Electronic Companion: Appendices

Appendix A summarizes the principal notation. Appendix B contains proofs of the formal claims in the main paper. Appendix C provides supplementary content for the paper: Appendix C.1 gives further details on discretization techniques that satisfy our assumption of a constant pmf, Appendix C.2 contains an alternate version of our model where the distribution of alternatives is assumed to be continuous, and Appendix C.3 provides additional results for the variable search intensity variant of our model, as given in Section 4.2.

Appendix A: Notation

Table EC.1 Notation

Notation	Definition
N	Maximum number of periods in the model (search horizon)
T	Stochastic stopping time of the search process (acceptance time or N, whichever is first)
p	Probability of finding an alternative within a single period when searching
$F(\cdot)$	Distribution from which search alternatives are drawn
$P(S)$	Probability of drawing an outcome in set S when drawing from F
$\mu(\cdot)$	Probability mass function of $F(\cdot)$; with property that $\mu(x) = 1/\lvert \operatorname{supp} F\rvert$, for all $x \in \operatorname{supp} F$
c	Search cost
e	Evaluation cost
α	Discount factor
$\{\bar{\xi}_n\}$	Canonical threshold search policy, equal to $\{\alpha V_n\}$, where V_n is given by (1)
$\{\omega_n\}$	Agent's search policy
$\{\Xi_n\}$	Agent's reporting policy
$\{\xi_n\}$	Agent's reporting policy in case it is a threshold policy ($\Xi_n = [\xi_n, \bar{u}] \cap \operatorname{supp} F, \forall n$)
$\{\gamma_n\}$	Agent's consumption policy
$\{\xi_n^{\mathrm{FB}}\}$	First-best (threshold) reporting policy given by Lemma A2
$\{\Psi_n\}$	Principal's acceptance policy
$\{\psi_n\}$	Principal's acceptance policy in case it is a threshold policy ($\Psi_n = [\psi_n, \bar{u}] \cap \operatorname{supp} F, \forall n$)
X_n	Common knowledge about the search in period n
X^n	Common knowledge about the search process up to the end of period n, i.e., $(X_0, ..., X_n)$
$\Rsh \cdot \Lsh$	Rounding up to an element in $\operatorname{supp} F$ ($\Rsh x \Lsh := \min\{u \in \operatorname{supp} F \mid u \geq x\}$)
$s = \{s_n(X^n)\}$	Agent's contract
$\{s_n^*(X^n)\}$	The optimal (second-best) contract, as given in Theorem 3
$V_n^P(\cdot)$ ($V_n^A(\cdot)$)	Principal's (agent's) value function in period n
r	Agent's risk aversion coefficient
W_n	Agent's wealth in period n, immediately following the contractual payments in that period
w	Agent's opportunity cost (per-period amount earned *absent* the principal)
$s_n^B(\Xi_n)$ ($s_n^B(\xi_n)$)	Single-period payment to the agent when an alternative of value in Ξ_n (at least ξ_n) is delivered to the principal under a Theorem 1-type contract that induces $\{\Xi_n\}$ ($\{\xi_n\}$)
$z(\xi_n)$	Expected single-period principal costs when inducing the agent to use ξ_n (Lemma 1)
ξ^{LC}	Value of ξ_n that minimizes the principal's expected single-period costs $z(\xi_n)$ (Lemma 1)
$\{\xi_n^*\}$ ($\{\psi_n^*\}$)	Optimal threshold reporting (acceptance) policy as given in Theorem 3

Appendix B: Proofs of Formal Statements

The following lemma, stated here in the appendix, establishes a property of search policies that will be used in several of the proofs that follow for results from the main paper.

Lemma A1. *In the canonical model, if there exists a period $n \in \{1,...,N\}$ in which it is optimal to pay the search cost, then it is optimal to pay the search cost in every period.*

Proof. Because $\{V_n\}$ is positive (from the boundary condition) and decreasing in n, it follows that $p\int_{\alpha V_{n+1}}^{\bar{u}} \bar{F}(x)\,dx$ is increasing in n. So if there exists a $k \in \{1,...,N\}$ such that $p\int_{\alpha V_{n+1}}^{\bar{u}} \bar{F}(x)\,dx - c \geq 0$, then $p\int_{\alpha V_{n+1}}^{\bar{u}} \bar{F}(x)\,dx - c \geq 0$ for all $h \in \{k,...,N\}$. Therefore, the optimality of paying the search cost in period k implies the optimality of paying it in every subsequent period.

We demonstrate that the same holds for preceding periods by contradiction. Assume there exists a period m such that (a) it is optimal for the agent not to pay the search cost in period m but (b) it is optimal to pay that cost in period $m+1$. Then, under an optimal policy, this agent's NPV in period m is αV_{n+1}. Yet, it is possible to use the same search strategy *as if* it were currently period $m+1$; doing so would yield a higher value (V_{n+1}) in expectation—contradicting the optimality of not searching in period m. □

Proof of Theorem 1.

Assume (temporarily) that the principal commits to an acceptance policy $\{\Psi_n\}$ when offering the contract to the agent. Then, the principal has no decisions to make once the contract is accepted, which reduces this problem to delegation of control over a Markov decision process. It follows from Plambeck and Zenios (2000, Thm. 1) that the problem of finding a contract that implements any policy pair $(\{\omega_n\},\{\Xi_n\})$ at the lowest possible cost to the principal can be decomposed into N single-period optimization problems (for every $n \in \{1,...,N\}$), where each problem is given by

$$\operatorname*{arg\,min}_{s_n^A(u),s_n^R(u),s_n^0} \quad p\omega_n \sum_{u\in\Xi_n\cap\Psi_n} s_n^A(u)\mu(u) + p\omega_n \sum_{u\in\Xi_n\setminus\Psi_n} s_n^R(u)\mu(u) + \left(1 - p\omega_n \sum_{u\in\Xi_n}\mu(u)\right)s_n^0; \tag{EC.1}$$

$$\begin{aligned} -p\omega_n \sum_{u\in\Xi_n\cap\Psi_n} \exp[-r(1-\alpha)(s_n^A(u)-\alpha^{-1}c)]\mu(u) - p\omega_n \sum_{u\in\Xi_n\setminus\Psi_n} \exp[-r(1-\alpha)(s_n^R(u)-\alpha^{-1}c)]\mu(u) \\ -\left(1 - p\omega_n \sum_{u\in\Xi_n}\mu(u)\right)\exp[-r(1-\alpha)(s_n^0-\alpha^{-1}c)] = -\exp[-r\alpha^{-1}(1-\alpha)w]; \end{aligned} \tag{EC.2}$$

$$\begin{aligned} -p \sum_{u\in\Xi\cap\Psi_n} \exp[-r(1-\alpha)(s_n^A(u)-\alpha^{-1}c)]\mu(u) - p \sum_{u\in\Xi\setminus\Psi_n} \exp[-r(1-\alpha)(s_n^R(u)-\alpha^{-1}c)]\mu(u) \\ -\left(1 - p \sum_{u\in\Xi}\mu(u)\right)\exp[-r(1-\alpha)(s_n^0-\alpha^{-1}c)] \leq -\exp[-r\alpha^{-1}(1-\alpha)w] \quad \forall \Xi \in \mathcal{F}; \end{aligned} \tag{EC.3}$$

$$-\exp[-r(1-\alpha)s_n^0] \leq -\exp[-r\alpha^{-1}(1-\alpha)w]. \tag{EC.4}$$

In these expressions, $s_n^A(u)$ is the agent's reward for bringing an alternative of value u to the principal when the principal accepts that alternative; his reward is $s_n^R(u)$ when delivering an alternative of value u that the principal rejects. Finally, s_n^0 is the agent's reward when he does not present the principal with any alternatives. The objective function (EC.1) minimizes the expected sum of these rewards conditional on the agent using the desired policy Ξ_n. Equation (EC.2) is the (binding) participation constraint. The incentive compatibility constraint (EC.3) ensures that the agent can not do better by searching and using any other reporting policy, while (EC.4) ensures that he can not do better by not searching at all (notice that the reporting policy is redundant if not searching). The recurring $-\exp[-r(1-\alpha)x]$ expression in the system (EC.1)–(EC.4) is the amount by which the agent's expected wealth-independent utility component changes (under an optimal consumption policy) if the agent receives payment x (this also follows from Theorem 1 of Plambeck and Zenios 2000).

If the principal would like to implement $\omega_n = 0$—i.e., induce the agent to *not* search—a simple solution of the problem (EC.1)–(EC.4) is $s_n^R(u) = s_n^A(u) = s_n^0 = w/\alpha, \forall u$.

We now turn to the case where $\omega_n = 1$. We can make several observations that will allow us to simplify the system of equations (EC.1)–(EC.4).

First of all: to solve (EC.1)–(EC.4) while ignoring the incentive compatibility constraint (EC.3), we will show that we can restrict our attention to $s_n^A(u)$ and $s_n^R(u)$, which are step functions that pay a fixed value s_n^A (resp. s_n^R) if the reported alternative is in Ξ_n and pay 0 otherwise. To do that, let $s_n^{A*}(u)$, $s_n^{R*}(u)$, and s_n^{0*} be solutions of the system (EC.1),(EC.2),(EC.4) and denote $\varphi(x) := -\exp -r(1-\alpha)(x - c/\alpha)$. We can define $s_n^{A**}(u)$ as $s_n^{A**}(u) := s_n^{A**}$ if $u \in \Xi_n$ and $s_n^{A**}(u) := 0$ otherwise, where the constant s_n^{A**} is the solution of $\sum_{u\in\Xi_n\cap\Psi_n} \varphi(s_n^{A**})\mu(u) = \sum_{u\in\Xi_n\cap\Psi_n} \varphi(s_n^{A*}(u))\mu(u)$. In other words, $s_n^{A**}(u)$ is constructed so that it is a step function of the desired form and so that the agent's expected payoff when using Ξ_n is the same under $s_n^{A*}(u)$ and $s_n^{A**}(u)$. Then, noticing that $\varphi(\cdot)$ is an increasing and concave function and using Jensen's inequality yields $\varphi(\mathbb{E}_u[s_n^{A**}(u)|u \in \Xi_n \cap \Psi_n]) = \phi(s_n^{A**}) = \mathbb{E}_u[\varphi(s_n^{A**}(u))|u \in \Xi_n \cap \Psi_n] = \mathbb{E}_u[\varphi(s_n^{A*}(u))|u \in \Xi_n \cap \Psi_n] \le \varphi(\mathbb{E}_u[s_n^{A*}(u)|u \in \Xi_n \cap \Psi_n]) \Rightarrow \mathbb{E}_u[s_n^{A**}(u)|u \in \Xi_n \cap \Psi_n] \le \mathbb{E}_u[s_n^{A*}(u)|u \in \Xi_n \cap \Psi_n]$. Observing that the objective function in (EC.1) depends on the choice of $s_n^A(u)$ only through its first term, which is equal to $pP(\Xi_n \cap \Psi_n)\mathbb{E}_u[s_n^A(u)|u \in \Xi_n \cap \Psi_n]$, the objective function is weakly higher under $(s_n^{A**}(u), s_n^{R*}(u), s_n^{0*})$ compared to $(s_n^{A*}(u), s_n^{R*}(u), s_n^{0*})$, while the constraints (EC.2) and (EC.4) are satisfied in both cases. Thus, the optimality of $(s_n^{A**}(u), s_n^{R*}(u), s_n^{0*})$ follows, showing that we can restrict attention to $s_n^A(u)$ functions of the desired form without loss of optimality. The same property for $s_n^R(u)$ follows analogously.

The intuition of the result in the above paragraph is straightforward. By choosing policy Ξ_n, the agent ensures that all alternatives in Ξ_n that are found are brought to the principal. Yet, the agent has no control over the exact value of these alternatives, and from this, it follows that basing his payment on such exact values—and not simply on whether an alternative's value is in Ξ_n—serves only as an additional source of risk for the agent, which in turn reduces his expected utility due to the agent's risk aversion.

Returning to the problem given by the system (EC.1),(EC.2),(EC.4): the participation constraint is binding, and so, the agent is indifferent between the principal accepting the alternative (and thereby ending the search process) or rejecting it (thus continuing the search). By the same Jensen's inequality argument as before, we also have $s_n^A = s_n^R$. With this in mind, a relaxed version of the problem (EC.1)–(EC.4) that drops constraint (EC.3) is the solution of

$$\arg\min_{s_n^A, s_n^0} pP(\Xi_n)s_n^A + (1 - pP(\Xi_n))s_n^0; \tag{EC.5}$$

$$-pP(\Xi_n)\exp[-r(1-\alpha)(s_n^A - \alpha^{-1}c)] - (1 - pP(\Xi_n))\exp[-r(1-\alpha)(s_n^0 - \alpha^{-1}c)] = -\exp[-r\alpha^{-1}(1-\alpha)w]; \tag{EC.6}$$

$$-\exp[-r(1-\alpha)s_n^0] \le -\exp[-r\alpha^{-1}(1-\alpha)w]. \tag{EC.7}$$

We proceed to solve this relaxed version of the problem, after which we will demonstrate that the solution does not violate (EC.3) and is thus also a solution to the original problem (EC.1)–(EC.4). Substituting

$u_n^A := \exp\left[-r(1-\alpha)s_n^A\right]$, $u_n^0 := \exp\left[-r(1-\alpha)s_n^0\right]$, $k := \exp\left[r(1-\alpha)\alpha^{-1}c\right]$, $o := \exp\left[-r\alpha^{-1}(1-\alpha)w\right]$, the problem (EC.5)–(EC.7) simplifies to

$$\arg\min_{u_n^A, u_n^0}(r(1-\alpha))^{-1}\left(-(1-pP(\Xi_n))\ln u_n^0 - pP(\Xi_n)\ln u_n^A\right); \tag{EC.8}$$

$$-(1-pP(\Xi_n))u_n^0 k - pP(\Xi_n)u_n^A k + o = 0; \tag{EC.9}$$

$$-u_n^0 + o \leq 0. \tag{EC.10}$$

Here, the objective function is convex and the constraints are affine, so the Karush-Kuhn-Tucker (KKT) conditions are both necessary and sufficient. From KKT conditions, the solution only exists if $1-k(1-pP(\Xi_n)) > 0$ (this is the implementability condition (10)), and is given by $u_n^0 = o, u_n^A = o(1-k(1-pP(\Xi_n)))/(kpP(\Xi_n))$. Reverting the substitution then gives

$$s_n^A = \frac{w+c}{\alpha} + \frac{1}{r(1-\alpha)}\ln\frac{pP(\Xi_n)}{1-(1-pP(\Xi_n))\exp\left[rc(1-\alpha)\alpha^{-1}\right]}, \qquad s_n^0 = \frac{w}{\alpha}, \tag{EC.11}$$

as the solution. We find it useful to resort to slightly different notation in the main text compared to this proof, writing $s_n^B(\Xi_n)$ rather than s_n^A in (11) to emphasize its dependence on Ξ_n as well as the fact that whether the bonus is awarded does not depend on whether the alternative is accepted (thus, the superscript A might be misleading). We still need to verify if this solution adheres to the dropped constraint. Plugging the solution of (EC.5)–(EC.7) into the constraint (EC.3) and expanding its RHS yields

$$pP(\Xi_n \cap \Xi)s_n^A + (1-pP(\Xi_n\cap\Xi))s_n^0 \leq pP(\Xi_n)s_n^A + (1-pP(\Xi_n))s_n^0, \forall \Xi \in \mathcal{F},$$

which holds because it follows from (EC.11) that $s_n^A \geq s_n^0$. Thus, this is the solution to the original problem (EC.1)–(EC.4) as well. That this contract is renegotiation-proof follows from Plambeck and Zenios (2000, Cor. 1).

