# Electronic Companion to

## "Joint Pricing and Inventory Management with Strategic Customers"

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## EC.1. Proofs of Lemmas 5, 6 and 7

*Proof of Lemma 5.*

**Part 1.**

For any $v \in [v^*, V)$, we have

$$\theta_v = W(v) - \frac{\bar{F}(v)}{f(v)} w(v) = W(v) - (v - g_v)\, w(v) = W(v)\left(1 - \frac{v w(v)}{W(v)}\right) + g_v w(v) > 0.$$

The inequality holds due to the following reasons. First, Assumption 1 Part 1 assumes $W(v) > 0$ for all $v \in [0, V)$. Second, Assumption 1 Parts 1 and 3 imply $\frac{vw(v)}{W(v)} < 1$. Third, the properties that $g_v$ is strictly increasing in $v$ and $g_{v^*} = 0$ imply $g_v \geq 0$ for $v \in [v^*, V)$. Fourth, Assumption 1 Part 3 that $W(v)$ is non-decreasing in $v$ implies $w(v) \geq 0$.

For any $v \in [v^*, V)$, we have

$$\theta_v = W(v) - \frac{\bar{F}(v)}{f(v)} w(v) \leq W(v) \leq W(V).$$

The first and the second inequalities follow from Assumption 1 Part 3 that $W(v)$ is non-decreasing in $v$.

**Part 2.**

For any $v, v' \in [0, V)$ with $v' > v$, we have

$$\theta_{v'} = W(v') - \frac{\bar{F}(v')}{f(v')} w(v') \geq W(v) - \frac{\bar{F}(v)}{f(v)} w(v) = \theta_v.$$

The inequality holds due to the following reasons. First, Assumption 1 Part 3 that $W(v)$ is non-decreasing and concave in $v$ and condition $v' > v$ imply $W(v') \geq W(v)$ and $0 \leq w(v') \leq w(v)$. Second, Assumption 2 and condition $v' > v$ imply $\frac{\bar{F}(v')}{f(v')} \leq \frac{\bar{F}(v)}{f(v)}$.

**Part 3.**

Consider the first case that $K = 0$.

We have $g_v - \theta_v K = g_v$. Note that $g_v$ is strictly increasing in $v$ and $g_{v^*} = 0$. Hence, for any $v^* \leq v < v' < V$, $g_{v'} > g_v \geq 0$.

Consider the second case that $K > 0$.

We notice that $g_{v^*} - \theta_{v^*} K = -\theta_{v^*} K < 0$, where the inequality follows from Part 1 in this lemma that $\theta_{v^*} > 0$. Hence, to prove the statement in this part, we only need to consider $v, v' \in (v^*, V)$ with $v < v'$.

Consider any $v, v' \in (v^*, V)$ with $v < v'$. Suppose $g_v - \theta_v K \geq 0$. We notice that

$$\frac{1}{v}\left(g_v - \theta_v K\right) = \left(1 - \frac{\bar{F}(v)}{vf(v)}\right)(1 - w(v)K) - \frac{W(v)}{v}\left(1 - \frac{vw(v)}{W(v)}\right)K.$$

First, we note that Assumption 1 Part 1 assumes that $W(v) > 0$ for $v \in [0, V)$, Assumption 1 Parts 1 and 3 imply $\frac{vw(v)}{W(v)} < 1$, and $K > 0$. Hence, the second term on the RHS is strictly positive. Second, we note that the properties that $g_v$ is strictly increasing in $v$, $g_{v^*} = 0$, and condition $v > v^*$ jointly imply $1 - \frac{\bar{F}(v)}{vf(v)} = \frac{g_v}{v} > \frac{g_{v^*}}{v^*} = 0$. Therefore, $1 - w(v)K > 0$.

For $v'$, we have

$$\begin{aligned}
\frac{1}{v'}\left(g_{v'} - \theta_{v'} K\right) &= \left(1 - \frac{\bar{F}(v')}{v'f(v')}\right)(1 - w(v')K) - \frac{W(v')}{v'}\left(1 - \frac{v'w(v')}{W(v')}\right)K \\
&> \left(1 - \frac{\bar{F}(v)}{vf(v)}\right)(1 - w(v)K) - \frac{W(v)}{v}\left(1 - \frac{vw(v)}{W(v)}\right)K \\
&= \frac{1}{v}\left(g_v - \theta_v K\right) \\
&\geq 0.
\end{aligned}$$

The first inequality holds due to following reasons. First, Assumption 2, condition $v' > v > v^*$, and the property that $g_v$ is strictly increasing in $v$ with $g_{v^*} = 0$ imply $1 - \frac{\bar{F}(v')}{v'f(v')} > 1 - \frac{\bar{F}(v)}{vf(v)} = \frac{g_v}{v} > \frac{g_{v^*}}{v} \geq 0$. Second, Assumption 1 Part 3 that $W(v)$ is concave in $v$ and condition $v' > v$ imply $1 - w(v')K \geq 1 - w(v)K > 0$. Third, Assumption 1 Part 1 that $W(v) > 0$ for $v \in [0, V)$ and the property that Assumption 1 Parts 1 and 3 imply $\frac{vw(v)}{W(v)} < 1$ jointly imply $\frac{d}{dv}\frac{W(v)}{v} = -\frac{W(v)}{v^2}\left(1 - \frac{vw(v)}{W(v)}\right) < 0$. Hence, condition $v' > v$ implies $\frac{W(v')}{v'} < \frac{W(v)}{v}$. Fourth, the property that Assumption 1 Parts 1 and 3 imply $\frac{vw(v)}{W(v)} < 1$, Assumption 1 Part 4 that $\frac{vw(v)}{W(v)}$ is non-decreasing in $v$ and condition $v' > v$ jointly imply $0 < 1 - \frac{v'w(v')}{W(v')} \leq 1 - \frac{vw(v)}{W(v)}$.

To summarize two cases above, for any $v, v' \in [v^*, V)$ with $v < v'$, if $g_v - \theta_v K \geq 0$, then $g_{v'} - \theta_{v'} K > 0$.

**Part 4.**

Consider any $K > 0$ and any $v \in (0, V)$. Suppose $v - W(v)K \geq 0$.

Consider the first case that $v' \in (v, V]$.

We have

$$(v' - W(v')K) - (v - W(v)K) = \frac{v'}{v}\left(v - v\frac{W(v')}{v'}K\right) - (v - W(v)K) > \left(\frac{v'}{v} - 1\right)(v - W(v)K) \geq 0.$$

The first inequality holds due the following reasons. First, Assumption 1 Part 1 that $W(v) > 0$ for $v \in [0, V)$ and the property that Assumption 1 Parts 1 and 3 imply $\frac{vw(v)}{W(v)} < 1$ jointly imply $\frac{d}{dv}\frac{W(v)}{v} = -\frac{W(v)}{v^2}\left(1 - \frac{vw(v)}{W(v)}\right) < 0$. Hence, condition $v' > v$ implies $\frac{W(v')}{v'} < \frac{W(v)}{v}$. Second, we have $K > 0$. The second inequality follows from condition $v' \geq v$ and property $v - W(v)K \geq 0$.

In addition, we have

$$(v' - W(v')K) - (v - W(v)K) = v' - v - (W(v') - W(v))K \leq v' - v,$$

where the inequality follows from Assumption 1 Part 3 that $W(v)$ is non-decreasing in $v$, condition $v' > v$, and condition $K > 0$.

Consider the second case that $v' \in [0, v)$.

If $v' > 0$, then we have

$$(v' - W(v')K) - (v - W(v)K) = \frac{v'}{v}\left(v - v\frac{W(v')}{v'}K\right) - (v - W(v)K) < \left(\frac{v'}{v} - 1\right)(v - W(v)K) \leq 0.$$

The first inequality holds due the following reasons. First, Assumption 1 Part 1 that $W(v) > 0$ for $v \in [0, V)$ and the property that Assumption 1 Parts 1 and 3 imply $\frac{vw(v)}{W(v)} < 1$ jointly imply $\frac{d}{dv}\frac{W(v)}{v} = -\frac{W(v)}{v^2}\left(1 - \frac{vw(v)}{W(v)}\right) < 0$. Hence, condition $v' < v$ implies $\frac{W(v')}{v'} > \frac{W(v)}{v}$. Second, we have $K > 0$. The second inequality follows from condition $v' < v$ and property $v - W(v)K \geq 0$.

If $v' = 0$, then we have

$$(v' - W(v')K) - (v - W(v)K) = -W(0)K - (v - W(v)K) \leq -W(0)K < 0,$$

where the first inequality follows from the assumption that $v - W(v)K \geq 0$, the second inequality follows from Assumption 1 Part 1 that $W(0) > 0$ and the condition $K > 0$.

The analysis of three cases above completes the proof of this part.

**Q.E.D.**

*Proof of Lemma 6.*

**Part 1.**

Because $g_v$ is strictly increasing and continuous in $v$, we have $v_t^h = g^{-1}(\min\{ht, g_V\})$, which is non-decreasing and continuous in $t \in [0, L]$, with $v_0^h = g^{-1}(0) = v^*$ and $v_t^h > g^{-1}(0) = v^*$ for all $t \in (0, L]$.

**Part 2.**

Consider any $t, t' \in [0, L]$ with $t > t'$. For any $v \in [v^*, V)$, if $g_v - \theta_v(L-t) < 0$, then $g_v - \theta_v(L-t') = g_v - \theta_v(L-t) - \theta_v(t-t') < 0$, where the inequality follows from Lemma 5 Part 1 that $\theta_v > 0$ for $v \in [v^*, V)$ and condition $t > t'$. Therefore, $v_t^\theta$ is non-increasing in $t \in [0, L]$.

Because $g_v$ is strictly increasing in $v$ and $g_{v^*} = 0$, we have $v_L^\theta = \sup\{v \in [v^*, V) : g_v < 0\} = v^*$.

Suppose there exists $t \in [0, L)$, such that $v_t^\theta = v^*$. Hence, the definition of $v_t^\theta$ and the property that $g_v - \theta_v(L-t)$ is continuous in $v$ imply $g_{v_t^\theta} - \theta_{v_t^\theta}(L-t) \geq 0$. We note that the condition $v_t^\theta = v^*$ implies $g_{v_t^\theta} - \theta_{v_t^\theta}(L-t) = g_{v^*} - \theta_{v^*}(L-t) = -\theta_{v^*}(L-t)$. Thus, $\theta_{v^*} \leq 0$. This contradicts with Lemma 5 Part 1 that $\theta_v > 0$ for $v \in [v^*, V)$. Therefore, we have $v_t^\theta > v^*$ for all $t \in [0, L)$.

Next, we prove that $v_t^\theta$ is continuous in $t \in [0, L]$.

First, we prove that $v_t^\theta$ is right-continuous in $t \in [0, L]$.

Consider any $t \in [0, L)$. Because $t < L$, $v_t^\theta > v^*$. Consider any $\epsilon \in (0, v_t^\theta - v^*]$ The definition of $v_t^\theta$ and the property that $g_v - \theta_v(L-t)$ is continuous in $v$ imply $g_{v_t^\theta} - \theta_{v_t^\theta}(L-t) \leq 0$. Hence, Lemma 5 Part 3 implies that for any $v \in [v^*, v_t^\theta)$, $g_v - \theta_v(L-t) < 0$. Define $\Delta \triangleq -\sup_{v \in [v^*, v_t^\theta - \epsilon]} (g_v - \theta_v(L-t)) > 0$. Define $\delta \triangleq \frac{\Delta}{2W(V)}$. Note that Assumption 1 Part 1 that $W(v) > 0$ for all $v \in [0, V)$ and the condition $V < \infty$ imply $W(V) \in (0, \infty)$. Hence, $\delta \in (0, \infty)$.

Therefore, for any $t' \in (t, \min\{t+\delta, L\}]$ and any $v \in [v^*, v_t^\theta - \epsilon]$,

$$g_v - \theta_v(L-t') = g_v - \theta_v(L-t) + \theta_v(t'-t) \leq -\Delta + W(V)\delta = -\frac{\Delta}{2} < 0,$$

where the inequality follows from Lemma 5 Part 1 that $\theta_v \leq W(V)$. Therefore, $v_t^\theta$ is right-continuous in $t$.

Second, we prove that $v_t^\theta$ is left-continuous in $t \in [0, L]$.

