A Appendix: Moment Matrices for Common Uncertainty Sets

This appendix elaborates on the calculation of the moment matrix $\mathbf{\Sigma}_A$ in the dual problem $\underline{\mathcal{P}}(\Xi_S)$. We first present the moment matrices $\mathbf{\Sigma}_A$ of several commonly used primitive uncertainty sets in Section A.1. Afterwards, Section A.2 shows how a moment matrix changes if particular classes of transformations are applied to an uncertainty set.

A.1 Primitive Uncertainty Sets

In the following, we list the moment matrices $\mathbf{\Sigma}_A$ of several commonly used primitive uncertainty sets. Since the derivations are basic but tedious, we omit them for the sake of brevity. In the following, we denote by $\lfloor x \rfloor$ and $\lceil x \rceil$ the largest (smallest) integer less than or equal to (larger than or equal to) $x \in \mathbb{R}$.

A.1.1 1-Norm Ball Uncertainty Sets

For a 1-norm ball uncertainty set of the form

$$\Xi = \left\{ \boldsymbol{\xi} \in \mathbb{R}^k \,:\, \|\boldsymbol{\xi}\|_1 \leq 1 \right\},$$

the set of extreme points satisfies

$$\operatorname{ext} \left\{ \boldsymbol{\xi} \in \mathbb{R}^k \,:\, \|\boldsymbol{\xi}\|_1 \leq 1 \right\} \;=\; \operatorname{ext} \left\{ \boldsymbol{\xi} \in \mathbb{R}^k \,:\, \sum_{i=1}^{k} |\xi_i| \leq 1 \right\} \;=\; \left\{ \pm \mathbf{e}_i \in \mathbb{R}^k \,:\, i = 1, \ldots, k \right\},$$

and we therefore obtain the moment matrix

$$\mathbf{\Sigma}_A \;=\; \begin{pmatrix} 2k & \mathbf{0}^\top \\ \mathbf{0} & 2 \cdot \boldsymbol{I} \end{pmatrix}.$$

A.1.2 ∞-Norm Ball Uncertainty Sets

We now consider an ∞-norm ball uncertainty set of the form

$$\Xi = \left\{ \boldsymbol{\xi} \in \mathbb{R}^k \,:\, \|\boldsymbol{\xi}\|_\infty \leq 1 \right\}.$$

Uncertainty sets of this type are commonly employed to describe the demand in operations management applications [8, 34] or when the uncertainty underlying the parameters $\boldsymbol{\xi}$ is described by a factor model [35]. The set of extreme points satisfies

$$\operatorname{ext}\left\{\boldsymbol{\xi} \in \mathbb{R}^k \,:\, \|\boldsymbol{\xi}\|_\infty \leq 1\right\} \;=\; \operatorname{ext}\left\{\boldsymbol{\xi} \in \mathbb{R}^k \,:\, \max_{i=1,\ldots,k} |\xi_i| \leq 1\right\} \;=\; \{-1, 1\}^k,$$

and we therefore obtain the moment matrix

$$\boldsymbol{\Sigma}_A \;=\; \begin{pmatrix} 2^k & \mathbf{0}^\top \\ \mathbf{0} & 2^k \cdot \boldsymbol{I} \end{pmatrix}.$$

A.1.3 $(1 \cap \infty)$-Norm Ball Uncertainty Sets

We now consider an uncertainty set that emerges from the intersection of a scaled 1-norm ball and an ∞-norm ball:

$$\Xi = \left\{\boldsymbol{\xi} \in \mathbb{R}^k \,:\, \|\boldsymbol{\xi}\|_1 \leq \kappa, \;\; \|\boldsymbol{\xi}\|_\infty \leq 1\right\}$$

For $\kappa \leq 1$ and $\kappa \geq k$, the uncertainty set reduces to a scaled 1-norm ball and an ∞-norm ball, respectively. We therefore assume that $\kappa \in (1, k)$. Uncertainty sets of this type are commonly used as polyhedral outer approximations of ellipsoidal uncertainty sets [29, 35].