Then, observe from (EC.11) that the solution depends neither on the acceptance policy nor on the agent's knowledge of that policy. Furthermore, the solution does not depend on the principal adhering to the announced policy; the reason is that, because the single-period participation constraint is binding, the agent is always indifferent between the principal accepting or rejecting the alternative he delivers.[1]

It remains to find the optimal consumption policy under contracts given by this theorem. Thus, let the agent hold the contract given by this theorem which implements policies $\{\omega_n\}$, $\{\Xi_n\}$, and let the principal use acceptance policy $\{\Psi_n\}$. From (5), it follows that the agent's optimal consumption policy solves

$$\begin{aligned} V_n^A(W_n) = \max_{\gamma_n} - \exp(-r\gamma_n) + \omega_n p \int\limits_{\Xi_n\cap\Psi_n} -\frac{\alpha}{1-\alpha}\exp\left[-r\left(w + (1-\alpha)\left(\frac{W_n-c-\gamma_n}{\alpha} + s_n^B(\Xi_n)\right)\right)\right] dF(u) \\ + \omega_n p \int\limits_{\Xi_n\setminus\Psi_n} \alpha V_{n+1}^A\left(\frac{W_n-\gamma_n-c}{\alpha} + s_n^B(\Xi_n)\right) dF(u) + \left(1-\omega_n pP(\Xi_n)\right)\alpha V_{n+1}^A\left(\frac{W_n-\gamma_n-\omega_n c}{\alpha} + s_n^0\right), \end{aligned} \tag{EC.12}$$

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[1] The optimal contract identified here is not unique. In particular, the contract is not sensitive to the payment linked to bringing the principal an undesirable alternative ($s_n^A(u)$ and $s_n^R(u)$ for $u \notin \Xi_n$). We arbitrarily set this value to 0 in the proof so as to simplify the expressions, but any payment that does not exceed w/α can accomplish the same result.

with boundary condition $V_{N+1}^A(W_{N+1}) = -\frac{1}{1-\alpha}\exp[-r(w+(1-\alpha)W_{N+1})]$. We will show—by backwards induction—that in any period, the optimal consumption is given by $\gamma_n = W_n(1-\alpha)+w$ and that the agent's value function is $V_n^A(W_n) = -\frac{1}{1-\alpha}\exp[-r(w+(1-\alpha)W_n)]$. Assume there is a period $n+1$ such that $V_{n+1}^A(W_{n+1}) = -\frac{1}{1-\alpha}\exp[-r(w+(1-\alpha)W_{n+1})]$. For the induction base, this property holds for $n=N$, directly from the boundary condition.

For the induction step, in period n, using $V_{n+1}^A(W_{n+1}) = -\frac{1}{1-\alpha}\exp[-r(w+(1-\alpha)W_{n+1})]$ yields

$$\begin{aligned} V_n^A(W_n) = \max_{\gamma_n} -\exp(-r\gamma_n) + \omega_n p \int\limits_{\Xi_n} -\frac{\alpha}{1-\alpha}\exp\left[-r\left(w+(1-\alpha)\left(\frac{W_n - c - \gamma_n}{\alpha} + s_n^B(\Xi_n)\right)\right)\right] dF(u) \\ -\big(1-\omega_n pP(\Xi_n)\big)\exp\left[-r\left(w+(1-\alpha)\left(\frac{W_n - \omega_n c - \gamma_n}{\alpha} + s_n^0\right)\right)\right]. \end{aligned} \tag{EC.13}$$

If $\omega_n = 0$, this reduces to $V_{n+1}^A(W_{n+1}) = \max_{\gamma_n} -\exp(-r\gamma_n) - \frac{\alpha}{1-\alpha}\exp\left[-r\left(w+(1-\alpha)\left(\frac{W_n+w-\gamma_n}{\alpha}\right)\right)\right]$. Here the objective is concave; solving for its FOC yields $\gamma_n = w+(1-\alpha)W_n$ as the sole maximizer and $V_n^A(W_n) = -\frac{1}{1-\alpha}\exp\left[-r\left(w+(1-\alpha)W_n\right)\right]$ as the maximum. For the other case, if $\omega_n = 1$, (EC.13) can be decomposed as

$$\begin{aligned} V_n^A(W_n) = \max_{\gamma_n} -\exp(-r\gamma_n) - \frac{\alpha}{1-\alpha}\exp\left[-r\left(w+(1-\alpha)\frac{W_n-\gamma_n}{\alpha}\right)\right] \\ \times\Big(pP(\Xi_n)\exp[-r(1-\alpha)(s_n^B(\Xi_n)-c/\alpha)] + \big(1-pP(\Xi_n)\big)\exp\left[-r(1-\alpha)\left(s_n^0-c/\alpha\right)\right]\Big). \end{aligned}$$

Using (EC.6), the second line of this expression is equal to $\exp[-r\alpha^{-1}(1-\alpha)w]$, which reduces the problem to $V_{n+1}^A(W_{n+1}) = \max_{\gamma_n} -\exp(-r\gamma_n) - \frac{\alpha}{1-\alpha}\exp\left[-r\left(w+(1-\alpha)\left(\frac{W_n+w-\gamma_n}{\alpha}\right)\right)\right]$, which is the same as in the $\omega_n = 0$ case. Thus, $\gamma_n = w+(1-\alpha)W_n$ and $V_n^A(W_n) = -\frac{1}{1-\alpha}\exp\left[-r\left(w+(1-\alpha)W_n\right)\right]$ hold in both cases, completing the induction. ☐

Proof of Theorem 2 and Lemma 1.

We prove these two results concurrently.

Part 1: the Cost Function. This part is an intermediate step that is needed for the other parts of the proof. Summing the different costs in (12), and using $s_n^0 = w/\alpha$ (shown in Theorem 1), the principal's expected single-period costs to induce the agent to search and use reporting policy Ξ_n are:

$$\zeta(\Xi_n) = w + pP(\Xi_n)(\alpha s_n^B(\Xi_n) + e - w). \tag{EC.14}$$

Applying (11) to expand $s_n^B(\Xi_n)$ then yields

$$\zeta(\Xi_n) = w + pP(\Xi_n)\left(c + \frac{\alpha}{r(1-\alpha)}\ln\frac{pP(\Xi_n)}{1-(1-pP(\Xi_n))\exp\left[rc(1-\alpha)\alpha^{-1}\right]} + e\right). \tag{EC.15}$$

As $\zeta(\Xi_n)$ depends on Ξ_n only through $P(\Xi_n)$, for any two sets $\Xi_n, \Xi_n' \in \mathcal{F}$ such that $P(\Xi_n) = P(\Xi_n')$, it also holds that $\zeta(\Xi_n) = \zeta(\Xi_n')$.

Part 2: Optimality of History-Independent Policies. This part establishes the first sentence of Theorem 2. Applying Theorem 1, the contracting problem (6)–(9) reduces to an MDP, after which from (12) we have that the principal's value function in any non-terminal state is given by:

$$V_n^P = \max_{\omega_n\in\{0,1\},\Xi_n,\Psi_n\in\mathcal{F}} \omega_n pP(\Xi_n)\left(\alpha s_n^0 - e - \alpha s_n^B(\Xi_n)\right) + \omega_n p\int\limits_{\Xi_n\cap\Psi_n}\left(u - \alpha V_{n+1}^P\right) dF(u) - \alpha s_n^0 + \alpha V_{n+1}^P,$$

with boundary condition $V_{N+1}^P = 0$. Here, the objective function is independent of the history, and thus, the optimal choices of ω_n, Ξ_n, Ψ_n need not depend on it either. Note that the property that the optimal policy of an MDP needs to depend only on the current state and not on the full history is universal to most dynamic programs (see the adequacy of Markov policies in ch. 1.1.3 of Bertsekas 2012); however, as in the canonical search problem with no recall, the optimal policy here can also be independent of the current state

Part 3: Optimality of Threshold Policies. This part establishes the second sentence of Theorem 2. If $\omega_n = 0$, the choice of Ξ_n and Ψ_n is redundant, so consider the case where $\omega_n = 1$. From (12), the choice of Ψ_n affects the principal's value function only via the $\sum_{u \in \Xi_n \cap \Psi_n} (u - \alpha V_{n+1}^P)\mu(u)$ term, thus the threshold acceptance policy $\Psi_n^* = [\alpha V_{n+1}^P, \bar{u}] \cap \operatorname{supp} F$ is optimal (as this sum is maximized by including all points for which the summands are positive).[2]

Assume, by contradiction, there exists an optimal Ξ_n^* that is not a threshold one, denote $k := |\Xi_n^*|$ ($|\cdot|$ is the cardinality of a set), and define Ξ_n^{**} as the set of k largest elements of $\operatorname{supp} F$. We then have $P(\Xi_n^*) = P(\Xi_n^{**})$, as $|\Xi_n^*| = |\Xi_n^{**}|$ and $\mu(u)$ is constant. Thus, part 1 yields $\zeta(\Xi_n^*) = \zeta(\Xi_n^{**})$. Denote by $V_n^P(\omega_n, \Xi_n, \Psi_n)$ the principal's expected value when using $(\omega_n, \Xi_n, \Psi_n)$ in period n and proceeding optimally. From the objective function in (12) we then have $V_n^P(1, \Xi_n^{**}, \Psi_n^*) - V_n^P(1, \Xi_n^*, \Psi_n^*) = p\mu \sum_{u \in \Xi_n^{**} \cap \Psi_n^*} (u - \alpha V_{n+1}^P) - p\mu \sum_{u \in \Xi_n^* \cap \Psi_n^*} (u - \alpha V_{n+1}^P) \geq 0$. (Here, the last inequality follows from noticing that the first sum has at least as many summands as the second, all of which are positive, and that the i-th biggest element of the first sum is greater than or equal to the i-th biggest element of the second.) Thus, either both Ξ_n^* and Ξ_n^{**} are optimal, or we have a contradiction to optimality of Ξ_n^*. Either way, the optimality of threshold reporting policies follows.

Part 4: the Cost Function for Threshold Policies. This part establishes the statement of Lemma 1. Defining $z(\xi_n) := \zeta([\xi_n, \bar{u}] \cap \operatorname{supp} F)$ yields (14). Because $\zeta(\Xi_n)$ depends on Ξ_n only through the probability of finding an alternative in Ξ_n, we can also consider a cost function $\Pi(\pi)$ defined directly on these probabilities, which is then given by

$$\Pi(\pi) = w + \pi \left(c + \frac{\alpha}{r(1-\alpha)} \ln \frac{\pi}{1 - (1-\pi)\exp\left[rc(1-\alpha)\alpha^{-1}\right]} + e \right).$$

From (14), $\Pi(p\bar{F}(\xi_n)) = z(\xi_n)$ for all ξ_n that satisfy the implementability condition of Theorem 1. Consider $\Pi(\pi)$ on the domain $(\tau, p]$ where $\tau := 1 - 1/\exp[rc(1-\alpha)/\alpha]$ (this is the implementability threshold). We then have

$$\frac{d^2\Pi(\pi)}{d\pi^2} = \frac{\alpha \left(e^{\left(\frac{1}{\alpha}-1\right)cr} - 1 \right)^2}{(1-\alpha) r \pi \left((\pi - 1) e^{\left(\frac{1}{\alpha}-1\right)cr} + 1 \right)^2} > 0,$$

thus $\Pi(\pi)$ is strictly convex. Consequently, as $\bar{F}(\xi_n)$ is weakly decreasing, $z(\xi_n) = \Pi(p\bar{F}(\xi_n))$ is quasi-convex. Because $\bar{F}|_{\operatorname{supp} F}(\xi_n)$ is strictly decreasing, it follows from the strict convexity of $\Pi(\pi)$ that $z|_{\operatorname{supp} F}(\xi_n)$ is strictly increasing on $\xi_n \leq \xi^{\mathrm{LC}}$ and strictly decreasing on $\xi_n > \xi^{\mathrm{LC}}$. $\square$

The following lemma, which characterizes the first-best policy, will be used in several of the proofs.

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[2] Note that because summation here is taken over $\Xi_n \cap \Psi_n$, Ψ_n^* is not *uniquely* optimal. Every Ψ_n' such that $\Xi_n \cap \Psi_n^* = \Xi_n \cap \Psi_n'$ will also be optimal. However, Ψ_n^* is the only policy that is optimal irrespective of which Ξ_n the agent uses, which is why we focus on it.

Lemma A2. *The principal's first-best value function is given by*

$$V_n^{\mathrm{FB}} = \left(p\int_{\alpha V_{n+1}^{\mathrm{FB}}+e}^{\bar{u}} \bar{F}(u)du - c\right)^+ - w + \alpha V_{n+1}^{\mathrm{FB}}, \tag{EC.16}$$

with boundary condition $V_{N+1}^{\mathrm{FB}} = 0$*. The first-best reporting and acceptance policies are* $\{\xi_n^{\mathrm{FB}}\} = \{\psi_n^{\mathrm{FB}}\} = \{\Rsh \alpha V_{n+1}^{\mathrm{FB}} + e \Lsh\}$*. For all* $n \in \{1, ..., N\}$*, the first-best search policy is*

$$\omega_n^{\mathrm{FB}} = \begin{cases} 1 & \text{if } p\int_{\alpha V_{n+1}^{\mathrm{FB}}+e}^{\bar{u}} \bar{F}(x)\,dx - c > 0; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Noting that under the FB, the agent's costs are just passed on to the principal, and the agent receives no additional compensation, from (4) we have that the FB policies solve the Bellman equation

$$V_n^{\mathrm{FB}} = \max_{\omega_n \in \{0,1\}, \xi_n, \psi_n \in \operatorname{supp} F} \omega_n \left(p\int_{\max\{\xi_n,\psi_n\}}^{\bar{u}} (u - e - \alpha V_{n+1}^{\mathrm{FB}})dF(u) - peP([\xi_n, \psi_n)) - c\right) - w + \alpha V_{n+1}^{\mathrm{FB}}, \tag{EC.17}$$

with boundary condition $V_{N+1}^{\mathrm{FB}} = 0$. The choice of ξ_n, ψ_n affects the objective function of (EC.17) in two ways: a) through the non-positive $-peP([\xi_n, \psi_n))$ term (which is then maximized by any $\xi_n \geq \psi_n$, and b) the integral term $\int_{\max\{\xi_n,\psi_n\}}^{\bar{u}} (u - e - \alpha V_{n+1}^{\mathrm{FB}})dF(u)$, which is maximized by $\max\{\xi_n, \psi_n\} = \alpha V_{n+1}^{\mathrm{FB}} + e$ (thus also for $\max\{\xi_n, \psi_n\} = \Rsh \alpha V_{n+1}^{\mathrm{FB}} + e \Lsh$, as $F(u)$ has no probability mass in $(\alpha V_{n+1}^{\mathrm{FB}} + e, \Rsh \alpha V_{n+1}^{\mathrm{FB}} + e \Lsh)$). Consequently, setting $\xi_n^{\mathrm{FB}} = \psi_n^{\mathrm{FB}} = \Rsh \alpha V_{n+1}^{\mathrm{FB}} + e \Lsh$ is optimal. Statement of the lemma then follows from using the identity $\int_{\Rsh \alpha V^{\mathrm{FB}}+e \Lsh}^{\bar{u}} (u - e - \alpha V_{n+1}^{\mathrm{FB}})dF(u) = \int_{\alpha V^{\mathrm{FB}}+e}^{\bar{u}} (u - e - \alpha V_{n+1}^{\mathrm{FB}})dF(u) = \int_{\alpha V_{n+1}^{\mathrm{FB}}+e}^{\bar{u}} \bar{F}(u)du$ to shorten the integral in (EC.17) and noticing that $\omega_n = 1$ is optimal if and only if the expression in large brackets in (EC.17) is positive. ☐

Proof of Theorem 3.

As shown in part 2 of the proof for Theorem 2, $\psi_n^* = \Rsh \alpha V_n^P \Lsh$ is an optimal acceptance policy, inserting which into (16) yields

$$V_n^P = \max_{\omega_n \in \{0,1\}, \xi_n \in \operatorname{supp} F} \omega_n \left(p\int_{\max\{\xi_n, \alpha V_{n+1}^P\}}^{\bar{u}} (x - \alpha V_{n+1}^P)\,dF(x) - z(\xi_n) + w\right) - w + \alpha V_{n+1}^P, \tag{EC.18}$$

with boundary condition $V_{N+1}^P = 0$.

First, we show that $v^* \geq 0 \Leftrightarrow V_n^P \geq 0$ ($v^* > 0 \Leftrightarrow V_n^P > 0$), $\forall n \in \{1, ..., N\}$. These equivalences follow directly from two bounds for V_n^P, which we establish next. The first of these is the upper bound $\max\{v^*, -w\} + \sum_{i=1}^{N-n} (\alpha^i \max\{v^*, -w\})^+$, that can be established using (16). This bound has the same sign as v^*. To establish the second bound, denote by $\hat{\xi}, \hat{\psi}$ the maximizers of (17). Then, directing the agent to search in every period and using policies $\hat{\xi}, \hat{\psi}$ yields to the principal an expectation of $\sum_{i=0}^{N-n} \alpha^i (1 - \bar{F}(\hat{\xi}, \hat{\psi}))^i v^*$ which is a lower bound to V_n^P, and also has the same sign as v^*.

