Electronic-Companion: "Dynamic Optimal Policy for an Inventory System of Two Substitutable Products with Positive Replenishment Leadtimes"

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We need Lemmas 3 and 4 below for the proof of Theorem 1.

Lemma 3. *If J_t is $L^\natural$-convex in $(\tilde{\mathbf{u}}, -\tilde{\mathbf{v}})$ and $J_t(\tilde{\mathbf{u}}-z\mathbf{e}_{L_1+1}, \tilde{\mathbf{v}}+z\mathbf{e}_{L_2+1})+zc_{12}$ is increasing in z when $z \geq u_0^+ \wedge v_0^-$, then (i) H_t is $L^\natural$-convex in $(\tilde{\mathbf{u}}, -\tilde{\mathbf{v}})$ and $H_t(\tilde{\mathbf{u}}, \tilde{\mathbf{v}})$ can be attained by allowing virtual substitutions; and (ii) G_t is $L^\natural$-convex in $(\tilde{\mathbf{u}}, -\tilde{\mathbf{v}})$.*

Proof. First, we assume that virtual substitutions are allowed in (3), by replacing the constraint $u_0^+ \wedge v_0^- \geq z \geq 0$ with $z \geq 0$, and consider the following problem:

$$\bar{H}_t(\tilde{\mathbf{u}}, \tilde{\mathbf{v}}) = \min_{z\geq 0}\{c_{12}z + J_t(\tilde{\mathbf{u}} - z\mathbf{e}_{L_1+1}, \tilde{\mathbf{v}} + z\mathbf{e}_{L_2+1})\}. \tag{A1}$$

Note that if $J_t(\tilde{\mathbf{u}} - z\mathbf{e}_{L_1+1}, \tilde{\mathbf{v}} + z\mathbf{e}_{L_2+1}) + zc_{12}$ is increasing in z when $z \geq u_0^+ \wedge v_0^-$, then having $z = u_0^+ \wedge v_0^-$ is better than having $z > u_0^+ \wedge v_0^-$. Hence, virtual substitutions can be precluded by the optimal substitution decision in (A1), which means that problems (A1) and (3) are equivalent and, as a result, $\bar{H}_t = H_t$. Thus, for the $L^\natural$-convexity of $H_t(\tilde{\mathbf{u}}, \tilde{\mathbf{v}})$, it suffices to consider (A1) and show that $\bar{H}_t$ is $L^\natural$-convex in $(\tilde{\mathbf{u}}, -\tilde{\mathbf{v}})$.

Clearly, for any $z' \geq 0$,

$$\begin{aligned}\bar{H}_t(\tilde{\mathbf{u}} - z'\mathbf{e}_{L_1+1}, \tilde{\mathbf{v}} + z'\mathbf{e}_{L_2+1}) &= \min_{z\geq 0}\{J_t(\tilde{\mathbf{u}} - (z+z')\mathbf{e}_{L_1+1}, \tilde{\mathbf{v}} + (z+z')\mathbf{e}_{L_2+1}) + c_{12}z\} \\ &= \min_{s\geq z'}\{J_t(\tilde{\mathbf{u}} - s\mathbf{e}_{L_1+1}, \tilde{\mathbf{v}} + s\mathbf{e}_{L_2+1}) + c_{12}(s-z')\}.\end{aligned}$$

As $\{(\tilde{\mathbf{u}}, \tilde{\mathbf{v}}, z', s) : (\tilde{\mathbf{u}} - s\mathbf{e}_{L_1+1}, \tilde{\mathbf{v}} + s\mathbf{e}_{L_2+1}) \in \mathcal{V}_2, s \geq z'\}$ is a sublattice, and

$$\Psi(\tilde{\mathbf{u}}, \tilde{\mathbf{v}}, z', s) := J_t(\tilde{\mathbf{u}} - s\mathbf{e}_{L_1+1}, \tilde{\mathbf{v}} + s\mathbf{e}_{L_2+1}) + c_{12}(s-z')$$

is submodular in $(\tilde{\mathbf{u}}, -\tilde{\mathbf{v}}, z', s)$ over this sublattice, it follows from Theorem 2.7.6 in Topkis (1998) that $\bar{H}_t(\tilde{\mathbf{u}} - z'\mathbf{e}_{L_1+1}, \tilde{\mathbf{v}} + z'\mathbf{e}_{L_2+1})$ is submodular in $(\tilde{\mathbf{u}}, -\tilde{\mathbf{v}}, z')$, and therefore $\bar{H}_t(\tilde{\mathbf{u}}, \tilde{\mathbf{v}})$ is $L^\natural$-convex in $(\tilde{\mathbf{u}}, -\tilde{\mathbf{v}})$ for $(\tilde{\mathbf{u}}, \tilde{\mathbf{v}}) \in \mathcal{V}_2$. Consequently, if the condition of the lemma holds, it is optimal not to have any virtual substitution and $H_t = \bar{H}_t$ is $L^\natural$-convex in $(\tilde{\mathbf{u}}, -\tilde{\mathbf{v}})$. The $L^\natural$-convexity of G_t immediately follows from the above conclusion and the fact that $L^\natural$-convexity is preserved under expectation. This completes the proof of the lemma. □

Lemma 4. *Under the conditions of Lemma 3, we have (i)* $H_t(\tilde{\mathbf{u}} - z\mathbf{e}_{L_1+1}, \tilde{\mathbf{v}} + z\mathbf{e}_{L_2+1}) + zc_{12}$ *is increasing in* z. *(ii)* f_t *is* $L^\natural$*-convex in* $(\mathbf{u}, -\mathbf{v})$, *and* $f_t(\mathbf{u} - z\mathbf{e}_{L_1}, \mathbf{v} + z\mathbf{e}_{L_2}) + zc_{12}$ *is increasing in* z.

Proof. (i) It is equivalent to show that

$$\sum_{l=0}^{L_1} \frac{\partial H_t(\tilde{\mathbf{u}}, \tilde{\mathbf{v}})}{\partial u_l} - \sum_{j=0}^{L_2} \frac{\partial H_t(\tilde{\mathbf{u}}, \tilde{\mathbf{v}})}{\partial v_j} \le c_{12} \qquad \text{for any } (\tilde{\mathbf{u}}, \tilde{\mathbf{v}}) \in \mathcal{V}_2; \tag{A2}$$

Note that for $u_0 \le 0$ or $v_0 \ge 0$, $H_t = J_t$, and thus (A2) holds automatically. Hence, it suffices to verify (A2) for $u_0 > 0$ and $v_0 < 0$. To this end, consider $u_0 > 0$ and $0 < \delta \le u_0 \wedge (-v_0)$. Suppose that there exists $0 \le z$ such that $z \le (u_0 - \delta) \wedge (-v_0 - \delta)$ and

$$H_t(\tilde{\mathbf{u}} - \delta\mathbf{e}_{L_1+1}, \tilde{\mathbf{v}} + \delta\mathbf{e}_{L_2+1}) = J_t(\tilde{\mathbf{u}} - (z+\delta)\mathbf{e}_{L_1+1}, \tilde{\mathbf{v}} + (z+\delta)\mathbf{e}_{L_2+1}) + c_{12}z.$$

As $z + \delta$ is a feasible substitution quantity for state $(\tilde{\mathbf{u}}, \tilde{\mathbf{v}})$,

$$\begin{aligned} H_t(\tilde{\mathbf{u}}, \tilde{\mathbf{v}}) &\le J_t(\tilde{\mathbf{u}} - (z+\delta)\mathbf{e}_{L_1+1}, \tilde{\mathbf{v}} + (z+\delta)\mathbf{e}_{L_2+1}) + c_{12}(z+\delta) \\ &= H_t(\tilde{\mathbf{u}} - \delta\mathbf{e}_{L_1+1}, \tilde{\mathbf{v}} + \delta\mathbf{e}_{L_2+1}) + c_{12}\delta, \end{aligned} \tag{A3}$$

which implies

$$\frac{H_t(\tilde{\mathbf{u}}, \tilde{\mathbf{v}}) - H_t(\tilde{\mathbf{u}} - \delta\mathbf{e}_{L_1+1}, \tilde{\mathbf{v}} + \delta\mathbf{e}_{L_2})}{\delta} \le c_{12}.$$

Letting $\delta \to 0$ in the left-hand side of the above inequality establishes (A2) at $(\tilde{\mathbf{u}}, \tilde{\mathbf{v}})$ with $u_0 > 0, v_0 < 0$.