If $\kappa \in \mathbb{N}$, we obtain the moment matrix

$$\boldsymbol{\Sigma}_A \;=\; \begin{pmatrix} 2^\kappa \binom{k}{\kappa} & \mathbf{0}^\top \\ \mathbf{0} & 2^\kappa \dbinom{k-1}{\kappa-1} \cdot \boldsymbol{I} \end{pmatrix}.$$

For fractional κ, on the other hand, the moment matrix is

$$\boldsymbol{\Sigma}_A \;=\; \begin{pmatrix} 2^{\lceil \kappa \rceil} \lceil \kappa \rceil \binom{k}{\lceil \kappa \rceil} & \mathbf{0}^\top \\ \mathbf{0} & 2^{\lceil \kappa \rceil} \dbinom{k-1}{\lfloor \kappa \rfloor} \left(\lfloor \kappa \rfloor + (\kappa - \lfloor \kappa \rfloor)^2\right) \cdot \boldsymbol{I} \end{pmatrix}.$$

A.1.4 Budget Uncertainty Sets

We next consider a budget uncertainty set of the form

$$\Xi = \left\{\boldsymbol{\xi} \in [0, 1]^k \,:\, \mathbf{e}^\top \boldsymbol{\xi} \leq B\right\},$$

where $B \in \mathbb{N}_0$. Note that $B = 0$ and $B \geq k$ correspond to the cases $\Xi = \{\mathbf{0}\}$ and $\Xi = [0,1]^k$ (the latter being a translation of the ∞-norm ball, which can be calculated using the results of Section A.1.2 and the transformations from Section A.2) and can therefore be omitted. Budget uncertainty sets have been popularized by [21] and have since been applied widely across domains.

For $B \in \{1, \ldots, k-1\}$, the moment matrix is

$$\boldsymbol{\Sigma}_A = \begin{pmatrix} \sum_{i=0}^{B} \binom{k}{i} & \left[\sum_{i=0}^{B-1} \binom{k-1}{i}\right] \cdot \mathbf{e}^\top \\ \left[\sum_{i=0}^{B-1} \binom{k-1}{i}\right] \cdot \mathbf{e} & \left[\sum_{i=0}^{B-1} \binom{k-1}{i} - \sum_{i=0}^{B-2} \binom{k-2}{i}\right] \cdot \boldsymbol{I} + \left[\sum_{i=0}^{B-2} \binom{k-2}{i}\right] \cdot \mathbf{e}\mathbf{e}^\top \end{pmatrix}.$$

A.1.5 Central Limit Theorem-Type Uncertainty Sets

We finally consider a central limit theorem-type uncertainty set of the form

$$\Xi = \left\{\boldsymbol{\xi} \in [-1,1]^k \,:\, -\Gamma \leq \left|\mathbf{e}^\top \boldsymbol{\xi}\right| \leq +\Gamma\right\},$$

where $\Gamma \in (0,k)$ [7]. In this case, we have $|\mathrm{ext}\,\Xi| = \eta_1 + \eta_2 + \eta_3$, where

$$\eta_1 = \begin{cases} \binom{k}{\frac{k}{2}} + 2\sum_{i=1}^{\lfloor \frac{\Gamma}{2} \rfloor} \binom{k}{\frac{k}{2}+i} & \text{if } k \text{ is even,} \\ 2\sum_{i=0}^{\lfloor \frac{\Gamma-1}{2} \rfloor} \binom{k}{\frac{k+1}{2}+i} & \text{if } k \text{ is odd;} \end{cases}$$

$$\eta_2 = \begin{cases} 2k\binom{k-1}{\frac{k+\Gamma-1}{2}} & \text{if } k \text{ is even and } \Gamma \text{ is odd, or vice versa,} \\ 0 & \text{otherwise;} \end{cases}$$

$$\eta_3 = \begin{cases} 2\binom{k}{\frac{k}{2}+\lfloor\frac{\Gamma}{2}\rfloor} \cdot \left(\frac{k}{2} - \left\lfloor \frac{\Gamma}{2} \right\rfloor\right) & \text{if } k \text{ is even and } \Gamma \text{ is fractional,} \\ 2\binom{k}{\frac{k+1}{2}+\lfloor\frac{\Gamma-1}{2}\rfloor} \cdot \left(\frac{k-1}{2} - \left\lfloor \frac{\Gamma-1}{2} \right\rfloor\right) & \text{if } k \text{ is odd and } \Gamma \text{ is fractional,} \\ 0 & \text{otherwise.} \end{cases}$$