We will now demonstrate that if $v^* \geq -w$, it is optimal to search (set $\omega_n = 1$) in every period. From $v^* \geq -w$, it follows that it is optimal to induce the agent to search in at least one period (searching in just the last period is at least as good as never searching). Assume there are two consecutive periods, k and $k+1$, such that under the optimal policies $\{\omega_n^*\}, \{\xi_n^*\}, \{\psi_n^*\}$ the agent searches in period $k+1$ but not in k. Then, from (EC.18), we have

$$V_k^P = -w + \alpha \left(p\int_{\max\{\xi_{k+1}^*, \psi_{k+1}^*\}}^{\bar{u}} (x - \alpha V_{k+2}^P)\,dF(x) - z(\xi_{k+1}^*) + \alpha V_{n+1}^P\right).$$

Using an alternate policy that switches the order of strategies for these two periods (searching and using $\xi^*_{k+1}, \psi^*_{k+1}$ in period k and then not searching in period $k+1$) and denoting its value function by $V_k^{P\dagger}$ yields

$$V_k^{P\dagger} := p \int_{\max\{\xi^*_{k+1}, \psi^*_{k+1}\}}^{\bar{u}} (x - \alpha^2 V^P_{k+2} + \alpha w)\, dF(x) - z(\xi^*_{k+1}) - \alpha w + \alpha^2 V^P_{n+1}.$$

Looking at the difference between these two value functions gives

$$\begin{aligned} V_k^{P\dagger} - V_k^P &= p \int_{\max\{\xi^*_{k+1}, \psi^*_{k+1}\}}^{\bar{u}} ((1-\alpha)x + \alpha w)\, dF(x) - (1-\alpha) z(\xi^*_{k+1}) + (1-\alpha) w, \\ &= (1-\alpha)\left[p \int_{\max\{\xi^*_{k+1}, \psi^*_{k+1}\}}^{\bar{u}} x\, dF(x) - z(\xi^*_{k+1}) \right] + (1 - \alpha + p\alpha \bar{F}(\max\{\xi^*_{k+1}, \psi^*_{k+1}\}))w, \\ &\geq (1-\alpha)\left[p\bar{F}(\max\{\xi^*_{k+1}, \psi^*_{k+1}\}) \alpha V^P_{k+2} - w \right] + (1 - \alpha + p\alpha \bar{F}(\max\{\xi^*_{k+1}, \psi^*_{k+1}\}))w, \\ &= p\alpha \bar{F}(\max\{\xi^*_{k+1}, \psi^*_{k+1}\}) \big((1-\alpha) V^P_{k+2} + w \big) > 0. \end{aligned}$$

Here, the first inequality follows from the assumed optimality of $\omega^*_{k+1}, \xi^*_{k+1}, \psi^*_{k+1}$, while the second follows from $-w/(1-\alpha)$ being a strict lower bound for the value function under an optimal policy (it is the NPV of paying $-w$ in every period ad infinitum). Consequently, $V_k^{P\dagger} > V_k^P$, contradictory to the optimality of $\{\omega^*_n\}, \{\xi^*_n\}, \{\psi^*_n\}$. Thus, if there are periods in which the agent does not search, they have to come at the end of the search process, or in other words, there exists $K \in \{1, ..., N\}$ such that $\omega^*_n = 1$ for $n \leq K$ and $\omega^*_n = 0$ for $n > K$.

Let $(\xi^*, \psi^*) \in \arg\max_{\xi, \psi \in \operatorname{supp} F} p \int_{\max\{\xi, \psi\}}^{\bar{u}} u\, dF(u) - z(\xi)$. If $K < N$, in period $K+1$, it follows from $v^* \geq -w$ that setting $\omega_{K+1} = 1$, $\xi_{K+1} = \xi^*$, and $\psi_{K+1} = \psi^*$ would yield an expectation at least as high as using $\omega^*_{K+1} = 0$, thus searching in period $K+1$ is optimal. Consequently, for each $n \in \{1, ..., N\}$, $\omega^*_n = 1$ is optimal.

Now to show the other direction of the 'if and only if' condition. Assume the $v^* < -w$ and the $\omega^*_n = 1$ is optimal $\forall n \in \{1, ..., N\}$. But then, using $\omega_N = 0$ in the last period gives an expectation of $-w$, whereas using $\omega_N = 1$ yields $v^* < -w$, contradictory to the optimality of searching in the last period.

Using $n \in \{1, ..., N\}$, $\omega^*_n = 1$, (EC.18) simplifies to

$$V_n^P = \max_{\xi_n \in \operatorname{supp} F} p \int_{\max\{\xi_n, \alpha V^P_{n+1}\}}^{\bar{u}} (x - \alpha V^P_{n+1})\, dF(x) - z(\xi_n) + \alpha V^P_{n+1}, \tag{EC.19}$$

with boundary condition $V^P_{N+1} = 0$. The optimal contract is then given by Theorem 1, and is the one that implements the above-given $\{\omega^*_n\}$ alongside with $\{\xi^*_n\}$ that solves (EC.19).

It remains to prove the properties stated in parts (i)–(v) of the theorem.

Part (i): $\min\{\xi^{\mathrm{LC}}, \Rsh \alpha V^P_{n+1} \Lsh\} \leq \xi^*_n \leq \min\{\xi^{\mathrm{LC}}, \Rsh \alpha V^P_{n+1} + e \Lsh\}$, $\forall n \in \{1, ..., N\}$. Consider the objective function in (EC.19) as a function of ξ_n on domain $\operatorname{supp} F$. In situations where $\xi^{\mathrm{LC}} \leq \alpha V^P_{n+1}$, the optimal policy is $\xi^*_n = \xi^{\mathrm{LC}}$—as is evident from (EC.19), where the integral term is constant in ξ_n for $\xi_n \leq \alpha V^P_{n+1}$ (and strictly decreasing in ξ_n afterwards) while the cost term $z(\xi_n)$ is minimized at $\xi_n = \xi^{\mathrm{LC}}$ (per Lemma 1). Consider now the situations where $\xi^{\mathrm{LC}} > \alpha V^P_{n+1}$. Then, it has to be that $\xi^*_n \leq \xi^{\mathrm{LC}}$ because higher values would both weakly increase the costs (as $z(\xi_n)$ is minimized at ξ^{LC}) and strictly decrease the value from search (the integral term). In that case, we also have $\xi^*_n \geq \alpha V^P_{n+1}$, because the integral term in (EC.19) is constant for $\xi_n \le \alpha V^P_{n+1}$, whereas the costs are strictly decreasing in ξ_n on the same region (by Lemma 1, as $\xi^{\text{LC}} > \alpha V^P_{n+1}$). Thus, in the case where $\xi^{\text{LC}} > \alpha V^P_{n+1}$, we can expand $z(\xi_n)$ and rewrite (EC.19) as

$$V^P_n = \max_{\xi_n \in \operatorname{supp} F \mid \alpha V^P_{n+1} \le \xi_n \le \xi^{\text{LC}}} p\int_{\xi_n}^{\bar{u}} (x - e - \alpha V^P_{n+1})\, dF(x) - w - p\bar{F}(\xi_n)(\alpha s^B_n(\xi_n) - w) + \alpha V^P_{n+1}, \tag{EC.20}$$

from which we see that the integral term is strictly increasing in ξ_n for $\xi_n \le \alpha V^P_{n+1} + e$ (and strictly decreasing afterwards), while the term $p\bar{F}(\xi_n)(\alpha s^B_n(\xi_n) - w)$ is strictly increasing in ξ_n. Therefore, if $[\alpha V^P_{n+1}, \alpha V^P_{n+1} + e] \neq \emptyset$ then $\xi^*_n \in [\alpha V^P_{n+1}, \alpha V^P_{n+1} + e] \cap \operatorname{supp} F$, otherwise $\xi^*_n = \Rsh \alpha V^P_{n+1} + e \Lsh$. Finally, because $\xi^{\text{LC}} \le \alpha V^P_{n+1}$ implies $\xi^*_n = \xi^{\text{LC}}$ and $\xi^{\text{LC}} > \alpha V^P_{n+1}$ implies $\xi^*_n \in [\alpha V^P_{n+1}, \min\{\xi^{\text{LC}}, \Rsh \alpha V^P_{n+1} + e \Lsh\}] \cap \operatorname{supp} F$, noticing that $\xi_n \ge x$ (for any x) also implies $\xi_n \ge \Rsh x \Lsh$ we obtain $\min\{\xi^{\text{LC}}, \Rsh \alpha V^P_{n+1} \Lsh\} \le \xi^*_n \le \min\{\xi^{\text{LC}}, \Rsh \alpha V^P_{n+1} + e \Lsh\}$.

Part (ii): $\{\psi^*_n\}, \{V^P_n\} \downarrow n$. At any time m, following the policy $(\{\xi^{\text{shifted}}_n\}, \{\psi^{\text{shifted}}_n\})$ given by $\forall n : \xi^{\text{shifted}}_n := \xi^*_{n+1}, \psi^{\text{shifted}}_n := \psi^*_{n+1}$ yields V^P_{m+1} in expectation,[3] implying that the optimal policy has to yield at least that much, i.e., $V^P_m \ge V^P_{m+1}$ and thus $\{V^P_n\} \downarrow n$. Then, $\{\psi^*_n\} \downarrow n$ follows from $\psi^*_n = \Rsh \alpha V^P_{n+1} \Lsh$.

Part (iii): $\{\xi^*_n\}$ *is constant and equal to* ξ^{LC} *up to a point, after which it is decreasing in* n. We have $\{V^P_n\} \downarrow n$ from part (ii) of the theorem and that $\alpha V^P_{n+1} \ge \xi^{\text{LC}}$ implies $\xi^*_n = \xi^{\text{LC}}$ from part (i). Hence there exists $K \ge 0$ such that the first K elements of $\{\xi^*_n\}$ are equal to ξ^{LC} and, for all $n > K$, $\xi^{\text{LC}} > \alpha V^P_{n+1}$. It follows that $\{\xi^*_n\}$ solves (EC.20) for $n > K + 1$. Denote the objective function of (EC.20) as $\text{ob}(\xi, n) := p\int_{\xi}^{\bar{u}} (x - e - \alpha V^P_{n+1})\, dF(x) - w - p\bar{F}(\xi)(\alpha s^B_n(\xi) - w) + \alpha V^P_{n+1}$. Thus defined $\text{ob}(\xi, n)$ is submodular, and so, by Topkis's theorem (Topkis 1978), $\{\xi^*_n\} \downarrow n$.

Part (iv): *for all* $n \in \{1, ..., N\}$, $\xi^*_n \le \xi^{\text{FB}}_n$ *and* $\psi^*_n < \xi^{\text{FB}}_n$. By Lemma A2, $\xi^{\text{FB}}_n = \Rsh \alpha V^{\text{FB}}_{n+1} + e \Lsh \ge \Rsh \alpha V^P_{n+1} + e \Lsh \ge \Rsh \alpha V^P_{n+1} \Lsh = \psi^*_n$. Here, the $V^{\text{FB}}_{n+1} \ge V^P_{n+1}$ inequality follows from the definition of the first-best and the principal of optimality (it can also be derived from a comparison of (13) to (EC.19) by noticing that $z(\xi_n) \ge w + c + p\bar{F}(\xi_n)e$). As shown in part (i), $\xi^*_n \le \Rsh \alpha V^P_{n+1} + e \Lsh$, so $\xi^{\text{FB}}_n = \Rsh \alpha V^{\text{FB}}_n + e \Lsh \ge \Rsh \alpha V^P_{n+1} + e \Lsh \ge \xi^*_n$.

Part (v): *Single crossing of* ψ^*_n and ξ^*_n. From part (iii), ξ^*_n is constant and equal to ξ^{LC} up to a certain period, after which it is weakly decreasing in n. Then, from part (i), in the non-constant region of ξ^*_n we have that $\xi^*_n \ge \alpha V^P_n$, thus also $\xi^*_n \ge \Rsh \alpha V^P_n \Lsh = \psi^*_n$ (so no crossings within this region). Hence, $\psi^*_n > \xi^*_n$ can only be in the region where $\xi^*_n = \xi^{\text{LC}}$, from which the $\xi^*_{M-1} = \xi^{\text{LC}}$ property immediately follows. Using part (ii), ψ^*_n is weakly decreasing and thus crosses the constant function at most once. $\square$

Proof of Proposition 1.

In the first-best solution (analogously, for the principal who delegates search), V^{FB}_n (resp., V^P_n) is the value of the rest of the search under an optimal policy. As $N \to \infty$, this value must be equal in all periods (Proposition 5.4.1 of Bertsekas 2017). Hence, as $\xi^{\text{FB}}_n = \Rsh \alpha V^{\text{FB}}_{n+1} + e \Lsh$ (Lemma A2) and $\psi^*_n = \Rsh \alpha V^{\text{FB}}_{n+1} \Lsh$ (Theorem 3), those policies are equal in all periods as well. Then, stationarity of ξ^*_n follows from (18), while $\xi^{\text{FB}}_n \ge \xi^*_n$ and $\xi^{\text{FB}}_n \ge \psi^*_n$ follow from Theorem 3(iv). Because $\psi^*_n = \Rsh \alpha V^P_{n+1} \Lsh$, we have $\Rsh \psi^*_n + e \Lsh = \Rsh \Rsh \alpha V^P_{n+1} \Lsh + e \Lsh \ge \Rsh \alpha V^P_{n+1} + e \Lsh \ge \xi^*_n$,

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[3] The shifted policy leaves undefined what to do in the last period (as ξ^*_{N+1} is undefined). However, we do not need to know what happens in the last period, only that there is some policy we can use there that does not result in a negative payoff. As we demonstrated earlier in the proof, a policy pair (ξ, ψ) exists for which $p\int_{\max\{\xi,\psi\}}^{\bar{u}} x\, dF(x) - z(\xi) > 0$ which can then be used in the last period.

where the last inequality follows from Theorem 3(i). The time invariance of s_n^B then follows from equation (11) because ξ_n^* is constant.

Now we turn to the risk-neutral case. Using the expression for s_n^B from (11) gives us the equality $\lim_{r\to 0} s_n^B(\xi_n) = w/\alpha + c/(p\bar{F}(\xi_n)\alpha)$. Observe that the expected single-period payment to the agent is now independent of ξ_n, which follows more explicitly from substituting this expression for s_n^B into (14). Thus the principal can induce any policy (including the first-best) for the same expected cost. Given that the participation constraint is binding (as shown in the proof of Theorem 1), the principal is also able to extract all value from the agent; hence the contract achieves the first-best.

For part (iii) of the proposition, $z(\xi)$ is an increasing function of ξ when $e = 0$, so $\xi^{\mathrm{LC}} = \min \operatorname{supp} F$ and thus also $\xi_n^* = \min \operatorname{supp} F$ by Theorem 3(i). First-best being attained for $p = 1$ (but not otherwise) follows from $s_n^B(\min \operatorname{supp} F) = (c + w)/\alpha$ when $p = 1$ (but not when $p < 1$), which then makes the first-best value function (13) equivalent to the one under the optimal contract, as given by (18).

Lastly, part (iv) of the proposition follows directly from Theorem 3 under parameter value $w = 0$. $\square$

Proof of Proposition 2.

Denote by V_n^{IN} the principal's value function (her NPV under an optimal policy) in period n if she searches on her own (in-house); V_n^{IN} is given by (1) with $c = c_P$, $p = p_P$, and $F(\cdot) = F_P(\cdot)$. From Lemma A1 it follows that the bracketed term in (1) is positive, so

$$V_n^{\mathrm{IN}} = p_P \int_{\alpha V_{n+1}^{\mathrm{IN}}}^{\bar{u}_P} \bar{F}_P(x)\,dx - c_P + \alpha V_{n+1}^{\mathrm{IN}}, \tag{EC.21}$$

with boundary condition $V_{N+1}^{\mathrm{IN}} = 0$. We similarly use V_n^{OUT} to denote the value function for the principal who delegates (outsources) search. Applying (18) now yields

$$V_n^{\mathrm{OUT}} = \max_{\xi_n \in \operatorname{supp} F} p_A \int_{\max\{\xi_n, \alpha V_{n+1}^{\mathrm{OUT}}\}}^{\bar{u}_A} (x - \alpha V_{n+1}^{\mathrm{OUT}})\,dF_A(x) - z(\xi_n) + \alpha V_{n+1}^{\mathrm{OUT}}, \tag{EC.22}$$

with boundary condition $V_{N+1}^{\mathrm{OUT}} = 0$. The rest of our proof proceeds in three steps: after showing that these two value functions do not cross more than once, we characterize situations in which they do not cross at all.

