right)^{-1}$ and $\mu'(\rho) = \frac{\mathrm{d}v_2}{\mathrm{d}s}\left(\frac{\mathrm{d}\lambda_2}{\mathrm{d}s}\right)^{-1}$, which holds because $\lambda_2(s)$ is decreasing. Using equations (C-19) and (C-22) we obtain that

$$\zeta' = \frac{h''(\eta)(h'(\eta) + \eta g'(\eta)) + h'(\eta)g'(\eta)}{(h'(\eta)+\eta g'(\eta))(h''(\eta) + \eta g''(\eta) + 2g'(\eta))}\,.$$

In the proof of item 7 of Proposition C.1 we showed that the denominator is bounded from below by a positive constant. Boundedness of the numerator follows by similar arguments. This implies that $\zeta'(\rho)$ is bounded.

Because $\eta(\rho) = (1-\gamma(\rho))/(1+\mu(\rho))$ and $\mu(\rho) \geq 0$, Lemma C.2 implies that $\eta(\rho)$ is Lipschitz continuous, while item 1 of the proof of Theorem 2 implies that $h'(\eta)$, $g'(\eta)$, $h''(\eta)$ and $g'(\eta)$ are Lipschitz continuous. Lipschitz continuity of $\zeta'(\rho)$ follows from the fact that composition, addition and product of functions preserves Lipschitz continuity.

D.4.3 Proof of Lemma D.4

We prove the result in three steps. We first leverage the Lipschitz continuity of the expected expenditure under the optimal mechanism to show that deviations from the expected path accumulate linearly. We then use a discrete version of Gronwall's Lemma to bound the absolute deviations in an almost sure sense. We conclude by taking expectations over every sample path and bounding the expected deviations by leveraging that the stochastic evolution of the mean-adjusted budget is a martingale.

Step 1. Fix a time period $1 \leq i < n$ (the result for $i = n$ is trivial) and consider a sample path under the event $\{i \geq \tau_\nu\}$. The latter implies that all time previous periods $i < j \leq n$ satisfy $(x_j, t_j) \in S_\nu$, which in turns implies that $\hat{M}[x_j, t_j] = M^*[x_j, t_j]$. For any period $j = n, \ldots, i+1$ the budget at period $j-1$ satisfies

$$x_{j-1} - \bar{x}(t_{j-1}) = x_j - \bar{x}(t_j) + \alpha_j\,,$$

where the difference is given by

$$\alpha_j = \int_{t_{j-1}}^{t_j} \dot{\bar{x}}(s)\ \mathrm{d}s - \delta Z^*[x_j, t_j](v_j)$$

with the integral representation following because the solution of the ODE is absolutely continuous. For the first term of α_j we have from the ODE that

$$\begin{aligned}\int_{t_{j-1}}^{t_j} \dot{\bar{x}}(s)\ \mathrm{d}s &= \int_{t_{j-1}}^{t_j} e(\bar{x}(s), s)\ \mathrm{d}s = (t_j - t_{j-1}) e(\bar{x}(\xi), \xi) \\ &= \delta e(x_j, t_j) + \delta\Big(e(\bar{x}(t_j), t_j) - e(x_j, t_j)\Big) + \delta\Big(e(\bar{x}(\xi), \xi) - e(\bar{x}(t_j), t_j)\Big)\end{aligned}$$

where the second equality follows from the Mean Value Theorem for some $\xi \in [t_{j-1}, t_j]$. Therefore we can write $\alpha_j = \beta_j + \Delta_j$, where Δ_j is a *stochastic error* given by

$$\Delta_j = \delta \left[e(x_j, t_j) - Z^*[x_j, t_j](v_j) \right] ,$$

and β_j is an *integration error* which is upper bounded by

$$\begin{aligned}|\beta_j| &\leq \delta \big|e(\bar{x}(t_j), t_j) - e(x_j, t_j)\big| + \delta \big|e(\bar{x}(\xi), \xi) - e(\bar{x}(t_j), t_j)\big| \\ &\leq \delta L_\nu \big|x_j - \bar{x}(t_j)\big| + \delta L_0 \big|\bar{x}(\xi) - \bar{x}(t_j)\big| + \delta L_0 |\xi - t_j| \\ &\leq \delta L_\nu \big|x_j - \bar{x}(t_j)\big| + \delta L_0 |\xi - t_j| \left(\big|e(\bar{x}(\xi'), \xi')\big| + 1 \right) \\ &\leq \delta L_\nu \big|x_j - \bar{x}(t_j)\big| + \delta^2 L_0 \left(\bar{v} + 1\right), \end{aligned} \tag{D-30}$$

where the second inequality follows from Lipschitz continuity of $e(\cdot, \cdot)$ in Lemma D.3 because the points (x_j, t_j) and $(\bar{x}(\xi), \xi)$ lie in the tubes $\mathcal{S}_\nu$ and $\mathcal{S}_0$ respectively, the third from the fact that by the Mean Value Theorem there exists $\xi' \in [\xi, t_j]$ such that $\bar{x}(\xi) = \bar{x}(t_j) + (\xi - t_j) e(\bar{x}(\xi'), \xi')$, and the last because $\xi \in [t_{j-1}, t_j]$ and the fact that payments are bounded by $\bar{v}$.

Summing over periods $j = n, \dots, i+1$ and using that the initial conditions coincide, that is, $x_n = \bar{x}(t_n) = x$, we obtain that the deviations are upper bounded by

$$\begin{aligned}\big|x_i - \bar{x}(t_i)\big| &= \Big|\sum_{j=i+1}^{n} \alpha_j\Big| \leq \sum_{j=i+1}^{n} |\beta_j| + \Big|\sum_{j=i+1}^{n} \Delta_j\Big| \\ &\leq \delta L_\nu \sum_{j=i+1}^{n} \big|x_j - \bar{x}(t_j)\big| + \underbrace{(t_n - t_i)\delta L_0 \left(\bar{v} + 1\right)}_{E_i} + \underbrace{\Big|\sum_{j=i+1}^{n} \Delta_j\Big|}_{D_i}, \end{aligned} \tag{D-31}$$

where the first inequality follows from the triangle inequality and the second from the bound on the integration error given in (D-30) and using that $\delta(n-i) = t_n - t_i$.

Step 2. We next apply Gronwall's Discrete Lemma to inequality (D-31) to bound the global error. We use the following version of Gronwall's Lemma: let ϵ_k be a non-negative sequence satisfying that $\epsilon_k \leq a + b \sum_{j=1}^{k-1} \epsilon_j$ where $a, b \geq 0$, then we have that $\epsilon_k \leq a \exp(bk)$. It follows then that the global error for every sample path under the event $\{i \geq \tau_\nu\}$ is upper bounded by

$$\big|x_i - \bar{x}(t_i)\big| \leq \big(E_i + \max_{i \leq j < n} |D_j|\big) e^{(t_n - t_i) L_\nu} ,$$

because the error term E_i is increasing as i decreases. The latter implies that

$$\sup_{i \leq j < n} \big|x_j - \bar{x}(t_j)\big| \mathbf{1}\{j \geq \tau_\nu\} = \sup_{i \vee \tau_\nu \leq j < n} \big|x_j - \bar{x}(t_j)\big| \leq \big(E_i + \max_{i \vee \tau_\nu \leq j < n} |D_j|\big) e^{(t_n - t_i) L_\nu} ,$$

where the first inequality follows because all terms are zero for $j < \tau_\nu$ and denoting by $x \vee y = \max\{x, y\}$ the maximum operator.