(ii) From Lemma 3, H_t, G_t are $L^\natural$-convex and by (i), (A2) holds. From (A3), for any sample of demand for two products, D_1 and D_2 respectively,

$$H_t(\tilde{\mathbf{u}} - D_1\mathbf{e}_{L_1+1}, \tilde{\mathbf{v}} - D_2\mathbf{e}_{L_2+1}) - H_t(\tilde{\mathbf{u}} - D_1\mathbf{e}_{L_1+1} - \delta\mathbf{e}_{L_1+1}, \tilde{\mathbf{v}} - D_2\mathbf{e}_{L_2+1} + \delta\mathbf{e}_{L_2+1}) \le \delta c_{12}.$$

With the expectation on the left-hand side, it is similar to (A2) to entail

$$\sum_{l=0}^{L_1} \frac{\partial G_t(\tilde{\mathbf{u}}, \tilde{\mathbf{v}})}{\partial u_l} - \sum_{j=0}^{L_2} \frac{\partial G_t(\tilde{\mathbf{u}}, \tilde{\mathbf{v}})}{\partial v_j} \le c_{12}.$$

It follows that $G_t(\tilde{\mathbf{u}} - z\mathbf{e}_{L_1+1}, \tilde{\mathbf{v}} + z\mathbf{e}_{L_2+1})$ is submodular in $(\tilde{\mathbf{u}}, -\tilde{\mathbf{v}}, z)$. From Theorem 2.7.6 in Topkis (1998), submodularity is preserved under minimization and thus, $f_t(\mathbf{u} - z\mathbf{e}_{L_1}, \mathbf{v} + z\mathbf{e}_{L_2})$ is submodular in $(\mathbf{u}, -\mathbf{v}, z)$, i.e. $f_t(\mathbf{u}, \mathbf{v})$ is $L^\natural$-convex in $(\mathbf{u}, -\mathbf{v})$.

To prove that $f_t(\mathbf{u} - z\mathbf{e}_{L_1}, \mathbf{v} + z\mathbf{e}_{L_2}) + zc_{12}$ is increasing in z, it suffices to show that

$$\sum_{l=0}^{L_1-1} \frac{\partial f_t(\mathbf{u}, \mathbf{v})}{\partial u_l} - \sum_{j=0}^{L_2-1} \frac{\partial f_t(\mathbf{u}, \mathbf{v})}{\partial v_j} \leq c_{12} \qquad \text{for any } (\mathbf{u}, \mathbf{v}) \in \mathcal{V}_1. \tag{A4}$$

To prove (A4), let $(\hat{u} - \delta, \hat{v} + \delta)$ be the optimal order-up-to inventory positions for products 1 and 2, respectively, at state $(\mathbf{u} - \delta\mathbf{e}_{L_1}, \mathbf{v} + \delta\mathbf{e}_{L_2})$, for $\delta > 0$. Thus, $u_{L_1-1} - \delta \leq \hat{u} - \delta \leq u_{L_1-1} - \delta + C_1$ and $v_{L_2-1} + \delta \leq \hat{v} + \delta \leq v_{L_2-1} + \delta + C_2$, which implies that $u_{L_1-1} \leq \hat{u} \leq u_{L_1-1} + C_1$ and $v_{L_2-1} \leq \hat{v} \leq v_{L_2-1} + C_2$. Thus, $(\hat{u}, \hat{v})$ are also feasible for state $(\mathbf{u}, \mathbf{v})$. In view of (A2),

$$\begin{aligned}
& f_t(\mathbf{u}, \mathbf{v}) - f_t(\mathbf{u} - \delta\mathbf{e}_{L_1}, \mathbf{v} + \delta\mathbf{e}_{L_2}) \\
\leq\ & G_t(\tilde{\mathbf{u}}, \tilde{\mathbf{v}}) + c_1\gamma^{L_1}(\hat{u} - u_{L_1-1}) + c_2\gamma^{L_2}(\hat{v} - v_{L_2-1}) \\
-\ & G_t(\tilde{\mathbf{u}} - \delta\mathbf{e}_{L_1+1}, \tilde{\mathbf{v}} + \delta\mathbf{e}_{L_2+1}) - \gamma^{L_1}c_1(\hat{u} - u_{L_1-1}) - c_2\gamma^{L_2}(\hat{v} - v_{L_2-1}) \\
=\ & \int_0^\delta \left[\sum_{l=0}^{L_1} \frac{\partial G_t(\tilde{\mathbf{u}} - z\mathbf{e}_{L_1+1}, \tilde{\mathbf{v}} + z\mathbf{e}_{L_2+1})}{\partial u_l} - \sum_{j=0}^{L_2} \frac{\partial G_t(\tilde{\mathbf{u}} - z\mathbf{e}_{L_1+1}, \tilde{\mathbf{v}} + z\mathbf{e}_{L_2+1})}{\partial v_j} \right] dz \\
\leq\ & c_{12}\delta,
\end{aligned}$$

where the first inequality is from the optimality of f_t and the second is due to (i). Therefore,

$$\frac{f_t(\mathbf{u}, \mathbf{v}) - f_t(\mathbf{u} - \delta\mathbf{e}_{L_1}, \mathbf{v} + \delta\mathbf{e}_{L_2})}{\delta} \leq c_{12}$$

holds for any $\delta > 0$ at state $(\mathbf{u}, \mathbf{v})$. We obtain (A4) by letting $\delta \to 0$. □

Proof of Theorem 1.

We use induction. Since

$$f_{T+L+1}(\mathbf{u}, \mathbf{v}) = -c_1u_0 - c_2v_0 \quad \text{and} \quad g(u_0, v_0) = h_1u_0^+ + b_1u_0^- + h_2v_0^+ + b_2v_0^-,$$

clearly,

$$J_{T+L}(\tilde{\mathbf{u}}, \tilde{\mathbf{v}}) = g(u_0, v_0) + \gamma f_{T+L+1}(u_1, u_2, \cdots, \hat{u}, v_1, v_2, \cdots, \hat{v})$$

is $L^\natural$-convex in $(\tilde{\mathbf{u}}, -\tilde{\mathbf{v}})$. In addition, under Assumption 1 we can easily verify that

$$\frac{\partial f_{T+L+1}(\mathbf{u}, \mathbf{v})}{\partial u_0} - \frac{\partial f_{T+L+1}(\mathbf{u}, \mathbf{v})}{\partial v_0} \leq c_{12}$$

and

$$\frac{\partial g(u_0, v_0)}{\partial u_0} - \frac{\partial g(u_0, v_0)}{\partial v_0} \le (1-\gamma)c_{12} \quad \text{for } u_0 \le 0 \text{ or } v_0 \ge 0.$$

Thus,

$$\sum_{l=0}^{L_1} \frac{\partial J_{T+L}(\tilde{\mathbf{u}}, \tilde{\mathbf{v}})}{\partial u_l} - \sum_{j=0}^{L_2} \frac{\partial J_{T+L}(\tilde{\mathbf{u}}, \tilde{\mathbf{v}})}{\partial v_j} \le c_{12} \qquad \text{for } u_0 \le 0 \text{ or } v_0 \ge 0.$$

From Lemmas 3 and 4, we further have (A2) and (A4) for $t = T + L$. In addition, f_{T+L} is $L^\natural$-convex in $(\mathbf{u}, -\mathbf{v})$, H_{T+L} and G_{T+L} are $L^\natural$-convex in $(\tilde{\mathbf{u}}, -\tilde{\mathbf{v}})$, and virtual substitution is not viable at $t = T + L$ even if it is allowed.

Assume that all conclusions, as well as (A4), are true for $t + 1$. Following from the assumption that f_{t+1} is $L^\natural$-convex in $(\mathbf{u}, -\mathbf{v})$,

$$J_t(\tilde{\mathbf{u}}, \tilde{\mathbf{v}}) = g(u_0, v_0) + \gamma f_{t+1}(u_1, u_2, \cdots, \hat{u}, v_1, v_2, \cdots, \hat{v})$$

is $L^\natural$-convex in $(\tilde{\mathbf{u}}, -\tilde{\mathbf{v}})$. Besides, with simple algebra we can show that

$$\sum_{l=0}^{L_1} \frac{\partial J_t(\tilde{\mathbf{u}}, \tilde{\mathbf{v}})}{\partial u_l} - \sum_{j=0}^{L_2} \frac{\partial J_t(\tilde{\mathbf{u}}, \tilde{\mathbf{v}})}{\partial v_j} \le c_{12} \qquad \text{for } u_0 \le 0 \text{ or } v_0 \ge 0; \tag{A5}$$

Therefore, from Lemmas 3 and 4 again, we have f_t is $L^\natural$-convex in $(\mathbf{u}, -\mathbf{v})$, H_t, G_t are $L^\natural$-convex in $(\tilde{\mathbf{u}}, -\tilde{\mathbf{v}})$, and (A2) and (A4) hold for t. In addition, from Lemma 3 again, virtual substitution can be precluded from the optimal policy even if it is allowed. Thus, allowing virtual substitution in the formulation will not result in any change to the optimal policies and optimal value functions. □

Proof of Lemma 1.