Step 1: Single-crossing property. Consider the problem faced by a principal who can, in each period, choose whether to search herself in that period or to delegate the search task. Thus she need not commit to one of these options at the beginning of the search and can mix and match as needed, getting the best of both options. (Such principal is also the subject of Remark 2 in the main text.) Our motive for considering such a principal is to establish that optimal single-period outsourcing decisions depend primarily on the value of the rest of the search (i.e., from period $n+1$ onward): delegation is optimal when that value is low, but an in-house search is optimal when it is high.

Let $\hat{V}_n$ denote this principal's value function. To express that function concisely, we first define

$$\hat{\xi}(V) = \min \arg\max_{\xi \in \operatorname{supp} F} p_A \int_{\max\{\xi, \alpha V\}}^{\bar{u}_A} (x - \alpha V)\,dF_A(x) - z(\xi) + \alpha V. \tag{EC.23}$$

This function gives the optimal threshold (reporting policy) $\hat{\xi}$ that the principal should induce when delegating, and her value function in the next period will be V. The property given in part (i) of Theorem 3 can also be extended, via an identical proof, to show that $\hat{\xi}(V) \in [\min\{\xi^{\mathrm{LC}}, \Rsh \alpha V \Lsh\}, \min\{\xi^{\mathrm{LC}}, \Rsh \alpha V + e \Lsh\}] \cap \operatorname{supp} F$. Using (EC.21) and (EC.22), the principal's value function is given by

$$\begin{aligned}\hat{V}_n = \max\Bigg\{ & p_P \int_{\alpha \hat{V}_{n+1}}^{\bar{u}_P} \bar{F}_P(x)\,dx - c_P, \\ & p_A \int_{\max\{\hat{\xi}(\hat{V}_{n+1}), \alpha\hat{V}_{n+1}\}}^{\bar{u}_A} (x - \alpha\hat{V}_{n+1})\,dF_A(x) - z(\hat{\xi}(\hat{V}_{n+1}))\Bigg\} + \alpha\hat{V}_{n+1}, \qquad \text{(EC.24)}\end{aligned}$$

with boundary condition $\hat{V}_{N+1} = 0$. We can also express (EC.24) as

$$\begin{aligned}\hat{V}_n = \max_{h_n \in \{0,1\}} h_n & \left[p_A \int_{\max\{\hat{\xi}(\hat{V}_{n+1}), \alpha\hat{V}_{n+1}\}}^{\bar{u}_A} (x - \alpha\hat{V}_{n+1}) dF_A(x) - p_P \int_{\alpha\hat{V}_{n+1}}^{\bar{u}_P} \bar{F}_P(x) dx + c_P - z(\hat{\xi}(\hat{V}_{n+1}))\right] \\ & + p_P \int_{\alpha\hat{V}_{n+1}}^{\bar{u}_P} \bar{F}_P(x)\,dx - c_P + \alpha\hat{V}_{n+1},\end{aligned}$$

where $h_n = 1$ (resp. $h_n = 0$) stands for the decision to delegate search (resp. not delegate) in period n. It is then optimal to delegate if and only if the expression in brackets next to h_n is positive. We abbreviate that expression as VD_n, the *value of delegation*, where

$$\mathrm{VD}_n(\hat{V}_{n+1}) = p_A \int_{\max\{\hat{\xi}(\hat{V}_{n+1}), \alpha\hat{V}_{n+1}\}}^{\bar{u}_A} (x - \alpha\hat{V}_{n+1})\,dF_A(x) - p_P \int_{\alpha\hat{V}_{n+1}}^{\bar{u}_P} \bar{F}_P(x)\,dx + c_P - z(\hat{\xi}(\hat{V}_{n+1})). \qquad \text{(EC.25)}$$

Next, we show that $\mathrm{VD}_n(\hat{V}_{n+1})$ is a weakly decreasing function of $\hat{V}_{n+1}$. Consider first the case when $\alpha\hat{V}_{n+1} \geq \xi^{\mathrm{LC}}$. In this case, we have that $\hat{\xi}(\hat{V}_{n+1}) = \xi^{\mathrm{LC}}$, which enables the following simplification:

$$\begin{aligned}\mathrm{VD}_n(\hat{V}_{n+1}) &= p_A \int_{\alpha\hat{V}_{n+1}}^{\bar{u}_A} (x - \alpha\hat{V}_{n+1})\,dF_A(x) - p_P \int_{\alpha\hat{V}_{n+1}}^{\bar{u}_P} \bar{F}_P(x)\,dx + c_P - z(\xi^{\mathrm{LC}}), \\ &= \int_{\alpha\hat{V}_{n+1}}^{\max\{\bar{u}_A, \bar{u}_P\}} (p_A \bar{F}_A(x) - p_P \bar{F}_P(x))\,dx + c_P - z(\xi^{\mathrm{LC}}),\end{aligned}$$

which is decreasing in $\hat{V}_{n+1}$ because the integrand is positive.

Now consider the case where $\alpha\hat{V}_{n+1} \leq \xi^{\mathrm{LC}}$ (thus $\hat{\xi}(\hat{V}_{n+1}) \geq \Rsh \alpha\hat{V}_{n+1} \Lsh$), and let $\xi^{\mathrm{LC}}/\alpha \geq \hat{V}^H_{n+1} > \hat{V}^L_{n+1}$. We will show that the following difference is positive

$$\begin{aligned}\mathrm{VD}_n(\hat{V}^L_{n+1}) - \mathrm{VD}_n(\hat{V}^H_{n+1}) &= p_A \int_{\hat{\xi}(\hat{V}^L_{n+1})}^{\bar{u}_A} (x - \alpha\hat{V}^L_{n+1}) dF_A(x) - p_A \int_{\hat{\xi}(\hat{V}^H_{n+1})}^{\bar{u}_A} (x - \alpha\hat{V}^H_{n+1}) dF_A(x) \\ &\quad - z(\hat{\xi}(\hat{V}^L_{n+1})) + z(\hat{\xi}(\hat{V}^H_{n+1})) - p_P \int_{\alpha\hat{V}^L_{n+1}}^{\alpha\hat{V}^H_{n+1}} \bar{F}_P(x) dx \\ &\geq p_A \int_{\alpha\hat{V}^L_{n+1}+e}^{\bar{u}_A} (x - \alpha\hat{V}^L_{n+1}) dF_A(x) - p_A \int_{\hat{\xi}(\hat{V}^H_{n+1})}^{\bar{u}_A} (x - \alpha\hat{V}^H_{n+1}) dF_A(x) \\ &\quad - z(\hat{\xi}(\hat{V}^L_{n+1})) + z(\hat{\xi}(\hat{V}^H_{n+1})) - p_P \int_{\alpha\hat{V}^L_{n+1}}^{\alpha\hat{V}^H_{n+1}} \bar{F}_P(x) dx \\ &= p_A \int_{\alpha\hat{V}^L_{n+1}+e}^{\bar{u}_A} (x - \alpha\hat{V}^L_{n+1} - e) dF_A(x) - p_A \int_{\hat{\xi}(\hat{V}^H_{n+1})}^{\bar{u}_A} (x - \alpha\hat{V}^H_{n+1} - e) dF_A(x) \\ &\quad + p_A e \bar{F}_A(\alpha\hat{V}^L_{n+1} + e) - p_A e \bar{F}_A(\hat{\xi}(\hat{V}^H_{n+1})) \\ &\quad - z(\hat{\xi}(\hat{V}^L_{n+1})) + z(\hat{\xi}(\hat{V}^H_{n+1})) - p_P \int_{\alpha\hat{V}^L_{n+1}}^{\alpha\hat{V}^H_{n+1}} \bar{F}_P(x) dx.\end{aligned}$$

Then, as $\hat{\xi}(\hat{V}^H_{n+1}) \leq \Rsh \alpha\hat{V}^H_{n+1} + e \Lsh$ we have $\int_{\hat{\xi}(\hat{V}^H_{n+1})}^{\bar{u}_A}(x - \alpha\hat{V}^H_{n+1} - e)dF_A(x) = \int_{\Rsh\alpha\hat{V}^H_{n+1}+e\Lsh}^{\bar{u}_A}(x - \alpha\hat{V}^H_{n+1} - e)dF_A(x) + \int_{x\in[\hat{\xi}(\hat{V}^H_{n+1}),\Rsh\alpha\hat{V}^H_{n+1}+e\Lsh)}(x - \alpha\hat{V}^H_{n+1} - e)dF_A(x) \leq \int_{\alpha\hat{V}^H_{n+1}+e}^{\bar{u}_A}(x - \alpha\hat{V}^H_{n+1} - e)dF_A(x) = \int_{\alpha\hat{V}^H_{n+1}+e}^{\bar{u}_A}\bar{F}_A(x)dx = \int_{\alpha\hat{V}^H_{n+1}}^{\bar{u}_A}\bar{F}_A(x+e)dx$. Inserting this inequality into the equation above yields

$$
\begin{aligned}
\mathrm{VD}_n(\hat{V}^L_{n+1}) - \mathrm{VD}_n(\hat{V}^H_{n+1}) \geq\, & p_A\int_{\alpha\hat{V}^L_{n+1}}^{\bar{u}_A}\bar{F}_A(x+e)dx - p_A\int_{\alpha\hat{V}^H_{n+1}}^{\bar{u}_A}\bar{F}_A(x+e)dx \\
& + p_A e\bar{F}_A(\alpha\hat{V}^L_{n+1}+e) - p_A e\bar{F}_A(\hat{\xi}(\hat{V}^H_{n+1})) \\
& - z(\hat{\xi}(\hat{V}^L_{n+1})) + z(\hat{\xi}(\hat{V}^H_{n+1})) - p_P\int_{\alpha\hat{V}^L_{n+1}}^{\alpha\hat{V}^H_{n+1}}\bar{F}_P(x)dx \\
=\, & \int_{\alpha\hat{V}^L_{n+1}}^{\alpha\hat{V}^H_{n+1}}\left(p_A\bar{F}_A(x+e) - p_P\bar{F}_P(x)\right)dx \\
& + p_A e\bar{F}_A(\alpha\hat{V}^L_{n+1}+e) - p_A e\bar{F}_A(\hat{\xi}(\hat{V}^H_{n+1})) - z(\hat{\xi}(\hat{V}^L_{n+1})) + z(\hat{\xi}(\hat{V}^H_{n+1})).
\end{aligned}
$$

Expanding the $z(\cdot)$ terms yields

$$
\begin{aligned}
\mathrm{VD}_n(\hat{V}^L_{n+1}) - \mathrm{VD}_n(\hat{V}^H_{n+1}) \geq\, & \int_{\alpha\hat{V}^L_{n+1}}^{\alpha\hat{V}^H_{n+1}}\left(p_A\bar{F}_A(x+e) - p_P\bar{F}_P(x)\right)dx \\
& - p_A\bar{F}_A(\hat{\xi}(\hat{V}^L_{n+1}))(\alpha s^B_n(\hat{\xi}(\hat{V}^L_{n+1})) - w) + p_A\bar{F}_A(\hat{\xi}(\hat{V}^H_{n+1}))(\alpha s^B_n(\hat{\xi}(\hat{V}^H_{n+1})) - w) \\
& + p_A e(\bar{F}_A(\alpha\hat{V}^L_{n+1}+e) - \bar{F}_A(\hat{\xi}(\hat{V}^L_{n+1}))).
\end{aligned}
$$

Here, the first line is weakly positive as the integrand is weakly positive, the second is weakly positive because $p_A\bar{F}_A(x)(\alpha s^B_n(x) - w) \uparrow x$ and $\hat{\xi}(V) \uparrow V$, and the last is also weakly positive as $\hat{\xi}(\hat{V}^L_{n+1})) \leq \Rsh \alpha\hat{V}^L_{n+1} + e \Lsh$, thus $\bar{F}_A(\hat{\xi}(\hat{V}^L_{n+1})) \geq \bar{F}_A(\Rsh \alpha\hat{V}^L_{n+1} + e \Lsh) = \bar{F}_A(\alpha\hat{V}^L_{n+1} + e)$. Consequently, $\mathrm{VD}_n(\hat{V}^L_{n+1}) - \mathrm{VD}_n(\hat{V}^H_{n+1}) \geq 0$, that is $\mathrm{VD}_n(\hat{V}_{n+1})$ is weakly decreasing.

As a result, the optimal control sequence $\{h^*_n\}$ is weakly increasing in n. So if there exists k such that $h^*_k = 1$, then for every $n \in \{k, ..., N\}$ we have $h^*_n = 1$ and hence $\hat{V}_n = V^{\mathrm{OUT}}_n$ for $n \geq k$. We conclude that, if the principal who solves (EC.24) should delegate, then (a) so should the main model's principal, and (b) their value functions are identical. So if there exists period n in which $V^{\mathrm{OUT}}_n \geq V^{\mathrm{IN}}_n$, then the same holds for any *subsequent* period. Conversely, if there exists n, in which $V^{\mathrm{OUT}}_n \leq V^{\mathrm{IN}}_n$, then the same holds in every *preceding* period, completing the single-crossing property.

Step 2: Situations where it is optimal not to delegate. Consider the problem in the last period N. The principal receives $V^{\mathrm{IN}}_N = p_P\int_{\underline{u}_P}^{\bar{u}_P}\bar{F}_P(x)\,dx - c_P$ if searching on her own or $V^{\mathrm{OUT}}_N = \max_{\xi\in\mathrm{supp}\,F} p_A\int_{\xi}^{\bar{u}_A} x\,dF_A(x) - z(\xi)$ if she delegates. So in the last period, the principal is better off conducting her own search if and only if $V^{\mathrm{IN}}_N \geq V^{\mathrm{OUT}}_N$; expanding this expression gives the condition (19). Finally, if this condition holds, then it is optimal to conduct an in-house search not only in the last period but in every preceding period as well, a result that follows from Step 1 of this proof and completes part (i) of the proposition.

Step 3: Situations where not delegating is optimal. Notice that $V^{\mathrm{IN}}_n \geq V^{\mathrm{OUT}}_n \Rightarrow V^{\mathrm{IN}}_{n-1} \geq V^{\mathrm{OUT}}_{n-1}$, which was shown in part 1, also implies that if it is optimal to delegate if there are N periods until the horizon, it is also optimal to delegate if there are K periods until the horizon, for every $K \in \{1, .., N\}$. Thus we can identify situations in which delegation is optimal—irrespective of the number of periods—by considering $N \to \infty$, in which case the optimal policies are stationary (see Lippman and McCall 1976). Then, by (EC.21) and (EC.22), the value functions in any period n are given by

$$V^{\text{IN}} = \frac{1}{1-\alpha}\left(p_P \int_{\alpha V^{\text{IN}}}^{\bar{u}_P} \bar{F}_P(x)\,dx - c_P\right) \quad \text{and}$$
$$V^{\text{OUT}} = \frac{1}{1-\alpha} \max_{\xi \in \operatorname{supp} F}\left(p_A \int_{\max\{\xi, \alpha V^{\text{OUT}}\}}^{\bar{u}_A} (x - \alpha V^{\text{OUT}})\,dF_A(x) - z(\xi)\right).$$

Therefore, if $V^{\text{OUT}} \geq V^{\text{IN}}$, then it is optimal to delegate for every possible horizon length $N \in \mathbb{N}$. This completes part (ii) of the proposition, and also part (iii) by applying the single-crossing property (Step 1 of this proof). □

Proposition A1 and Lemma A3 (to follow) are the analogues of (respectively) Theorems 1 and 2 and Lemma 1 for the setting in which u is not contractible. Both of these results are used in our proof of Theorem 4.