Step 3. Let $\|X\|_2 = \sqrt{\mathbb{E}[X^2]}$ be the L_2-norm. Taking expectations and using Minkowski's inequality we obtain that

$$\big\| \sup_{i \le j < n} |x_j - \bar{x}(t_j)| \mathbf{1}\{j \ge \tau_\nu\} \big\|_2 \le \big(E_i + \big\| \max_{i \vee \tau_\nu \le j < n} D_j \big\|_2 \big) e^{(t_n - t_i)L_\nu} ,$$

because the term E_i is deterministic. For the stochastic error we use that τ_ν is a stopping time and that the partial sum $D_i = \sum_{j=i+1}^n \Delta_j$ is a martingale with respect to the natural filtration to obtain via Doob's Martingale Inequality that

$$\begin{aligned} \Big\| \max_{i \vee \tau_\nu \le j < n} |D_j| \Big\|_2^2 &= \Big\| \max_{i \le j < n} |D_{j \vee \tau_\nu}| \Big\|_2^2 \le 4 \big\| D_{i \vee \tau_\nu} \big\|_2^2 = 4 \Big\| \sum_{j=i+1}^n \Delta_j \mathbf{1}\{j \ge \tau_\nu\} \Big\|_2^2 \\ &= 4 \sum_{j=i+1}^n \big\| \Delta_j \mathbf{1}\{j \ge \tau_\nu\} \big\|_2^2 \le 4 \sum_{j=i+1}^n \big\| \Delta_j \big\|_2^2 , \end{aligned}$$

where the second equality follows from writing the stopped martingale $D_{i \vee \tau_\nu} = \sum_{j=i+1}^n \Delta_j \mathbf{1}\{j \ge \tau_\nu\}$ and the third equality follows from the fact that the martingale differences $D_\ell - D_{\ell-1} = \Delta_\ell$ are orthogonal. In turn we can bound the stochastic error by

$$\begin{aligned} \big\| \Delta_j \big\|_2^2 &= \delta^2 \left\| e(x_j, t_j) - Z^*[x_j, t_j](v_j) \right\|_2^2 = \delta^2 \left\| \mathbb{E}_{v_j} \left[Z^*[x_j, t_j](v_j) \right] - Z^*[x_j, t_j](v_j) \right\|_2^2 \\ &= \delta^2 \mathbb{E}_{x_j} \left[\mathrm{Var}_{v_j} \left(Z^*[x_j, t_j](v_j) \mid x_j \right) \right] \le \delta^2 \frac{\bar{v}^2}{4} , \end{aligned}$$

where the last inequality follows because payments are bounded almost surely as follows $0 \le Z^*[x_i, t_i] \le \bar{v}$, and hence the variance is bounded by $\mathrm{Var}\left(Z^*[x_i, t_i](v_i) \mid x_i \right) \le \bar{v}^2/4$. Summing over we obtain that

$$\mathbb{E}\left[\max_{i \vee \tau_\nu \le j < n} D_j^2 \right] \le \delta(t_n - t_i)\bar{v}^2 .$$

Putting everything together we obtain that

$$\big\| \sup_{i \le j < n} |x_j - \bar{x}(t_j)| \mathbf{1}\{j \ge \tau_\nu\} \big\|_2 \le \Big(\delta(t_n - t_i) L_0 \left(\bar{v} + 1 \right) + \sqrt{\delta(t_n - t_i)} \bar{v} \Big) e^{(t_n - t_i)L_\nu} ,$$

and the result follows.

D.4.4 Proof of Corollary D.1

We have:

$$\mathbb{P}\left\{ i \le \tau_\nu \right\} = \mathbb{P}\left\{ \sup_{i \le j < n} |x_j - \bar{x}(t_j)| > \nu \right\} = \mathbb{P}\left\{ \sup_{i \le j < n} |x_j - \bar{x}(t_j)| \mathbf{1}\{j \ge \tau_\nu\} > \nu \right\} \le C\delta\nu^{-2} e^{2TL_\nu} ,$$

where the last follows from Markov's inequality.

D.4.5 Proof of Corollary D.2

To bound the fourth moment use that the partial sum $D_i = \sum_{j=i+1}^{n} \Delta_j$ is a martingale with respect to the natural filtration to obtain via Doob's Martingale Inequality that

$$\mathbb{E}\left[\max_{i\le j<n} |D_{j\vee\tau_\nu}|^4\right] \le \frac{16}{9}\mathbb{E}\left[|D_{i\vee\tau_\nu}|^4\right] = \frac{16}{9}\mathbb{E}\left[\Big|\sum_{j=i+1}^{n} \Delta_j \mathbf{1}\{j\ge\tau_\nu\}\Big|^4\right],$$

where the first equality follows from writing the stopped martingale as $D_{i\vee\tau_\nu} = \sum_{j=i+1}^{n}\Delta_j\mathbf{1}\{j \ge \tau_\nu\}$. Using that the martingale differences $D_\ell - D_{\ell-1} = \Delta_\ell$ are orthogonal we obtain

$$\mathbb{E}\left[\max_{i\le j<n} |D_{j\vee\tau_\nu}|^4\right] \le \frac{16}{9}\sum_{j=i+1}^{n}\mathbb{E}\left[\left|\Delta_j\right|^4\right] + \frac{16}{9}6\sum_{j=i+1}^{n}\sum_{\ell=i+1}^{j-1}\mathbb{E}\left[\left|\Delta_j\right|^2\cdot\left|\Delta_\ell\right|^2\right].$$

Using a similar bound as before we can bound the stochastic error by $\mathbb{E}\left[\left|\Delta_j\right|^4\right] = O(\delta^4)$ and $\mathbb{E}\left[\left|\Delta_j\right|^2\cdot\left|\Delta_\ell\right|^2\right] = O(\delta^4)$. Thus

$$\mathbb{E}\left[\max_{i\le j<n} |D_{j\vee\tau_\nu}|^4\right] = O((n-i)^2\delta^4) = O(\delta^2),$$

because $\delta(n-i) = t_n - t_i$. Therefore there exists some constant $C > 0$ such that

$$\Big\|\sup_{i\le j<n} |x_j - \bar{x}(t_j)|\mathbf{1}\{j\ge\tau_\nu\}\Big\|_4 \le \big(E_i + \big\|\max_{i\vee\tau_\nu\le j<n} D_j\big\|_4\big)e^{(t_n-t_i)L} \le C\sqrt{\delta}e^{(t_n-t_i)L_\nu},$$

and the result follows by taking fourth power and using that $t_n - t_i \le T$.

D.4.6 Proof of Lemma D.5

When the initial state (x, t_n) satisfies $x/t_n \ge \rho_0$ the result is trivial because the Myerson optimal auction is implemented throughout the horizon and budget is depleted at the deterministic rate $e_0 = \zeta(\rho_0)$. In the remainder of this proof we consider the case when the initial state (x, t_n) satisfies $x/t_n < \rho_0$.

We have that $\bar{x}(s)$ satisfies the ODE $\dot{\bar{x}}(s) = e(\bar{x}(s), s)$ with initial condition $\bar{x}(t_n) = x$. We first derive (D-29) heuristically. Taking partial derivatives in the previous ODE we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\bar{x}}{\partial x}(s) = \frac{\partial}{\partial x}\dot{\bar{x}}(s) = \frac{\partial}{\partial x}e(\bar{x}(s), s) = e_x(\bar{x}(s), s)\frac{\partial\bar{x}}{\partial x}(s).$$

Multiplying by $x - y$ and letting $\bar{q}(s) = (x-y)\frac{\partial\bar{x}}{\partial x}(s)$ we obtain the desired expression. To make this argument rigourous we need to account that $e_x(y, s)$ is not Lipschitz continuous when the state jumps over the line $\mathcal{L}$ defined in (D-28).

Let $t_0 = \sup\{s \in [0, t_n] : \frac{\bar{x}(s)}{s} \ge \rho_0\}$ be the first (deterministic) time that the budget-to-time ratio exceeds ρ_0. Proposition 1 implies that the budget-to-time ratio increases as time decreases, and a result $\bar{x}(s) < \rho_0$ for $s > t_0$ and $\bar{x}(s) > \rho_0$ for $s < t_0$. Lemma D.3 shows that the partial derivative $e_x(y, s)$ is Lipschitz continuous whenever $y/s < \rho_0$, and as result we obtain by the Differentiable Dependence Theorem for ODEs that $\frac{\partial\bar{x}}{\partial x}(s)$ exists and $\bar{q}(s) = (x-y)\frac{\partial\bar{x}}{\partial x}(s)$ satisfies the ODE (D-29) when $s > t_0$. We can extend the result up to $s = t_0$ by using the left derivative of the expenditure at $(y, s) = (\rho_0 t_0, t_0)$. For $s < t_0$ we can write the solution as

$$\bar{x}(s) = \bar{x}(t_0) - \int_s^{t_0} e(\bar{x}(s), s)\ \mathrm{d}s = \rho_0 t_0 - e_0(t_0 - s)\,. \tag{D-32}$$