Since $G_t(\tilde{\mathbf{u}}, \tilde{\mathbf{v}})$ is $L^\natural$-convex in $(\tilde{\mathbf{u}}, -\tilde{\mathbf{v}})$, for any $\xi > 0$, $G_t(\tilde{\mathbf{u}} - \xi \mathbf{e}_{L_1+1}, \tilde{\mathbf{v}} + \xi \mathbf{e}_{L_2+1})$ is submodular in $(\mathbf{u}, \hat{u}, -\mathbf{v}, -\hat{v}, \xi)$. Consider

$$\begin{aligned}
& f_t(\mathbf{u} - \xi \mathbf{e}_{L_1}, \mathbf{v} + \xi \mathbf{e}_{L_2}) \\
= & \min_{\substack{C_1 + u_{L_1-1} - \xi \ge \hat{u} \ge u_{L_1-1} - \xi \\ C_2 + v_{L_2-1} + \xi \ge \hat{v} \ge v_{L_2-1} + \xi}} \{ c_1 \gamma^{L_1} [\hat{u} - (u_{L_1-1} - \xi)] + c_2 \gamma^{L_2} [\hat{v} - (v_{L_2-1} + \xi)] \\
& \qquad\qquad + G_t(\mathbf{u} - \xi \mathbf{e}_{L_1}, \hat{u}, \mathbf{v} + \xi \mathbf{e}_{L_2}, \hat{v}) \} \\
= & \min_{\substack{C_1 + u_{L_1-1} \ge \bar{u} \ge u_{L_1-1} \\ C_2 + v_{L_2-1} \ge \bar{v} \ge v_{L_2-1}}} \{ c_1 \gamma^{L_1} (\bar{u} - u_{L_1-1}) + c_2 \gamma^{L_2} (\bar{v} - v_{L_2-1}) \\
& \qquad\qquad + G_t(\mathbf{u} - \xi \mathbf{e}_{L_1}, \bar{u} - \xi, \mathbf{v} + \xi \mathbf{e}_{L_2}, \bar{v} + \xi) \},
\end{aligned} \tag{A6}$$

where $\bar{u} := \hat{u} + \xi$ and $\bar{v} := \hat{v} - \xi$. It follows from Theorem 2.8.3 in Topkis (1998) that for any $\xi \ge 0$, there exists a greatest element of the minimizers for state $(\mathbf{u}, \mathbf{v}, \xi)$, say

$(\bar{u}(\mathbf{u},\mathbf{v},\xi),\bar{v}(\mathbf{u},\mathbf{v},\xi))$, such that $(\bar{u}(\mathbf{u},\mathbf{v},\xi),-\bar{v}(\mathbf{u},\mathbf{v},\xi))$ is increasing in $(\mathbf{u},-\mathbf{v},\xi)$. Since

$$(\bar{u}(\mathbf{u},\mathbf{v},\xi),\bar{v}(\mathbf{u},\mathbf{v},\xi)) = (\hat{u}(\mathbf{u},\mathbf{v}),\hat{v}(\mathbf{u},\mathbf{v})) \qquad \text{for } \xi = 0,$$

the statement (i) of the lemma is established. Besides,

$$\begin{aligned}
& (\hat{u}(\mathbf{u},\mathbf{v}),-\hat{v}(\mathbf{u},\mathbf{v})) - (\xi,\xi) \\
= \ & (\bar{u}(\mathbf{u},\mathbf{v},0),-\bar{v}(\mathbf{u},\mathbf{v},0)) - (\xi,\xi) && \text{(A7)} \\
\le \ & (\bar{u}(\mathbf{u},\mathbf{v},\xi),-\bar{v}(\mathbf{u},\mathbf{v},\xi)) - (\xi,\xi) && \text{(A8)} \\
= \ & (\hat{u}(\mathbf{u}-\xi\mathbf{e}_{L_1},\mathbf{v}+\xi\mathbf{e}_{L_2}),-\hat{v}(\mathbf{u}-\xi\mathbf{e}_{L_1},\mathbf{v}+\xi\mathbf{e}_{L_2})), && \text{(A9)}
\end{aligned}$$

where (A8) is due to the fact that $(\bar{u}(\mathbf{u},\mathbf{v},\xi),-\bar{v}(\mathbf{u},\mathbf{v},\xi))$ is increasing in ξ, and (A9) follows from the definitions of $(\bar{u}(\mathbf{u},\mathbf{v},\xi),\bar{v}(\mathbf{u},\mathbf{v},\xi))$ and $(\hat{u}(\mathbf{u}-\xi\mathbf{e}_{L_1},\mathbf{v}+\xi\mathbf{e}_{L_2}),\hat{v}(\mathbf{u}-\xi\mathbf{e}_{L_1},\mathbf{v}+\xi\mathbf{e}_{L_2}))$. Here in (A8), we slightly abuse the notation of "$\le$": the inequality simply means that the components in (A7) are less than their counterparts in (A8). Hence, (ii) is proved. □

Proof of Theorem 2. From Lemma 1, $\hat{u}(\mathbf{u},\mathbf{v})$ and $\hat{v}(\mathbf{u},\mathbf{v})$, and thus, $q_1(\mathbf{u},\mathbf{v})$ and $q_2(\mathbf{u},\mathbf{v})$, are differentiable almost everywhere. Easy to see that

$$\sum_{j=l}^{L_1-1} \frac{\partial q_1(\mathbf{u},\mathbf{v})}{\partial u_j} = \frac{\partial q_1(\mathbf{u},\mathbf{v})}{\partial x_l} \qquad \text{for } l = 0,1,\cdots,L_1-1,$$

and

$$\sum_{j=l}^{L_2-1} \frac{\partial q_2(\mathbf{u},\mathbf{v})}{\partial v_j} = \frac{\partial q_2(\mathbf{u},\mathbf{v})}{\partial y_l} \qquad \text{for } l = 0,1,\cdots,L_2-1.$$

Thus, (7) and (8) are respectively equivalent to

$$\begin{aligned}
& -1 \le \frac{\partial q_1(\mathbf{u},\mathbf{v})}{\partial u_{L_1-1}} \le \sum_{i=L_1-2}^{L_1-1} \frac{\partial q_1(\mathbf{u},\mathbf{v})}{\partial u_i} \le \cdots \le \sum_{i=0}^{L_1-1} \frac{\partial q_1(\mathbf{u},\mathbf{v})}{\partial u_i} \\
\le \ & \sum_{j=0}^{L_2-1} \frac{\partial q_1(\mathbf{u},\mathbf{v})}{\partial v_j} \le \sum_{j=1}^{L_2-1} \frac{\partial q_1(\mathbf{u},\mathbf{v})}{\partial v_j} \le \cdots \frac{\partial q_1(\mathbf{u},\mathbf{v})}{\partial v_{L_2-1}} \le 0, && \text{(A10)}
\end{aligned}$$

and

$$\begin{aligned}
& -1 \le \frac{\partial q_2(\mathbf{u},\mathbf{v})}{\partial v_{L_2-1}} \le \sum_{j=L_2-2}^{L_2-1} \frac{\partial q_2(\mathbf{u},\mathbf{v})}{\partial v_j} \le \cdots \le \sum_{j=0}^{L_2-1} \frac{\partial q_2(\mathbf{u},\mathbf{v})}{\partial v_j} \\
\le \ & \sum_{i=0}^{L_1-1} \frac{\partial q_2(\mathbf{u},\mathbf{v})}{\partial u_i} \le \sum_{i=1}^{L_1-1} \frac{\partial q_2(\mathbf{u},\mathbf{v})}{\partial u_i} \le \cdots \le \frac{\partial q_2(\mathbf{u},\mathbf{v})}{\partial u_{L_1-1}} \le 0. && \text{(A11)}
\end{aligned}$$

We prove these two inequalities below.