Due to the symmetry of Ξ, the sum of first moments satisfies $\sum_{\boldsymbol{\xi}\in\operatorname{ext}\Xi}\boldsymbol{\xi} = \mathbf{0}$. The sum of second moments, finally, satisfies $\sum_{\boldsymbol{\xi}\in\operatorname{ext}\Xi}\boldsymbol{\xi}\boldsymbol{\xi}^\top = \omega\cdot(\mathbf{e}\mathbf{e}^\top) + (\delta-\omega)\cdot\boldsymbol{I}$, where $\delta = \delta_1+\delta_2+\delta_3$ with

$$\delta_1 = \begin{cases} \binom{k}{\frac{k}{2}} + 2\sum_{i=1}^{\lfloor\frac{\Gamma}{2}\rfloor}\binom{k}{\frac{k}{2}+i} & \text{if } k \text{ is even,}\\ 2\sum_{i=0}^{\lfloor\frac{\Gamma-1}{2}\rfloor}\binom{k}{\frac{k+1}{2}+i} & \text{if } k \text{ is odd;}\end{cases}$$

$$\delta_2 = \begin{cases} 2(k-1)\binom{k-1}{\frac{k+\Gamma-1}{2}} & \text{if } k \text{ is even and } \Gamma \text{ is odd, or vice versa,}\\ 0 & \text{otherwise;}\end{cases}$$

$$\delta_3 = \begin{cases} 2\binom{k}{\frac{k}{2}+\lfloor\frac{\Gamma}{2}\rfloor}\left(\frac{k}{2}-\left\lfloor\frac{\Gamma}{2}\right\rfloor\right)\left[\frac{k-1}{k}+\frac{1}{k}\cdot\left(1-\left[\Gamma-2\left\lfloor\frac{\Gamma}{2}\right\rfloor\right]\right)^2\right] & \text{if } k \text{ is even and } \Gamma \text{ is fractional,}\\ 2\binom{k}{\frac{k+1}{2}+\lfloor\frac{\Gamma-1}{2}\rfloor}\left(\frac{k-1}{2}-\left\lfloor\frac{\Gamma-1}{2}\right\rfloor\right)\left[\frac{k-1}{k}+\frac{1}{k}\cdot\left(2-\Gamma+2\left\lfloor\frac{\Gamma-1}{2}\right\rfloor\right)^2\right] & \text{if } k \text{ is odd and } \Gamma \text{ is fractional,}\\ 0 & \text{otherwise,}\end{cases}$$

as well as $\omega = \omega_1+\omega_2+\omega_3$ with

$$\omega_1 = \begin{cases} 2\sum_{i=\max\{2-\frac{k}{2},-\lfloor\frac{\Gamma}{2}\rfloor\}}^{\min\{\frac{k}{2},\lfloor\frac{\Gamma}{2}\rfloor\}}\binom{k-2}{\frac{k}{2}+i-2} - 2\sum_{i=\max\{1-\frac{k}{2},-\lfloor\frac{\Gamma}{2}\rfloor\}}^{\min\{\frac{k}{2}-1,\lfloor\frac{\Gamma}{2}\rfloor\}}\binom{k-2}{\frac{k}{2}+i-1} & \text{if } k \text{ is even,}\\ 2\sum_{i=\max\{\frac{1}{2}-\frac{k}{2},-\lfloor\frac{\Gamma-1}{2}\rfloor-2\}}^{\min\{\frac{k}{2}-\frac{3}{2},\lfloor\frac{\Gamma-1}{2}\rfloor-1\}}\binom{k-2}{\frac{k-1}{2}+i} - 2\sum_{i=\max\{\frac{1}{2}-\frac{k}{2},-\lfloor\frac{\Gamma-1}{2}\rfloor-1\}}^{\min\{\frac{k}{2}-\frac{3}{2},\lfloor\frac{\Gamma-1}{2}\rfloor\}}\binom{k-2}{\frac{k-1}{2}+i} & \text{if } k \text{ is odd;}\end{cases}$$