where the second equation follows from $\bar{x}(t_0) = \rho_0 t_0$ and using that the expenditure rate is e_0 when $s < t_0$. By the Implicit Function Theorem we obtain that t_0 is differentiable w.r.t. the initial state x and satisfies (by taking derivatives in the expression $\bar{x}(t_0) = \rho_0 t_0$):

$$\frac{\mathrm{d}\bar{x}}{\mathrm{d}t}(t_0)\frac{\partial \bar{t}_0}{\partial x} + \frac{\partial \bar{x}}{\partial x}(t_0) = \rho_0 \frac{\partial \bar{t}_0}{\partial x}\,,$$

and a result $\frac{\partial t_0}{\partial x} = \frac{\partial \bar{x}}{\partial x}(t_0)/(\rho_0 - e_0)$ because $\frac{\mathrm{d}\bar{x}}{\mathrm{d}t}(t_0) = e(\bar{x}(t_0), t_0) = e_0$. Taking derivatives w.r.t. the initial state in equation (D-32) we obtain that

$$\frac{\partial \bar{x}}{\partial x}(s) = \frac{\partial t_0}{\partial x}\,(\rho_0 - e_0) = \frac{\partial \bar{x}}{\partial x}(t_0)\,,$$

which implies that $\bar{q}(s)$ is constant and equal to $\bar{q}(t_0)$ for $s < t_0$. This is consistent with ODE (D-29) because Lemma D.3 shows that $e_x(y, s) = 0$ when $y/s > \rho_0$.

A standard application of Grownall's inequality shows that $\bar{q}(s) = O(|x - y|)$.

D.4.7 Proof of Lemma D.6

Proof. We prove the result in three steps.

Step 1. Fix a time period $1 \le i < n$ (the result for $i = n$ is trivial) and consider a sample path under the event $\{i \ge \tau_\nu\}$. The latter implies that all previous time periods $i < j \le n$ satisfy $(x_j, t_j) \in S_\nu^x$ and $(y_j, t_j) \in S_\nu^y$ and the dynamic mechanism satisfies $\hat{M} = M^*$. Because the stochastic processes $\{x_i\}_i$ and $\{y_i\}_i$ are coupled by the realization of values $\{v_i\}_i$ we have that for any period $j = n, \ldots, i+1$ the difference $\{q_i\}_i$ evolves according to

$$q_{j-1} = q_j - \delta\big(Z^*[x_j, t_j](v_j) - Z^*[y_j, t_j](v_j)\big) = q_j - \alpha_j\,.$$

where $\alpha_j = \delta\big(Z^*[x_j, t_j](v_j) - Z^*[y_j, t_j](v_j)\big)$ is the difference in expenditures between the two systems. Summing over periods $j = n, \ldots, i+1$ and using that the initial condition is $q_n = x - y$ we obtain that the deviations are upper bounded by

$$|q_i| = |x - y| + \Big|\sum_{j=i+1}^{n} \alpha_j\Big| \le |x - y| + \sum_{j=i+1}^{n} |\alpha_j|\,,$$

where the first inequality follows from the triangle inequality. Taking expectations under the event $\mathbf{1}\{i \ge \tau_\nu\}$ we obtain that

$$\mathbb{E}|q_i \mathbf{1}\{i \ge \tau_\nu\}| \le |x - y| + \sum_{j=i+1}^{n} \mathbb{E}|\alpha_j \mathbf{1}\{i \ge \tau_\nu\}| \le |x - y| + \sum_{j=i+1}^{n} \mathbb{E}|\alpha_j \mathbf{1}\{j \ge \tau_\nu\}|\,, \tag{D-33}$$

where the last inequality follows from $\{i \ge \tau_\nu\} \subseteq \{j \ge \tau_\nu\}$ since $j \ge i$. We next bound the terms $\mathbb{E}|\alpha_j \mathbf{1}\{j \ge \tau_\nu\}|$.

Step 2. Let $\hat{r}(x,t) = r(x,t)/(1+\mu(x,t))$ be the payment of the buyer when the item is won. When $j \geq \tau_\nu$ the term α_j can be decomposed as follows

$$\begin{aligned}\alpha_j &= \delta\hat{r}(x_j,t_j)\mathbf{1}\{v_j > r(x_j,t_j)\} - \delta\hat{r}(y_j,t_j)\mathbf{1}\{v_j > r(y_j,t_j)\} \\ &= \delta\underbrace{\hat{r}(x_j,t_j)\Big(\mathbf{1}\{v_j > r(x_j,t_j)\} - \mathbf{1}\{v_j > r(y_j,t_j)\}\Big)}_{(I)} + \delta\underbrace{\Big(\hat{r}(x_j,t_j) - \hat{r}(y_j,t_j)\Big)\mathbf{1}\{v_j > r(y_j,t_j)\}}_{(II)}\,.\end{aligned}$$

We can bound the first term as

$$\begin{aligned}\mathbb{E}|(I)\mathbf{1}\{j \geq \tau_\nu\}| &\leq \bar{v}\mathbb{P}\{v_j \in [r(x_j,t_j), r(y_j,t_j)], j \geq \tau_\nu\} \\ &= \bar{v}\mathbb{E}_{x_j,y_j}\Big[\Big(F\big(r(x_j,t_j) \vee r(y_j,t_j)\big) - F\big(r(x_j,t_j) \wedge r(y_j,t_j)\big)\Big)\mathbf{1}\{j \geq \tau_\nu\}\Big] \\ &\leq \bar{v}\bar{f}\mathbb{E}_{x_j,y_j}\Big[\Big|r(x_j,t_j) - r(y_j,t_j)\Big|\mathbf{1}\{j \geq \tau_\nu\}\Big] \\ &\leq \bar{v}\bar{f}L\mathbb{E}|q_j\mathbf{1}\{j \geq \tau_\nu\}|\,,\end{aligned}$$

where the first inequality follows because payments are bounded by $0 \leq \hat{r}(x,t) \leq \bar{v}$ and the events that the item is won in each system differ whenever $v_j \in [r(x_j,t_j), r(y_j,t_j)]$, the equality because values are independent of the state and τ_ν is a stopping time, the second inequality because the probability density function of values is upper bounded by $\bar{f}$ according to Assumption 1, and the last because the threshold value is Lipschitz continuous from Lemma D.3. For the second term we obtain that

$$\mathbb{E}|(II)\mathbf{1}\{j \geq \tau_\nu\}| \leq \mathbb{E}_{x_j,y_j}\Big[\Big|\hat{r}(x_j,t_j) - \hat{r}(y_j,t_j)\Big|\mathbf{1}\{j \geq \tau_\nu\}\Big] \leq L\mathbb{E}|q_j\mathbf{1}\{j \geq \tau_\nu\}|\,,$$

because $\mathbf{1}\{v_j > r(y_j,t_j)\} \leq 1$ and payments are Lipschitz continuous. As a result we obtain that

$$\mathbb{E}|\alpha_j\mathbf{1}\{j \geq \tau_\nu\}| \leq \delta L(\bar{v}\bar{f}+1)\mathbb{E}|q_j\mathbf{1}\{j \geq \tau_\nu\}|\,.$$

Step 3. Using our previous bound on (D-33) we obtain

$$\mathbb{E}\big|q_i\mathbf{1}\{i \geq \tau_\nu\}\big| \leq |x-y| + \delta L(\bar{v}\bar{f}+1)\sum_{j=i+1}^{n}\mathbb{E}|q_j\mathbf{1}\{j \geq \tau_\nu\}|\,,$$

which implies by Gronwall's Inequality that

$$\mathbb{E}\big|q_i\mathbf{1}\{i \geq \tau_\nu\}\big| \leq |x-y|e^{L(\bar{v}\bar{f}+1)(t_n-t_i)}$$

because $\delta(n-i) = t_n - t_i$. ☐

D.4.8 Proof of Lemma D.7

Proof. We prove the result in three steps.