Note that $v_t^\theta$ is non-increasing in $t$ and $v_t^\theta \leq V$. Hence, if there exists $t \in [0, L]$, such that $v_t^\theta = V$. Then for all $t' \in [0, t)$, $v_{t'}^\theta = V$. This implies that $v_t^\theta$ is left-continuous in $t$.

Now, we consider $t \in (0, L]$ with $v_t^\theta < V$. Consider any $\epsilon \in (0, V - v_t^\theta)$. The definition of $v_t^\theta$ and the property that $g_v - \theta_v(L-t)$ is continuous in $v$ imply $g_{v_t^\theta} - \theta_{v_t^\theta}(L-t) \geq 0$. Hence, Lemma 5 Part 3 implies that for any $v \in (v_t^\theta, V)$, $g_v - \theta_v(L-t) > 0$. Define $\Delta \triangleq \inf_{v \in [v_t^\theta + \epsilon, V)} (g_v - \theta_v(L-t)) > 0$. Define $\delta \triangleq \frac{\Delta}{2W(V)}$. Note that Assumption 1 Part 1 that $W(v) > 0$ for all $v \in [0, V)$ and the condition $V < \infty$ imply $W(V) \in (0, \infty)$. Hence, $\delta \in (0, \infty)$.

Therefore, for any $t' \in \left[(t-\delta)^+, t\right)$ and any $v \in [v_t^\theta + \epsilon, V)$,

$$g_v - \theta_v(L-t') = g_v - \theta_v(L-t) - \theta_v(t-t') \geq \Delta - W(V)\delta = \frac{\Delta}{2} > 0,$$

where the inequality follows from Lemma 5 Part 1 that $\theta_v \leq W(V)$. Therefore, $v_t^\theta$ is left-continuous in $t$.

Because $v_t^\theta$ is both right-continuous and left-continuous in $t$, it is continuous in $t$.

**Q.E.D.**

*Proof of Lemma 7.*

**Part 1.**

Define $s \triangleq \sup\left\{t \in [0,L] : v_t^\theta = V\right\}$. Following from Lemma 6 Part 2, we have that $v_t^\theta \in (v^*, V)$ for all $t \in (s, L)$, and $v_t^\theta = V$ for all $t \in [0, s)$. Following from the definition of $v_t^\theta$ and the property that $g_v - \theta_v(L-t)$ is continuous in $v$, we have that for all $t \in (s, L)$, $v_t^\theta$ satisfies condition $g_{v_t^\theta} = \theta_{v_t^\theta}(L-t)$.

Therefore, for any $t \in (s, L)$, we have

$$v_t^\theta - W\left(v_t^\theta\right)(L-t) = v_t^\theta - W\left(v_t^\theta\right)\frac{g_{v_t^\theta}}{\theta_{v_t^\theta}} = v_t^\theta - \frac{v_t^\theta - \frac{\bar{F}(v_t^\theta)}{f(v_t^\theta)}}{1 - \frac{\bar{F}(v_t^\theta)}{f(v_t^\theta)}\frac{w(v_t^\theta)}{W(v_t^\theta)}} = \frac{\bar{F}(v_t^\theta)}{f(v_t^\theta)}\frac{1 - \frac{v_t^\theta w(v_t^\theta)}{W(v_t^\theta)}}{1 - \frac{\bar{F}(v_t^\theta)}{f(v_t^\theta)}\frac{w(v_t^\theta)}{W(v_t^\theta)}}.$$

The RHS term is non-decreasing in $t \in (s, L)$ due to following reasons. First, we have Assumption 2 that $\frac{f(v)}{\bar{F}(v)}$ is non-decreasing in $v$. Second, we have Assumption 1 Part 4 that $\frac{vw(v)}{W(v)}$ is non-decreasing in $v$ and the property that Assumption 1 Parts 1 and 3 imply $\frac{vw(v)}{W(v)} < 1$. Third, following from Assumption 1 Part 1 that $W(v) > 0$ for all $v \in [0, V)$ and Part 3 that $W(v)$ is concave in $v$, we have $\frac{d}{dv}\frac{w(v)}{W(v)} = \frac{W(v)w'(v) - w(v)^2}{W(v)^2} \leq 0$. Fourth, Assumption 1 Part 1 that $W(v) > 0$ for all $v \in [0, V)$ and Lemma 5 Part 1 that $\theta_v > 0$ for all $v \geq v^*$ imply that $1 - \frac{\bar{F}(v_t^\theta)}{f(v_t^\theta)}\frac{w(v_t^\theta)}{W(v_t^\theta)} = \frac{\theta_{v_t^\theta}}{W(v_t^\theta)} > 0$ for $v \geq v^*$. Fifth, we have Lemma 6 Part 2 that $v_t^\theta$ is non-increasing in $t$. Therefore, all properties above jointly imply that $v_t^\theta - W\left(v_t^\theta\right)(L-t)$ is non-decreasing in $t \in (s, L)$.

For $t \in [0, s)$, because $v_t^\theta = V$ and Assumption 1 Part 1 assumes that $W(v) \geq 0$ for all $v \in [0, V)$, we have that $v_t^\theta - W\left(v_t^\theta\right)(L-t) = V - W(V)(L-t)$ is non-decreasing in $t \in [0, s)$.

We also note from Lemma 6 Part 2 that $v_t^\theta$ is continuous at $t = s$ and $L$. Therefore, all results above imply that $v_t^\theta - W\left(v_t^\theta\right)(L-t)$ is non-decreasing in $t \in [0, L]$.

**Part 2.**

Following from Lemma 6 Part 1 that $v_t^h$ is non-decreasing in $t$ and Part 2 that $v_t^\theta$ is non-increasing in $t$ and the definition of $\underline{t}_L$, we have that $v_t^\theta < V$ for all $t \in (\underline{t}_L, L]$. In addition, we have Lemma 6

Part 2 that $v_t^\theta > v^*$ for $t < L$ and Lemma 1 that $\underline{t}_L < L$. Hence, $v_t^\theta \in (v^*, V)$ for all $t \in (\underline{t}_L, L)$. The definition of $v_t^\theta$ and the property that $g_v - \theta_v(L-t)$ is continuous in $v$ imply that $v_t^\theta$ satisfies the condition $g_{v_t^\theta} - \theta_{v_t^\theta}(L-t) = 0$.

We notice that

$$g_{v_t^\theta} - \theta_{v_t^\theta}(L-t) = \frac{1}{v_t^\theta} g_{v_t^\theta}\left(v_t^\theta - W(v_t^\theta)(L-t)\right) - \frac{\bar{F}(v_t^\theta)}{f(v_t^\theta)} \frac{W(v_t^\theta)}{v_t^\theta}\left(1 - \frac{v_t^\theta w(v_t^\theta)}{W(v_t^\theta)}\right)(L-t).$$

We note that the properties that $g_v$ is strictly increasing in $v$ and $g_{v^*} = 0$ imply $g_{v_t^\theta} > 0$. In addition, we have Assumption 1 Part 1 that $W(v) \geq 0$ for all $v \in [0, V)$ and the property that Assumption 1 Parts 1 and 3 imply $\frac{vw(v)}{W(v)} < 1$. Therefore, for any $t \in (\underline{t}_L, L)$, $v_t^\theta - W(v_t^\theta)(L-t) \geq 0$. Following from Lemma 6 Part 2 that $v_t^\theta$ is continuous in $t$, we have $v_{\underline{t}_L}^\theta - W\left(v_{\underline{t}_L}^\theta\right)(L - \underline{t}_L) \geq 0$.

**Q.E.D.**

## EC.2. Proof of Proposition 1

We provide the proof of the main result in §3.1, Proposition 1. We begin with proving the first inequality in Proposition 1.

Lemma EC.1. *We have*

$$J^* \leq \bar{J}.$$

*Proof of Lemma EC.1.*

Proving this lemma requires us to compare the policy defined in §2 and the mechanism design problem defined in §3. Note that some notation that appears in both contexts have different definitions, such as $z_\phi$. Therefore, to distinguish the differences of the same notation that appears in both contexts, throughout this proof, we add '∧' on the notation that appears in the mechanism design context.

We use the seller's any feasible policy $\pi \in \Pi$ and customer corresponding purchasing rules $z^\pi = (\tau^\pi, p^\pi, s^\pi)$ to construct the following mechanism $\left(\hat{z}^\pi, \hat{\mathcal{Q}}^{\infty,\pi}, \hat{q}^\pi\right)$:

$$\hat{\mathcal{Q}}^{\infty,\pi} = \mathcal{Q}^{\infty,\pi},\ \hat{z}_\phi^\pi = z_\phi^\pi\ \forall\ \phi,\ \hat{q}_t^\pi = q_t^\pi\ \forall\ t \geq 0.$$

Because mechanism $\left(\hat{z}^\pi, \hat{\mathcal{Q}}^{\infty,\pi}, \hat{q}^\pi\right)$ replicates the seller's decisions and customer purchase decisions under $\pi$, we have $\left(\hat{z}^\pi, \hat{\mathcal{Q}}^{\infty,\pi}, \hat{q}^\pi\right) \in \mathcal{M}$ and $\Pi\left(\hat{z}^\pi, \hat{\mathcal{Q}}^{\infty,\pi}, \hat{q}^\pi\right) = J^{\pi, z^\pi}$.

Now, we show $\left(\hat{z}^\pi, \hat{\mathcal{Q}}^{\infty,\pi}, \hat{q}^\pi\right)$ satisfies the (IC) and (IR) constraints in (2).

For any $\phi$ and $\phi_{v'}$, we have

$$U(\phi,\hat{z}^{\pi}_{\phi}) = U(\phi,z^{\pi}_{\phi}) \geq U(\phi,z^{\pi}_{\phi_{v'}}) = U(\phi,\hat{z}^{\pi}_{\phi_{v'}}),$$

where the first and the second equalities follow from property $\hat{z}^{\pi}=z^{\pi}$, the inequality follows from the property that $z^{\pi}_{\phi}$ is the optimal solution of customer $\phi$'s optimization problem. Therefore, mechanism $\left(\hat{z}^{\pi},\hat{\mathcal{Q}}^{\infty,\pi},\hat{q}^{\pi}\right)$ satisfies the (IC) constraints in (2).

For any $\phi$, define $z^{0}_{\phi}\triangleq\left(t_{\phi},p^{0}_{\phi},s^{0}_{\phi}\right)$. We have

$$U(\phi,\hat{z}^{\pi}_{\phi}) = U(\phi,z^{\pi}_{\phi}) \geq U(\phi,z^{0}_{\phi}) = 0,$$

where the first equality follows from property $\hat{z}^{\pi}=z^{\pi}$, the first inequality follows from the property that $z^{\pi}_{\phi}$ is the optimal solution of customer $\phi$'s optimization problem. Therefore, mechanism $\left(\hat{z}^{\pi},\hat{\mathcal{Q}}^{\infty,\pi},\hat{q}^{\pi}\right)$ satisfies the (IR) constraints in (2).

Therefore, for any $\pi\in\Pi$, we have $J^{\pi,z^{\pi}}=\Pi\left(\hat{z}^{\pi},\hat{\mathcal{Q}}^{\infty,\pi},\hat{q}^{\pi}\right)\leq\bar{J}$. **Q.E.D.**

Next, we prove a series of lemmas that will jointly imply the second inequality in Proposition 1. To lighten the notation in presenting and proving Lemmas EC.2-EC.5, we introduce the following notation

$$a_{\phi}\triangleq\mathbf{1}\left\{s_{\phi}<\infty\right\}.$$

We begin with establishing the following two lemmas.

Lemma EC.2. *If* (IC) *and* (IR) *hold, then for any* $\phi$*,*

$$p_{\phi} = \left(v_{\phi}-W\left(v_{\phi}\right)\left(s_{\phi}-t_{\phi}\right)\right)a_{\phi}-\int_{v'=0}^{v_{\phi}}\left(1-w(v')\left(s_{\phi_{v'}}-t_{\phi_{v'}}\right)\right)a_{\phi_{v'}}dv'.$$

*Proof of Lemma EC.2.*

First, we show that (IR) implies that

$$U\left(\phi_0,z_{\phi_0}\right)=0. \tag{EC.1}$$

To see this notice that by definition and Assumption 1 Part 1 that $W(0)>0$,

$$U\left(\phi_0,z_{\phi_0}\right)=\left(0-p_{\phi_0}-W(0)\left(s_{\phi_0}-t_{\phi_0}\right)\right)a_{\phi_0}\leq 0.$$

But since (IR) requires $U\left(\phi_0,z_{\phi_0}\right)\geq 0$, we must have (EC.1).