$\omega_2 = \omega_{21}+\omega_{22}+\omega_{23}$ if k is even and Γ is odd, or vice versa, and $\omega_2 = 0$ otherwise, where

$$\omega_{21} = \begin{cases} 2(k-2)\binom{k-3}{\frac{k+\Gamma-5}{2}} & \text{if } k+\Gamma\geq 5,\ k\geq\Gamma+1,\\ 0 & \text{otherwise;}\end{cases}$$

$$\omega_{22} = \begin{cases} 2(k-2)\binom{k-3}{\frac{k-\Gamma-5}{2}} & \text{if } k\geq\Gamma+5,\\ 0 & \text{otherwise;}\end{cases} \qquad \omega_{23} = \begin{cases} -4\cdot(k-2)\binom{k-3}{\frac{k+\Gamma-3}{2}} & \text{if } k\geq\Gamma+3,\\ 0 & \text{otherwise;}\end{cases}$$

and finally $\omega_3 = \omega_{31} - \omega_{32} + \omega_{33} - \omega_{34}$ if k is even and Γ is fractional, where

$$\omega_{31} = \begin{cases} 2\binom{k-2}{\frac{k}{2} + \lfloor \frac{\Gamma}{2} \rfloor - 2} \cdot \left(\frac{k}{2} - \left\lfloor \frac{\Gamma}{2} \right\rfloor\right) + 2\binom{k-2}{\frac{k}{2} - \lfloor \frac{\Gamma}{2} \rfloor - 2} \cdot \left(\frac{k}{2} - \left\lfloor \frac{\Gamma}{2} \right\rfloor - 2\right) & \text{if } \frac{k}{2} \geq \left\lfloor \frac{\Gamma}{2} \right\rfloor + 2, \\ 0 & \text{otherwise;} \end{cases}$$

$$\omega_{32} = 4\binom{k-2}{\frac{k}{2} + \lfloor \frac{\Gamma}{2} \rfloor - 1} \cdot \left(\frac{k}{2} - \left\lfloor \frac{\Gamma}{2} \right\rfloor - 1\right);$$

$$\omega_{33} = \begin{cases} 4\binom{k-2}{\frac{k}{2} - \lfloor \frac{\Gamma}{2} \rfloor - 2} & \text{if } \frac{k}{2} \geq \left\lfloor \frac{\Gamma}{2} \right\rfloor + 2, \\ 0 & \text{otherwise;} \end{cases} \qquad \omega_{34} = \begin{cases} 4\binom{k-2}{\frac{k}{2} - \lfloor \frac{\Gamma}{2} \rfloor - 1} & \text{if } \frac{k}{2} \geq \left\lfloor \frac{\Gamma}{2} \right\rfloor + 1, \\ 0 & \text{otherwise;} \end{cases}$$

as well as $\omega_3 = \omega_{35} + \omega_{36} - \omega_{37} + \omega_{38} - \omega_{39}$ if k is odd and Γ is fractional, where

$$\omega_{35} = \begin{cases} 2\binom{k-2}{\frac{k-3}{2} + \lfloor \frac{\Gamma-1}{2} \rfloor} \cdot \left(\frac{k-1}{2} - \left\lfloor \frac{\Gamma-1}{2} \right\rfloor\right) & \text{if } \frac{k-1}{2} \geq \left\lfloor \frac{\Gamma-1}{2} \right\rfloor, \\ 0 & \text{otherwise;} \end{cases}$$

$$\omega_{36} = \begin{cases} 2\binom{k-2}{\frac{k-5}{2} - \lfloor \frac{\Gamma-1}{2} \rfloor} \cdot \left(\frac{k-5}{2} - \left\lfloor \frac{\Gamma-1}{2} \right\rfloor\right) & \text{if } \frac{k-5}{2} \geq \left\lfloor \frac{\Gamma-1}{2} \right\rfloor, \\ 0 & \text{otherwise;} \end{cases}$$

$$\omega_{37} = \begin{cases} 4\binom{k-2}{\frac{k-1}{2} + \lfloor \frac{\Gamma-1}{2} \rfloor} \cdot \left(\frac{k-3}{2} - \left\lfloor \frac{\Gamma-1}{2} \right\rfloor\right) & \text{if } \frac{k-3}{2} \geq \left\lfloor \frac{\Gamma-1}{2} \right\rfloor, \\ 0 & \text{otherwise;} \end{cases}$$