Step 1. Fix a time period $1 \leq i < n$ (the result for $i = n$ is trivial) and consider a sample path under the event $\{i \geq \tau_\nu\}$. The latter implies that all previous time periods $i < j \leq n$ satisfy $(x_j,t_j) \in S^x_\nu$ and $(y_j,t_j) \in S^y_\nu$ and the dynamic mechanism satisfies $\hat{M} = M^*$. Because the stochastic processes $\{x_i\}_i$ and $\{y_i\}_i$ are coupled by the realization of values $\{v_i\}_i$ we have that for any period $j = n, \dots, i+1$ the difference $\{q_i\}_i$ evolves according to

$$q_{j-1} = q_j - \delta\big(Z^*[x_j, t_j](v_j) - Z^*[y_j, t_j](v_j)\big) = q_j - \alpha_j \,.$$

where $\alpha_j = \delta\big(Z^*[x_j, t_j](v_j) - Z^*[y_j, t_j](v_j)\big)$ is the difference in expenditures between the two systems. We can decompose the term α_j as follows

$$\alpha_j = \beta_j + \Delta_j \,,$$

where $\beta_j = \mathbb{E}[\alpha_j | x_j, y_j] = \delta\big(e(x_j, t_j) - e(y_j, t_j)\big)$ is the expected difference in expenditures between the two systems and $\Delta_j = \alpha_j - \mathbb{E}[\alpha_j | x_j, y_j]$. Summing over periods $j = n, \dots, i+1$ and using that the initial condition is $q_n = x - y$ we obtain that the deviations are upper bounded by

$$|q_i| = |x - y| + \Big| \sum_{j=i+1}^{n} \alpha_j \Big| \le |x - y| + \sum_{j=i+1}^{n} |\beta_j| + \Big| \sum_{j=i+1}^{n} \Delta_j \Big| \,,$$

where the first inequality follows from the triangle inequality. Taking expectations under the event $\mathbf{1}\{i \ge \tau_\nu\}$ and using Minkowski's Inequality we obtain that the L_2 norm of q_i is upper bounded by

$$\|q_i \mathbf{1}\{i \ge \tau_\nu\}\|_2 \le |x - y| + \sum_{j=i+1}^{n} \|\beta_j \mathbf{1}\{j \ge \tau_\nu\}\|_2 + \Big\| \underbrace{\Big(\sum_{j=i+1}^{n} \Delta_j \Big)}_{D_i} \mathbf{1}\{i \ge \tau_\nu\} \Big\|_2 \,, \tag{D-34}$$

where the last inequality follows from $\{i \ge \tau_\nu\} \subseteq \{j \ge \tau_\nu\}$ since $j \ge i$. We next bound the terms $\|\beta_j \mathbf{1}\{j \ge \tau_\nu\}\|_2$ and $\|D_i \mathbf{1}\{i \ge \tau_\nu\}\|_2$.

Step 2. Because the expenditure is Lipschitz continuous when $j \ge \tau_\nu$ we obtain that

$$|\beta_j| = \delta |e(x_j, t_j) - e(y_j, t_j)| \le \delta L |x_j - y_j| = \delta L |q_j| \,,$$

and thus $\|\beta_j \mathbf{1}\{j \ge \tau_\nu\}\|_2 \le L\delta \|q_j \mathbf{1}\{j \ge \tau_\nu\}\|_2$. For the second term we use that the partial sum D_i is a martingale to obtain that

$$\begin{aligned}
\|D_i \mathbf{1}\{i \ge \tau_\nu\}\|_2^2 \le \|D_{i \vee \tau_\nu}\|_2^2 &= \Big\| \sum_{j=i+1}^{n} \Delta_j \mathbf{1}\{j \ge \tau_\nu\} \Big\|_2^2 \\
&= \sum_{j=i+1}^{n} \|\Delta_j \mathbf{1}\{j \ge \tau_\nu\}\|_2^2 = \sum_{j=i+1}^{n} \big\| \big(\alpha_j - \beta_j\big) \mathbf{1}\{j \ge \tau_\nu\} \big\|_2^2 \\
&\le \sum_{j=i+1}^{n} \left(\|\alpha_j \mathbf{1}\{j \ge \tau_\nu\}\|_2 + \|\beta_j \mathbf{1}\{j \ge \tau_\nu\}\|_2 \right)^2 \,,
\end{aligned}$$

where the first inequality follows because $D_i \mathbf{1}\{i \ge \tau_\nu\} = 0$ if $i < \tau_\nu$, the second equality follows because martingale differences are orthogonal, and the last inequality from Minkowski. Following the steps of Lemma D.6 we obtain that $\|\alpha_j \mathbf{1}\{j \ge \tau_\nu\}\|_2 \le \delta \|(I) \mathbf{1}\{j \ge \tau_\nu\}\|_2 + \delta \|(II) \mathbf{1}\{j \ge \tau_\nu\}\|_2$ where

$$\|(I) \mathbf{1}\{j \ge \tau_\nu\}\|_2 \le \bar{v} \sqrt{\mathbb{P}\{v_j \in [r(x_j, t_j), r(y_j, t_j)], j \ge \tau_\nu\}} \le \bar{v} \sqrt{\bar{f} L \mathbb{E} |q_j \mathbf{1}\{j \ge \tau_\nu\}|} \le C \sqrt{|x - y|} \,,$$

for some constant $C > 0$ by Lemma D.6. Similarly it not hard to see that $\|(II)\mathbf{1}\{j \geq \tau_\nu\}\|_2 \leq L\|q_j 1\{j \geq \tau_\nu\}\|_2$. Therefore we obtain that

$$\|D_i \mathbf{1}\{i \geq \tau_\nu\}\|_2^2 \leq \sum_{j=i+1}^{n} \left(2\delta L \|q_j \mathbf{1}\{j \geq \tau_\nu\}\|_2 + \delta C\sqrt{|x-y|}\right)^2 .$$

Step 3. Using our previous bounds on (D-34) we obtain

$$\|q_i \mathbf{1}\{i \geq \tau_\nu\}\|_2 \leq |x-y| + \delta L \sum_{j=i+1}^{n} \|q_j \mathbf{1}\{j \geq \tau_\nu\}\|_2 + \delta \sqrt{\sum_{j=i+1}^{n} \left(2L\|q_j \mathbf{1}\{j \geq \tau_\nu\}\|_2 + C\sqrt{|x-y|}\right)^2},$$

which implies by Lemma D.12 (with $\epsilon_0 = |x-y|$, $a = \delta L$, $b = 2L\delta$, $c = C\delta\sqrt{x-y}$) that

$$\|q_i \mathbf{1}\{i \geq \tau_\nu\}\|_2 = O\left(|x-y| + \sqrt{\delta|x-y|}\right) . \qquad \square$$

D.4.9 Proof of Lemma D.8

Proof. Let $E_j^1 = \mathbf{1}\{\bar{A}_j^1\}$, $E_j^2 = \sqrt{\mathbb{P}\{\bar{A}_j^2, j \geq \tau_\nu\}}$ and $E_j^3 = \sqrt{\mathbb{P}\{\bar{A}_j^3, j \geq \tau_\nu\}}$. In this notation the result can be written as

$$\sum_{j=i+1}^{n} E_j = \sum_{j=i+1}^{n} E_j^1 + \sum_{j=i+1}^{n} E_j^2 + \sum_{j=i+1}^{n} E_j^3 .$$

We analyze the first and second sum. The bound for the third sum follows mutatis mutandis.

First sum. For the first sum, note that $\bar{A}_j^1$ implies that $\frac{\bar{x}(t_j)}{t_j} \leq \rho_0 \leq \frac{\bar{x}(t_{j-1})}{t_{j-1}}$ since the budget-to-time ratio increases as time decreases. Therefore this event occurs at most once and $\sum_{j=i+1}^{n} E_j^1 \leq 1$ for all $i < n$.