Now, define $u(\phi,z) \triangleq \frac{\partial}{\partial v_\phi} U(\phi,z)$. Applying the envelope theorem, we have:

$$\begin{aligned} U\left(\phi, z_\phi\right) &= \int_{v'=0}^{v_\phi} u\left(\phi_{v'}, z_{\phi_{v'}}\right) dv' + U\left(\phi_0, z_{\phi_0}\right) \\ &= \int_{v'=0}^{v_\phi} \left(1 - w(v')\left(s_{\phi_{v'}} - t_{\phi_{v'}}\right)\right) a_{\phi_{v'}} dv' + U\left(\phi_0, z_{\phi_0}\right) \\ &= \int_{v'=0}^{v_\phi} \left(1 - w(v')\left(s_{\phi_{v'}} - t_{\phi_{v'}}\right)\right) a_{\phi_{v'}} dv'. \end{aligned} \tag{EC.2}$$

The first equality follows from Fubini's theorem and the envelope theorem (specifically, Theorem 2 of Milgrom and Segal (2002)). The second equality follows from the definition of $u(\cdot)$, and the final equality follows from (EC.1). Consequently,

$$\begin{aligned} p_\phi &= \left(v_\phi - W\left(v_\phi\right)\left(s_\phi - t_\phi\right)\right) a_\phi - U(\phi, z_\phi) \\ &= \left(v_\phi - W\left(v_\phi\right)\left(s_\phi - t_\phi\right)\right) a_\phi - \int_{v'=0}^{v_\phi} \left(1 - w(v')\left(s_{\phi_{v'}} - t_{\phi_{v'}}\right)\right) a_{\phi_{v'}} dv'. \end{aligned}$$

The first equality follows from the definition of $U(\cdot)$. The second equality follows from our application of the envelope theorem above. **Q.E.D.**

The next lemma establishes a second implication of the constraints (IC) and (IR).

Lemma EC.3. *If* (IC) *and* (IR) *hold, then for any* $\phi$ *with* $v_\phi > 0$*, we have:*

$$\left(1 - w\left(v_\phi\right)\left(s_\phi - t_\phi\right)\right) a_\phi \geq 0.$$

*Proof of Lemma EC.3.*

Consider any $\phi$ and any $v, v' \in [0, V)$. (IC) implies

$$\begin{aligned} U\left(\phi_v, z_{\phi_v}\right) &\geq U\left(\phi_v, z_{\phi_{v'}}\right), \\ U\left(\phi_{v'}, z_{\phi_{v'}}\right) &\geq U\left(\phi_{v'}, z_{\phi_v}\right). \end{aligned}$$

Adding these two inequalities, and writing them explicitly (using the definition of $U(\cdot)$), yields:

$$\begin{aligned} (v - v')\left(a_{\phi_v} - a_{\phi_{v'}}\right) &\geq \left(W(v) - W(v')\right)\left(s_{\phi_v} - t_\phi\right) a_{\phi_v} - \left(W(v) - W(v')\right)\left(s_{\phi_{v'}} - t_\phi\right) a_{\phi_{v'}} \\ &\geq w(v)(v - v')\left(s_{\phi_v} - t_\phi\right) a_{\phi_v} - w(v')(v - v')\left(s_{\phi_{v'}} - t_\phi\right) a_{\phi_{v'}} \\ &= (v - v')\left(w(v)\left(s_{\phi_v} - t_\phi\right) a_{\phi_v} - w(v')\left(s_{\phi_{v'}} - t_\phi\right) a_{\phi_{v'}}\right), \end{aligned}$$

where the second inequality follows from Assumption 1 Part 3 that $W(v)$ is concave in $v$. Thus,

$$(v - v')\left(\left(1 - w(v)\left(s_{\phi_v} - t_\phi\right)\right) a_{\phi_v} - \left(1 - w(v')\left(s_{\phi_{v'}} - t_\phi\right)\right) a_{\phi_{v'}}\right) \geq 0.$$

Therefore, $\left(1-w(v)\left(s_{\phi_v}-t_\phi\right)\right)a_{\phi_v}$ is non-decreasing in $v$. But (EC.2) and (IR) imply that for any $v_\phi \geq 0$,

$$U\left(\phi, z_\phi\right)=\int_{v'=0}^{v_\phi}\left(1-w(v')\left(s_{\phi_{v'}}-t_\phi\right)\right)a_{\phi_{v'}}dv' \geq 0,$$

which with the fact that $\left(1-w(v)\left(s_{\phi_v}-t_\phi\right)\right)a_{\phi_v}$ is non-decreasing in $v$ immediately lets us conclude that

$$\left(1-w(v)\left(s_{\phi_v}-t_\phi\right)\right)a_{\phi_v} \geq 0$$

for all $v>0$. **Q.E.D.**

Now, we use Lemmas EC.2 and EC.3 to establish an upper bound of the optimal mechanism design problem (2). This upper bound only requires us to solve a pure dynamic optimization problem without (IC) constraints.

Lemma EC.4. *If* (IC) *and* (IR) *hold, then*

$$\Pi\left(z,\mathcal{Q},q\right) \leq \liminf_{T\to\infty}\frac{1}{T}\left(\int_{\phi\in\mathcal{H}^T}\left(g_{v_\phi}-\theta_{v_\phi}\left(s_\phi-t_\phi\right)\right)a_\phi - h\int_{t=0}^{T}I_t dt - K|\mathcal{Q}^T|\right).$$

*Proof of Lemma EC.4.*

For any $T>0$, we have

$$\begin{aligned}&\int_{\phi\in\mathcal{H}^T}p_\phi \mathbf{1}\left\{\tau_\phi \leq T\right\}\\ \leq &\int_{\phi\in\mathcal{H}^T}p_\phi\\ = &\int_{\phi\in\mathcal{H}^T}\left(\left(v_\phi - W\left(v_\phi\right)\left(s_\phi-t_\phi\right)\right)a_\phi - \int_{v'=0}^{v_\phi}a_{\phi_{v'}}dv' + \int_{v'=0}^{v_\phi}w(v')\left(s_{\phi_{v'}}-t_{\phi_{v'}}\right)a_{\phi_{v'}}dv'\right)\end{aligned} \tag{EC.3}$$

where the inequality follows from the properties that $p_\phi \geq 0$ and $\tau_\phi \geq t_\phi$, the second equality follows from Lemma EC.2.

For the third term in (EC.3), we have

$$\begin{aligned}\int_{\phi\in\mathcal{H}^T}\int_{v'=0}^{v_\phi}a_{\phi_{v'}}dv' &= \int_{t=0}^{T}\lambda\int_{v=0}^{V}f(v)\int_{v'=0}^{v}a_{(t,v')}dv'dvdt\\ &= \int_{t=0}^{T}\lambda\int_{v'=0}^{V}a_{(t,v')}\int_{v=v'}^{V}f(v)dvdv'dt\\ &= \int_{t=0}^{T}\lambda\int_{v'=0}^{V}\frac{\bar{F}(v')}{f(v')}a_{(t,v')}f(v')dv'dt\\ &= \int_{\phi\in\mathcal{H}^T}\frac{\bar{F}(v_\phi)}{f(v_\phi)}a_\phi.\end{aligned}$$

Here the first equality follows from the fact that $v_\phi$ is independent of $t_\phi$, the second equality follows from an exchange in the order of integration, and the fourth equality again employs the fact that $v_\phi$ is independent of $t_\phi$.

Following the similar analysis, for the fourth term in (EC.3), we have

$$\int_{\phi\in\mathcal{H}^T}\int_{v'=0}^{v_\phi} w(v')\left(s_{\phi_{v'}}-t_{\phi_{v'}}\right)a_{\phi_{v'}}dv' = \int_{\phi\in\mathcal{H}^T}\frac{\bar{F}(v_\phi)}{f(v_\phi)}w\left(v_\phi\right)\left(s_\phi-t_\phi\right)a_\phi.$$

Therefore,

$$\Pi\left(z,\mathcal{Q},q\right)\leq\liminf_{T\to\infty}\frac{1}{T}\left(\int_{\phi\in\mathcal{H}^T}\left(g_{v_\phi}-\theta_{v_\phi}\left(s_\phi-t_\phi\right)\right)a_\phi - h\int_{t=0}^{T}I_t dt - K|\mathcal{Q}^T|\right).$$

**Q.E.D.**

We denote by $\hat{J}$ the optimal value of the following optimization problem:

$$\begin{aligned} \max\nolimits_{(z,\mathcal{Q},q)\in\mathcal{M}} \quad & \liminf_{T\to\infty}\frac{1}{T}\left(\int_{\phi\in\mathcal{H}^T}\left(g_{v_\phi}-\theta_{v_\phi}\left(s_\phi-t_\phi\right)\right)a_\phi - h\int_{t=0}^{T}I_t dt - K|\mathcal{Q}^T|\right) \\ \text{subject to} \quad & \left(1-w\left(v_\phi\right)\left(s_\phi-t_\phi\right)\right)a_\phi\geq 0,\ \forall\ \phi \text{ with } v_\phi>0. \end{aligned} \tag{EC.4}$$

Therefore, Lemmas EC.3 and EC.4 immediately imply $\bar{J}\leq\hat{J}$.

In the next step, we establish an upper bound of $\hat{J}$. We introduce the following notation. We denote by $L_n\left(\mathcal{Q}^\infty\right)$ the $n$-th inventory replenishment time in the replenishment schedule $\mathcal{Q}^\infty$. We make a convention that $L_0\left(\mathcal{Q}^\infty\right)=0$. We denote by $\mathcal{H}_n\triangleq\{\phi: t_\phi\in[L_{n-1}\left(\mathcal{Q}^\infty\right),L_n\left(\mathcal{Q}^\infty\right))\}$ the collection of customers who arrive between the $(n-1)$-th and the $n$-th inventory replenishment times. To ease notation, in the rest of this paper, unless we cause confusion, we suppress the argument $\mathcal{Q}^\infty$ in $L_n\left(\mathcal{Q}^\infty\right)$.

Lemma EC.5. *We have*

$$\hat{J}\leq\max_{\mathcal{Q}^\infty}J^{\mathcal{Q}^\infty},$$

*where*

$$J^{\mathcal{Q}^\infty}\triangleq\liminf_{N\to\infty}\frac{1}{L_N}\left(\sum_{n=1}^{N}\int_{\phi\in\mathcal{H}_n}\left(g_{v_\phi}-\min\left\{h\left(t_\phi-L_{n-1}\right),\theta_{v_\phi}\left(L_n-t_\phi\right)\right\}\right)^+\mathbf{1}\left\{v_\phi\geq v^*\right\}-KN\right).$$

*Proof of Lemma EC.5.*

We complete the proof by taking the following steps.