$$\omega_{38} = \begin{cases} 4\binom{k-2}{\frac{k-5}{2} - \lfloor \frac{\Gamma-1}{2} \rfloor} & \text{if } \frac{k-5}{2} \geq \left\lfloor \frac{\Gamma-1}{2} \right\rfloor, \\ 0 & \text{otherwise;} \end{cases} \qquad \omega_{39} = \begin{cases} 4\binom{k-2}{\frac{k-3}{2} - \lfloor \frac{\Gamma-1}{2} \rfloor} & \text{if } \frac{k-3}{2} \geq \left\lfloor \frac{\Gamma-1}{2} \right\rfloor, \\ 0 & \text{otherwise.} \end{cases}$$

A.2 Transformations of Primitive Uncertainty Sets

In the following, we show how a moment matrix $\mathbf{\Sigma}_A$ for an uncertainty set Ξ changes if Ξ is transformed by an injective affine map, or if Ξ is composed of the cross product of primitive uncertainty sets Ξ_i with known moment matrices $\mathbf{\Sigma}_{A,i}$.

A.2.1 Injective Affine Maps

It follows from Theorem 9.2.3 in [48] that extreme points are preserved under injective affine maps. Let us assume that $\Xi' = f(\Xi)$ for an injective affine map $f : \mathbb{R}^k \overset{A}{\mapsto} \mathbb{R}^{k'}$. We then obtain that

$$\boldsymbol{\Sigma}'_A \;=\; \begin{pmatrix} |\mathrm{ext}\,\Xi| & \sum_{\boldsymbol{\xi}\in\mathrm{ext}\,\Xi} f(\boldsymbol{\xi})^\top \\ \sum_{\boldsymbol{\xi}\in\mathrm{ext}\,\Xi} f(\boldsymbol{\xi}) & \sum_{\boldsymbol{\xi}\in\mathrm{ext}\,\Xi} f(\boldsymbol{\xi}) f(\boldsymbol{\xi})^\top \end{pmatrix}.$$

Let us now additionally assume that $f(\boldsymbol{\xi}) = \boldsymbol{F}\boldsymbol{\xi} + \boldsymbol{f}$. From the previous assumption that f is injective, we conclude that the matrix $\boldsymbol{F}$ has full column rank. We then have

$$\sum_{\boldsymbol{\xi}\in\mathrm{ext}\,\Xi} f(\boldsymbol{\xi}) \;=\; \sum_{\boldsymbol{\xi}\in\mathrm{ext}\,\Xi} (\boldsymbol{F}\boldsymbol{\xi} + \boldsymbol{f}) \;=\; \boldsymbol{F}\left[\sum_{\boldsymbol{\xi}\in\mathrm{ext}\,\Xi} \boldsymbol{\xi}\right] + \boldsymbol{f} + |\mathrm{ext}\,\Xi|\boldsymbol{f}.$$

In other words, we can calculate $\sum_{\boldsymbol{\xi}\in\mathrm{ext}\,\Xi} f(\boldsymbol{\xi})$ efficiently from the quantities $|\mathrm{ext}\,\Xi|$ and $\sum_{\boldsymbol{\xi}\in\mathrm{ext}\,\Xi} \boldsymbol{\xi}$. In a similar way, we obtain that

$$\begin{aligned} \sum_{\boldsymbol{\xi}\in\mathrm{ext}\,\Xi} f(\boldsymbol{\xi}) f(\boldsymbol{\xi})^\top \;=\; \sum_{\boldsymbol{\xi}\in\mathrm{ext}\,\Xi} (\boldsymbol{F}\boldsymbol{\xi} + \boldsymbol{f})(\boldsymbol{F}\boldsymbol{\xi} + \boldsymbol{f})^\top \;=\; &\left(\boldsymbol{F}\left[\sum_{\boldsymbol{\xi}\in\mathrm{ext}\,\Xi} \boldsymbol{\xi}\right]\boldsymbol{f}^\top + \left[\boldsymbol{F}\left[\sum_{\boldsymbol{\xi}\in\mathrm{ext}\,\Xi} \boldsymbol{\xi}\right]\boldsymbol{f}^\top\right]^\top\right) \\ &+ \boldsymbol{F}\left[\sum_{\boldsymbol{\xi}\in\mathrm{ext}\,\Xi} \boldsymbol{\xi}\boldsymbol{\xi}^\top\right]\boldsymbol{F}^\top + |\mathrm{ext}\,\Xi|\boldsymbol{f}\boldsymbol{f}^\top. \end{aligned}$$