Second sum. For the second sum, note that $\bar{A}_j^2$ implies that either $\frac{x_j}{t_j} \leq \rho_0 \leq \frac{\bar{x}(t_j)}{t_j}$ or $\frac{\bar{x}(t_j)}{t_j} \leq \rho_0 \leq \frac{x_j}{t_j}$. It is not hard to see that $\bar{A}_j^2 \subseteq \{|x_j - \bar{x}(t_j)| \geq |\rho_0 t_j - \bar{x}(t_j)|\}$. Therefore,

$$\begin{aligned}\left(E_j^2\right)^2 = \mathbb{P}\{\bar{A}_j^2, j \geq \tau_\nu\} &\leq \mathbb{P}\left\{|x_j - \bar{x}(t_j)| \geq |\rho_0 t_j - \bar{x}(t_j)|, j \geq \tau_\nu\right\} \\ &\leq \frac{\mathbb{E}\left[|x_j - \bar{x}(t_j)|^4 \mathbf{1}\{j \geq \tau_\nu^x\}\right]}{|\rho_0 t_j - \bar{x}(t_j)|^4} \leq C\frac{\delta^2}{|\rho_0 t_j - \bar{x}(t_j)|^4} ,\end{aligned}$$

where the second inequality follows from the conditional Markov's inequality and because $\{j \geq \tau_\nu\} \subseteq \{j \geq \tau_\nu^x\}$ since $\tau_\nu = \max(\tau_\nu^x, \tau_\nu^y)$, and the third from Corollary D.2 for some constant $C > 0$. Recall that $t_0 = \sup\{s \in [0, t_n] : \frac{\bar{x}(s)}{s} \geq \rho_0\}$ is the first (deterministic) time that the budget-to-time ratio exceeds ρ_0. We claim that, for every time s, we have that

$$|\rho_0 s - \bar{x}(s)| \geq |s - t_0|(\rho_0 - e_0), \qquad \text{(D-35)}$$

where $e_0 < \rho_0$ is the Myerson expenditure rate. Setting $n_0 \triangleq t_0/\delta$ and using that $t_j = \delta j$, we obtain that

$$E_j^2 \leq 1 \wedge \frac{C'}{\delta |j - n_0|^2} \,,$$

for some constant $C' > 0$.

We now prove the claim in (D-35). For $s > t_0$, we have that

$$\begin{aligned}\rho_0 s - \bar{x}(s) &= \rho_0(s - t_0) + \rho_0 t_0 - \bar{x}(t_0) + \bar{x}(t_0) - \bar{x}(s) = \rho_0(s - t_0) - \int_{t_0}^{s} e(\bar{x}(s'), s') \,\mathrm{d}s' \\ &= (s - t_0)(\rho_0 - e(\bar{x}(\xi), \xi)) \geq (s - t_0)(\rho_0 - e_0) \,,\end{aligned}$$

where the second equality follows because $\rho_0 t_0 = \bar{x}(t_0)$ from Proposition 1 together with the ODE for the budget evolution, the last equality from the Mean Value Theorem for some $\xi \in (t_0, s)$, and the inequality because $e(y, s) = \zeta(y/s) \leq \zeta(\rho_0) = e_0$ since the expenditure rate $\zeta(\rho)$ is always dominated by the Myerson expenditure rate e_0. The claim follows because $\rho_0 > e_0$ and taking absolute values (because both sides have the same sign). For $s < t_0$, the budget is depleted at the Myerson expenditure rate e_0, and the claim follows because

$$\bar{x}(s) - \rho_0 s = \bar{x}(t_0) - e_0(t_0 - s) - \rho_0 s \geq (t_0 - s)(\rho_0 - e_0) \,,$$

where we used that $\bar{x}(t_0) \geq \rho_0 t_0$.

We are now in position to bound the second sum. Let $f(j) = 1 \wedge C'/(\delta j^2)$. We have that

$$\begin{aligned}\sum_{j=i+1}^{n} E_j^2 \leq \sum_{j=1}^{n} E_j^2 &\leq 1 + \sum_{j=1}^{n_0-1} E_j^2 + \sum_{j=n_0+1}^{n} E_j^2 \\ &\leq 1 + \sum_{j=1}^{n_0-1} f(j - n_0) + \sum_{j=n_0+1}^{n} f(j - n_0) \\ &= 1 + \sum_{j=1}^{n_0-1} f(j) + \sum_{j=1}^{n-n_0} f(j) \leq 1 + 2\sum_{j=1}^{\infty} f(j) \,,\end{aligned}$$

where the first inequality follows from adding summands $1, \ldots, i$ because summands are positive, the second inequality from partitioning the sum and using that $E_j^2 \leq 1$ together with $n \geq n_0$, and the equality from changing indices in the sums and using that the function $f(j)$ is even. Letting $\tilde{n} = \sup\{n \in \mathbb{R}_+ : \frac{C'}{\delta n^2} \geq 1\}$ we obtain that

$$\sum_{j=1}^{\infty} f(j) \leq \int_0^{\infty} f(j) \,\mathrm{d}j = \tilde{n} + \int_{\tilde{n}}^{\infty} \frac{C'}{\delta j^2} \,\mathrm{d}j = \tilde{n} + \frac{C'}{\delta \tilde{n}} = \frac{2\sqrt{C'}}{\sqrt{\delta}} \,.$$

where the first inequality follows from the integral bound for decreasing sums, and the last equality because $\tilde{n} = \sqrt{C'/\delta}$. ☐

D.4.10 Proof of Lemma D.9

Proof. We prove the result in three steps.

Step 1. Fix a time period $1 \leq i < n$ (the result for $i = n$ is trivial) and consider a sample path under the event $\{i \geq \tau_\nu\}$. The latter implies that all time previous periods $i < j \leq n$ satisfy $(x_j, t_j) \in S^x_\nu$ and $(y_j, t_j) \in S^y_\nu$. Because the stochastic processes $\{x_i\}_i$ and $\{y_i\}_i$ are coupled by the realization of values $\{v_i\}_i$ we have that for any period $j = n, \ldots, i+1$ the deviation evolves according

$$q_{j-1} - \bar{q}(t_{j-1}) = q_j - \bar{q}(t_j) - \alpha_j \,.$$

where the difference is given by

$$\alpha_j = \delta\big(Z^*[x_j, t_j](v_j) - Z^*[y_j, t_j](v_j)\big) - \int_{t_{j-1}}^{t_j} e_x(\bar{x}(s), s)\bar{q}(s)\ \mathrm{d}s\,, \tag{D-36}$$

with the integral representation following because the solution of the ODE (D-29) is absolutely continuous and using that $\dot{\bar{q}}(s) = e_x(\bar{x}(s), s)\bar{q}(s)$.

We have

$$\begin{aligned}
\big|\mathbb{E}\left[(q_i - \bar{q}(t_i))\,\mathbf{1}\{i \geq \tau_\nu\}\right]\big| &= \Big| - \sum_{j=i+1}^{n} \mathbb{E}\left[\alpha_j \mathbf{1}\{i \geq \tau_\nu\}\right]\Big| \\
&\leq \sum_{j=i+1}^{n} \Big|\mathbb{E}\left[\alpha_j \mathbf{1}\{j \geq \tau_\nu\}\right]\Big| + \mathbb{E}\left[|\alpha_j| \mathbf{1}\{j \geq \tau_\nu > i\}\right] \\
&\leq \sum_{j=i+1}^{n} \underbrace{\Big|\mathbb{E}\left[\alpha_j \mathbf{1}\{j \geq \tau_\nu\}\right]\Big|}_{(I)} + \underbrace{\mathbb{E}\left[|\alpha_j| \mathbf{1}\{i < \tau_\nu\}\right]}_{(II)},
\end{aligned}$$

where the first inequality follows from $\mathbf{1}\{i \geq \tau_\nu\} = \mathbf{1}\{j \geq \tau_\nu\} - \mathbf{1}\{j \geq \tau_\nu > i\}$ for $j > i$ and the triangle inequality together with Jensen's, and the last inequality because $\mathbf{1}\{j \geq \tau_\nu > i\} \leq \mathbf{1}\{\tau_\nu > i\}$. We bound each term at a time.