From Lemma 1, we have

$$\frac{\partial \hat{u}(\mathbf{u},\mathbf{v})}{\partial u_j} \geq 0, \quad \frac{\partial \hat{v}(\mathbf{u},\mathbf{v})}{\partial v_k} \geq 0, \quad \frac{\partial \hat{u}(\mathbf{u},\mathbf{v})}{\partial v_k} \leq 0, \quad \frac{\partial \hat{v}(\mathbf{u},\mathbf{v})}{\partial u_j} \leq 0$$

for $j = 0, 1, \cdots, L_1 - 1$ and $k = 0, 1, \cdots, L_2 - 1$. In addition,

$$\hat{u}(\mathbf{u},\mathbf{v}) - \xi \leq \hat{u}(\mathbf{u} - \xi\mathbf{e}_{L_1}, \mathbf{v} + \xi\mathbf{e}_{L_2}) \leq \hat{u}(\mathbf{u},\mathbf{v})$$

and

$$-\hat{v}(\mathbf{u},\mathbf{v}) - \xi \leq -\hat{v}(\mathbf{u} - \xi\mathbf{e}_{L_1}, \mathbf{v} + \xi\mathbf{e}_{L_2}) \leq -\hat{v}(\mathbf{u},\mathbf{v}),$$

which imply, respectively and conjugately,

$$0 \geq -\sum_{i=0}^{L_1-1} \frac{\partial \hat{u}(\mathbf{u},\mathbf{v})}{\partial u_i} + \sum_{j=0}^{L_2-1} \frac{\partial \hat{u}(\mathbf{u},\mathbf{v})}{\partial v_j} \geq -1 \tag{A12}$$

and

$$0 \geq \sum_{i=0}^{L_1-1} \frac{\partial \hat{v}(\mathbf{u},\mathbf{v})}{\partial u_i} - \sum_{j=0}^{L_2-1} \frac{\partial \hat{v}(\mathbf{u},\mathbf{v})}{\partial v_j} \geq -1. \tag{A13}$$

As $\frac{\partial \hat{u}(\mathbf{u},\mathbf{v})}{\partial u_j} \geq 0$ and $\frac{\partial \hat{u}(\mathbf{u},\mathbf{v})}{\partial v_k} \leq 0$, from (A12) we have

$$\sum_{i=i_1}^{L_1-1} \frac{\partial \hat{u}(\mathbf{u},\mathbf{v})}{\partial u_i} - 1 \leq \sum_{j=j_1}^{L_2-1} \frac{\partial \hat{u}(\mathbf{u},\mathbf{v})}{\partial v_j} \leq 0 \leq \sum_{i=i_2}^{L_1-1} \frac{\partial \hat{u}(\mathbf{u},\mathbf{v})}{\partial u_i} \leq 1 + \sum_{j=j_2}^{L_2-1} \frac{\partial \hat{u}(\mathbf{u},\mathbf{v})}{\partial v_j} \tag{A14}$$

for $0 \leq i_1,\ i_2 \leq L_1 - 1$ and $0 \leq j_1,\ j_2 \leq L_2 - 1$. Similarly, from (A13) we have

$$\sum_{j=k_1}^{L_2-1} \frac{\partial \hat{v}(\mathbf{u},\mathbf{v})}{\partial v_j} - 1 \leq \sum_{i=l_1}^{L_1-1} \frac{\partial \hat{v}(\mathbf{u},\mathbf{v})}{\partial u_i} \leq 0 \leq \sum_{j=k_2}^{L_2-1} \frac{\partial \hat{v}(\mathbf{u},\mathbf{v})}{\partial v_j} \leq 1 + \sum_{i=l_2}^{L_1-1} \frac{\partial \hat{v}(\mathbf{u},\mathbf{v})}{\partial u_i} \tag{A15}$$

for $0 \leq l_1,\ l_2 \leq L_1 - 1$ and $0 \leq k_1,\ k_2 \leq L_2 - 1$.

As $\hat{u}(\mathbf{u},\mathbf{v}) = u_{L_1-1} + q_1(\mathbf{u},\mathbf{v})$, clearly $\frac{\partial q_1}{\partial u_{L_1-1}} = \frac{\partial \hat{u}}{\partial u_{L_1-1}} - 1$, and for $i = 1, \cdots, L_1 - 2$ and $j = 1, \cdots, L_2 - 1$, $\frac{\partial q_1}{\partial u_i} = \frac{\partial \hat{u}}{\partial u_i}$ and $\frac{\partial q_1}{\partial v_j} = \frac{\partial \hat{u}}{\partial v_j}$. Therefore, from (A14) we obtain (A10). Similarly, as $\hat{v}(\mathbf{u},\mathbf{v}) = v_{L_2-1} + q_2(\mathbf{u},\mathbf{v})$, from (A15) we obtain (A11). □

Proof of Lemma 2.

From Theorem 1,

$$\begin{aligned}
& H_t(\tilde{\mathbf{u}} - \xi \mathbf{e}_{L_1+1}, \tilde{\mathbf{v}} + \xi \mathbf{e}_{L_2+1}) \\
= & \min_{z \geq 0} \{c_{12} z + J_t(\tilde{\mathbf{u}} - \xi \mathbf{e}_{L_1+1} - z \mathbf{e}_{L_1+1}, \tilde{\mathbf{v}} + \xi \mathbf{e}_{L_2+1} + z \mathbf{e}_{L_2+1})\} \\
= & \min_{z \geq \xi} \{c_{12}(z - \xi) + J_t(\tilde{\mathbf{u}} - z \mathbf{e}_{L_1+1}, \tilde{\mathbf{v}} + z \mathbf{e}_{L_2+1})\}. \qquad \text{(A16)}
\end{aligned}$$

As J_t is $L^\natural$-convex in $(\tilde{\mathbf{u}}, -\tilde{\mathbf{v}})$, the objective function of (A16) is submodular in a sublattice of $(\tilde{\mathbf{u}}, -\tilde{\mathbf{v}})$ and (ξ, z). Let us denote by $\bar{z}(\tilde{\mathbf{u}}, \tilde{\mathbf{v}}, \xi)$ the largest minimizer of (A16). It follows from Theorem 2.8.3 in Topkis (1998) that $\bar{z}(\tilde{\mathbf{u}}, \tilde{\mathbf{v}}, \xi)$ is increasing in $(\tilde{\mathbf{u}}, -\tilde{\mathbf{v}})$ and ξ. Note that $z^*(\tilde{\mathbf{u}}, \tilde{\mathbf{v}}) = \bar{z}(\tilde{\mathbf{u}}, \tilde{\mathbf{v}}, 0)$, which is increasing in $(\tilde{\mathbf{u}}, -\tilde{\mathbf{v}})$. In addition, for $\xi > 0$,

$$\begin{aligned}
z^*(\tilde{\mathbf{u}}, \tilde{\mathbf{v}}) &= \bar{z}(\tilde{\mathbf{u}}, \tilde{\mathbf{v}}, 0) \leq \bar{z}(\tilde{\mathbf{u}}, \tilde{\mathbf{v}}, \xi) = \bar{z}(\tilde{\mathbf{u}} - \xi \mathbf{e}_{L_1+1}, \tilde{\mathbf{v}} + \xi \mathbf{e}_{L_2+1}, 0) + \xi \\
&= z^*(\tilde{\mathbf{u}} - \xi \mathbf{e}_{L_1+1}, \tilde{\mathbf{v}} + \xi \mathbf{e}_{L_2+1}) + \xi.
\end{aligned}$$

This completes the proof. □

Proof of Theorem 3. We can easily see that

$$\frac{\partial z^*(\tilde{\mathbf{u}}, \tilde{\mathbf{v}})}{\partial q_1} + \sum_{i=l}^{L_1-1} \frac{\partial z^*(\tilde{\mathbf{u}}, \tilde{\mathbf{v}})}{\partial u_i} = \frac{\partial z^*(\tilde{\mathbf{u}}, \tilde{\mathbf{v}})}{\partial x_l}, \qquad \text{for } l = 0, \cdots, L_1;$$

and

$$\frac{\partial z^*(\tilde{\mathbf{u}}, \tilde{\mathbf{v}})}{\partial q_2} + \sum_{i=k}^{L_2-1} \frac{\partial z^*(\tilde{\mathbf{u}}, \tilde{\mathbf{v}})}{\partial v_i} = \frac{\partial z^*(\tilde{\mathbf{u}}, \tilde{\mathbf{v}})}{\partial y_k} \qquad \text{for } k = 0, \cdots, L_2.$$

Hence, to prove (i) and (ii), it is sufficient to prove (a) and (b) below:

(a)

$$0 \leq \frac{\partial z^*(\tilde{\mathbf{u}}, \tilde{\mathbf{v}})}{\partial q_1} + \sum_{i=l}^{L_1-1} \frac{\partial z^*(\tilde{\mathbf{u}}, \tilde{\mathbf{v}})}{\partial u_i} \leq 1 + \frac{\partial z^*(\tilde{\mathbf{u}}, \tilde{\mathbf{v}})}{\partial q_2} + \sum_{i=k}^{L_2-1} \frac{\partial z^*(\tilde{\mathbf{u}}, \tilde{\mathbf{v}})}{\partial v_i} \leq 1; \qquad \text{(A17)}$$

(b) $\frac{\partial z^*(\tilde{\mathbf{u}}, \tilde{\mathbf{v}})}{\partial q_1} + \sum_{i=l}^{L_1-1} \frac{\partial z^*(\tilde{\mathbf{u}}, \tilde{\mathbf{v}})}{\partial u_i}$ is decreasing in l, and $\frac{\partial z^*(\tilde{\mathbf{u}}, \tilde{\mathbf{v}})}{\partial q_2} + \sum_{i=k}^{L_2-1} \frac{\partial z^*(\tilde{\mathbf{u}}, \tilde{\mathbf{v}})}{\partial v_i}$ is increasing in k.