In other words, we can calculate $\sum_{\boldsymbol{\xi}\in\mathrm{ext}\,\Xi} f(\boldsymbol{\xi}) f(\boldsymbol{\xi})^\top$ efficiently from the quantities $|\mathrm{ext}\,\Xi|$, $\sum_{\boldsymbol{\xi}\in\mathrm{ext}\,\Xi} \boldsymbol{\xi}$ and $\sum_{\boldsymbol{\xi}\in\mathrm{ext}\,\Xi} \boldsymbol{\xi}\boldsymbol{\xi}^\top$.

A.2.2 Cross Products

Assume that $\Xi = \Xi_1 \times \Xi_2$ with $\Xi_1 \subseteq \mathbb{R}^{k_1}$ and $\Xi_2 \subseteq \mathbb{R}^{k_2}$ such that $k_1 + k_2 = k$. We then have

$$|\mathrm{ext}\,\Xi| = |\mathrm{ext}\,\Xi_1| \cdot |\mathrm{ext}\,\Xi_2|,$$

$$\sum_{\boldsymbol{\xi} \in \mathrm{ext}\,\Xi} \boldsymbol{\xi} = \begin{pmatrix} |\mathrm{ext}\,\Xi_2| \cdot \displaystyle\sum_{\boldsymbol{\xi}_1 \in \mathrm{ext}\,\Xi_1} \boldsymbol{\xi}_1 \\ |\mathrm{ext}\,\Xi_1| \cdot \displaystyle\sum_{\boldsymbol{\xi}_2 \in \mathrm{ext}\,\Xi_2} \boldsymbol{\xi}_2 \end{pmatrix},$$

$$\sum_{\boldsymbol{\xi} \in \mathrm{ext}\,\Xi} \boldsymbol{\xi}\boldsymbol{\xi}^\top = \begin{pmatrix} |\mathrm{ext}\,\Xi_2| \cdot \displaystyle\sum_{\boldsymbol{\xi}_1 \in \mathrm{ext}\,\Xi_1} \boldsymbol{\xi}_1\boldsymbol{\xi}_1^\top & \left(\displaystyle\sum_{\boldsymbol{\xi}_1 \in \mathrm{ext}\,\Xi_1} \boldsymbol{\xi}_1\right)\left(\displaystyle\sum_{\boldsymbol{\xi}_2 \in \mathrm{ext}\,\Xi_2} \boldsymbol{\xi}_2\right)^\top \\ \left(\displaystyle\sum_{\boldsymbol{\xi}_2 \in \mathrm{ext}\,\Xi_2} \boldsymbol{\xi}_2\right)\left(\displaystyle\sum_{\boldsymbol{\xi}_1 \in \mathrm{ext}\,\Xi_1} \boldsymbol{\xi}_1\right)^\top & |\mathrm{ext}\,\Xi_1| \cdot \displaystyle\sum_{\boldsymbol{\xi}_2 \in \mathrm{ext}\,\Xi_2} \boldsymbol{\xi}_2\boldsymbol{\xi}_2^\top \end{pmatrix}.$$

In other words, we can calculate $|\mathrm{ext}\,\Xi|$, $\sum_{\boldsymbol{\xi} \in \mathrm{ext}\,\Xi} \boldsymbol{\xi}$ and $\sum_{\boldsymbol{\xi} \in \mathrm{ext}\,\Xi} \boldsymbol{\xi}\boldsymbol{\xi}^\top$ efficiently from the quantities $|\mathrm{ext}\,\Xi_i|$, $\sum_{\boldsymbol{\xi} \in \mathrm{ext}\,\Xi_i} \boldsymbol{\xi}$ and $\sum_{\boldsymbol{\xi} \in \mathrm{ext}\,\Xi_i} \boldsymbol{\xi}\boldsymbol{\xi}^\top$, $i = 1, 2$.