Step 2: term (I). We can decompose α_i in the following terms $\alpha_j = \alpha_j^{(1)} + \alpha_j^{(2)} + \alpha_j^{(3)} + \alpha_j^{(4)}$ where

$$\begin{aligned}
\alpha_j^{(1)} &= \delta e_x(\bar{x}(t_j), t_j)\bar{q}(t_j) - \int_{t_{j-1}}^{t_j} e_x(\bar{x}(s), s)\bar{q}(s)\ \mathrm{d}s\,, \\
\alpha_j^{(2)} &= \delta e_x(\bar{x}(t_j), t_j)\Big(q_j - \bar{q}(t_j)\Big)\,, \\
\alpha_j^{(3)} &= \delta\Big(e_x(x_j, t_j) - e_x(\bar{x}(t_j), t_j)\Big)q_j\,, \\
\alpha_j^{(4)} &= \delta\Big(Z^*[x_j, t_j](v_j) - Z^*[y_j, t_j](v_j)\Big) - \delta e_x(x_j, t_j) q_j\,.
\end{aligned}$$

Let $\mathbb{E}_j[X] \triangleq \mathbb{E}[X\mathbf{1}\{j \geq \tau_\nu\}]$ be the expectation under the event $j \geq \tau_\nu$. We have

$$(I) = \Big|\mathbb{E}\left[\alpha_j \mathbf{1}\{i \geq \tau_\nu\}\right]\Big| = \Big|\mathbb{E}_j\left[\alpha_j\right]\Big| \leq \Big|\mathbb{E}_j\left[\alpha_j^{(1)}\right]\Big| + \Big|\mathbb{E}_j\left[\alpha_j^{(2)}\right]\Big| + \Big|\mathbb{E}_j\left[\alpha_j^{(3)}\right]\Big| + \Big|\mathbb{E}_j\left[\alpha_j^{(4)}\right]\Big|,$$

where the first inequality follows from the triangle inequality. We next bound each term at a time.

Step 2.1: term ($\alpha_j^{(1)}$). For the first term, under the deterministic event A_j^1 (the deterministic event that the state under the fluid model does not jump over the line $\mathcal{L}$ defined in (D-28)), we obtain via an application of the Mean Value Theorem that there exists some $\xi \in [t_{j-1}, t_j]$ such that

$$\begin{aligned}
|\alpha_j^{(1)}| &= \delta \big| e_x(\bar{x}(t_j), t_j)\bar{q}(t_j) - e_x(\bar{x}(\xi), \xi)\bar{q}(\xi) \big| \\
&\leq \underbrace{\delta |\bar{q}(t_j)| \cdot \big| e_x(\bar{x}(t_j), t_j) - e_x(\bar{x}(\xi), \xi) \big|}_{(\clubsuit)} + \underbrace{\delta |e_x(\bar{x}(\xi), \xi)| \cdot \big| \bar{q}(t_j) - \bar{q}(\xi) \big|}_{(\spadesuit)},
\end{aligned}$$

where the equality follows from the ODE (D-29) for $\bar{q}(t)$ and the inequality follows from the triangle inequality. In turn, the first term can be bounded by

$$\begin{aligned}
(\clubsuit) &= \delta |\bar{q}(t_j)| \cdot \big| e_x(\bar{x}(t_j), t_j) - e_x(\bar{x}(\xi), \xi) \big| \leq \delta L |\bar{q}(t_j)| \big(|\bar{x}(t_j) - \bar{x}(\xi)| + |t_j - \xi| \big) \\
&\leq \delta L |t_j - \xi| \cdot |\bar{q}(t_j)| \cdot \big(|e(\bar{x}(\xi'), \xi')| + 1 \big) \leq \delta^2 |x - y| L C \leq C\delta^3 ,
\end{aligned}$$

where the first inequality follows from the Lipschitz continuity of $e_x(x,t)$ from item 4 of Lemma D.3, the second from the fact that by the Mean Value Theorem there exists $\xi' \in [\xi, t_j]$ such that $\bar{x}(\xi) = \bar{x}(t_j) + (\xi - t_j) e(\bar{x}(\xi'), \xi')$, the third because $\xi \in [t_{j-1}, t_j]$ and using that $|\bar{q}(t_j)| = O(|x - y|)$ from Lemma D.5 and $|e(\bar{x}(\xi'), \xi')|$ is bounded from item 5 of Lemma D.3, and the last because $|x - y| = O(\delta)$. For the second term we have

$$\begin{aligned}
(\spadesuit) &= \delta |e_x(\bar{x}(\xi), \xi)| \cdot \big| \bar{q}(t_j) - \bar{q}(\xi) \big| \\
&\leq \delta |t_j - \xi| \cdot |e_x(\bar{x}(\xi), \xi)| \cdot |e_x(\bar{x}(\xi''), \xi'')| \cdot |\bar{q}(\xi'')| \leq C\delta^2 |x - y| \leq C\delta^3 ,
\end{aligned}$$

where the first inequality follows from the fact that by the Mean Value Theorem there exists $\xi'' \in [\xi, t_j]$ such that $\bar{q}(\xi) = \bar{q}(t_j) + (\xi - t_j)\dot{\bar{q}}(\xi'')$ and using the ODE (D-29), and the second because $\xi \in [t_{j-1}, t_j]$, using that the partial derivatives $e_x(x,t)$ and $e_x(x,t)$ are bounded, and $|\bar{q}(t_j)| = O(|x - y|)$.

When the deterministic event A_j^1 does not hold, we still have $|\alpha_j^{(1)}| = O(\delta^2)$ because $|\bar{q}(t_j)| = O(|x - y|) = O(\delta)$ from Lemma D.5. Therefore,

$$|\alpha_j^{(1)}| \leq C\delta^3 \mathbf{1}\{A_j^1\} + C\delta^2 \mathbf{1}\{\bar{A}_j^1\} \leq C\delta^3 + C\delta^2 \mathbf{1}\{\bar{A}_j^1\} ,$$

because $\mathbf{1}\{A_j^1\} \leq 1$.

Step 2.2: term ($\alpha_j^{(2)}$). For the second term we have that

$$\left| \mathbb{E}_j \left[\alpha_j^{(2)} \right] \right| = \left| \delta e_x(\bar{x}(t_j), t_j) \mathbb{E}_j \left[q_j - \bar{q}(t_j) \right] \right| \leq C\delta \left| \mathbb{E}_j \left[q_j - \bar{q}(t_j) \right] \right| ,$$

where the equality follows from extracting deterministic terms from the expectation and because the partial derivative $e_x(x,t)$ is bounded from item 5 of Lemma D.3 (if $\bar{x}(t_j)/t_j = \rho_0$ it suffices to consider the left derivative).

Step 2.3: term ($\alpha_j^{(3)}$). For the third term we condition on the event A_j^2 (the stochastic event that the state under the fluid model and the stochastic system starting at x lie in the same side of the line $\mathcal{L}$ defined in (D-28)) to obtain

$$\left| \mathbb{E}_j \left[\alpha_j^{(3)} \right] \right| \leq \mathbb{E}_j \left| \alpha_j^{(3)} \right| \leq \mathbb{E}_j \left| \alpha_j^{(3)} \mathbf{1}\{A_j^2\} \right| + \mathbb{E}_j \left| \alpha_j^{(3)} \mathbf{1}\{\bar{A}_j^2\} \right| ,$$

where the first inequality follows from Jensen's inequality because $|\cdot|$ is convex. Using that $e_x(x,t)$ is Lipschitz continuous under A_j^2 from item 4 of Lemma D.3, we bound the first term as

$$\begin{aligned}\mathbb{E}_j \left|\alpha_j^{(3)} \mathbf{1}\{A_j^2\}\right| &\le \delta L \mathbb{E}_j\big[|x_j - \bar{x}(t_j)| \cdot |q_j| \mathbf{1}\{A_j^2\}\big] \le \delta L \mathbb{E}\big[|x_j-\bar{x}(t_j)|\cdot|q_j|\mathbf{1}\{j\ge\tau_\nu\}\big] \\ &\le \delta L \big\| (x_j - \bar{x}(t_j))\,\mathbf{1}\{j\ge\tau_\nu\}\big\|_2 \cdot \big\|q_j \mathbf{1}\{j\ge \tau_\nu\}\big\|_2 \le C\delta^2\sqrt{\delta}\,,\end{aligned}$$

where the second inequality follows because $\mathbf{1}\{A_j^2\} \le 1$ and $\mathbb{E}_j[X] = \mathbb{E}[X\mathbf{1}\{j \ge \tau_\nu\}]$, the third from Cauchy-Schwartz, and the last from Lemma D.4 and Lemma D.7 with $|x-y| = O(\delta)$. For the second term we have

$$\mathbb{E}_j \left|\alpha_j^{(3)}\mathbf{1}\{\bar{A}_j^2\}\right| \le C\delta \mathbb{E}_j\left|q_j \mathbf{1}\{\bar{A}_j^2\}\right| \le C\delta \big\|q_j\mathbf{1}\{j\ge\tau_\nu\}\big\|_2 \sqrt{\mathbb{P}\{\bar{A}_j^2, j\ge\tau_\nu\}} \le C\delta^2\sqrt{\mathbb{P}\{\bar{A}_j, j\ge \tau_\nu\}}\,,$$

where the first inequality follows because the partial derivatives are bounded from item 5 of Lemma D.3 (taking directional derivatives if necessary), the second from Cauchy-Schwartz, and the last from Lemma D.7 because $|x-y| = O(\delta)$. Combining both bounds we obtain that

$$\left|\mathbb{E}_j\left[\alpha_j^{(3)}\right]\right| \le C\delta^2\sqrt{\delta} + C\delta^2\sqrt{\mathbb{P}\{\bar{A}_j^2, j\ge\tau_\nu\}}\,.$$

