Electronic Companion

All notations follow from the main text. When calling upon the equations in the main text, we refer to their indices therein.

EC.1. Proofs

Proof of Proposition 1. Jackson and Rogers (2007) provide valuable information regarding the degree distributions, but they do not explicitly lay out the relevant neighbor degree distributions. We now build upon their analysis and characterize the neighbor degree distributions in detail to serve our purpose. Consistent with Jackson and Rogers (2007), let $d_i(t)$ denote the in-degree of node i at time t. Under mean field approximation, the network formation process described in Section 3 leads to the following differential equation:

$$\frac{d}{dt}d_i(t) = \frac{o_n}{tm}d_i(t) + \frac{o_r}{t}, \tag{EC.1}$$

where the initial in-degree of nodes upon birth is denoted by d_0. To understand (EC.1), note when $o_n = p_n m_n, o_r = p_r m_r$ it is equivalent to equation (1) in Jackson and Rogers (2007), for which one should refer to the explanation in their paper. [18]

Equation (EC.1) pins down the in-degree function:

$$d_i(t) = (d_0 + rm)\left(\frac{t}{i}\right)^{\frac{1}{1+r}} - rm. \tag{EC.2}$$

For completeness we keep a general value of d_0, and at times derive the explicit formulas for the case $d_0 = 0$ (as does the majority of analysis in Jackson and Rogers (2007)). Under the mean field approximation, there is a unique mapping between node identity i, time t, and node in-degree k, so that one can use two of the variables to infer the third one. Let $i_t(d)$ be the birthdate (or the identity) of the node that has degree-d at time t. (EC.2) implies:

$$i_t(d) = t\left\{\frac{d_0 + rm}{d + rm}\right\}^{1+r}. \tag{EC.3}$$

[18] Same as (1) of Jackson and Rogers (2007), (EC.1) is not an exact calculation since it ignores the potential overlapping of search ranges. However, it remains a reasonable approximation for large networks ($N >> o_r, o_n$) and o_n is small relative to $o_r m$. See footnote 20 of Jackson and Rogers (2007).

In-neighbors. For now, denote by $F_i^t(\cdot)$ the cdf. of the in-degree distribution of in-neighbors of node i at time t, and we will later make the expression time-invariant (i.e. removing t). Recall (Jackson and Rogers 2007, p.911, under the proof of Theorem 4):

$$1-F_i^t(d)=\frac{d_i(i_t(d))}{d_i(t)},$$

Substituting the expression of $d_i(t)$, we obtain

$$d_i(i_t(d))=(d_0+rm)\left(\frac{i_t(d)}{i}\right)^{\frac{1}{1+r}}-rm=(d_0+rm)\left(\frac{t}{i}\right)^{\frac{1}{1+r}}\left\{\frac{d_0+rm}{d+rm}\right\}-rm,$$

so that

$$1-F_i^t(d)=\frac{d_i(i_t(d))}{d_i(t)}=\frac{(d_0+rm)\left(\frac{t}{i}\right)^{\frac{1}{1+r}}\left\{\frac{d_0+rm}{d+rm}\right\}-rm}{(d_0+rm)\left(\frac{t}{i}\right)^{\frac{1}{1+r}}-rm}, \tag{EC.4}$$

for $d<d_i(t)$ (one's in-neighbor's in-degree must be lower than the in-degree of oneself).

As an important step, we will transform the degree distribution to be independent of time. For this purpose, observe by (EC.2) that for a degree-k player at time t,

$$\left(\frac{t}{i}\right)^{\frac{1}{1+r}}=\frac{k+rm}{d_0+rm}, \tag{EC.5}$$

and hereby rewrite (EC.4):

$$1-F_i^t(d)=\frac{(k+rm)\frac{d_0+rm}{d+rm}-rm}{k}. \tag{EC.6}$$

Since the distribution is now time-invariant, we denote it by $F_k(\cdot)$, indicating the in-neighbor degree distribution of a degree-k node.

When $d_0=0$, we obtain from (EC.6) that:

$$1-F_k(d)=\frac{(k+rm)\frac{rm}{d+rm}-rm}{k},$$

which was included in the presentation of Proposition 1 in the main text.