First, under the demand fulfillment commitment, for any $N \in \mathbb{N}$, we have

$$\int_{t=L_{N-1}}^{L_N} I_t dt \geq \int_{\phi \in \mathcal{H}_n} (s_\phi - L_{n-1}) \mathbf{1}\{s_\phi < L_n\}.$$

Second, we show that for any $v_\phi \in (0, v^*)$, under the constraint that $(1 - w(v_\phi)(s_\phi - t_\phi)) a_\phi \geq 0$, we have

$$\left(g_{v_\phi} - \theta_{v_\phi}(s_\phi - t_\phi)\right) a_\phi \leq 0.$$

Suppose this statement is not true for some $\phi$ with $v_\phi \in (0, v^*)$. Hence, we have

$$\begin{aligned}\left(g_{v_\phi} - \theta_{v_\phi}(s_\phi - t_\phi)\right) a_\phi &= g_{v_\phi}(1 - w(v_\phi)(s_\phi - t_\phi)) a_\phi - W(v_\phi)\left(1 - \frac{v_\phi w(v_\phi)}{W(v_\phi)}\right)(s_\phi - t_\phi) a_\phi \\ &> 0.\end{aligned}$$

Note that we have the following properties. First, following from Assumption 2, we have that $g_{v_\phi}$ is strictly increasing in $v_\phi$. Hence, for $v_\phi < v^*$, $g_{v_\phi} < 0$. Second, Assumption 1 Part 1 that $W(v) > 0$ for $v \geq 0$ and the property that Assumption 1 Parts 1 and 3 imply $\frac{v_\phi w(v_\phi)}{W(v_\phi)} < 1$ jointly imply $W(v_\phi)\left(1 - \frac{v_\phi w(v_\phi)}{W(v_\phi)}\right) > 0$. Hence, the two properties above imply $(1 - w(v_\phi)(s_\phi - t_\phi)) a_\phi < 0$. This contradicts with the property $(1 - w(v_\phi)(s_\phi - t_\phi)) a_\phi \geq 0$. Therefore, for any $v_\phi \in (0, v^*)$, under the constraint that $(1 - w(v_\phi)(s_\phi - t_\phi)) a_\phi \geq 0$, we have

$$\left(g_{v_\phi} - \theta_{v_\phi}(s_\phi - t_\phi)\right) a_\phi \leq 0.$$

Third, following from the results that we derive from the previous two steps, under the constraint that $(1 - w(v_\phi)(s_\phi - t_\phi)) a_\phi \geq 0$ for any $v_\phi > 0$, we have

$$\begin{aligned} J^{\mathcal{Q}^\infty} &= \liminf_{T \to \infty} \frac{1}{T}\left(\int_{\phi \in \mathcal{H}^T} \left(g_{v_\phi} - \theta_{v_\phi}(s_\phi - t_\phi)\right) a_\phi - h\int_{t=0}^{T} I_t dt - K|\mathcal{Q}^T|\right) \\ &= \liminf_{N \to \infty} \frac{1}{L_N}\left(\sum_{n=1}^{N} \int_{\phi \in \mathcal{H}_n} \left(g_{v_\phi} - \theta_{v_\phi}(s_\phi - t_\phi)\right) a_\phi - h\int_{t=L_{N-1}}^{L_N} I_t dt - KN\right) \\ &\leq \liminf_{N \to \infty} \frac{1}{L_N}\left(\sum_{n=1}^{N} \int_{\phi \in \mathcal{H}_n} \left(g_{v_\phi} - \theta_{v_\phi}(s_\phi - t_\phi)\right) a_\phi - h(s_\phi - L_{n-1})\mathbf{1}\{s_\phi < L_n\} - KN\right) \\ &\leq \liminf_{N \to \infty} \frac{1}{L_N}\left(\sum_{n=1}^{N} \int_{\phi \in \mathcal{H}_n} \Big(\left(g_{v_\phi} - \theta_{v_\phi}(s_\phi - t_\phi)\right) a_\phi \right. \\ &\qquad \left. - h(s_\phi - L_{n-1})\mathbf{1}\{s_\phi < L_n\}\Big)\mathbf{1}\{v_\phi \geq v^*\} - KN\right) \\ &= \liminf_{N \to \infty} \frac{1}{L_N}\left(\sum_{n=1}^{N} \int_{\phi \in \mathcal{H}_n} \Big(\left(g_{v_\phi} - \theta_{v_\phi}(s_\phi - t_\phi)\right) a_\phi \right. \end{aligned}$$

$$
\begin{aligned}
&\quad -h\left(s_\phi - L_{n-1}\right)\mathbf{1}\left\{s_\phi < L_n\right\}\Big)\mathbf{1}\left\{v_\phi \geq v^*\right\} - KN\Big) \\
&\leq \liminf_{N\to\infty}\frac{1}{L_N}\left(\sum_{n=1}^{N}\int_{\phi\in\mathcal{H}_n}\Big(\left(g_{v_\phi} - \theta_{v_\phi}\left(s_\phi - t_\phi\right)\right)a_\phi\right. \\
&\quad -h\left(s_\phi - L_{n-1}\right)\mathbf{1}\left\{s_\phi < L_n\right\}\Big)\mathbf{1}\left\{v_\phi \geq v^*\right\} - KN\Big) \\
&\leq \liminf_{N\to\infty}\frac{1}{L_N}\left(\sum_{n=1}^{N}\int_{\phi\in\mathcal{H}_n}\left(g_{v_\phi} - \min\left\{h\left(t_\phi - L_{n-1}\right), \theta_{v_\phi}\left(L_n - t_\phi\right)\right\}\right)^+\mathbf{1}\left\{v_\phi \geq v^*\right\} - KN\right).
\end{aligned}
$$

The first inequality follows from the result that we derive in the first step above. The second inequality follows from the result that we derive in the second step above and property $h > 0$. The third equality follows from the law of total expectation. The third and the fourth inequalities follow from Lemma 5 Part 1 that $\theta_{v_\phi} > 0$ for $v_\phi \geq v^*$ and the property $a_\phi \in \{0,1\}$.

**Q.E.D.**

We notice that $J^{\mathcal{Q}^\infty}$ is the function of the inventory replenishment time schedule, $\mathcal{Q}^\infty$. Next, we explore an inventory replenishment time schedule that maximizes $J^{\mathcal{Q}^\infty}$.

Lemma EC.6. *For any $\mathcal{Q}^\infty$, we have*

$$
J^{\mathcal{Q}^\infty} \leq \max_{L>0} J^L = J^{L^*}.
$$

*Proof of Lemma EC.6.*

For any $\mathcal{Q}^\infty$, we have

$$
J^{\mathcal{Q}^\infty} = \liminf_{N\to\infty}\frac{1}{L_N}\left(\sum_{n=1}^{N}\left(L_n - L_{n-1}\right)J^{L_n - L_{n-1}}\right) \leq \max_{L>0} J^L = J^{L^*}.
$$

**Q.E.D.**

## EC.3. Proofs of Lemmas 1 and 2 and Proposition 2

*Proof of Lemma 1.*

Following from Lemma 6 Parts 1 and 2, we have that $v_t^h - v_t^\theta$ is non-decreasing and continuous in $t \in [0, L]$, with $v_0^h - v_0^\theta = v^* - v_0^\theta < 0$ and $v_L^h - v_L^\theta = v_L^h - v^* > 0$. Therefore, the definition of $\underline{t}_L$ implies

$$
\underline{t}_L \in (0, L), \qquad \begin{cases} v_t^h \leq v_t^\theta & \text{if } t \in [0, \underline{t}_L) \\ v_t^h = v_t^\theta & \text{if } t = \underline{t}_L \\ v_t^h > v_t^\theta & \text{if } t \in (\underline{t}_L, L] \end{cases}.
$$

Following from Assumption 1 Part 1 that $W(v) > 0$ for $v \geq 0$ and the property $V < \infty$, we have $W(V) \in (0,\infty)$. The definition of $\bar{t}_L$ and the properties that $h, W(V) \in (0,\infty)$ imply $\bar{t}_L \in [\underline{t}_L, L)$.

Next, we characterize the properties of $\tilde{v}_t$.

First, we show that $v_t^h < v_t^\theta$ implies $\tilde{v}_t \leq v_t^h$.

Following from the definition of $\tilde{v}_t$, condition $v_t^h < v_t^\theta$ implies $\tilde{v}_t = \sup\{v \in [v^*, V) : \theta_v(L-t) < ht\}$. Consider any $t$ with $v_t^h < v_t^\theta$. Consider any $v \in (v_t^h, v_t^\theta)$. The definition of $v_t^h$ and the property that $g_v$ is strictly increasing in $v$ imply $g_v - ht > 0$. The definition of $v_t^\theta$, property $v_t^\theta > v_t^h \geq v^*$, and the property that $g_v - \theta_v(L-t)$ is continuous in $v$ imply $g_{v_t^\theta} - \theta_{v_t^\theta}(L-t) \leq 0$. Hence, condition $v < v_t^\theta$ and Lemma 5 Part 3 imply $g_v - \theta_v(L-t) < 0$. Therefore, for any $v \in (v_t^h, v_t^\theta)$, $\theta_v(L-t) > ht$. Following from Lemma 5 Part 2 that $\theta_v$ is non-decreasing in $v$, we have that $\theta_v(L-t) > ht$ for all $v \in (v_t^h, V)$. Therefore, $\tilde{v}_t \leq v_t^h$.

Second, we show that $v_t^h > v_t^\theta$ implies $\tilde{v}_t \geq v_t^h$.

Following from the definition of $\tilde{v}_t$, condition $v_t^h > v_t^\theta$ implies $\tilde{v}_t = \sup\{v \in [v^*, V) : \theta_v(L-t) < ht\}$. Consider any $t$ with $v_t^h > v_t^\theta$. Consider any $v \in (v_t^\theta, v_t^h)$. The definition of $v_t^h$ and the property that $g_v$ is strictly increasing in $v$ imply $g_v - ht < 0$. The definition of $v_t^\theta$ implies $g_v - \theta_v(L-t) \geq 0$. Hence, $\theta_v(L-t) < ht$ for all $v \in (v_t^\theta, v_t^h)$. Therefore, $\tilde{v}_t \geq v_t^h$.

Third, we show that for any $t \in (\bar{t}_L, L]$, $\tilde{v}_t = V$.

Consider any $t \in (\bar{t}_L, L]$. Because $\bar{t}_L \geq \underline{t}_L$, we note from the result that we prove earlier in this lemma that $v_t^h > v_t^\theta$ for $t \in (\bar{t}_L, L]$. Following from the definition of $\tilde{v}_t$, condition $v_t^h < v_t^\theta$ implies $\tilde{v}_t = \sup\{v \in [v^*, V) : \theta_v(L-t) < ht\}$.

Consider any $v \in [v^*, V)$, we have

$$\begin{aligned} &\theta_v(L-t) - ht \leq \theta_V(L-t) - ht = W(V)(L-t) - ht \\ &= W(V)L - (h + W(V))\,t < W(V)L - (h + W(V))\,\bar{t}_L \leq 0, \end{aligned}$$

where the first inequality follows from Lemma 5 Part 2 that $\theta_v$ is non-decreasing in $v$, the second inequality follows from Assumption 1 Part 1 that $W(v) > 0$, property $h > 0$, and condition $t > \bar{t}_L$, the third inequality follows from the definition of $\bar{t}_L$. Therefore, for $t \in (\bar{t}_L, L]$, we have $\tilde{v}_t = V$.

**Q.E.D.**

The following lemma will be used to prove Lemma 2.

Lemma EC.7. *For any customer* $(t,v)\in[0,L)\times[v^*,V)$*, we have*

$$(g_v-\min\{ht,\theta_v(L-t)\})^+=\begin{cases} g_v-ht & \text{if } t\in[0,\underline{t}_L],v\in[v_t^h,V),\ \text{ or } t\in(\underline{t}_L,\bar{t}_L],v\in[\tilde{v}_t,V)\\ g_v-\theta_v(L-t) & \text{if } t\in(\underline{t}_L,\bar{t}_L],v\in[v_t^\theta,\tilde{v}_t),\ \text{ or } t\in(\bar{t}_L,L),v\in[v_t^\theta,V)\\ 0 & \text{otherwise}\end{cases}.$$

*Proof of Lemma EC.7.*

First, we analyze the order relation of $g_v$ and $ht$.

Consider the first case that $v\in[v_t^h,V)$. Following from the definition of $v_t^h$ and the property that $g_v$ is continuous in $v$, we have $g_v\geq ht$.

Consider the second case that $v\in[v^*,v_t^h)$. Following from the definition of $v_t^h$ and the property that $g_v$ is strictly increasing in $v$, we have $g_v<ht$.

Second, we analyze the order relation of $g_v$ and $\theta_v(L-t)$.

Consider the first case that $v\in[v_t^\theta,V)$. Following from the definition of $v_t^\theta$ and the property that $g_v$ and $\theta_v$ are continuous in $v$, we have $g_v\geq\theta_v(L-t)$.

Consider the second case that $v\in[v^*,v_t^\theta)$. Following from the definition of $v_t^\theta$ and Lemma 5 Part 3, we have $g_v<\theta_v(L-t)$.

Third, we analyze the order relation of $\theta_v(L-t)$ and $ht$.

We analyze two scenarios that $v_t^h\neq v_t^\theta$ and $v_t^h=v_t^\theta$, respectively. Consider the first scenario that $v_t^h\neq v_t^\theta$. Following from the definition of $\tilde{v}_t$, we have $\tilde{v}_t=\sup\{v\in[v^*,V):\theta_v(L-t)<ht\}$. In this scenario, we consider two cases that $v\in[\tilde{v}_t,V)$ and $v\in[v^*,\tilde{v}_t)$, respectively.

Consider the first case that $v\in[\tilde{v}_t,V)$. Following from the definition of $\tilde{v}_t$ and the property that $\theta_v$ is continuous in $v$, we have $\theta_v(L-t)\geq ht$.