Step 2.4: term ($\alpha_j^{(4)}$). For the fourth term we obtain

$$\left|\mathbb{E}_j\left[\alpha_j^{(4)}\right]\right| \le \mathbb{E}_{x_i,y_i}\left[\left|\mathbb{E}_j\left[\alpha_j^{(4)} \mid x_i, y_i\right]\right|\right] = \delta\mathbb{E}_j\left[\left|\big(e(x_j,t_j) - e(y_j,t_j)\big) - e_x(x_j,t_j)q_j\right|\right]\,,$$

where the first inequality follows from the tower rule for expectations and Jensen's inequality and the last equality because v_i is independent from the event $j \ge \tau_\nu$ because τ_ν is a stopping time. Under A_j^3 (the stochastic event that the state under the stochastic systems starting at x and y lie in the same side of the line $\mathcal{L}$ defined in (D-28)), from the Mean Value theorem there exists some $\xi \in [x_j, y_j]$ such that

$$\left|\big(e(x_j,t_j) - e(y_j,t_j)\big) - e_x(x_j,t_j)q_j\right| = \left|\big(e_x(\xi,t_j) - e_x(x_j,t_j)\big)q_j\right| \le L|\xi - x_j|\cdot|q_j| \le Lq_j^2$$

where the first inequality follows because $e_x(x,t)$ is Lipschitz continuous from item 4 of Lemma D.3 and the last because $\xi \in [x_j,y_j]$. Under $\bar{A}_j^3$, we have that $\left|\big(e(x_j,t_j)-e(y_j,t_j)\big)-e_x(x_j,t_j)q_j\right| \le C|q_j|$ because $e(x,t)$ is Lipschitz continuous under $j \ge \tau_\nu$ and the partial derivative is bounded from item 5 of Lemma D.3. Therefore we obtain that the fourth term can be bounded as

$$\begin{aligned}\left|\mathbb{E}_j\left[\alpha_j^{(4)}\right]\right| &\le \delta C\mathbb{E}_j\left[|q_j|^2\mathbf{1}\{A_j^3\}\right] + \delta C\mathbb{E}_j\left[|q_j|\mathbf{1}\{\bar{A}_j^3\}\right] \\ &\le \delta C\big\|q_j\mathbf{1}\{j\ge\tau_\nu\}\big\|_2^2 + \delta C\big\|q_j\mathbf{1}\{j\ge\tau_\nu\}\big\|_2\sqrt{\mathbb{P}\{\bar{A}_j^3, j\ge\tau_\nu\}} \\ &\le C\delta^2\sqrt{\delta} + C\delta^2\sqrt{\mathbb{P}\{\bar{A}_j^3, j\ge\tau_\nu\}}\,,\end{aligned}$$

where the second inequality follows because $\mathbf{1}\{A_j\} \le 1$ and from Cauchy-Schwartz, and the last from Lemma D.7 because $|x-y| = O(\delta)$.

Step 2.5: Combining bounds. Combining the previous results we obtain

$$(I) \leq C\delta \big| \mathbb{E}_j \left[q_j - \bar{q}(t_j) \right] \big| + C\delta^2\sqrt{\delta} + C\delta^2 \Big(\underbrace{\mathbf{1}\{\bar{A}_j^1\} + \sqrt{\mathbb{P}\{\bar{A}_j^2, j \geq \tau_\nu\}} + \sqrt{\mathbb{P}\{\bar{A}_j^3, j \geq \tau_\nu\}}}_{E_j} \Big) .$$

Step 3: term (II). For the second term we have

$$(II) = \mathbb{E}\left[|\alpha_j| \mathbf{1}\{i < \tau_\nu\} \right] \leq C\delta \mathbb{P}\{i < \tau_\nu\} \leq C\delta \Big(\mathbb{P}\{i < \tau_\nu^x\} + \mathbb{P}\{i < \tau_\nu^y\} \Big) = O(\delta^3) ,$$

where the first inequality follows because $|\alpha_j| \leq C\delta$ for some $C > 0$, the second from the union bound because $\tau_\nu = \max(\tau_\nu^x, \tau_\nu^y)$, and the last because $\mathbb{P}\{i < \tau_\nu^x\} = O(\delta^2)$ and $\mathbb{P}\{i < \tau_\nu^y\} = O(\delta^2)$ from Corollary D.3.

Step 4. Our previous bounds together with the fact that $|x-y| = O(\delta)$ and $\delta(n-i) = t_n - t_i = O(1)$ imply that

$$\big| \mathbb{E}\left[q_i - \bar{q}(t_i) \right] \big| \leq C\delta^{3/2} + C\delta^2 \sum_{j=i+1}^{n} E_j + C\delta \sum_{j=i+1}^{n} \big| \mathbb{E}\left[q_j - \bar{q}(t_j) \right] \big| .$$

Lemma D.8 implies that $\sum_{j=i+1}^n E_j = O\left(\frac{1}{\sqrt{\delta}} \right)$. Thus we obtain by Gronwall's Inequality that

$$\big| \mathbb{E}\left[q_i - \bar{q}(t_i) \right] \big| \leq C\delta^{3/2} e^{C(t_n - t_i)} . \qquad \square$$

D.4.11 Proof of Lemma D.10

Proof. The difference in performance between the discrete and fluid model can be written as

$$\big| \mathbb{E}\left[\gamma_i \right] \big| \leq \underbrace{\big| \mathbb{E}\left[\gamma_i \mathbf{1}\{i \geq \tau_\nu\} \right] \big|}_{(I)} + \underbrace{\mathbb{E}\left[|\gamma_i| \mathbf{1}\{i < \tau_\nu\} \right]}_{(II)} ,$$

where the first inequality follows from the triangle inequality together with Jensen's inequality because $|\cdot|$ is convex. We bound each term at a time.

Step 1: term (I). Following the steps of Lemma D.9 we can decompose the difference

$$\gamma_i = \delta \Big(u(x_i, t_i) - u(y_i, t_i) \Big) - \int_{t_{i-1}}^{t_i} u_x(\bar{x}(s), s) \bar{q}(s) \, \mathrm{d}s ,$$

in the following terms $\gamma_i = \gamma_i^{(1)} + \gamma_i^{(2)} + \gamma_i^{(3)} + \gamma_i^{(4)}$ where

$$\begin{aligned}
\gamma_i^{(1)} &= \delta u_x(\bar{x}(t_i), t_i)\bar{q}(t_i) - \int_{t_{i-1}}^{t_i} u_x(\bar{x}(s), s)\bar{q}(s)\,\mathrm{d}s\,,\\
\gamma_i^{(2)} &= \delta u_x(\bar{x}(t_i), t_i)\Big(q_i - \bar{q}(t_i)\Big)\,,\\
\gamma_i^{(3)} &= \delta\Big(u_x(x_i, t_i) - \delta u_x(\bar{x}(t_i), t_i)\Big)q_i\,,\\
\gamma_i^{(4)} &= \delta\Big(u(x_i, t_i) - u(y_i, t_i)\Big) - \delta u_x(x_i, t_i)q_i\,.
\end{aligned}$$