In principle, by the reasoning with which we verified the last inequality in the proof of Lemma 2, we can prove that

$$\begin{aligned}
z^*(\tilde{\mathbf{u}} - \mathbf{a}, \tilde{\mathbf{v}} + \mathbf{b}) &\leq z^*(\tilde{\mathbf{u}} - \xi \mathbf{e}_{L_1+1}, \tilde{\mathbf{v}} + \xi \mathbf{e}_{L_2+1}) + \xi \\
&\leq z^*(\tilde{\mathbf{u}} - \mathbf{a}, \tilde{\mathbf{v}} + \mathbf{b}) + \xi,
\end{aligned}$$

where $\mathbf{a}$ is a non-negative (L_1+1)-vector and $\mathbf{b}$ is a non-negative (L_2+1)-vector, satisfying $\mathbf{a} \le \xi \mathbf{e}_{L_1+1}$ and $\mathbf{b} \le \xi \mathbf{e}_{L_2+1}$. With $\mathbf{a} = \xi \sum_{i \neq l} \mathbf{e}^i_{L_1+1}$ and $\mathbf{b} = \xi \mathbf{e}_{L_2+1}$, we may obtain

$$\begin{aligned} z^*(\tilde{\mathbf{u}} - \xi \sum_{i\neq l} \mathbf{e}^i_{L_1+1}, \tilde{\mathbf{v}} + \xi \mathbf{e}_{L_2+1}) &\le z^*(\tilde{\mathbf{u}} - \xi \mathbf{e}_{L_1+1}, \tilde{\mathbf{v}} + \xi \mathbf{e}_{L_2+1}) + \xi \\ &\le z^*(\tilde{\mathbf{u}} - \xi \sum_{i\neq l} \mathbf{e}^i_{L_1+1}, \tilde{\mathbf{v}} + \xi \mathbf{e}_{L_2+1}) + \xi. \end{aligned}$$

Thus,

$$-\xi \le z^*(\tilde{\mathbf{u}} - \xi \mathbf{e}_{L_1+1}, \tilde{\mathbf{v}} + \xi \mathbf{e}_{L_2+1}) - z^*(\tilde{\mathbf{u}} - \xi \sum_{i\neq l} \mathbf{e}^i_{L_1+1}, \tilde{\mathbf{v}} + \xi \mathbf{e}_{L_2+1}) \le 0. \tag{A18}$$

For convenience we denote $u_{L_1} := \hat{u}$ and $v_{L_2} := \hat{v}$. Dividing by ξ on both sides of (A18) and then letting $\xi \to 0$ establishes

$$0 \le \frac{\partial z^*(\tilde{\mathbf{u}}, \tilde{\mathbf{v}})}{\partial u_l} \le 1 \qquad \text{for } 0 \le l \le L_1. \tag{A19}$$

Similarly, let $\mathbf{b} = \xi \sum_{i\neq k} \mathbf{e}^i_{L_2+1}$ and $\mathbf{a} = \xi \mathbf{e}_{L_1+1}$ to obtain

$$-1 \le \frac{\partial z^*(\tilde{\mathbf{u}}, \tilde{\mathbf{v}})}{\partial v_k} \le 0 \qquad \text{for } 0 \le k \le L_2. \tag{A20}$$

Next, for $0 \le k < L_1$ and $0 \le m < L_2$, letting $\mathbf{a} = \xi \sum_{i=0}^{k} \mathbf{e}^i_{L_1+1}$ and $\mathbf{b} = \xi \sum_{i=0}^{l} \mathbf{e}^i_{L_2+1}$, by the same token we can obtain

$$-1 \le -\sum_{i=k+1}^{L_1} \frac{\partial z^*(\tilde{\mathbf{u}}, \tilde{\mathbf{v}})}{\partial u_i} + \sum_{i=l+1}^{L_2} \frac{\partial z^*(\tilde{\mathbf{u}}, \tilde{\mathbf{v}})}{\partial v_i} \le 0. \tag{A21}$$

Note that $\frac{\partial z^*(\tilde{\mathbf{u}}, \tilde{\mathbf{v}})}{\partial u_{L_1}} = \frac{\partial z^*(\tilde{\mathbf{u}}, \tilde{\mathbf{v}})}{\partial q_1}$, and $\frac{\partial z^*(\tilde{\mathbf{u}}, \tilde{\mathbf{v}})}{\partial v_{L_2}} = \frac{\partial z^*(\tilde{\mathbf{u}}, \tilde{\mathbf{v}})}{\partial q_2}$. (a) and (b) follow immediately from (A19), (A20) and (A21). □

Proof of Theorem 4.

We let $r = u - z$ be an amount of inventory reserved in stock after deleting a substitution z from an on-hand stock u of product 1, and used to transfer $J_t(u-z, v+z) = J_t(r, u+v-r)$ into a submodular function of $(r, u+v)$. Deeming r as the reservation of product 1, we can take r as the equivalent decision variable in minimizing $c_{12}z + J_t(u-z, v+z)$, in lieu of substitution z, such that

$$\begin{aligned} H_t(u,v) &= \min_{u\ge z\ge 0, v+z\le 0} \{c_{12}z + J_t(u-z, v+z)\} \\ &= \min_{u\ge r\ge 0, u+v\le r} \{-c_{12}r + J_t(r, u+v-r)\} + c_{12}u. \end{aligned} \tag{A22}$$

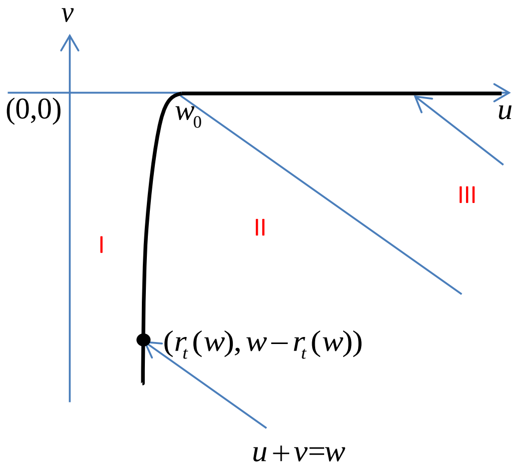

Figure A1: Optimal substitution policy for the case $L_i = 0$, $i = 1, 2$.

In particular, for $u > 0$ and $v < 0$, we can designate equivalent optimal reservation $r_t^*(u,v)$ with a nonnegative optimal reservation reference given by

$$r_t(w) = \operatorname{argmin}_{r \geq 0}\{-c_{12}r + J_t(r, w-r)\},$$

such that for $w = u+v$, $r_t^*(u,v) = [r_t(u+v) \vee (u+v)] \wedge u = [r_t(u+v) \vee (u+v)^+] \wedge u$ and $z_t^*(u,v) = u - r_t^*(u,v) = [u - r_t(u+v) \vee (u+v)^+] \vee 0 = [u - r_t(u+v)]^+ \wedge [u - (u+v)^+]$ are equivalent optimal reservation and substitution respectively. By (A22), the objective function of its right-hand side, $-c_{12}r + J_t(r, u+v-r)$, is submodular in $(r, u+v)$. By Topkis (1998), $r_t(u+v)$ is increasing in and dependent on $u+v$ only. Furthermore, as $-c_{12}r + f_{t+1}(r, w-r) = -c_{12}(r-w) - c_{12}w + f_{t+1}(r-w+w, -(r-w))$ is submodular in $(r-w, -w)$, optimal $r_t(w) - w$ is decreasing in w. Therefore, if $u+v \leq w_0$, then, $r_t(u+v) \geq u+v$, optimal substitution $z_t^*(u,v) = [u - r_t(u+v)]^+$; otherwise, for $u+v > w_0$, $z_t^*(u,v) = u - (u+v) = -v = v^-$.