Proposition A1 (Implementable policies under unspecified search) *If Y^n is the sole contractible variable and the principal commits to an acceptance policy $\{\Psi_n\}$ when offering the contract, then the principal can implement any policy pair $(\{\omega_n\}, \{\Xi_n\})$ that satisfies the following conditions for every period $n \in \{1, ..., N\}$:* either $\omega_n = 0$ or $\Psi_n \subseteq \Xi_n$ and $pP(\Psi_n)^{\mathbb{1}_{\Xi_n \neq \operatorname{supp} F}} > 1 - 1/\exp[rc(1-\alpha)\alpha^{-1}]$. *The contract that implements $\{\Xi_n\}$ at the lowest possible cost to the principal has two possible components. In periods during which the agent does not bring all alternatives to the principal* ($\Xi_n \neq \operatorname{supp} F$), *the contract pays the agent a bonus of $s_n^B(\Psi_n)$ if the principal accepts the delivered alternative or of s_n^0 if she does not accept it or no alternative was delivered; here $s_n^B(\cdot)$ and s_n^0 are given by* (11). *In periods where* ($\Xi_n = \operatorname{supp} F$), *the bonus is equal to $s_n^B(\operatorname{supp} F)$ and is paid for any alternative presented to the principal—that is, irrespective of her decision to accept or reject it. This contract is renegotiation-proof.*

Proof. From Theorem 1 of Plambeck and Zenios (2000) it follows that the problem of finding a contract that implements any admissible search policy $\{\omega_n\}$ and reporting policy $\{\Xi_n\}$ at the lowest possible cost to the principal can be decomposed into N single-period problems, each given by

$$\arg\min_{s_n^A, s_n^R, s_n^0} p\omega_n P(\Psi_n \cap \Xi_n) s_n^A + p\omega_n P(\Xi_n \backslash \Psi_n) s_n^R + (1 - p\omega_n P(\Xi_n)) s_n^0; \tag{EC.26}$$

$$\begin{aligned} -p\omega_n P(\Psi_n \cap \Xi_n) \exp[-r(1-\alpha)(s_n^A - \alpha^{-1}c)] - p\omega_n P(\Xi_n \backslash \Psi_n) \exp[-r(1-\alpha)(s_n^R - \alpha^{-1}c)] \\ -(1 - p\omega_n P(\Xi_n)) \exp[-r(1-\alpha)(s_n^0 - \alpha^{-1}c)] = -\exp[-r\alpha^{-1}(1-\alpha)w]; \end{aligned} \tag{EC.27}$$

$$\begin{aligned} -pP(\Psi_n \cap \Xi) \exp[-r(1-\alpha)(s_n^A - \alpha^{-1}c)] - pP(\Xi \backslash \Psi_n) \exp[-r(1-\alpha)(s_n^R - \alpha^{-1}c)] \\ -(1 - pP(\Xi)) \exp[-r(1-\alpha)(s_n^0 - \alpha^{-1}c)] \leq -\exp[-r\alpha^{-1}(1-\alpha)w] \quad \forall \Xi \in \mathcal{F}; \end{aligned} \tag{EC.28}$$

$$-\exp[-r(1-\alpha)s_n^0] \leq -\exp[-r\alpha^{-1}(1-\alpha)w]. \tag{EC.29}$$

As before, s_n^A (resp. s_n^R) is the agent's reward for bringing an alternative to the principal that she accepts (resp. rejects), and s_n^0 is his reward if he does not bring her any alternative. The objective function (EC.26) minimizes the expected sum of these rewards conditional on the agent using the desired policies ω_n and Ξ_n. Equation (EC.27) is the (binding) participation constraint. The incentive compatibility constraints (EC.28) and (EC.29) ensure that the agent could not do better by choosing a different reporting policy or by declining to pay the search cost. As in Proposition 1, if $\omega_n = 0$ then (EC.26)–(EC.29) has a trivial solution of $s_n^A = s_n^R = s_n^0 = w/\alpha$, so we proceed to examine the case when $\omega_n = 1$.

Notice from the incentive compatibility constraint (EC.28) that if $s_n^A, s_n^R > s_n^0$, then it is strictly optimal for the agent to report all alternatives, if $s_n^A > s_n^0 > s_n^R$, it is strictly optimal for him to report only the ones that the principal will accept (and the exact value of s_n^R is redundant), and if $s_n^R > s_n^0 > s_n^A$, it is strictly optimal for the agent to report all that the principal will *not* accept (and the exact value of s_n^A is redundant). Thus, unless $s_n^A = s_n^0$ or $s_n^R = s_n^0$, the only search policies which can be induced are $\Xi_n = \operatorname{supp} F$, $\Xi_n = \Psi_n$, and $\Xi_n = \operatorname{supp} F \backslash \Psi_n$. Thus, we will separate this problem into 5 exhaustive cases and solve each case individually.

Case 1. Consider first the contract parameters $s_n^R = s_n^0$. Then, we also need $s_n^A > s_n^R$, otherwise the contract would induce $\omega_n = 0$. Such contract parameters will induce the agent to report any alternative in Ψ_n to the principal, and make him indifferent about reporting other alternatives. We first solve a relaxed version of the problem, where the constraint (EC.28) is replaced by $s_n^R = s_n^0$, which is then given by

$$\arg\min_{s_n^A, s_n^0} pP(\Psi_n)s_n^A + (1 - pP(\Psi_n))s_n^0; \tag{EC.30}$$

$$-pP(\Psi_n)\exp[-r(1-\alpha)(s_n^A - \alpha^{-1}c)] - (1 - pP(\Psi_n))\exp[-r(1-\alpha)(s_n^0 - \alpha^{-1}c)] = -\exp[-r\alpha^{-1}(1-\alpha)w]; \tag{EC.31}$$

$$-\exp[-r(1-\alpha)s_n^0] \le -\exp[-r\alpha^{-1}(1-\alpha)w]. \tag{EC.32}$$

This problem is identical to (EC.5)–(EC.7), which was solved in our proof of Theorem 1. It is thus solvable only if the implementability condition $pP(\Psi_n) > 1 - 1/\exp[rc(1-\alpha)\alpha^{-1}]$ holds, and its solution is

$$s_n^A = \frac{w+c}{\alpha} + \frac{1}{r(1-\alpha)} \ln \frac{pP(\Psi_n)}{1 - (1 - pP(\Psi_n))\exp[rc(1-\alpha)\alpha^{-1}]}, \qquad s_n^0 = s_n^R = \frac{w}{\alpha}. \tag{EC.33}$$

Plugging this solution back into the incentive compatibility constraint (EC.28), we obtain

$$\begin{aligned} &-pP(\Psi_n \cap \Xi)\exp[-r(1-\alpha)(s_n^A - \alpha^{-1}c)] \\ &-(1 - pP(\Psi_n \cap \Xi))\exp[-r(1-\alpha)(s_n^0 - \alpha^{-1}c)] \\ &\le -\exp[-r\alpha^{-1}(1-\alpha)w], \forall \Xi \in \mathcal{F}. \end{aligned}$$

This condition is satisfied for any Ξ_n such that $\Psi_n \subseteq \Xi_n$. This is because the LHS of the condition is maximized at $\Xi = \Psi_n$ (as $s_n^A > s_n^0$) and because, by (EC.31), the value of the left-hand side when evaluated at $\Xi = \Psi_n$ is equal to that of the right-hand side. Hence (EC.33) is a *candidate* solution to the original system (EC.26)–(EC.29) for any Ξ_n that satisfies $\Psi_n \subseteq \Xi_n$. The solution is only a candidate solution as the optimization restricts the contract parameters to those that satisfy $s_n^R = s_n^0 < s_n^A$. Thus, we still need to verify that none of the other cases are capable of implementing the same reporting policy at a lower cost. Note that the expression in (EC.33) for optimal payments is independent of the desired policy Ξ_n; in fact, the optimal incentive structure renders the agent indifferent between all policies that satisfy $\Psi_n \subseteq \Xi_n$.

Case 2. Consider the contract parameters $s_n^R < s_n^0$. Then we also need $s_n^A > s_n^0$, otherwise the contract would induce $\omega_n = 0$. As noted earlier, an optimal strategy for the agent would then be to report everything that the principal will accept and nothing else; this is a situation which can be solved analogously to Case 1 and yields the same candidate solution for s_n^A and s_n^0 as in (EC.33), while s_R^0 can be any $s_R^0 < s_n^0$. Another difference from Case 1 is that here we can only implement $\Xi_n = \Psi_n$. This similarity of solutions between

Case 1 and 2 is driven by insensitivity to parameter s_n^R when $s_n^R < s_n^0$, as under any induced policy reporting something that the principal will reject is a zero probability event.

Case 3. Consider the contract parameters $s_n^A = s_n^0$; then we also need $s_n^R > s_n^A$, lest we induce $\omega_n = 0$. We can solve this analogously to Case 1, which is possible if the implementability condition $pP(\operatorname{supp} F \backslash \Psi_n) > 1 - 1/\exp[rc(1-\alpha)\alpha^{-1}]$ holds, and yields

$$s_n^A = \frac{w+c}{\alpha} + \frac{1}{r(1-\alpha)} \ln \frac{pP(\operatorname{supp} F \backslash \Psi_n)}{1-(1-pP(\operatorname{supp} F \backslash \Psi_n))\exp[rc(1-\alpha)\alpha^{-1}]}, \qquad s_n^0 = s_n^R = \frac{w}{\alpha}. \tag{EC.34}$$

This is a candidate solution to problem (EC.26)–(EC.29) for any Ξ_n that satisfies $\Xi_n \subseteq \operatorname{supp} F \backslash \Psi_n$. Note that the contract from this case will never be a part of an optimal contract, as it generates an acceptable outcome for the principal with probability 0, but is more expensive to implement than $\omega_n = 0$.

Case 4. Consider the contract parameters $s_n^A < s_n^0$; then we also need $s_n^R > s_n^A$, lest we induce $\omega_n = 0$. Solving it (analogously to Case 1) yields s_n^R and s_n^0 given by (EC.34) as the candidate solution (s^A can be anything as long as $s_n^A < s_n^0$), and can only implement reporting policies such that $\Xi_n = \operatorname{supp} F \backslash \Psi_n$. As in Case 3, the contract from this case will never be a part of an optimal contract, as it generates an acceptable outcome for the principal with probability 0, but is more expensive to implement than $\omega_n = 0$.

Case 5. Consider the contract parameters $s_n^A, s_n^R > s_n^0$. This contract will induce the agent to report *all* found alternatives to the principal. Using the same Jensen's inequality argument as in Theorem 1, the contract then also has to satisfy $s_n^A = s_n^R$; otherwise, it would impose an additional source of risk on the risk-averse agent, requiring the principal to pay an increased risk premium in expectation. This is also solvable analogously to Case 1, which is possible if the implementability condition $p > 1 - 1/\exp[rc(1-\alpha)\alpha^{-1}]$ holds, and yields

$$s_n^A = s_n^R = \frac{w+c}{\alpha} + \frac{1}{r(1-\alpha)} \ln \frac{p}{1-(1-p)\exp[rc(1-\alpha)\alpha^{-1}]} \quad \text{and} \quad s_n^0 = \frac{w}{\alpha}. \tag{EC.35}$$

This is the solution to problem (EC.26)–(EC.29) for $\Xi_n = \operatorname{supp} F$—even though the contract from Case 1 can also implement this reporting policy—as it can be observed from inserting (EC.35) and (EC.33) into (EC.26) that (EC.35) implements it for a lower cost.

Finally, the renegotiation-proof nature of this contract in all cases follows from Corollary 1 of Plambeck and Zenios (2000), and the statement of the proposition follows from combining the exhaustive cases. □

Lemma A3. *If Y^n is the sole contractible variable and the principal commits to an acceptance policy, we can restrict attention to history-independent policies without loss of optimality and under an optimal contract, threshold reporting and accepting policies are optimal, so $\Psi_n^{\mathrm{US}*} = [\psi_n^{\mathrm{US}*}, \bar{u}] \cap \operatorname{supp} F$ and $\Xi_n^{\mathrm{US}*} = [\xi_n^{\mathrm{US}*}, \bar{u}] \cap \operatorname{supp} F$, for all $n \in \{1,..,N\}$. The optimal contract implements either $\xi_n^{\mathrm{US}*} = \psi_n^{\mathrm{US}*}$ or $\xi^{\mathrm{US}*} = \min \operatorname{supp} F$ for all $n \in \{1,..,N\}$. The expected single-period costs incurred by the principal when implementing reporting policy $\xi_n^{\mathrm{US}*}$ are $z(\xi_n^{\mathrm{US}*})$ as given by (14).*

Proof. *Part 1: The agent should report either all alternatives or all acceptable ones.* If $\omega_n = 0$, the choice of threshold and reporting policies is redundant (any policy is optimal), so consider periods where $\omega_n = 1$. Notice that for any implementable Ξ_n other than $\Xi_n = \operatorname{supp} F$, the amount paid to the agent is the same (from Proposition A1), as is the principal's upside (she will accept any alternative with a value in Ψ_n), but the principal's evaluation costs are incurred every time an alternative is delivered to her and are thus increasing in $P(\Xi_n)$. Consequently, under an optimal contract, it has to be that $\Xi_n^{\mathrm{US}*} = \Psi_n^{\mathrm{US}*}$ or $\Xi_n^{\mathrm{US}*} = \operatorname{supp} F$, and similarly to the proof of Lemma 1, the principal's expected single-period costs to induce the agent to search and use reporting policy Ξ_n^{US} are given by $\zeta(\Xi_n^{\mathrm{US}})$ as defined in (EC.14).

Part 2: Optimality of threshold policies. Assume, by contradiction, there exists an optimal $\Psi_n^{\mathrm{US}*}$ that is not a threshold one in a period where $\omega_n^{US} = 1$, and let $\psi_n^{\mathrm{US}*}$ be such that $P([\psi_n^{\mathrm{US}*}, \bar{u}] \cap \operatorname{supp} F) = P(\Psi_n^{\mathrm{US}*})$ (the existence of such $\psi_n^{\mathrm{US}*}$ is guaranteed by the probability mass function μ being constant). Then, from (EC.14), the costs to the principal will be the same under both policies, acceptable alternatives will be delivered to the principal with the same probability, but the value of delivered alternatives will be better (in terms of weak first-order stochastic dominance) under $[\psi_n^{\mathrm{US}*}, \bar{u}] \cap \operatorname{supp} F$. Thus, using $[\psi_n^{\mathrm{US}*}, \bar{u}] \cap \operatorname{supp} F$ gives a higher expectation to the principal, in contradiction to the optimality of $\Psi_n^{\mathrm{US}*}$, so optimality of threshold acceptance policies follows. Then, the optimality of threshold reporting policies follows from part 1 of this proof. Thus, either $\Xi_n^{US} = \Psi_n^{US}$ is optimal or $\Xi_n^{US} = \operatorname{supp} F$ is. Lastly, the principal cost function being given by (14) follows by inserting the optimal threshold policies into (EC.14).

Part 3: Optimality of History-Independent Policies. Proposition A1 reduces the contracting problem to an MDP in which the principal's choice is over the policies $\{\Xi_n\}, \{\Psi_n\}, \{\omega_n\}$. Applying part 1 of this proof to (4), the resulting MDP can be expressed as

$$V_n^{\mathrm{US}} = \max_{\omega_n \in \{0,1\}, \Psi_n \in \mathcal{F}, \Xi_n \in \{\Psi_n, \operatorname{supp} F\}} \left(\omega_n p \int_{\Psi_n} (x - \alpha V_{n+1}^{\mathrm{US}})\, dF(x) - \omega_n \zeta(\Xi_n) - (1-\omega_n) w + \alpha V_{n+1}^{\mathrm{US}} \right),$$

with boundary condition $V_{N+1}^{\mathrm{US}} = 0$. Here, the objective function is independent of the state history, including the current state, thus, so are the optimal choices of ω_n, Ξ_n, and Ψ_n. $\square$

Proof of Theorem 4.