Consider the second case that $v\in[v^*,\tilde{v}_t)$. Following from the definition of $\tilde{v}_t$ and Lemma 5 Part 2 that $\theta_v$ is non-decreasing in $v$, we have $\theta_v(L-t)<ht$.

Consider the second scenario that $v_t^h=v_t^\theta\triangleq v^{**}$. If $v^{**}=V$, then the analysis in this proof above implies that for any $v\in[0,V)$, $(g_v-\min\{ht,\theta_v(L-t)\})^+=0$. Therefore, we only need to analyze the case that $v^{**}<V$. Following from the definition of $\tilde{v}_t$, we have $\tilde{v}_t=v^{**}$. Following from Lemma 6 Parts 1 and 2, we have $v^{**}>v^*$. Therefore, the definition of $v_t^h$, the property that $g_v$ is continuous in $v$, and property $v^{**}\in(v^*,V)$ imply $g_{v^{**}}-ht=0$. The definition of $v_t^\theta$, the property that $g_v$

and $\theta_v$ are continuous in $v$, and property $v^{**} \in (v^*, V)$ imply $g_{v^{**}} - \theta_{v^{**}}(L-t) = 0$. Therefore, $\theta_{v^{**}}(L-t) = ht$. Therefore, following from Lemma 5 Part 2 that $\theta_v$ is non-decreasing in $v$, we have

$$\begin{cases} \theta_v(L-t) \geq ht & \text{if } v \in [\tilde{v}_t, V) \\ \theta_v(L-t) \leq ht & \text{if } v \in [v^*, \tilde{v}_t) \end{cases}.$$

Therefore, all results above complete the proof.

**Q.E.D.**

*Proof of Lemma 2.*

Following from the definition of $J^L$, (3), we have

$$\begin{aligned}
J^L &= \frac{\lambda}{L} \int_{t=0}^{L} \int_{v=v^*}^{V} \left(g_v - \min\left\{ht, \theta_v\left(L-t\right)\right\}\right)^+ f(v) dv dt - \frac{K}{L} \\
&= \frac{\lambda}{L} \Bigg( \int_{t=0}^{\underline{t}_L} \int_{v=v_t^h}^{V} \left(g_v - ht\right) f(v) dv dt \\
&\quad + \int_{t=\underline{t}_L}^{\bar{t}_L} \left( \int_{v=\tilde{v}_t}^{V} \left(g_v - ht\right) f(v) dv + \int_{v=v_t^\theta}^{\tilde{v}_t} \left(g_v - \theta_v(L-t)\right) f(v) dv \right) dt \\
&\quad + \int_{t=\bar{t}_L}^{L} \int_{v=v_t^\theta}^{V} \left(g_v - \theta_v(L-t)\right) f(v) dv dt \Bigg) - \frac{K}{L} \\
&= \frac{\lambda}{L} \Bigg( \int_{t=0}^{\underline{t}_L} \left(v_t^h - ht\right) \bar{F}\left(v_t^h\right) dt + \int_{t=\underline{t}_L}^{\bar{t}_L} \Big( \left(\tilde{v}_t - ht\right) \bar{F}\left(\tilde{v}_t\right) \\
&\quad + \left(v_t^\theta - W\left(v_t^\theta\right)(L-t)\right) \bar{F}\left(v_t^\theta\right) - \left(\tilde{v}_t - W\left(\tilde{v}_t\right)(L-t)\right) \bar{F}\left(\tilde{v}_t\right) \Big) dt \\
&\quad + \int_{t=\bar{t}_L}^{L} \left(v_t^\theta - W\left(v_t^\theta\right)(L-t)\right) \bar{F}\left(v_t^\theta\right) dt \Bigg) - \frac{K}{L} \\
&= \frac{\lambda}{L} \Bigg( \int_{t=0}^{\underline{t}_L} \left(v_t^h - ht\right) \bar{F}\left(v_t^h\right) dt + \int_{t=\underline{t}_L}^{\bar{t}_L} \Big( \left(v_t^\theta - W\left(v_t^\theta\right)(L-t) + W\left(\tilde{v}_t\right)(L-t) - ht\right) \bar{F}\left(\tilde{v}_t\right) \\
&\quad + \left(v_t^\theta - W\left(v_t^\theta\right)(L-t)\right) \left(\bar{F}\left(v_t^\theta\right) - \bar{F}\left(\tilde{v}_t\right)\right) \Big) dt \\
&\quad + \int_{t=\bar{t}_L}^{L} \left(v_t^\theta - W\left(v_t^\theta\right)(L-t)\right) \bar{F}\left(v_t^\theta\right) dt \Bigg) - \frac{K}{L} \\
&= \frac{\lambda}{L} \Bigg( \int_{t=0}^{\underline{t}_L} \left(p_t^{L,h} - ht\right) \bar{F}\left(v_t^h\right) dt + \int_{t=\underline{t}_L}^{\bar{t}_L} \left( \left(p_t^{L,h} - ht\right) \bar{F}\left(\tilde{v}_t\right) + p_t^{L,\theta} \left(\bar{F}\left(v_t^\theta\right) - \bar{F}\left(\tilde{v}_t\right)\right) \right) dt \\
&\quad + \int_{t=\bar{t}_L}^{L} p_t^{L,\theta} \bar{F}\left(v_t^\theta\right) dt \Bigg) - \frac{K}{L}.
\end{aligned}$$

The second equality follows from Lemma EC.7.

**Q.E.D.**

*Proof of Proposition 2.*

Following from the definition of $J^L$, (3), we have

$$J^L = \frac{\lambda}{L}\int_{t=0}^{L}\int_{v=v^*}^{V}\left(g_v - \min\left\{ht, \theta_v\left(L-t\right)\right\}\right)^+ f(v)dvdt - \frac{K}{L}.$$

Therefore,

$$\frac{\partial^2 J^L}{\partial L \partial K} = \frac{1}{L^2} \geq 0.$$

Therefore, following from Topkis's theorem (see Topkis (2011)), we have that $L^*$ is non-decreasing in $K$.

Consider one extreme case that $K = 0$.

For any $L > 0$, we have

$$\begin{aligned} J^L &= \frac{\lambda}{L}\int_{t=0}^{L}\int_{v=v^*}^{V}\left(g_v - \min\left\{ht, \theta_v\left(L-t\right)\right\}\right)^+ f(v)dvdt \\ &\leq \frac{\lambda}{L}\int_{t=0}^{L}\int_{v=v^*}^{V} g_v f(v)dvdt \\ &= \lambda\int_{v=v^*}^{V} g_v f(v)dv \\ &= J^0. \end{aligned}$$

The inequality holds due to following reasons. First, the properties that $g_v$ is strictly increasing in $v$ and $g_{v^*} = 0$ imply $g_v > 0$ for any $v \in [v^*, V)$. Second, we have $h > 0$. Third, we have Lemma 5 Part 1 that $\theta_v > 0$ for $v \geq v^*$.

Consider the other extreme case that $K \to \infty$.

Define $T \triangleq \max\left\{\frac{V}{h}, \frac{V}{\theta_{v^*}}\right\}$. Hence, if $L \geq 2T$, then for any $t \in [T, L-T]$ and any $v \geq v^*$, we have

$$\left(g_v - \min\left\{ht, \theta_v\left(L-t\right)\right\}\right)^+ \leq \left(V - \min\left\{hT, \theta_v T\right\}\right)^+ \leq \left(V - \min\left\{hT, \theta_{v^*} T\right\}\right)^+ = 0,$$

where the first inequality follows from property $g_v = v - \frac{\bar{F}(v)}{f(v)} \leq v \leq V$, condition $t \in [T, L-T]$, property $h > 0$ and Lemma 5 Part 1 that $\theta_v > 0$ for $v \geq v^*$, the second inequality follows from Lemma 5 Part 2 that $\theta_v$ is non-decreasing in $v$ and condition $v \geq v^*$.

For any $L > 0$, define

$$G^L \triangleq \lambda\int_{t=0}^{L}\int_{v=v^*}^{V}\left(g_v - \min\left\{ht, \theta_v\left(L-t\right)\right\}\right)^+ f(v)dvdt.$$

Hence, for $L > 2T$, we have $G^L = G^{2T}$.

Consider any $K > G^{2T}$.

For $L \leq 2T$, we have $J^L = \frac{1}{L}\left(G^L - K\right) \leq \frac{1}{L}\left(G^{2T} - K\right) < 0$, where the first inequality follows from the property that $G^L$ is non-decreasing in $L$.

For $L > 2T$, we have $J^L = \frac{1}{L}\left(G^L - K\right) = \frac{1}{L}\left(G^{2T} - K\right) < 0$, where the second equality follows from the property that $G^L = G^{2T}$ for $L > 2T$.

In addition, we note that $\lim_{L\to\infty} J^L = \lim_{L\to\infty} \frac{1}{L}\left(G^{2T} - K\right) = 0$.

Therefore, for $K > G^{2T}$, $L^* = \infty$.

**Q.E.D.**

## EC.4. Proofs of Lemma 4 and Theorem 2

*Proof of Lemma 4.*

Due to the cyclic nature of policy $\tilde{\pi}_L$, without loss of generality, we only need to characterize the equilibrium behaviors of customers who arrive during the first cycle $[0, L)$.

In this proof, we introduce notation $\phi_{t'} \triangleq (t', v_\phi)$. We extend the definition of $p_t^{L,h}$ to the support $t \in (\underline{t}_L, L)$ as $p_t^{L,h} \triangleq p_t^{L,\theta} + W\left(\tilde{v}_t\right)(L - t)$.

We make the proof by taking the following steps.

**Step 1**: We show that for any customer $\phi \in (0, L) \times [0, V)$, if at time $t_\phi$, $\Omega_{t_\phi}^{\tilde{\pi}_L} = \left\{\left(p^{L,\theta}_{\max\{t_\phi, \underline{t}_L\}}, L\right)\right\}$, then for any $z_\phi$ with $\tau_\phi \in [t_\phi, L)$, $U\left(\phi, z_\phi\right) \leq U\left(\phi, z_\phi^{\tilde{\pi}_L}\right)$.

In this scenario, at any time $t \in [t_\phi, L)$, the seller only offers the delayed delivery option $\Omega_t^{\tilde{\pi}_L} = \left\{\left(p^{L,\theta}_{\max\{t, \underline{t}_L\}}, L\right)\right\}$. This entails that either $t_\phi \in (0, \bar{t}_L]$ and $I_{t_\phi -} = 0$ or $t_\phi \in (\bar{t}_L, L)$. Therefore,

$$\begin{aligned} U\left(\phi, z_\phi\right) &= \left(v_\phi - p^{L,\theta}_{\max\{\tau_\phi, \underline{t}_L\}} - W\left(v_\phi\right)\left(L - t_\phi\right)\right)^+ \\ &\leq \left(v_\phi - p^{L,\theta}_{\max\{t_\phi, \underline{t}_L\}} - W\left(v_\phi\right)\left(L - t_\phi\right)\right)^+. \end{aligned}$$

The inequality follows from Lemma 7 Part 1 that $p_t^{L,\theta} = v_t^\theta - W\left(v_t^\theta\right)(L - t)$ is non-decreasing in $t$.

Now, we show that the RHS is equal to $U\left(\phi, z_\phi^{\tilde{\pi}_L}\right)$. We prove this result by taking the following steps.

**Step 1.1**: Consider the case that $t_\phi \in (\underline{t}_L, L)$.

We have

$$v_\phi - p^{L,\theta}_{\max\{t_\phi, \underline{t}_L\}} - W\left(v_\phi\right)\left(L - t_\phi\right) = v_\phi - p^{L,\theta}_{t_\phi} - W\left(v_\phi\right)\left(L - t_\phi\right)$$

$$
\begin{aligned}
&= \left(v_\phi - W\left(v_\phi\right)\left(L-t_\phi\right)\right) - \left(v^\theta_{t_\phi} - W\left(v^\theta_{t_\phi}\right)\left(L-t_\phi\right)\right) \\
&\begin{cases} \geq 0 & \text{if } v_\phi \geq v^\theta_{t_\phi} \\ \leq 0 & \text{if } v_\phi < v^\theta_{t_\phi} \end{cases},
\end{aligned}
$$

where the first equality follows from condition $t_\phi > \underline{t}_L$, the second equality follows from the definition of $p^{L,\theta}_t$, the inequality follows from the property that Lemma 7 Parts 1 and 2 and condition $t_\phi > \underline{t}_L$ imply $v^\theta_{t_\phi} - W\left(v^\theta_{t_\phi}\right)\left(L-t_\phi\right) \geq 0$, condition $L - t_\phi > 0$, and Lemma 5 Part 4.