Optimal substitution actions is shown in Figure A1. We can see that reference reservation $r_t(u+v)$ keeps constant for (u,v) on line $w = u+v$ and is represented by curve $(r_t(w), w - r_t(w))$. In area I, as $u \leq r_t(u+v)$, no substitution is made for any point, in area II, $u > r_t(u+v)$, every point must be substituted down to the border of the two areas, whereas in area III, every point must be substituted down to a segment of u axis to the right of $(w_0, 0)$. □

Proof of Theorem 5. (i) This is obvious, since $(u,v) = (\bar{u}_t, \bar{v}_t)$ minimizes $\phi_t(u,v)$ and is feasible to (9).

(ii) Suppose $(x,y) \geq (\bar{u}_t, \bar{v}_t)$. Assume $(\tilde{u}_t, \tilde{v}_t)$ is any optimal solution to (9). Consider two cases: (a) $\tilde{u}_t > x$, and (b) $\tilde{u}_t = x$. Suppose $\tilde{u}_t > x \geq \bar{u}_t$. Note that following from Theorem 1, $\phi_t(u,v)$ is submodular and convex in $(u,-v)$. Since $\tilde{u}_t > x \geq \bar{u}_t$ and

$\tilde{v}_t \geq y \geq \bar{v}_t$ (which implies $-\tilde{v}_t \leq -\bar{v}_t$), we have

$$\begin{aligned} \phi_t(\bar{u}_t, \bar{v}_t) + \phi_t(\tilde{u}_t, \tilde{v}_t) &\geq \phi_t(\tilde{u}_t, \bar{v}_t) + \phi_t(\bar{u}_t, \tilde{v}_t) \\ &\geq \phi_t(\tilde{u}_t, \bar{v}_t) + \phi_t(x, \tilde{v}_t), \qquad \text{(A23)} \end{aligned}$$

where the first inequality is from the submodularity of $\phi_t(u, v)$, and the second inequality is due to the facts that $\phi_t(u, \tilde{v}_t)$ is convex in u with $u = \tilde{u}_t$ as a minimum point and that $\tilde{u}_t > x \geq \bar{u}_t$. But from the definitions of $(\bar{u}_t, \bar{v}_t)$ and $(\tilde{u}_t, \tilde{v}_t)$, we also have

$$\phi_t(\bar{u}_t, \bar{v}_t) \leq \phi_t(\tilde{u}_t, \bar{v}_t) \qquad \text{and} \qquad \phi_t(\tilde{u}_t, \tilde{v}_t) \leq \phi_t(x, \tilde{v}_t).$$

These, together with (A23), imply $\phi_t(\bar{u}_t, \bar{v}_t) = \phi_t(\tilde{u}_t, \bar{v}_t)$ and $\phi_t(\tilde{u}_t, \tilde{v}_t) = \phi_t(x, \tilde{v}_t)$. Hence, $(x, \tilde{v}_t)$ is also an optimal solution to (9), in both cases (a) and (b) above. With $(x, \tilde{v}_t)$ being an optimal solution, by similar argument, we can show that if $\tilde{v}_t > y$, then (x, y) is also an optimal solution to (9). Therefore, $(\hat{u}_t, \hat{v}_t) = (x, y)$ is an optimal solution.

(iii) & (iv) We will only prove (iii) below. (iv) can be obtained similarly. Assume $x \geq \bar{u}_t$ and $y < \bar{v}_t$, and suppose $(\tilde{u}_t, \tilde{v}_t)$ is an optimal solution to (9). If $\tilde{v}_t > \bar{v}_t$, then with the same token as for (ii), $(\tilde{u}_t, \bar{v}_t)$ is also a pair of optimal order-up-to levels. Below we assume $\tilde{v}_t \leq \bar{v}_t$. Consider the following two cases: $\tilde{u}_t = x$ and $\tilde{u}_t > x$. If $\tilde{u}_t = x$, then $(\hat{u}_t, \hat{v}_t) = (x, \tilde{v}_t)$ with $\tilde{v}_t \leq \bar{v}_t$ is the optimal solution we want. Suppose $\tilde{u}_t > x$. In this case, let

$$\lambda = \frac{x - \bar{u}_t}{\tilde{u}_t - \bar{u}_t} \qquad \text{and} \qquad v' = \lambda \tilde{v}_t + (1 - \lambda)\bar{v}_t.$$

Then $v' \leq \bar{v}_t$, and

$$\lambda(\tilde{u}_t, -\tilde{v}_t) + (1 - \lambda)(\bar{u}_t, -\bar{v}_t) = (x, -v').$$

Since $\phi_t(u, v)$ is convex in $(u, -v)$,

$$\phi_t(x, v') \leq \lambda \phi_t(\tilde{u}_t, \tilde{v}_t) + (1 - \lambda)\phi_t(\bar{u}_t, \bar{v}_t) \leq \phi_t(\tilde{u}_t, \tilde{v}_t).$$

Therefore, $(\hat{u}_t, \hat{v}_t) = (x, v')$ is an optimal solution with $v' \leq \bar{v}_t$. □

Lemma 5 to Lemma 8 below are for the proof of Theorem 6.

Lemma 5. *Suppose both $\psi_1(x)$ and $\psi_2(y)$ are convex functions, and $\psi(x, y) = \psi_1(x) + \psi_2(y)$. Then $\psi(x, y)$ is $L^\natural$-convex in (x, y) and in $(x, -y)$ respectively.*

Proof. To get the $L^\natural$-convexity in (x, y), we need to show that $\psi(x - z, y - z)$ is submodular in (x, y, z). Obviously, $\psi(x - z, y - z)$ is submodular in (x, y). Besides, note that

$$\frac{\partial}{\partial z}\psi(x - z, y - z) = -\psi_1'(x - z) - \psi_2'(y - z).$$

Due to the convexity of $\psi_1(\cdot)$ and $\psi_2(\cdot)$, $\frac{\partial}{\partial z}\psi(x-z, y-z)$ decreases in x and in y respectively, which implies that $\psi(x-z, y-z)$ is submodular in both (x, z) and (y, z). Thus, $\psi(x-z, y-z)$ is submodular in (x, y, z).

To see the $L^\natural$-convexity in $(x, -y)$, we need to show that, with $\tilde{y} = -y$,

$$\psi(x, y) = \psi_1(x) + \psi_2(y) = \psi_1(x) + \psi_2(-\tilde{y})$$

is $L^\natural$-convex in $(x, \tilde{y})$. But this follows directly from the conclusion above and the fact that $\psi_2(-\tilde{y})$ is also convex in $\tilde{y}$ if $\psi_2(y)$ is convex in y. □

For convenience, we denote

$$\pi^{(1)}(u, v) := \frac{\partial \pi(u, v)}{\partial u} \qquad \text{and} \qquad \pi^{(2)}(u, v) := \frac{\partial \pi(u, v)}{\partial v}$$

for any function $\pi(u, v)$.

Lemma 6. *For $t = 1, \cdots, T$, if $f_{t+1}(x, y)$ is $L^\natural$-convex in $(x, -y)$ and $h_i + b_i + \gamma f_{t+1}^{(i)}(x, y) \geq 0$ for $i = 1, 2$, then $J_t(u, v)$ is $L^\natural$-convex in $(u, -v)$ and $h_i + b_i + \gamma J_t^{(i)}(u, v) \geq 0$ for $i = 1, 2$.*

Proof. Note that $x^+ = x + x^-$, $x^- = \min\{q \geq 0 : q \geq -x\}$, and $x^+ = \min\{x + q \geq 0 : q \geq 0\}$. Thus, for any increasing function $\psi(x)$, we can write $\psi(x^+) = \min_{q \geq 0, x+q \geq 0} \psi(x + q)$. Therefore, for any $z \geq 0$, we have

$$\begin{aligned}
& J_t(u-z, v+z) \\
= \;& g(u-z, v+z) + \gamma f_{t+1}((u-z)^+, (v+z)^+) \\
= \;& (h_1 + b_1)(u-z)^+ + (h_2 + b_2)(v+z)^+ - b_1(u-z) - b_2(v+z) + \gamma f_{t+1}((u-z)^+, (v+z)^+) \\
= \;& \min_{\substack{q \geq 0, u-z+q \geq 0 \\ q' \geq 0, v+z+q' \geq 0}} \{(h_1 + b_1)(u - z + q) + (h_2 + b_2)(v + z + q') + \gamma f_{t+1}(u - z + q, v + z + q')\} \\
& -b_1(u-z) - b_2(v+z) \\
= \;& \min_{\substack{\bar{u} \geq u, \bar{u}-z \geq 0 \\ \bar{v} \geq v, \bar{v}+z \geq 0}} \{(h_1 + b_1)(\bar{u} - z) + (h_2 + b_2)(\bar{v} + z) + \gamma f_{t+1}(\bar{u} - z, \bar{v} + z)\} \\
& -b_1(u-z) - b_2(v+z),
\end{aligned}$$

where the third equality is due to the fact that, from the assumption of $h_i + b_i + \gamma f_{t+1}^{(i)}(x, y) \geq 0$, $(h_1 + b_1)x + (h_2 + b_2)y + \gamma f_{t+1}(x, y)$ is increasing in x and y. Since $f_{t+1}(x, y)$ is $L^\natural$-convex in $(x, -y)$, $f_{t+1}(\bar{u} - z, \bar{v} - z)$ is submodular in $(\bar{u}, -\bar{v}, z)$, and thus, from Theorem 2.7.6 of Topkis (1998), $J_t(u - z, v + z)$ is submodular in $(u, -v, z)$, and $J_t(u, v)$ is $L^\natural$-convex in $(u, -v)$.