By Lemma A3, history-independent threshold policies are optimal, and in every period, it is optimal for the principal to induce a reporting policy that is equal to the announced acceptance policy ($\xi_n = \psi_n^{\mathrm{US}}$) or to induce $\xi_n = \operatorname{supp} F$. We can therefore express the principal's problem as the dynamic program

$$V_n^{\mathrm{US}} = \max_{\omega_n \in \{0,1\}, \psi_n \in \operatorname{supp} F} \left(\omega_n p \int_{\psi_n}^{\bar{u}} (x - \alpha V_{n+1}^{\mathrm{US}})\, dF(x) - \omega_n \min\{z(\psi_n), z(\min \operatorname{supp} F)\} - (1-\omega_n) w + \alpha V_{n+1}^{\mathrm{US}} \right),$$

with boundary condition $V_{N+1}^{\mathrm{US}} = 0$. Here, the optimality of setting $\omega_n = 1$ in every period follows analogously to the same statement in the proof of Theorem 3, simplifying the equation to

$$V_n^{\mathrm{US}} = \max_{\psi_n \in \operatorname{supp} F} \left(p \int_{\psi_n}^{\bar{u}} (x - \alpha V_{n+1}^{\mathrm{US}})\, dF(x) - \min\{z(\psi_n), z(\min \operatorname{supp} F)\} + \alpha V_{n+1}^{\mathrm{US}} \right). \tag{EC.36}$$

Note that this reduces the problem to a univariate one: the choice of acceptance policy. Denoting $\mathrm{ob}(\psi, n) := p \int_{\psi}^{\bar{u}} \left(x - \alpha V_{n+1}^{\mathrm{US}}\right) dF(x) - \min\{z(\psi), z(\min \operatorname{supp} F)\} + \alpha V_{n+1}^{\mathrm{US}}$, it is easily verifiable that $\mathrm{ob}(\psi, n)$ is submodular, thus by Topkis's theorem, the optimal $\{\psi_n^{\mathrm{US}}\}$ is weakly decreasing in n. From the shape of the cost function z given in the proof of Lemma 1 (it is quasi-convex with a minimum at ξ^{LC}, such that it is strictly decreasing on $[\underline{u}, \xi^{\mathrm{LC}}] \cap \operatorname{supp} F$ and strictly increasing and unbounded for $\xi_n > \xi^{\mathrm{LC}}$), it follows that $\exists \psi^{\mathrm{crit}} \in \operatorname{supp} F$ s. t. $\psi^{\mathrm{crit}} \geq \xi^{\mathrm{LC}}, z(\psi^{\mathrm{crit}}) \geq z(\min \operatorname{supp} F)$ and $\forall \psi \in \operatorname{supp} F$ such that $\psi > \psi^{\mathrm{crit}} : z(\psi) >$

$z(\min \operatorname{supp} F)$, and $\forall \underline{\psi} \in \operatorname{supp} F$ s. t. $\underline{\psi} < \psi^{\text{crit}} : z(\psi) < z(\min \operatorname{supp} F)$. Thus, there exists $0 \leq K \leq N$ such that for K periods, the principal induces policy $\xi_n = \min \sup F$ in the agent, while in the remaining periods, she induces ψ_n^{US}. Parts (i) and (ii) of the theorem then follow from Proposition A1, which gives the optimal contracts for implementing those policies. □

Proof of Proposition 3.

Consider the setting where the agent always reports truthfully, i.e., $\mathcal{R}_n(u) = u, \forall u, n$. This eliminates the adverse selection dimension of the contracting problem, but the moral hazard part remains. It is then optimal for the principal to never evaluate the agent's reports ($\phi_n(x) = 0, \forall x, n$), and from Theorem 1 of Plambeck and Zenios (2000) it follows that the problem of finding a contract that implements any admissible search policy $\{\omega_n\}$ at the lowest possible cost to the principal can be decomposed into N single-period problems, each given by

$$\operatorname*{arg\,min}_{s_n^A(u), s_n^R(u), s_n^0} \quad p\omega_n \sum_{u \in \Psi_n} \big(s_n^A(u)\mu(u)\big) + p\omega_n \sum_{u \in \operatorname{supp} F \setminus \Psi_n} \big(s_n^R(u)\mu(u)\big) + (1 - p\omega_n)\, s_n^0; \tag{EC.37}$$

$$-p\omega_n \sum_{u \in \Psi_n} \big(\exp[-r(1-\alpha)(s_n^A(u) - \alpha^{-1}c)]\mu(u)\big) - p\omega_n \sum_{u \in \operatorname{supp} F \setminus \Psi_n} \big(\exp[-r(1-\alpha)(s_n^R(u) - \alpha^{-1}c)]\mu(u)\big) \\ - (1 - p\omega_n)\exp[-r(1-\alpha)(s_n^0 - \alpha^{-1}c)] = -\exp[-r\alpha^{-1}(1-\alpha)w]; \tag{EC.38}$$

$$-\exp[-r(1-\alpha)s_n^0] \leq -\exp[-r\alpha^{-1}(1-\alpha)w]. \tag{EC.39}$$

Notice that the system above is equivalent to the system (EC.1),(EC.2),(EC.4) (solved as part of the proof for Theorem 1) when $\Xi_n = \operatorname{supp} F$. Thus, the solution is also the same: for $\omega_n = 1$ the solution exists only if $p > 1 + v(c/\alpha)$ and it is to pay the agent a bonus $s_n^B(\operatorname{supp} F)$ as given by (11) when any alternative is reported, or $s_n^0 = w/\alpha$ when agent reports that no alternative was found. For $\omega_n = 0$, the solution is to pay the agent w/α irrespective of the report. By Theorem 3, the same contract is optimal in the Section 2 model when $e = 0$, and the principal's value function in that setting is equal to V_n^T.

Now we will demonstrate how to achieve the same with an agent which is not necessarily truthful. Consider a principal who conducts evaluations with frequency ϕ. Applying Theorem 1 of Plambeck and Zenios (2000) it follows that the problem of finding a contract that implements any admissible search policy $\{\omega_n\}$ and a truthful reporting policy ($\mathcal{R}_n(u) = u, \forall u, n$) at the lowest possible cost to the principal can be decomposed into N single-period problems, each given by

$$\operatorname*{arg\,min}_{s_n(x,u)} \left[(1 - \omega_n p)\Big(\phi s_n(\emptyset,\emptyset) + (1-\phi)s_n(\emptyset,\text{NE})\Big) + \omega_n p \sum_{u \in \operatorname{supp} F} \Big(\phi s_n(u,u) + (1-\phi)s_n(u,\text{NE})\Big)\mu(u)\right]; \tag{EC.40}$$

$$(1 - \omega_n p)\Big(\phi v(s_n(\emptyset,\emptyset)) + (1-\phi)v(s_n(\emptyset,\text{NE}))\Big) + \omega_n p \sum_{u \in \operatorname{supp} F} \Big(\phi v(s_n(u,u)) + (1-\phi)v(s_n(u,\text{NE}))\Big)\mu(u) \\ = -v(w/\alpha)/v(-c/\alpha); \tag{EC.41}$$

$$p \sum_{u \in \operatorname{supp} F} \Big(\phi v(s_n(\mathcal{R}(u),u)) + (1-\phi)v(s_n(\mathcal{R}(u),\text{NE}))\Big)\mu(u) \\ + (1-p)\Big(\phi v(s_n(\mathcal{R}(\emptyset),\emptyset)) + (1-\phi)v(s_n(\mathcal{R}(\emptyset),\text{NE}))\Big) \\ \leq -v(w/\alpha)/v(-c/\alpha) \quad \forall \mathcal{R} : \operatorname{supp} F \cup \{\emptyset\} \to \operatorname{supp} F \cup \{\emptyset\}; \tag{EC.42}$$

$$\phi v(s_n(\emptyset,\emptyset)) + (1-\phi)v(s_n(\emptyset,\text{NE})) \leq v(w/\alpha). \tag{EC.43}$$

Here, $s_n(x,u)$ denotes the payment to the agent in the scenario where he reports that an alternative with value x was found (or that no alternative was found, in which case $x=\emptyset$), and the principal's evaluation finds the value of the alternative to be u ($u=\mathrm{NE}$ if the principal did not conduct an evaluation). Note that the formulation (EC.40)–(EC.43) does not allow for payment to differ based on the principal's decision to accept or reject the alternative. This restriction is made without loss of optimality, which follows from the same Jensen's inequality argument as in the proof of Theorem 1. (Intuition: doing otherwise would impose an additional source of risk on the agent which is outside his control, necessitating a higher payment in expectation.) The objective function (EC.40) minimizes the principal's expected costs, subject to the binding participation constraint (EC.41), and the incentive compatibility constraints (EC.42)–(EC.43). For $\omega_n=0$, (EC.40)–(EC.43) has a trivial solution of $s(x,u)=w/\alpha, \forall x,u$.

For $\omega_n=1$, consider a contract that pays $s_n(u,u)=s_n(u,\mathrm{NE})=s_n^B(\operatorname{supp}F), \forall u\in\operatorname{supp}F$ (this is the bonus s_n^E given in part (i) of the proposition) and $s_n(\emptyset,\emptyset)=s_n(\emptyset,\mathrm{NE})=w/\alpha$ (this is the fixed pay given in part (iii) of the proposition), where $s_n^B(\operatorname{supp}F)$ is given by (11). Notice that this contract solves the problem (EC.40),(EC.41),(EC.43), irrespective of how $s_n(x,u)$ are defined in the case where $x\neq u$ and $u\neq\mathrm{NE}$. This is essentially the same solution as for (EC.37)–(EC.39), which ensures that the agent is incentivized to search if he is truthful, but not necessarily that truthfulness is incentive compatible, as it ignores (EC.42).

We will now show that this contract can be made incentive compatible, so also to satisfy (EC.42), by properly defining $s_n(x,u)$ in the case where $x\neq u$. Consider the point-wise incentive compatibility constraint

$$\phi v(s_n(u,u))+(1-\phi)v(s_n(u,\mathrm{NE}))\geq\phi v(s_n(x,u))+(1-\phi)v(s_n(x,\mathrm{NE})),\quad \forall x,u\in\operatorname{supp}F\cup\emptyset,\ \text{s.t. } x\neq u. \tag{EC.44}$$

This constraint ensures that for every possible u, the agent gets higher expected utility by honestly reporting it (the LHS of (EC.44)), than by making any dishonest report x (the RHS). Notice that the condition (EC.44) implies (EC.42). Setting $s_n(x,u)=s_n^P, \forall x\neq u, u\neq\mathrm{NE}$, where s_n^P is given by (22), ensures that (EC.44) holds. This is the reduced pay if caught making dishonest reports, as given in part (ii) of the proposition. Because (EC.44) is satisfied, this contract solves (EC.40)–(EC.43) and is thus optimal.

Finally, the identity $\lim_{\phi\to 0}V_n^{\mathrm{ARV}}(x;\phi)=V_n^T$ follows from comparing the solution of (EC.37)–(EC.39) to the one of (EC.40)–(EC.43) and noticing that these contracts induce the same actions and pay the same amount to the agent; yet in the second of these two cases, the principal bears an additional cost from random evaluations, but this cost vanishes as $\phi\to 0$. $\square$

Lemma A4. *Let the agent hold contract $\{s_n(x)\}\in\mathscr{F}\cup\mathscr{P}$ and the principal use threshold acceptance policy $\{\psi_n\}$. Then, it is optimal for the agent to use policies $\{\tilde\gamma_n\}$, $\{\tilde\omega_n\}$, $\{\tilde\xi_n\}$ which solve the recursion:*

$$\begin{aligned}U_n^A(\{s_n(x)\},\{\psi_n\})=&\max_{\gamma_n,\mathbb{R},\xi_n\in\{\xi\in\operatorname{supp}F|\xi\geq\psi_n\},\omega_n\in\{0,1\}}-\exp[-r\gamma_n]\\&+\omega_n p\alpha\int_{\xi_n}^{\bar u}\exp[-r(1-\alpha)(s_n(x)-(\gamma_n+c)/\alpha)]dF(x)\,U_{N+1}^A(\{s_n(x)\},\{\psi_n\})\\&+(1-\omega_n p\bar F(\xi_n))\alpha\exp[-r(1-\alpha)(-(\gamma_n+\omega_n c)/\alpha)]U_{n+1}^A(\{s_n(x)\},\{\psi_n\})\end{aligned}$$

with boundary condition

$$U_{N+1}^A(\{s_n(x)\},\{\psi_n\})=-\frac{1}{1-\alpha}\exp(-rw).$$

If $\{s_n(x)\}\in\mathscr{F}$, the agent reports only those alternatives that the principal is willing to accept ($\{\tilde\xi_n\}=\{\psi_n\}$).

Proof. Follows directly from applying Lemma 1 of Plambeck and Zenios (2000) to the agent's value function (5). $\square$

Lemma A5. *Let $\mathscr{C} \in \{\mathscr{F}, \mathscr{P}\}$ and $\{s_n(x)\} \in \mathscr{C}$. It will then be optimal for the principal to use a threshold acceptance policy $\psi_n(\{s_n(x)\}) = \ulcorner \alpha V^P_{n+1}(\{s_n(x)\}) + \alpha s \urcorner$ if $\mathscr{C} = \mathscr{F}$, or $\psi_n(\{s_n(x)\}) = \ulcorner \alpha V^P_{n+1}(\{s_n(x)\})/(1-\alpha q) \urcorner$ if $\mathscr{C} = \mathscr{P}$. Here, $V^P_n(\{s_n(x)\})$ is the principal's value function. Denoting by $\{\tilde{\omega}_n\}$ the agent's optimal search policy, as given by Lemma A4, $V^P_n(\{s_n(x)\})$ is given recursively by*

$$V^P_n(\{s_n(x)\}) = \alpha V^P_{n+1}(\{s_n(x)\}) + p\tilde{\omega}_n \int_{\psi_n(\{s_n(x)\})}^{\bar{u}} \left(x - \alpha s_n(x) - e - \alpha V^P_{n+1}(\{s_n(x)\})\right) dF(x),$$

with boundary condition $V^P_{N+1}(\{s_n(x)\}) = 0$.[4] *The optimal contract then solves*

$$\max_{\{s_n(x)\} \in \mathscr{C}} V^P_1(\{s_n(x)\}) \tag{EC.45}$$

$$s.t. \quad U^A_0(\{s_n(x)\}, \{\alpha V^P_{n+1}(\{s_n(x)\})\}) \geq -\frac{1}{1-\alpha}\exp(-rw). \tag{EC.46}$$

This optimization reduces to optimizing over $s \in \mathbb{R}$ if $\mathscr{C} = \mathscr{F}$ or optimizing over $q \in [0,1]$ if $\mathscr{C} = \mathscr{P}$.

Proof. Let $\{s_n(x)\} \in \mathscr{C}$ (resp. $\{s_n(x)\} \in \mathscr{F}$). Each time the principal is presented with an alternative of value x, the evaluation cost is already sunk when she finds out x, which leaves the principal with two options: a) accept the alternative, receiving its value x, but paying a bonus s (resp. qx) at the beginning of the next period, or b) reject it, in which case the search proceeds, which will yield $\alpha V^P_n(\{s_n(x)\})$ in expectation. Optimality of threshold acceptance policy $\psi_n(\{s_n(x)\}) = \ulcorner \alpha V^P_{n+1}(\{s_n(x)\}) + \alpha s \urcorner, \forall n$ (resp. $\psi_n(\{s_n(x)\}) = \ulcorner \alpha V^P_{n+1}(\{s_n(x)\})/(1-\alpha q) \urcorner, \forall n$) directly follows. The expression for the principal's value function then follows by inserting the optimal $\{\psi_n\} = \{\xi_n\}$ into (4). The system (EC.45)–(EC.46) is then a simplified version of the contracting problem (6)–(9), which follows from inserting the expressions for optimal policies for any given $\{s_n\}$ (as given by Lemma A4 and this proof) and from applying Lemma 1 of Plambeck and Zenios (2000) to replace the agent's participation constraint by its wealth-independent component. $\square$

Proof of Proposition 4

Part 1: $V^{\mathscr{F}}_n$, $V^{\mathscr{P}}_n \leq V^{US}_n$. Consider two different principals, each with her own agent. The first principal's agent holds $\{s_n(x)\} \in \mathscr{F} \cup \mathscr{P}$. Denote by $\{\tilde{\psi}_n\}$ the first principal's optimal acceptance policy and by V^1_n her value function (both given by Lemma A5). Also, denote the optimal searching and reporting policy for this principal's agent by $\{\tilde{\omega}_n\}$ and $\{\tilde{\xi}_n\}$ (both given by Lemma A4).

The second principal commits to the same acceptance policy $\{\tilde{\psi}_n\}$ and desires to induce the agent to use the same reporting and search policies $\{\tilde{\omega}_n\}$, $\{\tilde{\xi}_n\}$ but can optimize over which contract $\{s_n(Y^n)\}$ to use, without being bound to a particular contract class (thus, the second principal is in the setting of Section 4.1). Then, denoting the second principal's value function by V^2_n, it follows from Proposition A1 that if $\{s_n(x)\} \in \mathscr{F}$, then $V^2_n \geq V^1_n, \forall n$ as in any single period, the contract of Proposition A1 is the most cost-efficient way of inducing those policies, optimized over the space of all contracts, including the ones in $\mathscr{F}$.

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[4] Note that $\tilde{\omega}_n$ here depends on $\{\psi_n\}$ and $\{s_n(x)\}$, thus this recursion needs to be solved jointly with the agent's problem in Lemma A4.