Hence,

$$
\begin{aligned}
\left(v_\phi - p^{L,\theta}_{\max\{t_\phi,\underline{t}_L\}} - W\left(v_\phi\right)\left(L-t_\phi\right)\right)^+ &= \left(v_\phi - p^{L,\theta}_{t_\phi} - W\left(v_\phi\right)\left(L-t_\phi\right)\right)\mathbf{1}\left\{v_\phi \geq v^\theta_{t_\phi}\right\} \\
&= U\left(\phi, z^{\tilde{\pi}_L}_\phi\right).
\end{aligned}
$$

**Step 1.2**: Consider the case that $t_\phi \in (0, \underline{t}_L]$ and $v^b_{t_\phi} \in (v^*, V)$.

Following from the definition of $v^b_t$, condition $v^b_{t_\phi} \in (v^*, V)$, and the property that $v - W(v)K$ is continuous in $v$, we have $v^b_{t_\phi} - W\left(v^b_{t_\phi}\right)\left(L-t_\phi\right) = p^{L,\theta}_{\underline{t}_L}$. Following from Lemma 7 Part 2 that $p^{L,\theta}_{\underline{t}_L} = v^\theta_{\underline{t}_L} - W\left(v^\theta_{\underline{t}_L}\right)\left(L-\underline{t}_L\right) \geq 0$, we have $v^b_{t_\phi} - W\left(v^b_{t_\phi}\right)\left(L-t_\phi\right) = p^{L,\theta}_{\underline{t}_L} \geq 0$.

Therefore,

$$
\begin{aligned}
v_\phi - p^{L,\theta}_{\max\{t_\phi,\underline{t}_L\}} - W\left(v_\phi\right)\left(L-t_\phi\right) &= v_\phi - p^{L,\theta}_{\underline{t}_L} - W\left(v_\phi\right)\left(L-t_\phi\right) \\
&= \left(v_\phi - W\left(v_\phi\right)\left(L-t_\phi\right)\right) - \left(v^b_{t_\phi} - W\left(v^b_{t_\phi}\right)\left(L-t_\phi\right)\right) \\
&\quad \begin{cases} \geq 0 & \text{if } v_\phi \geq v^b_{t_\phi} \\ \leq 0 & \text{if } v_\phi < v^b_{t_\phi} \end{cases},
\end{aligned}
$$

where the first equality follows from condition $t_\phi \leq \underline{t}_L$, the inequality follows from property $v^b_{t_\phi} - W\left(v^b_{t_\phi}\right)\left(L-t_\phi\right) \geq 0$, the property that Lemma 1 that $\underline{t}_L < L$ and condition $t_\phi \leq \underline{t}_L$ jointly imply $L - t_\phi > 0$, and Lemma 5 Part 4.

Hence,

$$
\begin{aligned}
\left(v_\phi - p^{L,\theta}_{\max\{t_\phi,\underline{t}_L\}} - W\left(v_\phi\right)\left(L-t_\phi\right)\right)^+ &= \left(v_\phi - p^{L,\theta}_{\underline{t}_L} - W\left(v_\phi\right)\left(L-t_\phi\right)\right)\mathbf{1}\left\{v_\phi \geq v^b_{t_\phi}\right\} \\
&= U\left(\phi, z^{\tilde{\pi}_L}_\phi\right).
\end{aligned}
$$

**Step 1.3**: Consider the case that $t_\phi \in (0, \underline{t}_L]$ and $v^b_{t_\phi} = V$.

Following from the definition of $v^b_t$, condition $v^b_{t_\phi} = V$, and the property that $v - W(v)K$ is continuous in $v$, we have $v^b_{t_\phi} - W\left(v^b_{t_\phi}\right)\left(L-t_\phi\right) \leq p^{L,\theta}_{\underline{t}_L}$.

Suppose there exists $\phi$ with $t_\phi \in (0, \underline{t}_L]$, such that $v_\phi - W(v_\phi)(L - t_\phi) > p^{L,\theta}_{\underline{t}_L}$. Following from Lemma 7 Part 2 that $p^{L,\theta}_{\underline{t}_L} = v^\theta_{\underline{t}_L} - W\left(v^\theta_{\underline{t}_L}\right)(L - \underline{t}_L) \geq 0$, we have $v_\phi - W(v_\phi)(L - t_\phi) > 0$. Therefore,

$$\begin{aligned} v^b_{t_\phi} - W\left(v^b_{t_\phi}\right)(L - t_\phi) &= (V - W(V)(L - t_\phi)) \\ &> (v_\phi - W(v_\phi)(L - t_\phi)) \\ &> p^{L,\theta}_{\underline{t}_L}, \end{aligned}$$

where the first equality follows from condition $v^b_{t_\phi} = V$, the first inequality follows from property $v_\phi - W(v_\phi)(L - t_\phi) > 0$, the property that Lemma 1 that $\underline{t}_L < L$ and condition $t_\phi \leq \underline{t}_L$ jointly imply $L - t_\phi > 0$, property $v_\phi < V$, and Lemma 5 Part 4. This result contradicts with property $v^b_{t_\phi} - W\left(v^b_{t_\phi}\right)(L - t_\phi) \leq p^{L,\theta}_{\underline{t}_L}$.

Therefore,

$$\begin{aligned} \left(v_\phi - p^{L,\theta}_{\max\{t_\phi, \underline{t}_L\}} - W(v_\phi)(L - t_\phi)\right)^+ &= \left(v_\phi - p^{L,\theta}_{\underline{t}_L} - W(v_\phi)(L - t_\phi)\right)^+ \\ &= 0 \\ &= \left(v_\phi - p^{L,\theta}_{\underline{t}_L} - W(v_\phi)(L - t_\phi)\right) \mathbf{1}\{v_\phi \geq V\} \\ &= \left(v_\phi - p^{L,\theta}_{\underline{t}_L} - W(v_\phi)(L - t_\phi)\right) \mathbf{1}\left\{v_\phi \geq v^b_{t_\phi}\right\} \\ &= U\left(\phi, z^{\tilde{\pi}_L}_\phi\right), \end{aligned}$$

where the first equality follows from condition $t_\phi \leq \underline{t}_L$, the second equality follows from the property that we prove above that $v_\phi - W(v_\phi)(L - t_\phi) \leq p^{L,\theta}_{\underline{t}_L}$ if $t_\phi \in (0, \underline{t}_L]$ and $v^b_{t_\phi} = V$, the third equality follows from property $v_\phi < V$, the fourth equality follows from condition $v^b_{t_\phi} = V$.

**Step 1.4**: We show that if $t_\phi \in (0, \underline{t}_L]$, then we must have $v^b_{t_\phi} > v^*$.

For any $t \in (0, \underline{t}_L]$, we have

$$v^* - W(v^*)(L - t) \leq v^* - W(v^*)(L - \underline{t}_L) < v^\theta_{\underline{t}_L} - W\left(v^\theta_{\underline{t}_L}\right)(L - \underline{t}_L) = p^{L,\theta}_{\underline{t}_L},$$

where the first inequality follows from condition $t \leq \underline{t}_L$ and Assumption 1 Part 1 that $W(v) > 0$, the second inequality follows from Lemma 7 Part 2 that $v^\theta_{\underline{t}_L} - W\left(v^\theta_{\underline{t}_L}\right)(L - \underline{t}_L) \geq 0$, Lemma 1 that $\underline{t}_L < L$, Lemma 6 Part 2 that $v^\theta_t > v^*$ for $t \in [0, L)$, property $v^* > 0$, and Lemma 5 Part 4(b).

Therefore, this property, the property that $v - W(v)(L - t)$ is continuous in $v$, and the definition of $v^b_t$ jointly imply $v^b_{t_\phi} > v^*$.

Therefore, all results in Steps 1.1-1.4 jointly imply $U(\phi, z_\phi) \leq U\left(\phi, z^{\tilde{\pi}_L}_\phi\right)$.

**Step 2**: We show that for any customer $\phi \in (\underline{t}_L, \bar{t}_L] \times [0, V)$, if at time $t_\phi$, $\Omega_{t_\phi}^{\tilde{\pi}_L} = \left\{ \left( p_{t_\phi}^{L,h}, t_\phi \right), \left( p_{t_\phi}^{L,\theta}, L \right) \right\}$, then for any $z_\phi$ with $\tau_\phi \in [t_\phi, L)$, $U(\phi, z_\phi) \le U\left(\phi, z_\phi^{\tilde{\pi}_L}\right)$.

In this scenario, because the seller is able to offer a purchase option with the instantaneous product delivery, $\left( p_{t_\phi}^{L,h}, t_\phi \right)$, following from the definition of $\tilde{\pi}_L$, we must have $I_{t_\phi -} > 0$.

We have

$$
\begin{aligned}
U(\phi, z_\phi) &\le \max\left\{ \left( v_\phi - p_{\tau_\phi}^{L,h} - W(v_\phi)(\tau_\phi - t_\phi) \right)^+, \left( v_\phi - p_{\tau_\phi}^{L,\theta} - W(v_\phi)(L - t_\phi) \right)^+ \right\} \\
&= \max\left\{ \left( v_\phi - p_{\tau_\phi}^{L,\theta} - W(\tilde{v}_{\tau_\phi})(L - \tau_\phi) - W(v_\phi)(\tau_\phi - t_\phi) \right)^+, \left( v_\phi - p_{\tau_\phi}^{L,\theta} - W(v_\phi)(L - t_\phi) \right)^+ \right\} \\
&= \begin{cases} \left( v_\phi - p_{\tau_\phi}^{L,\theta} - W(\tilde{v}_{\tau_\phi})(L - \tau_\phi) - W(v_\phi)(\tau_\phi - t_\phi) \right)^+ & \text{if } v_\phi \ge \tilde{v}_{\tau_\phi} \\ \left( v_\phi - p_{\tau_\phi}^{L,\theta} - W(v_\phi)(L - t_\phi) \right)^+ & \text{if } v_\phi < \tilde{v}_{\tau_\phi} \end{cases} \\
&\le \begin{cases} \left( v_\phi - p_{\tau_\phi}^{L,\theta} - W(\tilde{v}_{\tau_\phi})(L - t_\phi) \right)^+ & \text{if } v_\phi \ge \tilde{v}_{\tau_\phi} \\ \left( v_\phi - p_{\tau_\phi}^{L,\theta} - W(v_\phi)(L - t_\phi) \right)^+ & \text{if } v_\phi < \tilde{v}_{\tau_\phi} \end{cases} \\
&= \left( v_\phi - p_{t_\phi}^{L,\theta} - W(\tilde{v}_{t_\phi})(L - t_\phi) \right)^+ \mathbf{1}\{ v_\phi \ge \tilde{v}_{t_\phi} \} + \left( v_\phi - p_{t_\phi}^{L,\theta} - W(v_\phi)(L - t_\phi) \right)^+ \mathbf{1}\{ v_\phi < \tilde{v}_{t_\phi} \} \\
&= \left( v_\phi - p_{t_\phi}^{L,\theta} - W(\tilde{v}_{t_\phi})(L - t_\phi) \right) \mathbf{1}\{ v_\phi \ge \tilde{v}_{t_\phi} \} + \left( v_\phi - p_{t_\phi}^{L,\theta} - W(v_\phi)(L - t_\phi) \right) \mathbf{1}\left\{ v_\phi \in \left[ v_{t_\phi}^\theta, \tilde{v}_{t_\phi} \right) \right\} \\
&= \left( v_\phi - p_{t_\phi}^{L,h} \right) \mathbf{1}\{ v_\phi \ge \tilde{v}_{t_\phi} \} + \left( v_\phi - p_{t_\phi}^{L,\theta} - W(v_\phi)(L - t_\phi) \right) \mathbf{1}\left\{ v_\phi \in \left[ v_{t_\phi}^\theta, \tilde{v}_{t_\phi} \right) \right\} \\
&= U\left(\phi, z_\phi^{\pi_L}\right).
\end{aligned}
$$

The first equality follows from the definition of $p_t^{L,h}$. The second equality, the second inequality and the third equality follow from Assumption 2 Part 1 that $W(\cdot) > 0$, Part 3 that $W(v)$ is nondecreasing in $v$, and the property that $t_\phi \le \tau_\phi \le L$. The fourth equality follows from the property that Lemma 7 Parts 1 and 2 and condition $t_\phi > \underline{t}_L$ imply $p_{t_\phi}^{L,\theta} = v_{t_\phi}^\theta - W\left(v_{t_\phi}^\theta\right)(L - t_\phi) \ge 0$, the property that Lemma 1 that $\bar{t}_L < L$ and condition $t_\phi \le \bar{t}_L$ imply $L - t_\phi > 0$, and Lemma 5 Part 4. The fifth equality follows from the definition of $p_t^{L,h}$.