Let $\mathbb{E}_i[X] \triangleq \mathbb{E}[X\mathbf{1}\{i \geq \tau_\nu\}]$ be the expectation under the event $i \geq \tau_\nu$. We have

$$(I) = \Big|\mathbb{E}\left[\gamma_i \mathbf{1}\{i \geq \tau_\nu\}\right]\Big| = \Big|\mathbb{E}_i\left[\gamma_i\right]\Big| \leq \Big|\mathbb{E}_i\left[\gamma_i^{(1)}\right]\Big| + \Big|\mathbb{E}_i\left[\gamma_i^{(2)}\right]\Big| + \Big|\mathbb{E}_i\left[\gamma_i^{(3)}\right]\Big| + \Big|\mathbb{E}_i\left[\gamma_i^{(4)}\right]\Big|\,,$$

where the first inequality follows from the triangle inequality. Following the steps of the proof of Lemma D.9 and leveraging that $u_x(x,t)$ satisfies the same properties that $e_x(x,t)$ according to Lemma D.3 we have

$$\begin{aligned}
|\gamma_i^{(1)}| &\leq C\delta^3 + C\delta^2\mathbf{1}\{\bar{A}_i^1\}\,,\\
\Big|\mathbb{E}_i\left[\gamma_i^{(2)}\right]\Big| &\leq C\delta\Big|\mathbb{E}_i\left[q_i - \bar{q}(t_i)\right]\Big|\,,\\
\Big|\mathbb{E}_i\left[\gamma_i^{(3)}\right]\Big| &\leq C\delta^2\sqrt{\delta} + C\delta^2\sqrt{\mathbb{P}\{\bar{A}_i^2, i \geq \tau_\nu\}}\,,\\
\Big|\mathbb{E}_i\left[\gamma_i^{(4)}\right]\Big| &\leq C\delta^2\sqrt{\delta} + C\delta^2\sqrt{\mathbb{P}\{\bar{A}_i^3, i \geq \tau_\nu\}}\,,
\end{aligned}$$

and thus combining the previous bounds yields

$$(I) \leq C\delta\Big|\mathbb{E}_i\left[q_i - \bar{q}(t_i)\right]\Big| + C\delta^2\sqrt{\delta} + C\delta^2\Big(\underbrace{\mathbf{1}\{\bar{A}_i^1\} + \sqrt{\mathbb{P}\{\bar{A}_i^2, i \geq \tau_\nu\}} + \sqrt{\mathbb{P}\{\bar{A}_i^3, i \geq \tau_\nu\}}}_{E_i}\Big)\,.$$

Step 2: term (II). For the second term we have

$$(II) = \mathbb{E}\left[|\gamma_i|\mathbf{1}\{i < \tau_\nu\}\right] \leq C\delta\mathbb{P}\{i < \tau_\nu\} = O(\delta^3)\,,$$

because $|\gamma_i| \leq C\delta$ for some $C > 0$ and $\mathbb{P}\{i < \tau_\nu\} = O(\delta^2)$ from Corollary D.3.

Step 3. Combining both bounds we obtain that

$$\Big|\mathbb{E}\left[\gamma_i\right]\Big| \leq (I) + (II) \leq C\delta\Big|\mathbb{E}_i\left[q_i - \bar{q}(t_i)\right]\Big| + C\delta^2\sqrt{\delta} + C\delta^2 E_i\,,$$

and the result follows. □

D.5 Auxiliary Results

Lemma D.11. *The function $f(x,t) = \min(x/t, \rho_0)$ satisfies*

$$|f(x,t) - f(y,s)| \leq \frac{\rho_0}{\max(s,t)}\,|t-s| + \frac{1}{\max(s,t)}\,|x-y|\,,$$

for all $(x,t),(y,s) \in \mathcal{S}$*.*

Proof. Assume without loss of generality that $t < s$. We want to show:

$$|f(x,t) - f(y,s)| \le \frac{\rho_0}{s}|t-s| + \frac{1}{s}|x-y| \,.$$

When $x/t \le \rho_0$ we use that $\ell(\rho) = \min(\rho,\rho_0)$ is $1-$Lipschitz continuous in ρ to obtain:

$$\begin{aligned} |f(x,t) - f(y,s)| &= \left|\ell\left(\frac{x}{t}\right) - \ell\left(\frac{y}{s}\right)\right| \le \left|\frac{x}{t} - \frac{y}{s}\right| \le \left|\frac{x}{t} - \frac{x}{s}\right| + \left|\frac{x}{s} - \frac{y}{s}\right| \\ &= \frac{x}{ts}|s-t| + \frac{1}{s}|x-y| \le \frac{\rho_0}{s}|t-s| + \frac{1}{s}|x-y| \,, \end{aligned}$$

where the second inequality follows from the triangle inequality, and the last because $x/t \le \rho_0$. When $x/t > \rho_0$, we need to consider whether $y/s \le \rho_0$ or $y/s > \rho_0$. The latter is trivial since the difference is zero when both ratios are above ρ_0. In the remainder of the proof we consider the former, that is, $y/s \le \rho_0$. Here we have:

$$\begin{aligned} |f(x,t) - f(y,s)| &= \left|\ell\left(\frac{x}{t}\right) - \ell\left(\frac{y}{s}\right)\right| = \rho_0 - \frac{y}{s} = \rho_0 - \frac{x}{s} + \frac{x}{s} - \frac{y}{s} \\ &= \frac{\rho_0}{s}\left(s - \frac{x}{\rho_0}\right) + \frac{1}{s}(x-y) \le \frac{\rho_0}{s}(s-t) + \frac{1}{s}(x-y) \\ &\le \frac{\rho_0}{s}|t-s| + \frac{1}{s}|x-y| \,, \end{aligned}$$

where the first inequality follows because $x/t > \rho_0$, and the last from taking absolute values and using the triangle inequality. ☐

Lemma D.12 (Generalized Gronwall's Inequality)**.** *Suppose that the non-negative sequence* $(\epsilon_k)_{k\ge 0}$ *satisfies*

$$\epsilon_k \le \epsilon_0 + a\sum_{j=1}^{k-1}\epsilon_j + \left(\sum_{j=1}^{k-1}(b\epsilon_j + c)^2\right)^{\frac{1}{2}} \,,$$

where $a,b,c > 0$*. Then we have*

$$\epsilon_k \le d_k\epsilon_0 + \left(d_k\epsilon_0 + \frac{c}{b}\right)\frac{f_k}{1-f_k} \,,$$

where $d_k = e^{ak}$ *and* $f_k = \sqrt{1-(1+b^2d_k^2)^{-k-1}} \le bd_k\sqrt{k-1}$*.*

Proof. Let $v_k = \left(\sum_{j=1}^{k-1}(b\epsilon_j + c)^2\right)^{\frac{1}{2}}$. We can write the inequality in the statement as $\epsilon_k \le \epsilon_0 + v_k + a\sum_{j=1}^{k-1}\epsilon_j$. Because v_k is non-decreasing we obtain by Gronwall's Inequality that

$$\epsilon_k \le (\epsilon_0 + v_k)e^{ak} \,.$$

Letting $y_k = b\epsilon_k + c$, $d_k = e^{ak}$ and using that $b \ge 0$ we obtain the following series of inequalities:

$$y_k \le bd_k\epsilon_0 + c + bd_kv_k = \underbrace{bd_k\epsilon_0 + c}_{\tilde{a}_k} + \underbrace{bd_k}_{\tilde{b}_k}\left(\sum_{j=1}^{k-1} y_k^2\right)^{\frac{1}{2}} \,,$$

because $v_k = \left(\sum_{j=1}^{k-1} y_k^2\right)^{\frac{1}{2}}$. Gronwall's L_p inequality with $p = 2$ (Willett and Wong, 1965, Theorem 3) together with the fact that the sequences $(\tilde{a}_k)_k$ and $(\tilde{b}_k)_k$ are non-decreasing gives that

$$y_k \leq \tilde{a}_k + \tilde{b}_k \frac{\left(\tilde{a}_k^2 \sum_{j=1}^{k-1} e(j)\right)^{1/2}}{1 - (1 - e(k-1))^{1/2}} ,$$

where $e(j) = (1 + \tilde{b}_k^2)^{-j}$. Letting $f_k = \sqrt{1 - (1 + \tilde{b}_k^2)^{-k+1}}$ we obtain that $\sum_{j=1}^{k-1} e(j) = (f_k/\tilde{b}_k)^2$ and as a result

$$y_k \leq \tilde{a}_k \left(1 + \frac{f_k}{1 - f_k}\right) .$$

The result follows because $\epsilon_k = (y_k - c)/b$. Finally note that $f_k \leq b d_k \sqrt{k-1}$ because $(1+\epsilon)^{-n} \geq 1 - \epsilon n$ for $n \in \mathbb{N}$ and $\epsilon \geq 0$. $\square$

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