To entail the second assertion, we envisage, for $\delta > 0$,

$$\begin{aligned}
& J_t(u+\delta, v) - J_t(u,v) \\
= \;& (h_1+b_1)(u+\delta)^+ - b_1\delta + \gamma f_{t+1}((u+\delta)^+, v^+) - [(h_1+b_1)u^+ + \gamma f_{t+1}(u^+, v^+)] \\
\geq \;& -b_1\delta \\
\geq \;& -\gamma^{-1}(h_1+b_1)\delta,
\end{aligned}$$

where the first inequality holds because by the assumption, $(h_1+b_1)x + \gamma f_{t+1}(x,y)$ is an increasing function of x. Dividing both sides with δ and letting δ go to zero leads to $h_1 + b_1 + \gamma J_t^{(1)}(u,v) \geq 0$. Analogously, $h_2 + b_2 + \gamma J_t^{(2)}(u,v) \geq 0$. This assures the two assertions of the lemma. □

Lemma 7. *$H_T(u,v)$ remains the same, no matter whether or not virtual substitutions are allowed in (12).*

Proof. We claim that when $v \geq 0$ or $u < 0$, any substitution in (12) is virtual. We will consider these two cases separately.

For convenience we denote $V_T(u,v,z) = c_{12}z + J_T(u-z, v+z)$. Then it suffices to show that

$$\frac{\partial}{\partial z} V_T(u,v,z) \geq 0 \qquad \text{for any } z \geq 0$$

when $v \geq 0$ or $u < 0$.

Case 1. $v \geq 0$. In this case, if $u \leq 0$, or $u > 0$ and $z \geq u$, we have

$$V_T(u,v,z) = c_{12}z + h_2(v+z) - b_1(u-z) - \gamma c_2(v+z),$$

and hence,

$$\begin{aligned}
\frac{\partial}{\partial z} V_T(u,v,z) &= c_{12} + h_2 + b_1 - \gamma c_2 \\
&\geq c_{12} + h_2 + b_2 - \gamma c_2 \\
&\geq 0,
\end{aligned}$$

where the second inequality is from $b_2 \geq c_2$. If $u > 0$ and $z < u$, then

$$V_T(u,v,z) = c_{12}z + h_1(u-z) + h_2(v+z) - \gamma c_1(u-z) - \gamma c_2(v+z),$$

and

$$\begin{aligned}
\frac{\partial}{\partial z} V_T(u,v,z) &= c_{12} - h_1 + h_2 + \gamma c_1 - \gamma c_2 \\
&\geq \gamma c_{12} + \gamma c_1 - \gamma c_2 \\
&\geq 0,
\end{aligned}$$

where the first inequality is from Assumption 1.

Case 2. $u < 0$. In this case, if $v < 0$ and $z \geq -v$, then

$$V_T(u, v, z) = c_{12}z + h_2(v + z) - b_1(u - z) - \gamma c_2(v + z),$$

and

$$\frac{\partial}{\partial z} V_T(u, v, z) = c_{12} + h_2 + b_1 - \gamma c_2 \geq 0,$$

as in case 1. If $v < 0$ and $z < -v$, then

$$V_T(u, v, z) = c_{12}z - b_1(u - z) - b_2(v + z),$$

and

$$\frac{\partial}{\partial z} V_T(u, v, z) = c_{12} + b_1 - b_2 \geq 0.$$

Note that the sub-case of $v \geq 0$ and $u < 0$ has been proved in Case 1. □

Lemma 8. *Under Assumption 1, for $t = 1, \cdots, T$, we have*

- **(i)** *$H_t(u, v)$, $G_t(u, v)$ and $J_t(u, v)$ are $L^\natural$-convex in $(u, -v)$, $f_t(x, y)$ is $L^\natural$-convex in $(x, -y)$, and $H_t(u, v)$ remains the same if virtual substitutions are allowed in (12);*
- **(ii)** *$f_t^{(1)}(x, y) - f_t^{(2)}(x, y) \leq c_{12}$, and $f_t^{(2)}(x, y) \geq -c_{12}$;*
- **(iii)** *for $\eta_t = H_t$, G_t, J_t, and f_t,*

$$h_j + b_j + \gamma \eta_t^{(j)}(u, v) \geq 0 \qquad \text{for } j = 1,\ 2.$$

Proof. First, consider that case of $t = T$. Note that

$$\begin{aligned} J_T(u, v) &= g(u, v) + \gamma f_{T+1}(u^+, v^+) \\ &= h_1 u^+ + h_2 v^+ + b_1 u^- + b_2 v^- - \gamma c_1 u^+ - \gamma c_2 v^+ \\ &= (h_1 + b_1 - \gamma c_1) u^+ + (h_2 + b_2 - \gamma c_2) v^+ - b_1 u - b_2 v, \end{aligned}$$

which is $L^\natural$-convex in $(u, -v)$ following from Lemma 5 and the assumption $b_i \geq c_i, i = 1, 2$.

From Lemma 7,

$$H_T(u, v) = \min_{z \geq 0} \{c_{12}z + J_T(u - z, v + z)\}.$$

Hence, from Theorem 2.7.6 of Topkis (1998), $H_T(u,v)$ is also $L^\natural$-convex in $(u,-v)$, which further implies that $G_T(u,v)$ is $L^\natural$-convex in $(u,-v)$ and $f_T(x,y)$ is $L^\natural$-convex in $(x,-y)$.

To see (ii) for $t=T$, suppose $\bar{z}$ is the optimal substitution quantity at $(u-\delta, v+\delta)$ for $\delta>0$. Noting that $\delta+\bar{z}$ is a feasible substitution quantity at state (u,v) since virtual substitution is allowed, we have

$$\begin{aligned}
& H_T(u-\delta,v+\delta)-H_T(u,v) \\
\geq\ & [g(u-\delta-\bar{z},v+\delta+\bar{z})+c_{12}\bar{z}-\gamma c_1(u-\delta-\bar{z})^+-\gamma c_2(v+\delta+\bar{z})^+] \\
& -[g(u-\delta-\bar{z},v+\delta+\bar{z})+c_{12}(\delta+\bar{z})-\gamma c_1(u-\delta-\bar{z})^+-\gamma c_2(v+\delta+\hat{z})^+] \\
=\ & -c_{12}\delta,
\end{aligned}$$

which implies

$$H_T^{(1)}(u,v)-H_T^{(2)}(u,v)\leq c_{12}.$$

As the inequality above is preserved under expectation, we also have $G_T^{(1)}(u,v)-G_T^{(2)}(u,v)\leq c_{12}$.

To show $f_T^{(1)}(x,y)-f_T^{(2)}(x,y)\leq c_{12}$, let $(\hat{u}-\delta,\hat{v}+\delta)$ be the optimal order-up-to level for state $(x-\delta,y+\delta)$. Then, $(\hat{u},\hat{v})$ is a feasible order-up-to level for state (x,y) and thus,

$$\begin{aligned}
& f_T(x-\delta,y+\delta)-f_T(x,y) \\
\geq\ & [c_1(\hat{u}-x)+c_2(\hat{v}-y)+G_T(\hat{u}-\delta,\hat{v}+\delta)]-[c_1(\hat{u}-x)+c_2(\hat{v}-y)+G_T(\hat{u},\hat{v})] \\
=\ & G_T(\hat{u}-\delta,\hat{v}+\delta)-G_T(\hat{u},\hat{v}) \\
\geq\ & -c_{12}\delta,
\end{aligned}$$

which implies $f_T^{(1)}(x,y)-f_T^{(2)}(x,y)\leq c_{12}$. With a same token we can get $f_T(x,y+\delta)-f_T(x,y)\geq -c_{12}\delta$, and hence, $f_T^{(2)}(x,y)\geq -c_{12}$.