**Step 3**: We show that for any customer $\phi \in [0, \underline{t}_L] \times [0, V)$, if at time $t_\phi$, $\Omega_{t_\phi}^{\tilde{\pi}_L} = \left\{ \left( p_{t_\phi}^{L,h}, t_\phi \right) \right\}$, then for any $z_\phi$ with $\tau_\phi \in [t_\phi, L)$, $U(\phi, z_\phi) \le U\left(\phi, z_\phi^{\tilde{\pi}_L}\right)$.

In this scenario, because the seller is able to offer a purchase option with the instantaneous product delivery, $\left( p_{t_\phi}^{L,h}, t_\phi \right)$, following from the definition of $\tilde{\pi}_L$, we must have $I_{t_\phi -} > 0$.

We prove this result by taking the following steps.

**Step 3.1**: Consider the case that $\tau_\phi \in [t_\phi, \underline{t}_L]$.

We have

$$\begin{aligned}
U\left(\phi, z_\phi\right) &\leq \max\left\{\left(v_\phi - p_{\tau_\phi}^{L,h} - W\left(v_\phi\right)\left(\tau_\phi - t_\phi\right)\right)^+, \left(v_\phi - p_{\underline{t}_L}^{L,\theta} - W\left(v_\phi\right)\left(L - t_\phi\right)\right)^+\right\} \\
&= \max\left\{\left(v_\phi - v_{\tau_\phi}^{h} - W\left(v_\phi\right)\left(\tau_\phi - t_\phi\right)\right)^+, \left(v_\phi - p_{\underline{t}_L}^{L,\theta} - W\left(v_\phi\right)\left(L - t_\phi\right)\right)^+\right\} \\
&\leq \max\left\{\left(v_\phi - v_{t_\phi}^{h}\right)^+, \left(v_\phi - p_{\underline{t}_L}^{L,\theta} - W\left(v_\phi\right)\left(L - t_\phi\right)\right)^+\right\} \\
&= \max\left\{\left(v_\phi - v_{t_\phi}^{h}\right)^+, \left(\left(v_\phi - W\left(v_\phi\right)\left(L - t_\phi\right)\right) - \left(v_{\underline{t}_L}^{\theta} - W\left(v_{\underline{t}_L}^{\theta}\right)\left(L - \underline{t}_L\right)\right)\right)^+\right\} \\
&\leq \max\left\{\left(v_\phi - v_{t_\phi}^{h}\right)^+, \left(\left(v_\phi - W\left(v_\phi\right)\left(L - \underline{t}_L\right)\right) - \left(v_{\underline{t}_L}^{\theta} - W\left(v_{\underline{t}_L}^{\theta}\right)\left(L - \underline{t}_L\right)\right)\right)^+\right\} \\
&\leq \max\left\{\left(v_\phi - v_{t_\phi}^{h}\right)^+, \left(v_\phi - v_{\underline{t}_L}^{\theta}\right)^+\right\} \\
&= \max\left\{\left(v_\phi - v_{t_\phi}^{h}\right)^+, \left(v_\phi - v_{\underline{t}_L}^{h}\right)^+\right\} \\
&= \left(v_\phi - v_{t_\phi}^{h}\right)^+ \\
&= U\left(\phi, z_\phi^{\tilde{\pi}_L}\right).
\end{aligned}$$

The first equality follows from the definition of $p_t^{L,h}$. The second inequality follows from Lemma 6 Part 1 that $v_t^h$ is non-decreasing in $t$, property $\tau_\phi \geq t_\phi$, and Assumption 1 Part 1 that $W(v) > 0$. The second equality follows from the definition of $p_t^{L,\theta}$. The third inequality follows from Assumption 1 Part 1 that $W(v) > 0$ and condition $t_\phi \leq \underline{t}_L$. The fourth inequality follows from Lemma 7 Part 2 that $v_{\underline{t}_L}^{\theta} - W\left(v_{\underline{t}_L}^{\theta}\right)\left(L - \underline{t}_L\right) \geq 0$, Lemma 1 that $L - \underline{t}_L > 0$, and Lemma 5 Part 4. The third equality follows from Lemma 1 that $v_{\underline{t}_L}^{h} = v_{\underline{t}_L}^{\theta}$. The fourth equality follows from Lemma 6 Part 1 that $v_t^h$ is non-decreasing in $t$ and property $t_\phi \leq \underline{t}_L$.

**Step 3.2**: Consider the case that $\tau_\phi \in (\underline{t}_L, L)$ and $t_\phi = \underline{t}_L$.

For any $t' \in (\underline{t}_L, \tau_\phi]$, we have

$$\begin{aligned}
U\left(\phi, z_\phi\right) &= U\left(\phi_{t'}, z_\phi\right) - W\left(v_\phi\right)\left(t' - \underline{t}_L\right) \\
&\leq U\left(\phi_{t'}, z_{\phi_{t'}}^{\tilde{\pi}_L}\right) - W\left(v_\phi\right)\left(t' - \underline{t}_L\right) \\
&\leq U\left(\phi_{t'}, z_{\phi_{t'}}^{\tilde{\pi}_L}\right) \\
&\leq \max\left\{\left(v_\phi - p_{t'}^{L,h}\right)^+, \left(v_\phi - p_{t'}^{L,\theta} - W\left(v_\phi\right)\left(L - t'\right)\right)^+\right\} \\
&= \max\left\{\left(v_\phi - p_{t'}^{L,\theta} - W\left(\tilde{v}_{t'}\right)\left(L - t'\right)\right)^+, \left(v_\phi - p_{t'}^{L,\theta} - W\left(v_\phi\right)\left(L - t'\right)\right)^+\right\} \\
&\leq \max\left\{\left(v_\phi - p_{t'}^{L,\theta} - W\left(v_{t'}^{h}\right)\left(L - t'\right)\right)^+, \left(v_\phi - p_{t'}^{L,\theta} - W\left(v_\phi\right)\left(L - t'\right)\right)^+\right\}.
\end{aligned}$$

The first inequality follows from results that we prove in Steps 1-2 above. The second inequality follows from Assumption 1 Part 1 that $W(v) > 0$. The second equality follows from the definition of $p_t^{L,h}$. The fourth inequality follows from Lemma 1 that $\tilde{v}_t \geq v_t^h$ if $t > \underline{t}_L$ and Assumption 1 Part 3 that $W(v)$ is non-decreasing in $v$.

Therefore,

$$
\begin{aligned}
U(\phi, z_\phi) &\leq \limsup_{t' \to \underline{t}_L+} \max\left\{\left(v_\phi - p_{t'}^{L,\theta} - W\left(v_{t'}^h\right)(L - t')\right)^+, \left(v_\phi - p_{t'}^{L,\theta} - W(v_\phi)(L - t')\right)^+\right\} \\
&= \max\left\{\left(v_\phi - p_{\underline{t}_L}^{L,\theta} - W\left(v_{\underline{t}_L}^h\right)(L - \underline{t}_L)\right)^+, \left(v_\phi - p_{\underline{t}_L}^{L,\theta} - W(v_\phi)(L - \underline{t}_L)\right)^+\right\} \\
&= \max\left\{\left(v_\phi - v_{\underline{t}_L}^h\right)^+, \left((v_\phi - W(v_\phi)(L - \underline{t}_L)) - \left(v_{\underline{t}_L}^\theta - W\left(v_{\underline{t}_L}^\theta\right)(L - \underline{t}_L)\right)\right)^+\right\} \\
&\leq \max\left\{\left(v_\phi - v_{\underline{t}_L}^h\right)^+, \left(v_\phi - v_{\underline{t}_L}^\theta\right)^+\right\} \\
&= \max\left\{\left(v_\phi - v_{\underline{t}_L}^h\right)^+, \left(v_\phi - v_{\underline{t}_L}^h\right)^+\right\} \\
&= \left(v_\phi - v_{\underline{t}_L}^h\right)^+ \\
&= U\left(\phi, z_\phi^{\tilde{\pi}_L}\right).
\end{aligned}
$$

The first equality follows from Lemma 6 Part 1 that $v_t^h$ is continuous in $t$, Part 2 that $v_t^\theta$ is continuous in $t$, and Assumption 1 Part 2 that $W(v)$ is continuous in $v$. The second equality follows from the definition of $p_{\underline{t}_L}^{L,\theta}$ and Lemma 1 that $v_{\underline{t}_L}^h = v_{\underline{t}_L}^\theta$. The second inequality follows from Lemma 7 Part 2 that $v_{\underline{t}_L}^\theta - W\left(v_{\underline{t}_L}^\theta\right)(L - \underline{t}_L) \geq 0$, Lemma 1 that $L - \underline{t}_L > 0$, and Lemma 5 Part 4(a). The third equality follows from Lemma 1 that $v_{\underline{t}_L}^h = v_{\underline{t}_L}^\theta$.

**Step 3.3**: Consider the case that $\tau_\phi \in (\underline{t}_L, L)$ and $t_\phi \in [0, \underline{t}_L)$.

We have

$$
\begin{aligned}
U(\phi, z_\phi) &= U\left(\phi_{\underline{t}_L}, z_\phi\right) - W(v_\phi)(\underline{t}_L - t_\phi) \\
&\leq U\left(\phi_{\underline{t}_L}, z_{\phi_{\underline{t}_L}}^{\tilde{\pi}_L}\right) - W(v_\phi)(\underline{t}_L - t_\phi) = U\left(\phi, z_{\phi_{\underline{t}_L}}^{\tilde{\pi}_L}\right) \leq U\left(\phi, z_\phi^{\tilde{\pi}_L}\right).
\end{aligned}
$$

The first inequality follows from the result that we prove in Step 3.2 above. The second inequality follows from the result that we prove in Step 3.1 above.

Therefore, all results in Steps 3.1-3.3 jointly imply $U(\phi, z_\phi) \leq U\left(\phi, z_\phi^{\tilde{\pi}_L}\right)$.

**Step 4**: We show that for any customer $\phi \in [0, L) \times [0, V)$, for any $z_\phi$ with $\tau_\phi = L$, $U(\phi, z_\phi) \leq U\left(\phi, z_\phi^{\tilde{\pi}_L}\right)$.

We have

$$\lim_{t\to L-} p_t^{L,\theta} = \lim_{t\to L-} v_t^\theta - W\left(v_t^\theta\right)(L-t) = v^* = v_0^h = p_0^{L,h},$$

where the first equality follows from the definition of $p_t^{L,\theta}$, the second equality follows from Lemma 6 Part 2 that $v_t^\theta$ is continuous in $t$ and $v_L^\theta = v^*$, the third equality follows from Lemma 6 Part 1, the fourth equality follows from the definition of $p_0^{L,h}$.

Therefore, this result, the definition of $\tilde{\pi}_L$ that $\Omega_L^{\tilde{\pi}_L} = \left\{\left(p_0^{L,h}, L\right)\right\}$, and all results that we prove in Steps 1-3 jointly imply $U(\phi, z_\phi) \le U\left(\phi, z_\phi^{\tilde{\pi}_L}\right)$.

**Step 5**: We show that for any customer $\phi \in [0,L)\times[0,V)$, for any $z_\phi$ with $\tau_\phi = nL+\tau$ for $n\in\mathbb{N}$ and $\tau\in(0,L]$, $U(\phi, z_\phi) \le U\left(\phi, z_\phi^{\tilde{\pi}_L}\right)$.

First, for any $n\in\mathbb{N}$, we have

$$U(\phi, z_\phi) = U(\phi_{nL}, z_\phi) + W(v_\phi)nL \le U\left(\phi_{nL}, z_{\phi_{nL}}^{\tilde{\pi}_L}\right) + W(v_\phi)nL = U\left(\phi, z_{\phi_{nL}}^{\tilde{\pi}_L}\right),$$

where the inequality follows from all results that we prove in Steps 1-4 above.