However, the percentage-bonus contracts of $\mathscr{P}$ require contracting on the value of the alternative, which the second principal is not able to do (as Y^n is his sole contractable variable, and it does not include that information). Thus, to show $V_n^2 \geq V_n^1$ in the case where $\{s_n(x)\} \in \mathscr{P}$, an additional step is needed: to construct a contract $\{s_n(Y^n)\}$ (in the setting of Proposition A1) which induces $\{\tilde{\omega}_n\}$ and $\{\tilde{\xi}_n\}$ (so the same policies as $\{s_n(x)\} \in \mathscr{P}$ with percentage bonus q), and yields at least V_n^1 to the principal. To do that, let the principal instead commit to acceptance policy $\{\tilde{\psi}_n^*\}$ given by $\tilde{\psi}_n^* = \max\{\tilde{\psi}_n, \tilde{\xi}_n\}, \forall n \in \{1, ..., N\}$, and use a contract in $\{s_n^*(x)\} \in \mathscr{F}$ with fixed bonus size s^* that solves

$$\int_{\tilde{\psi}_n^*}^{\bar{u}} \exp[-r(1-\alpha)qx]dF(x) = \int_{\tilde{\psi}_n^*}^{\bar{u}} \exp[-r(1-\alpha)s^*]dF(x). \tag{EC.47}$$

Solving it yields $s^* = -\frac{1}{r(1-\alpha)} \ln\left(\int_{\tilde{\psi}_n^*}^{\bar{u}} \exp[-r(1-\alpha)qx]dF(x)/\bar{F}(\tilde{\psi}_n^*)\right)$. Then, applying Lemma A4, because of (EC.47), $\{s_n^*(x)\}$ also induces $\{\tilde{\omega}_n\}$ and $\{\tilde{\xi}_n\}$ and yields the same value to the agent as $\{s_n(x)\}$. However the expected cost for the principal in period n under $\{s_n(x)\}$ is $\tilde{\omega}_n p \int_{\tilde{\psi}_n^*}^{\bar{u}} qx dF(x)$ while under $\{s_n^*(x)\}$ it is $\tilde{\omega}_n p s^* \bar{F}(\tilde{\psi}_n^*)$. Here, from concavity of $\exp[-r(1-\alpha)x]$ in x and (EC.47) we have that $\int_{\tilde{\psi}_n^*}^{\bar{u}} qx dF(x) \geq s^* \bar{F}(\tilde{\psi}_n^*)$ (as for every concave u, random variable X, and constant y, it holds that $\mathbb{E}[u(X)] = u(y) \Rightarrow \mathbb{E}[X] \geq y$), Thus, denoting the principal's value function under $\{s_n^*(x)\}$ by V_n^{1*}, we have that $V_n^2 \geq V_n^{1*} \geq V_n^1, \forall n$.[5]

Lastly, because $V_n^2 \geq V_n^1, \forall n$ holds in both cases, that the truly optimal contract yields an even higher expectation ($V_n^{US} \geq V_n^2 \geq V_n^1, \forall n$) follows from Bellman's principle of optimality.

Part 2: $V_n^P - V_n^{\mathscr{F}}$ is unbounded. The idea of the proof is to construct a situation in which no contract in $\mathscr{F}$ can create positive value for the principal, but the optimal contract can generate unboundedly high value. Consider a single period model ($N = 1$) where $\underline{u} = \min \operatorname{supp} F = 0$, $\max \operatorname{supp} F < \bar{u}$, $p = 1$, $e = 0$, and $w = 0$. Under a contract that pays a fixed bonus s, the principal will use threshold acceptance policy $\psi_1 = \Rsh \alpha s \Lsh$, and the agent will use reporting policy $\xi_1 = \Rsh \alpha s \Lsh$ (by Lemmas A5 and A4, respectively). As there is no opportunity cost, this contract will always satisfy the participation constraint. Using Proposition 5 of Plambeck and Zenios (2000), we can more concisely formulate the agent's search decision: he should search if and only if the following inequality holds

$$-\bar{F}(\alpha s) \exp(-r(1-\alpha)(s - c/\alpha)) - (1 - \bar{F}(\alpha s)) \exp(-r(1-\alpha)(-c/\alpha)) \geq -1. \tag{EC.48}$$

Here the first term is the agent's instantaneous utility of paying search cost c but then finding an alternative which results in bonus s, the second term is the scenario where the search cost is expanded, but no suitable alternative is found, and the RHS is the utility from not searching. Rearranging (EC.48) yields

$$\bar{F}(\alpha s) \exp(-r(1-\alpha)s) \leq -(1 - \bar{F}(\alpha s)) + \frac{1}{\exp(-r(1-\alpha)(-c/\alpha))}.$$

Here, the LHS is positive, so the RHS also needs to be positive, which yields the following necessary (but not sufficient) condition for the contract to induce the agent to search:

$$1 - \bar{F}(\alpha s) \leq \frac{1}{\exp(-r(1-\alpha)(-c/\alpha))}. \tag{EC.49}$$

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[5] From this construction, it may appear that the best contract in $\mathscr{F}$ outperforms all contracts in $\mathscr{P}$ (so $V_n^{\mathscr{F}} \geq V_n^{\mathscr{P}}$). However, this is not necessarily true (a counterexample is provided in Figure 4), as attaining V_n^{1*} also requires commitment to an acceptance policy whereas attaining $V_n^{\mathscr{F}}$ does not.

Define $c^* := \max\{c \in \mathbb{R}^+ | 1 - \bar{F}(\alpha c) \leq 1/\exp(-r(1-\alpha)(-c/\alpha))\}$. The existence and uniqueness of c^* are guaranteed by $1-\bar{F}(\alpha c)$ being an weakly increasing left-continuous function with $1-\bar{F}(0)=0$ and $1-\bar{F}(\bar{u}) = 1$, while $1/\exp(-r(1-\alpha)(-c/\alpha))$ is strictly decreasing, continuous, and evaluates to 1 at $c=0$. Let the agent's costs be c^*. Then, setting the bonus to any amount $s > c^*$ will violate the condition (EC.49) and thus induce the agent not to search, but setting $s \leq c^*$ will trivially violate (EC.48) (bonus is not enough to recoup even the cost of search), thus also induce the agent not to search. Consequently, $V_n^{\mathscr{F}} = 0$.

Now consider $F_y(\cdot)$ such that $F_y(u) = F(u), \forall u \leq c^*$ and the expectation of drawing from that distribution is $y > c^*$. Notice that the optimal contract s^* given by Theorem 3 remains the same if the agent is drawing from F_y instead of F, and the $V_n^{\mathscr{F}} = 0$ property is preserved. Applying Theorem 3, the optimal contract, in this case, is to pay the agent c^*/α for any alternative *delivered*, rather than only accepted alternatives. Using this contract gives the principal expectation of $y - c^*$, which can be unboundedly high as y increases.

Part 3: $V_n^P - V_n^{\mathscr{P}}$ is unbounded. As in part 2, we aim to construct a situation in which no contract in $\mathscr{P}$ can create positive value for the principal, but the optimal contract can generate unboundedly high value. Consider a single period model ($N=1$) where $\underline{u} = \min \operatorname{supp} F = 0$, $p=1$, $e=0$, and $w=0$. Applying Lemmas A4 and A5, under any contract in $\mathscr{P}$ we have $\xi_1 = \psi_1 = 0$. Analogously to (EC.48), it will be optimal for the agent holding such a contract to search if and only if the following equation holds:

$$-\int_0^{\bar{u}} \exp(-r(1-\alpha)(qu - c/\alpha)) dF(u) \geq -1.$$

Here the LHS is the instantaneous utility from searching, while the RHS is the utility of not searching. Now consider a contract $s_b(u)$, which pays the agent $c/(2\alpha)$ for delivery of any alternative with a value below $c/2$, and pays $b > \bar{u}$ for delivery of alternatives with a value at least equal to $c/2$. Note that this contract does not belong to $\mathscr{P}$, but is preferred by the agent over any contract in $\mathscr{P}$ as the distribution of payouts to the agent when searching will dominate the distribution of payouts under all contracts in $\mathscr{P}$ by first-order stochastic dominance. Under $\hat{s}(u)$, the agent should search if and only if

$$-\bar{F}(c/2)\exp(-r(1-\alpha)(b - c/\alpha)) - (1-\bar{F}(c/2))\exp(-r(1-\alpha)(-c/(2\alpha))) \geq -1.$$

The utility on the LHS is increasing in b, and we can find its supremum by considering $\lim_{b\to\infty}$ which yields

$$-(1-\bar{F}(c/2))\exp(-r(1-\alpha)(-c/(2\alpha))) \geq -1,$$

$$1 - \bar{F}(c/2) \leq 1/\exp(r(1-\alpha)c/(2\alpha)). \tag{EC.50}$$

This condition is of interest to us, as the agent not being incentivized to search here will also imply that no contract in $\mathscr{P}$ will incentivize him to search either. Because the RHS of (EC.50) is < 1, there are many distributions for which this condition is not satisfied. So, let $F(\cdot)$ be one such distribution for which $1 - \bar{F}(c/2) > 1/\exp(r(1-\alpha)c/(2\alpha))$. Then, no matter the value of b, $s_b(u)$ will incentivize the agent not to search; no contract in $\mathscr{P}$ will incentivize him to search either; and thus, we have $V_n^{\mathscr{P}} = 0$.

The intuition for why this occurs is as follows: the scenarios where $u < c/2$ create a loss (and the agent is loss averse) as the payment is not enough to recoup cost c, the situations where $u \geq c/2$ create a gain, but this gain is bounded as exponential utility is bounded on the right. Consequently, if the unfavorable scenario is common enough and the favorable scenario is rare enough, there is no payment for the favorable scenario that is high enough to make this an attractive prospect. (The same phenomenon is the driver of implementability condition (10) in Theorem 1.)

Now consider a distribution $F_y(\cdot)$ such that $F_y(u) = F(u), \forall u \leq c/2$ and the expectation of drawing from that distribution is $y > c/2$. Notice that if (EC.50) does not hold for $F(\cdot)$, then it does not hold for $F_y(\cdot)$ either. Applying Theorem 3, the optimal contract, in this case, is to pay the agent c/α for any alternative *delivered*, rather than only accepted alternatives. Using the optimal contract yields the principal expectation $y - c$, which can be unboundedly high as y increases. Thus, when the distribution is $F_y(u)$, we have that V_n^P can be unboundedly high (as $y \to \infty$), but $V_n^{\mathscr{P}} = 0$. $\square$

Appendix C: Additional results

C.1. Discretization techniques

Here, we illustrate bracket-mean and bracket-median, which are two commonly used discrete approximations of continuous distributions (Miller III and Rice 1983, Smith 1993); these algorithms are also used in computer science, where they are referred to as equal-frequency binning (Kotsiantis and Kanellopoulos 2006). Denote by $G(\cdot)$ a continuous probability distribution with support $[\underline{u}, \bar{u}]$, which we will approximate with a discrete distribution $F_n(\cdot)$. The bracket-mean technique starts with dividing $G(\cdot)$ into n equally probable intervals. Denoting $x_0^n := \underline{u}$, $x_n^n := \bar{u}$, the separation into intervals consists of finding points $x_1^n, x_2^n, ..., x_{n-1}^n$ such that $G(x_i^n) - G(x_{i-1}^n) = 1/n$, for all $i \in \{1, ..., n\}$. The discrete approximation $F_n(\cdot)$ is then given as the one with n elements in its support, each being equal to the mean of one of the n just-defined subintervals of $G(x)$, and each having a probability mass of $1/n$. Formally, for a random variable X distributed according to $G(\cdot)$, for every $i \in \{1, ..., n\}$, let $y_i^n := \mathbb{E}(X | x_{i-1}^n \leq X \leq x_i^n)$; the discrete approximation $F_n(\cdot)$ is then defined by its support $\{y_1^n, ..., y_n^n\}$ and a constant pmf $f_n(x) := 1/n$.

We can also define the discretization procedure via the discretization function $D_n : [\underline{u}, \bar{u}] \to [\underline{u}, \bar{u}]$, given by

$$D_n(x) := \begin{cases} y_1^n & \text{if } x \in [x_0^n, x_1^n) \\ y_2^n & \text{if } x \in [x_1^n, x_2^n) \\ \cdots & \quad \cdots \\ y_n^n & \text{if } x \in [x_{n-1}^n, x_n^n]. \end{cases}$$

The bracket-median technique is almost identical, differing only in the way y_i are defined: in bracket-median, they are medians (rather than means) of each of the n intervals. Importantly for our purposes, both of these approximation techniques yield discrete approximations that satisfy our assumption that all elements in the support of the distribution of alternatives have the same probability mass. In the lemma below, we show that for both these approximation techniques, the approximation can be made arbitrarily close to the true distribution by increasing n.

Lemma A6 (Convergence of approximations) *Let $\{F_n\}$ be a sequence of increasingly fine approximations of a G obtained through either bracket-mean or bracket-median. This sequence converges in distribution to G: $\lim_{n\to\infty} F_n(x) = G(x), \forall x \in [\underline{u}, \bar{u}]$. Let X be a random variable with cdf G. Then, $D_n(X)$ has cdf F_n and the sequence $\{D_n(X)\}$ converges to X in the L^1 norm: $\lim_{n\to\infty} \mathbb{E}[|D_n(X) - X|] = 0$.*

Proof Sketch. Denote $x^n_{y\downarrow} := \max\{x \in \{x^n_1, x^n_2, ..., x^n_n\} | x \le y\}$ and $x^n_{y\uparrow} := \min\{x \in \{x^n_1, x^n_2, ..., x^n_n\} | x \ge y\}$. From the construction of F_n, we have that $G(x^n_{y\downarrow}) \le F_n(y) \le G(x^n_{y\uparrow})$ and $G(x^n_{y\downarrow}) \le G(y) \le G(x^n_{y\uparrow}), \forall y \in [\underline{u}, \bar{u}]$. Then, $\lim_{n\to\infty} G(x^n_{y\uparrow}) - G(x^n_{y\downarrow}) = \lim_{n\to\infty} 1/n = 0$ so $\lim_{n\to\infty} F_n(y) = G(y)$, thus convergence in distributions. To show L^1 convergence, we have $0 \le \lim_{n\to\infty} \mathbb{E}[|D_n(X) - X|] \le \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n (x_i - x_{i-1}) = \lim_{n\to\infty}(\bar{u} - \underline{u})/n = 0$. $\square$

C.2. Continuous distribution of alternatives

Consider a version of our main model (as in Section 2), such that the distribution of alternatives $F(\cdot)$ is replaced by a continuous distribution $G(\cdot)$, with $\operatorname{supp} G = [\underline{u}, \bar{u}]$. Thus, in this model, $P(S)$ is the probability of drawing an outcome in set S, when drawing from a distribution with cdf G. This change will cause the state variable X_n will become uncountably infinite, which gives rise to measurability-related technical issues. We can ensure our model is well-behaved and eliminate the measurability-related problems by tightening some of the other assumptions. Thus, assume that $\mathcal{F}$ (the choice set for Ξ_n and Ψ_n) is the set of all sets that can be formed by finite unions of closed intervals in $[\underline{u}, \bar{u}]$, and that for every n, $s_n(X^n)$ is bounded and almost-everywhere continuous.[6] A deeper issue still remains in that the uncountably-infinite state space violates the assumptions of several papers whose results we leverage in the main paper: Plambeck and Zenios (2000), Fudenberg et al. (1990), and Smith (1998). Yet, there are considerable results we can obtain *without* relying on that literature. First, the proposition below serves as an analogue of Theorem 1 for this setting.