To prove (iii) for $t=T$, first note that since $f_{T+1}(x,y)=-c_1x-c_2y$,

$$\gamma f_{T+1}^{(i)}(x,y)=-\gamma c_i\geq -(h_i+b_i),$$

From Lemma 6, $\gamma J_T^{(i)}(u,v)\geq -(b_i+h_i)$ for $i=1,2$.

For $\delta>0$, let $\bar{z}$ denote optimal substitution quantity at state $(u+\delta,v)$. Then

$$\begin{aligned}
& H_T(u+\delta,v)-H_T(u,v) \\
=\ & \min_{z\geq 0}[c_{12}z+J_T(u+\delta-z,v+z)]-\min_{z\geq 0}[c_{12}z+J_T(u-z,v+z)] \\
\geq\ & [c_{12}\bar{z}+J_T(u+\delta-\bar{z},v+\bar{z})]-[c_{12}\bar{z}+J_T(u-\bar{z},v+\bar{z})] \\
\geq\ & -\gamma^{-1}(h_1+b_1)\delta, \qquad \text{(A24)}
\end{aligned}$$

where the second inequality is due to the fact that $\gamma J_T^{(1)}(u,v) \geq -(b_1+h_1)$. It is entailed by (A24) that $\gamma H_T^{(1)}(u,v) \geq -(h_1+b_1)$. By the same token, $\gamma H_T^{(2)}(u,v) \geq -(h_2+b_2)$. These further imply that $\gamma G_T^{(i)}(u,v) \geq -(h_i+b_i)$ for $i = 1$, 2.

Suppose all conclusions in (i) - (iii) hold for $t+1$. We consider the case of t below.

We first show that $H_t(u,v)$ remains the same if virtual substitutions are allowed in (12). Similar to the proof of Lemma 7, it suffices to show that

$$\frac{\partial}{\partial z} V_t(u,v,z) \geq 0 \qquad \text{for any } z \geq 0$$

when $v \geq 0$ or $u < 0$, with $V_t(u,v,z) = c_{12}z + J_t(u-z, v+z)$.

Case 1. $v \geq 0$. In this case, if $u \leq 0$, or $u > 0$ and $z \geq u$, we have

$$V_t(u,v,z) = c_{12}z + h_2(v+z) - b_1(u-z) + \gamma f_{t+1}(0, v+z),$$

and hence,

$$\begin{aligned} \frac{\partial}{\partial z} V_t(u,v,z) &= c_{12} + h_2 + b_1 + \gamma f_{t+1}^{(2)}(0, v+z) \\ &\geq c_{12} + h_2 + b_2 - \gamma c_{12} \\ &\geq 0, \end{aligned}$$

where the first inequality is due to the hypothesis assumption of $f_{t+1}^{(2)}(0, v+z) \geq -c_{12}$. If $u > 0$ and $z < u$, then

$$V_t(u,v,z) = c_{12}z + h_1(u-z) + h_2(v+z) + \gamma f_{t+1}(u-z, v+z),$$

and

$$\begin{aligned} \frac{\partial}{\partial z} V_t(u,v,z) &= c_{12} - h_1 + h_2 - \gamma[f_{t+1}^{(1)}(u-z, v+z) - f_{t+1}^{(2)}(u-z, v+z)] \\ &\geq c_{12} - h_1 + h_2 - \gamma c_{12} \\ &\geq 0, \end{aligned}$$

where the first inequality is again from hypothesis assumption.

Case 2. $u < 0$. In this case, if $v < 0$ and $z \geq -v$, then

$$V_t(u,v,z) = c_{12}z + h_2(v+z) - b_1(u-z) + \gamma f_{t+1}(0, v+z),$$

and

$$\frac{\partial}{\partial z} V_t(u,v,z) = c_{12} + h_2 + b_1 + \gamma f_{t+1}^{(2)}(0, v+z) \geq 0,$$

as in Case 1. If $v < 0$ and $z < -v$, then

$$V_t(u,v,z) = c_{12}z - b_1(u-z) - b_2(v+z) + \gamma f_{t+1}(0,0),$$

and

$$\frac{\partial}{\partial z}V_t(u,v,z) = c_{12} + b_1 - b_2 \geq 0.$$

Hence, we always have $\frac{\partial}{\partial z}V_t(u,v,z) \geq 0$ when $v \geq 0$ or $u < 0$, and $H_t(u,v)$ remains the same if virtual substitutions are allowed. Thus, we can write

$$H_t(u,v) = \min_{z\geq 0}\{c_{12}z + J_t(u-z, v+z)\}.$$

By following exactly the same argument as that for the case of T above we can obtain the remaining conclusions for t. □

Proof of Theorem 6. Note that (i) of Theorem 6 has been obtained in Lemma 8, while conclusions in (ii) can be proved by the same token as that for the backlog model, noting the $L^\natural$-convexity in (i).

Next, we prove assertion (iii) by backward induction. From assertions (i) and (ii), in period t ($t = 1, \cdots, T$), $f_t(u,v)$ is $L^\natural$-convex in $(u,-v)$ and virtual substitutions are not optimal even we allow them. Therefore, it is sufficient to designate optimal substitution explicitly for real substitutions when $u > 0$ and $v < 0$. In period T, by Lemma 8, for $u > 0$ and $v < 0$,

$$\begin{aligned} H_T(u,v) &= \min_{z\geq 0, u-z\geq 0, v+z\leq 0}\{c_{12}z + h_1(u-z) + b_2(-v-z) - \gamma c_1(u-z)\} \\ &= \min_{u\geq r\geq 0, u+v\leq r}\{(h_1+b_2-c_{12}-\gamma c_1)r\} + c_{12}u - b_2(u+v), \qquad \text{(A25)} \end{aligned}$$

where (A25) takes into account $r = u - z \geq 0$ as a nonnegative reservation, because a negative r will lead to a virtual substitution. Thereby, to optimize the right-hand side of (A25), we denote by

$$r_T = \operatorname{argmin}_{r\geq 0}\{(h_1+b_2-c_{12}-\gamma c_1)r\}$$

a nonnegative minimizer. Then, because $r_T \geq 0$, the minimizer of (A25) is $r_T^*(u,v) = [(u+v)\vee r_T]\wedge u = [(u+v)^+ \vee r_T]\wedge u$. In particular, if $h_1 + b_2 - \gamma c_1 - c_{12} \geq 0$, it is straightforward that $r_T = 0$; otherwise, $r_T = \infty$. Consequently, if $h_1 + b_2 + \gamma c_1 - c_{12} \geq 0$, optimal reservation for (A25) is $r_T^*(u,v) = (u+v)^+ \wedge u = (u+v)^+$; and otherwise, $r_T^*(u,v) = u$.

In period t ($t = 1, \cdots, T-1$), we assume $f_{t+1}(u,v)$ is $L^\natural$-convex in $(u,-v)$ and in the period, no virtual substitution is optimal. For $u > 0, v < 0$,

$$\begin{aligned} H_t(u,v) &= \min_{u \geq z \geq 0, v+z \leq 0} \{c_{12}z + h_1(u-z) + b_2(-v-z) + \gamma f_{t+1}((u-z)^+, (v+z)^+)\} \\ &= \min_{u \geq r \geq 0, u+v \leq r} \{(h_1 + b_2 - c_{12})r + \gamma f_{t+1}(r^+, (u+v-r)^+)\} \\ &\quad + c_{12}u - b_2(u+v). \end{aligned} \tag{A26}$$

As $f_{t+1}(r,0)$ is convex in r, there exists a nonnegative minimizer

$$r_t = \operatorname{argmin}_{r \geq 0} \{(h_1 + b_2 - c_{12})r + \gamma f_{t+1}(r,0)\}$$

where we limit optimal choices to nonnegative values again. Consequently, a minimizer of (A26) and an optimal reservation is $r_t^*(u,v) = [r_t \vee (u+v)] \wedge u = [r_t \vee (u+v)^+] \wedge u$, where the second equality holds obviously as $r_t \geq 0$. In addition, $r_t^*(u,v) = r_t \wedge u$ for $r_t \geq u+v$ and $r_t^*(u,v) = u+v$, otherwise. Thus, optimal substitution is $z_t^*(u,v) = u - r_t^*(u,v) = u - r_t \wedge u = (u - r_t)^+$ for $u + v \leq r_t$, and $z_t^*(u,v) = u - (u+v) = -v$ for $u + v \geq r_t$. $\square$