Second, for any $n\in\mathbb{N}$, we have

$$U\left(\phi, z_{\phi_{(n+1)L}}^{\tilde{\pi}_L}\right) = U\left(\phi_{nL}, z_{\phi_{(n+1)L}}^{\tilde{\pi}_L}\right) + W(v_\phi)nL \le U\left(\phi_{nL}, z_{\phi_{nL}}^{\tilde{\pi}_L}\right) + W(v_\phi)nL = U\left(\phi, z_{\phi_{nL}}^{\tilde{\pi}_L}\right),$$

where the inequality follows from the result that we prove in Step 4 above.

Therefore, for any $z_\phi$ with $\tau_\phi = nL+\tau$ for $n\in\mathbb{N}$ and $\tau\in(0,L]$, we have

$$U(\phi, z_\phi) \le U\left(\phi, z_{\phi_{nL}}^{\tilde{\pi}_L}\right) \le U\left(\phi, z_{\phi_L}^{\tilde{\pi}_L}\right) \le U\left(\phi, z_\phi^{\tilde{\pi}_L}\right),$$

where the first inequality follows from the first result that we prove in this step, the second inequality follows from the second result that we prove in this step, the third inequality follows from the result that we prove in Step 4 above.

All results that we prove in Steps 1-5 complete the proof of this lemma.

**Q.E.D.**

*Proof of Theorem 2.*

We use the mechanism design methodology to prove the first inequality of this theorem, $\tilde{J}^* \le J^{L^*}$. The mechanism design problem used in this proof is very similar to the mechanism design problem (2) studied in the deterministic setting, except that we replace every customer's dominant

(IC) and (IR) constraints by his Bayesian (IC) and (IR) constraints that are only conditional on this customer's type and all public information at time 0. The proof is analogous to the proof of Theorem 1. Therefore, we omit the proof.

Now, we prove the second inequality of this theorem.

Consider policy $\tilde{\pi}_L$ with any $L > 0$. Due to the cyclic nature of policy $\tilde{\pi}_L$, the seller's long-run average expected profit under policy $\tilde{\pi}_L$ is the same as her average expected profit over $[0, L)$. Therefore, without loss of generality, we only need to compute the seller's average expected profit over the first cycle $[0, L)$.

For each $t \in [0, \bar{t}_L]$, we denote by

$$N_t^h \triangleq \begin{cases} \sum_\phi \mathbf{1}\left\{t_\phi \in [0,t], v_\phi \geq v_t^h\right\} & \text{if } t \in [0, \underline{t}_L] \\ \sum_\phi \mathbf{1}\left\{t_\phi \in [0,\underline{t}_L], v_\phi \geq v_t^h\right\} + \mathbf{1}\left\{t_\phi \in (\underline{t}_L, t], v_\phi \geq \tilde{v}_t\right\} & \text{if } t \in (\underline{t}_L, \bar{t}_L] \end{cases}$$

the total number of customers who arrive no later than time $t$ and choose the instantaneous delivery option, assuming that the seller has inventory on hand up to time $t$. Thus, $N_t^h$ is a Poisson random variable with parameter

$$\lambda_t^h = \begin{cases} \lambda \int_{t'=0}^{t} \bar{F}\left(v_{t'}^h\right) dt' & \text{if } t \in [0, \underline{t}_L] \\ \lambda \int_{t'=0}^{\underline{t}_L} \bar{F}\left(v_{t'}^h\right) dt' + \lambda \int_{t'=\underline{t}_L}^{t} \bar{F}\left(\tilde{v}_{t'}\right) dt' & \text{if } t \in (\underline{t}_L, \bar{t}_L] \end{cases}.$$

For each $t \in (\underline{t}_L, L)$, we denote by

$$N_t^\theta \triangleq \begin{cases} \mathbf{1}\left\{t_\phi \in (\underline{t}_L, t], v_\phi \in [v_t^\theta, \tilde{v}_t)\right\} & \text{if } t \in (\underline{t}_L, \bar{t}_L] \\ \mathbf{1}\left\{t_\phi \in (\underline{t}_L, \bar{t}_L], v_\phi \in [v_t^\theta, \tilde{v}_t)\right\} + \mathbf{1}\left\{t_\phi \in (\bar{t}_L, t], v_\phi \geq v_t^\theta\right\} & \text{if } t \in (\bar{t}_L, L) \end{cases}$$

the total number of customers who arrive after time $\underline{t}_L$ and no later than time $t$ and choose the delayed delivery option, assuming that the seller has inventory on hand up to time $t$. Thus, $N_t^\theta$ is a Poisson random variable with parameter

$$\lambda_t^\theta = \begin{cases} \lambda \int_{t'=\underline{t}_L}^{t} \left(\bar{F}\left(v_{t'}^\theta\right) - \bar{F}\left(\tilde{v}_{t'}\right)\right) dt' & \text{if } t \in (\underline{t}_L, \bar{t}_L] \\ \lambda \int_{t'=\underline{t}_L}^{\bar{t}_L} \left(\bar{F}\left(v_{t'}^\theta\right) - \bar{F}\left(\tilde{v}_{t'}\right)\right) dt' + \lambda \int_{t'=\bar{t}_L}^{t} \bar{F}\left(v_{t'}^\theta\right) dt' & \text{if } t \in (\bar{t}_L, L) \end{cases}.$$

Therefore, under policy $\tilde{\pi}_L$, given customer behaviors characterized by Lemma 4, we have

$$\begin{aligned} \tilde{J}^L \geq{} & \frac{1}{L} \mathsf{E}\Big[\int_{t=0}^{\bar{t}_L} p_t^{L,h} dN_t^h - V\left(N_{\bar{t}_L}^h - q_0^{\tilde{\pi}_L}\right)^+ + \int_{t=\underline{t}_L}^{L} p_t^{L,\theta} dN_t^\theta - h \int_{t=0}^{\bar{t}_L} \left(q_0^{\tilde{\pi}_L} - N_t^h\right)^+ dt \\ & -h\left(L - \bar{t}_L\right)\left(q_0^{\tilde{\pi}_L} - N_{\bar{t}_L}^h\right)^+\Big] - \frac{K}{L} \\ ={} & \frac{1}{L} \mathsf{E}\Big[\int_{t=0}^{\bar{t}_L} p_t^{L,h} d\lambda_t^h - V\left(N_{\bar{t}_L}^h - q_0^{\tilde{\pi}_L}\right)^+ + \int_{t=\underline{t}_L}^{L} p_t^{L,\theta} d\lambda_t^\theta - h \int_{t=0}^{\bar{t}_L} \left(q_0^{\tilde{\pi}_L} - N_t^h\right)^+ dt \end{aligned}$$

$$
\begin{aligned}
&\quad -h\left(L-\bar{t}_L\right)\left(q_0^{\tilde{\pi}_L}-N_{\bar{t}_L}^h\right)^+\Bigg]-\frac{K}{L}\\
&=\frac{1}{L}\mathsf{E}\Bigg[\int_{t=0}^{\bar{t}_L} p_t^{L,h}d\lambda_t^h-V\left(N_{\bar{t}_L}^h-\lambda_{\bar{t}_L}^h\right)^++\int_{t=\underline{t}_L}^{L} p_t^{L,\theta}d\lambda_t^\theta-h\int_{t=0}^{\bar{t}_L}\left(\lambda_{\bar{t}_L}^h-N_t^h\right)^+dt\\
&\quad -h\left(L-\bar{t}_L\right)\left(\lambda_{\bar{t}_L}^h-N_{\bar{t}_L}^h\right)^+\Bigg]-\frac{K}{L}.
\end{aligned}
$$

The first inequality follows from the property that the RHS of this inequality does not take into account the revenue from those customers who intend to select the instantaneous delivery purchase option but eventually select the delayed delivery purchase option due to inventory unavailability. The first equality follows from Brémaud (1981) Theorem II. The second equality follows from property $q_0^{\tilde{\pi}_L}=\lambda_{\bar{t}_L}^h$.

We establish upper bounds of the second, fourth, and fifth terms on the RHS. For the second term, we have

$$
\mathsf{E}\left[\left(N_{\bar{t}_L}^h-\lambda_{\bar{t}_L}^h\right)^+\right]\le\frac{\sqrt{\lambda_{\bar{t}_L}^h}}{2}\le\frac{\sqrt{\lambda L}}{2},
$$

where the first inequality follows from Gallego and Van Ryzin (1994) Equation (18).

For the fourth and the fifth terms, for any $t\in[0,\bar{t}_L]$, we have

$$
\begin{aligned}
&\mathsf{E}\left[\left(\lambda_{\bar{t}_L}^h-N_t^h\right)^+\right]=\mathsf{E}\left[\left(N_t^h-\lambda_{\bar{t}_L}^h\right)^+\right]+\mathsf{E}\left[\lambda_{\bar{t}_L}^h-N_t^h\right]=\mathsf{E}\left[\left(N_t^h-\lambda_{\bar{t}_L}^h\right)^+\right]+\lambda_{\bar{t}_L}^h-\lambda_t^h\\
&\le\tfrac{1}{2}\left(\sqrt{\lambda_t^h+\left(\lambda_{\bar{t}_L}^h-\lambda_t^h\right)^2}-\left(\lambda_{\bar{t}_L}^h-\lambda_t^h\right)\right)+\lambda_{\bar{t}_L}^h-\lambda_t^h\le\tfrac{\sqrt{\lambda_t^h}}{2}+\lambda_{\bar{t}_L}^h-\lambda_t^h\le\tfrac{\sqrt{\lambda L}}{2}+\lambda_{\bar{t}_L}^h-\lambda_t^h,
\end{aligned}
$$

where the first inequality follows from Gallego and Van Ryzin (1994) Equation (18), the second inequality follows from the property that for $K\ge 0$ and $\sigma\ge 0$, $\frac{\sqrt{\sigma^2+K^2}-K}{2}=\frac{\sigma^2}{2(\sqrt{\sigma^2+K^2}+K)}\le\frac{\sigma^2}{2\sigma}=\frac{\sigma}{2}$.

Therefore,

$$
\begin{aligned}
\tilde{J}^L&\ge\frac{1}{L}\left(\int_{t=0}^{\bar{t}_L}p_t^{L,h}d\lambda_t^h+\int_{t=\underline{t}_L}^{L}p_t^{L,\theta}d\lambda_t^\theta-h\int_{t=0}^{\bar{t}_L}\left(\lambda_{\bar{t}_L}^h-\lambda_t^h\right)dt\right)-\frac{K}{L}-\frac{V}{2}\sqrt{\frac{\lambda}{L}}-\frac{hL}{2}\sqrt{\frac{\lambda}{L}}\\
&=\frac{1}{L}\left(\int_{t=0}^{\bar{t}_L}p_t^{L,h}d\lambda_t^h+\int_{t=\underline{t}_L}^{L}p_t^{L,\theta}d\lambda_t^\theta-h\int_{t=0}^{\bar{t}_L}td\lambda_t^h\right)-\frac{K}{L}-\frac{V}{2}\sqrt{\frac{\lambda}{L}}-\frac{hL}{2}\sqrt{\frac{\lambda}{L}}\\
&=\frac{1}{L}\left(\int_{t=0}^{\bar{t}_L}\left(p_t^{L,h}-ht\right)d\lambda_t^h+\int_{t=\underline{t}_L}^{L}p_t^{L,\theta}d\lambda_t^\theta\right)-\frac{K}{L}-\frac{V+hL}{2}\sqrt{\frac{\lambda}{L}}\\
&=J^L-\frac{V+hL}{2}\sqrt{\frac{\lambda}{L}},
\end{aligned}
$$

where the first equality follows from the formula for integration by parts.

Therefore,

$$\frac{\tilde{J}^{L^*}}{J^{L^*}} \geq 1 - \frac{V + hL^*}{2J^{L^*}} \sqrt{\frac{\lambda}{L^*}}.$$

Now, we consider the sequence of instances defined in this theorem. Because $\lambda^{(i)} = i\lambda$ and $K^{(i)} = iK$, for any $L > 0$, $J^{L,(i)} = iJ^L$. Hence, $L^{*,(i)} = L^*$. Therefore,

$$\frac{\tilde{J}^{L^*}(i)}{J^{L^*}(i)} \geq 1 - O\left(\frac{1}{\sqrt{i}}\right).$$

**Q.E.D.**