Proposition A2 (Implementable policies under a continuous distribution) *The principal can implement any policy pair* $(\{\omega_n\}, \{\Xi_n\})$*, provided it satisfies the implementability condition* (10)*, for all* $n \in \{1, ..., N\}$*. This can be done by offering the agent a contract that pays a base pay of* s^0_n *in every period, replaced by bonus* $s^B_n(\Xi_n)$ *if the agent delivers an alternative with a value in* Ξ_n *in a period in which the principal would like him to search* ($\omega_n = 1$)*, where* s^0_n *and* $s^B_n(\Xi_n)$ *are given by* (11)*. Under any such contract, the agent's optimal consumption policy is given by* $\gamma_n(X^{n-1}, W_n) = (1-\alpha)W_n + w$*, for all* $n \in \{1, ..., N\}$*. In all periods, the agent's value function under such contract is* $V^A_n(W_n, X^{n-1}) = -\frac{1}{1-\alpha}\exp[-r(w + (1-\alpha)W_n)]$*, rendering the agent indifferent between termination and continuation.*

Proof Sketch. Consider the agent holding the contract given by this proposition, i.e., one designed to induce policies $(\{\omega_n\}, \{\Xi_n\})$. From (5), we have that the agent's optimal policies solve the recursion

$$\begin{aligned}
V^A_n(W_n, X^{n-1}) = &\max_{\bar{\Xi}_n, \bar{\omega}_n, \bar{\gamma}_n} -\exp[-r\bar{\gamma}_n] \\
&+ \bar{\omega}_n \omega_n p P\left(\bar{\Xi}_n \cap \Xi_n(X^{n-1}) \cap \Psi_n(X^{n-1})\right)\left(-\frac{\alpha}{1-\alpha}\exp\left[-r\left(w + (1-\alpha)\left(\frac{W_n - c - \bar{\gamma}_n}{\alpha} + s^B_n(\Xi_n(X^{n-1}))\right)\right)\right]\right) \\
&+ \bar{\omega}_n \omega_n p P\left(\bar{\Xi}_n \cap \Xi_n(X^{n-1}) \backslash \Psi_n(X^{n-1})\right) \alpha V^A_n\left(\frac{W_n - c - \bar{\gamma}_n}{\alpha} + s^B_n(\Xi_n(X^{n-1}))\right) \\
&+ \bar{\omega}_n \omega_n p P\left(\bar{\Xi}_n \cap \Psi_n(X^{n-1}) \backslash \Xi_n(X^{n-1})\right)\left(-\frac{\alpha}{1-\alpha}\exp\left[-r\left(w + (1-\alpha)\left(\frac{W_n - c - \bar{\gamma}_n}{\alpha} + s^0_n\right)\right)\right]\right) \\
&+ \left(1 - \bar{\omega}_n \omega_n p P(\bar{\Xi}_n \cap (\Psi_n(X^{n-1}) \cup \Xi_n(X^{n-1}))\right) \alpha V^A_n\left(\frac{W_n - c\bar{\omega}_n - \bar{\gamma}_n}{\alpha} + s^0_n\right),
\end{aligned} \tag{EC.51}$$

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[6] The measurability issues can also be resolved using weaker assumptions by following Bertsekas and Shreve (2004), i.e., through the use of either outer integration or by constraining all policy choices to universally measurable ones. Using that approach requires the introduction of considerable addition notation and a relatively heavy measure-theoretic apparatus.

with boundary condition $V_{N+1}^A(W_{N+1}, X^N) = -\frac{1}{1-\alpha}\exp[-r(w + (1-\alpha)W_{N+1})]$. Note that s_n^B is well-defined if and only if (10) holds. The proof is based on showing by induction that $V_n^A(W_n, X^{n-1}) = -\frac{1}{1-\alpha}\exp[-r(w + (1-\alpha)W_n)]$ (this will also make the participation constraint satisfied and binding). All of the other results—i.e., that this contract really induces the desired policies $(\{\omega_n\}, \{\Xi_n\})$ as well as $\gamma_n(X^{n-1}, W_n) = (1-\alpha)W_n + w$—are obtained as steps in this proof. The boundary condition of (EC.51) also serves as the induction base. For the induction step, assume $V_{n+1}^A(W_{n+1}, X^n) = -\frac{1}{1-\alpha}\exp[-r(w + (1-\alpha)W_{n+1})]$, using which, from (EC.51) we have

$$\begin{aligned} V_n^A(W_n, X^{n-1}) = & \max_{\bar{\Xi}_n, \bar{\omega}_n, \bar{\gamma}_n} -\exp[-r\bar{\gamma}_n] \\ & + \bar{\omega}_n \omega_n p P\left(\bar{\Xi}_n \cap \Xi_n(X^{n-1})\right)\left(-\frac{\alpha}{1-\alpha}\exp\left[-r\left(w + (1-\alpha)\left(\frac{W_n - c - \bar{\gamma}_n}{\alpha} + s_n^B(\Xi_n(X^{n-1}))\right)\right)\right]\right) \\ & + \left(1 - \bar{\omega}_n \omega_n p P\left(\bar{\Xi}_n \cap \Xi_n(X^{n-1})\right)\right)\left(-\frac{\alpha}{1-\alpha}\exp\left[-r\left(w + (1-\alpha)\left(\frac{W_n - \bar{\omega}_N c - \bar{\gamma}_n}{\alpha} + s_n^0\right)\right)\right]\right). \end{aligned} \tag{EC.52}$$

Here, $\bar{\Xi}_n = \Xi_n[X^{n-1}]$ is optimal as V_n^A depends on the choice of $\bar{\Xi}_n$ only through $P\left(\bar{\Xi}_n \cap \Xi_n(X^{n-1})\right)$, and is increasing in that probability as $s_n^B(\Xi_n(X^{n-1})) > s_n^0$ by (11). If $\omega_n = 0$, the value function becomes $V_n^A(W_n, X^{n-1}) = \max_{\bar{\omega}_n, \bar{\gamma}_n} -\exp[-r\bar{\gamma}_n] + (-\alpha(1-\alpha)^{-1}\exp\left[-r\left(w + (1-\alpha)\left((W_n - \bar{\omega}_n c - \bar{\gamma}_n)/\alpha + s_n^0\right)\right)\right])$, where $\bar{\omega}_n = 0$ is optimal (it only appears in the $-\bar{\omega}_n c$ term), reducing the equation to

$$V_n^A(W_n, X^{n-1}) = \max_{\bar{\gamma}_n} -\exp[-r\bar{\gamma}_n] + \left(-\frac{\alpha}{1-\alpha}\exp\left[-r\left(w + (1-\alpha)\left(\frac{W_n - \bar{\gamma}_n}{\alpha} + s_n^0\right)\right)\right]\right), \tag{EC.53}$$

Solving (EC.53) for $\bar{\gamma}_n$ yields $\bar{\gamma}_n = (1-\alpha)W_n + w$; plugging that back into (EC.53) gives $V_n^A(W_n, X^{n-1}) = -\frac{1}{1-\alpha}\exp[-r(w + (1-\alpha)W_n)]$.

Now we turn to the case where $\omega_n = 1$. Here, from (EC.52), not searching also yields the RHS of (EC.53), so $-\frac{1}{1-\alpha}\exp[-r(w + (1-\alpha)W_n)]$. Expanding the expressions for bonus sizes in (EC.52) using (11) and simplifying the resulting expression yields that searching (choosing $\bar{\omega}_n = 1$) also gives the agent the RHS of (EC.53). So searching is optimal (weakly, as the agent is indifferent), the optimal consumption is $\bar{\gamma}_n = (1-\alpha)W_n + w$ and $V_n^A(W_n, X^{n-1}) = -\frac{1}{1-\alpha}\exp[-r(w + (1-\alpha)W_n)]$ in this case as well, completing the induction. $\square$

Proposition A2 shows that the contracts of Theorem 1 are incentive compatible and individually rational in this setting as well. They are also subject to the same implementability condition, induce the same consumption policy, and have the same property that the agent's participation constraint is binding in all periods, not just at contract signing. The important gap from Theorem 1 is that, here, these contracts are not guaranteed to be the lowest cost ones over the space of all contracts (even the existence of such minimum is not guaranteed in this setup).

The results of Proposition A2, combined with those that we can approximate the true distribution with arbitrarily fine precision with discrete approximations (shown in Lemma A6) and that for all the discrete approximations, the contract class of Proposition A2 is guaranteed to be optimal (shown in Theorem 1), provide a reasonable justification to focus on this contract class.

Then, as in (12), restricting the principal's choice of contracts to ones given in Proposition A2 reduces to the principal's problem to

$$V_n^P = \max_{\omega_n \in \{0,1\}, \Xi_n, \Psi_n \in \mathcal{F}} \omega_n p P(\Xi_n) \left(\alpha s_n^0 - e - \alpha s_n^B(\Xi_n)\right) + \omega_n p \int\limits_{\Xi_n \cap \Psi_n} \left(u - \alpha V_{n+1}^P\right) dG(u) - \alpha s_n^0 + \alpha V_{n+1}^P, \tag{EC.54}$$

with boundary condition $V_{N+1}^P = 0$. This is now a well-behaved dynamic program that is amenable to the same methods of analysis used in the main paper, thus our main results (Lemma 1, Theorem 2, Theorem 3, Proposition 1, and Proposition 2) should be obtainable in this setting as well, by following analogous proofs—of course, with the limitation that the principal be restricted in the form of contracts she can offer to the contracts of Proposition A2.

As a final reality check that the discrete approximation is reasonable, we show that as the precision of the approximation increases, the principal's value function converges to the value function in the continuous version of the model.

Lemma A7 (Convergence of value functions) *Let V_k^P be the value function of a principal with a continuous distribution of alternatives G with $\operatorname{supp} G = [\underline{u}, \bar{u}]$, when she uses acceptance policy $\{\Psi_k\}$ and induces the agent to use policies $\{\Xi_k\}$,$\{\omega_n\}$ using a contract given by Proposition A2. Let ${}^nV_k^P$ be the value function of a principal with a discrete distribution F_n (approximation of continuous distribution G obtained through either bracket-mean or bracket-median method, with n brackets), who uses acceptance policy $\{\Psi_k \cap \operatorname{supp} F_n\}$ and induces the agent to use policies $\{\Xi_k \cap \operatorname{supp} F_n\}$,$\{\omega_n\}$ using a contract given by Theorem 1. Then, $\lim_{n\to\infty} {}^nV_k^P = V_k^P$.*

Proof Sketch. The proof is by induction. The basis ${}^nV_{N+1}^P = V_{N+1}^P = 0$ follows directly from boundary conditions. Assume by the way of induction that $\lim_{n\to\infty} {}^nV_{k+1}^P = V_{k+1}^P$. Denote by $P(S;D)$ the probability of drawing an outcome in set S when drawing from a distribution with cdf D. Then, from (4) and Theorem 1 we have ${}^nV_k^P = \omega_k p \int\limits_{\Xi_k \cap \Psi_k \cap \operatorname{supp} F_n} (u - e - \alpha s_k^B(\Xi_k \cap \operatorname{supp} F_n))\, dF_n(u) + \omega_k p \int\limits_{(\Xi_k \setminus \Psi_k) \cap \operatorname{supp} F_n} \left(-e - \alpha s_k^B(\Xi_n \cap \operatorname{supp} F_n) + {}^nV_{k+1}^P \alpha\right) dF_n(u) + (1 - \omega_n p P(\Xi_n; F_n)) \left(-\alpha s_k^0 + {}^nV_{k+1}^P \alpha\right) = \omega_k p \int\limits_{\Xi_k \cap \Psi_k \cap \operatorname{supp} F_n} \left(u - {}^nV_{k+1}^P \alpha\right) dF_n(u) + \omega_k p P(\Xi_k; F_n)(-e - \alpha s_k^B(\Xi_k \cap \operatorname{supp} F_n) + \alpha s_k^0) - \alpha s_k^0 + {}^nV_{k+1}^P \alpha$. Taking a limit of this expression when $n \to \infty$ and using the convergence in distribution result of Lemma A6 yields $\lim_{n\to\infty} {}^nV_k^P = V_k^P$. $\square$

C.3. Variable search intensity model

Here we provide more detailed results for the model of Section 4.2, where the agent decides on search intensity. Proposition A3 is an analogue of Theorem 1 for this setting.

Proposition A3 *Any implementable policy pair $(\{\omega_n\}, \{\xi_n\})$ satisfies the condition $p\bar{F}(\xi_n|\omega_n) > 1 - 1/\exp[rc(i_n)(1-\alpha)\alpha^{-1}]$ for all $n \in \{1, ..., N | \omega_n \neq 0\}$.*[7] *The contract which implements $(\{\omega_n\}, \{\xi_n\})$ at the lowest possible cost to the principal consists of a sequence of outcome-dependent payments. Such a contract*

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[7] This implementability condition used to be both sufficient and necessary in the basic model, but here it is only necessary, as there are additional conditions on implementability of i_H: either (EC.57) needs to hold or the argument of the logarithm in (EC.59) needs to be positive.

pays $s_n^H(\omega_n,\xi_n)$ *for period* n *if the agent delivers an alternative with a value of at least* $\max\{\xi_n,\overline{u}_L\}$*, pays* $s_n^L(\omega_n,\xi_n)$ *if the agent delivers an alternative with a value in* $[\xi_n,\max\{\xi_n,\overline{u}_L\}]$ *and pays* s_n^0 *in any other scenario. The exact values of those payments depend on the policy which they are designed to induce as follows:*

(i) *If* $\omega_n=0$*, then* $s_n^H(0,\xi_n)=s_n^L(0,\xi_n)=s_n^0=\frac{w}{\alpha}$.

(ii) *If* $\omega_n=i_L$ *then*

$$s_n^L(i_L,\xi_n)=\frac{w+c_L}{\alpha}+\frac{1}{r(1-\alpha)}\ln\frac{p\bar{F}(\xi_n|i_L)}{1-(1-p\bar{F}(\xi_n|i_L))\exp[rc_L(1-\alpha)\alpha^{-1}]}\quad,\quad s_n^H(i_L,\xi_n)=s_n^0=\frac{w}{\alpha}. \tag{EC.55}$$

(iii) *If* $\omega_n=i_H$ *then first consider the contract with parameters*

$$s_n^L(i_H,\xi_n)=s_n^H(i_H,\xi_n)=\frac{w+c_H}{\alpha}+\frac{1}{r(1-\alpha)}\ln\frac{p\bar{F}(\xi_n|i_H)}{1-(1-p\bar{F}(\xi_n|i_H))\exp[rc_H(1-\alpha)\alpha^{-1}]}\quad,\quad s_n^0=\frac{w}{\alpha}. \tag{EC.56}$$

This is the solution if the contract really induces i_H*, that is, if it satisfies the constraint*

$$\frac{\overline{u}_H+1-\xi_n}{\overline{u}_H+1}v\left(s_n^H(i_H,\xi_n)-\frac{c_H}{\alpha}\right)+\frac{\xi_n}{\overline{u}_H+1}s_n^0\geq\frac{(\overline{u}_L+1-\xi_n)^+}{\overline{u}_L+1}v\left(s_n^H(i_H,\xi_n)-\frac{c_L}{\alpha}\right)+\frac{\min\{\xi_n,\overline{u}_L+1\}}{\overline{u}_L+1}s_n^0. \tag{EC.57}$$

Otherwise, the solution is given by

$$s_n^L(i_H,\xi_n)=\frac{w+c_L}{\alpha}+\frac{1}{r(1-\alpha)}\ln\frac{p\bar{F}(\xi_n|i_L)}{1-(1-p\bar{F}(\xi_n|i_L))\exp[rc_L(1-\alpha)\alpha^{-1}]}\quad,\quad s_n^0=\frac{w}{\alpha}, \tag{EC.58}$$

$$s_n^H(i_H,\xi_n)=\frac{w}{\alpha}-\frac{1}{r(1-\alpha)}\ln\left(-\frac{1-p}{p}+\frac{v(c_L/\alpha)(\overline{u}_L+1)-v(c_H/\alpha)(\overline{u}_H+1)}{(\overline{u}_H-\overline{u}_L)p}\right). \tag{EC.59}$$

Proof Sketch. The proof follows the same steps as the proof of Theorem 1, with the following key differences. After applying Theorem 1 of Plambeck and Zenios (2000), the problem reduces to (EC.1)–(EC.4) with an additional IC constraint that ensures that the agent cannot do better by choosing a different search intensity. For the case where the principal desires to induce i_H, this constraint is given by (EC.57). Separation into three different bonus levels follows from the observation that all alternatives in $\{0,1,...,\bar{u}_L\}$ carry the same information about the agent's search intensity, as do ones in $\{\bar{u}_L+1,...,\bar{u}_H\}$. Part (i) of the proposition is the same as in Theorem 1, as the new constraint is irrelevant when $\omega_n=0$. Part (ii) follows from the observation that changing s_n^H while keeping other contract parameters the same only affects the agent's payoff if he uses i_H, which reduces the problem to the one solved in Theorem 1 if the principal desires to induce i_L. For part (iii), the constrained problem is solved using KKT conditions (KKT is both sufficient and necessary here, as in Theorem 1), which yields (EC.56) if (EC.57) is not binding, or (EC.58)–(EC.59) if it is. □

This result breaks into a number of cases, yet, most of them closely parallel our main model: (EC.55) and (EC.56) are slight variations of our basic result. The sole situation where the results here substantially differ from the main model arises when the principal desires to induce $\omega_n=i_H$ combined with $\xi_n\leq\overline{u}_L$. This difference occurs because here, it is possible that the traditional flat contract for everything delivered, as given by (EC.56), induces $\omega_n=i_L$ instead.

If this happens, the solution involves modifying the values in (EC.56) simultaneously rising $s_n^H(i_H, \xi_n)$ (which makes the contract more attractive for the agent who is using high intensity) and reducing the $s_n^L(i_H, \xi_n)$ component (which makes the contract less attractive to both high and low intensity, but more so for the low intensity, who will experience these outcomes more often).

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