Out-neighbors. Above we have solved for the degree distribution of in-neighbors. Next we study that of out-neighbors. First note that the foregoing approach of calculating neighbor degree distribution based on the equation

$$1-F_i^t(d)=\frac{d_i(i_t(d))}{d_i(t)}, \tag{EC.7}$$

is no longer useful, because one's out-degree does not cumulate in time; in other words, one's out-neighbors are all formed at one shot upon arrival at the system (so that the fraction of neighbors whose degree is higher than d cannot be derived by comparing the birth dates).

To proceed, we need to analyze the formation of out-neighbors upon a *now*-degree-k player's arrival at the system. There are two types of out-neighbors obtained at that moment: tier-1 out-neighbors who are connected by the player in question by random connection, and tier-2 out-neighbors who are reached via the out-degree links from tier-1 out-neighbors. Moreover, we can infer from equation (1) of Jackson and Rogers (2007) the probability of being a tier-1 or tier-2 out-neighbor of the node conditional on the node's own degree is k at time t as follows.

$$
\begin{aligned}
Pr\{\text{tier-1 out-neighbor}|\text{degree} = d\} &= \frac{o_r}{i_t(k)} \\
&\underbrace{=}_{(EC.3)} \frac{o_r}{t\left\{\frac{d_0+rm}{k+rm}\right\}^{1+r}} \\
&\underbrace{=}_{d_0=0} \frac{o_r}{t\left\{\frac{rm}{k+rm}\right\}^{1+r}}
\end{aligned}
\tag{EC.8}
$$

where (EC.8) is the probability that the degree-d node is found at random by the new born node who has now degree k at time t, under mean field approximation. Note this probability is independent of d, given that the selection of tier-1 out-neighbors is random rather than degree-based.

As for the tier-2 out-neighbors,

$$
\begin{aligned}
& Pr\{\text{tier-2 out-neighbor}|\text{degree} = d\} \\
= & \left(\frac{o_r d_j(i_t(k))}{i_t(k)}\right)\left(\frac{o_n}{o_r m}\right) \\
= & \frac{d_j(i_t(k))}{i_t(k)}\frac{o_n}{m} \\
\underbrace{=}_{j \text{ is such that } d_j(t)=d,(EC.3),(EC.2)} & \left(\frac{\frac{d+rm}{k+rm}(d_0+rm)-rm}{t\left(\frac{d_0+rm}{k+rm}\right)^{1+r}}\right)\left(\frac{o_n}{m}\right) \\
\underbrace{=}_{d_0=0} & \left(\frac{\frac{d-k}{k+rm}rm}{t\left(\frac{rm}{k+rm}\right)^{1+r}}\right)\left(\frac{o_n}{m}\right) \\
= & \frac{r o_n (k+rm)^r}{t(rm)^{1+r}}(d-k)
\end{aligned}
\tag{EC.9}
$$

To understand (EC.9), one should refer to the explanation of equation (1) in Jackson and Rogers (2007), and note the changes we made to equation (1) of Jackson and Rogers (2007) to adapt it for our purpose. Here we will briefly describe the intuition: Refer to node i the node in question, who has degree-k at time t, and we trace back to his birthdate, $i_t(k)$, to investigate the likelihood that he gets another node j, who has degree-d at time t, as his tier-2 out-neighbor. Note 1) The term $\frac{o_r d_j(i_t(k))}{i_t(k)}$ is the probability that some node with a link to j, is reached by node i as tier-1 out-neighbor, so that j has the potential of being met in this way. [19] 2) The term $\frac{o_n}{o_r m}$ is then the probability that j is found, given that some in-neighbor of him has been met randomly in 1)[20]. [21]

Now, what we have obtained is the probability of connecting to a node with certain degree, yet the concept of neighbor degree distribution is about, given the connection, the probability of the connected node having certain degree. This gap can be filled by applying the Bayes' theorem:

$$\begin{aligned} f_k^q(d) &= \frac{Pr\{\text{type-}q\text{ out-neighbor}|\text{degree}=d\}f(d|d>k)}{\int_k^{\bar{k}} Pr\{\text{type-}q\text{ out-neighbor}|\text{degree}=d\}f(d|d>k)\mathrm{d}d}, q\in\{o1,o2\} \\ &= \frac{Pr\{\text{type-}q\text{ out-neighbor}|\text{degree}=d\}f(d)/(1-F(k))}{\int_k^{\bar{k}} Pr\{\text{type-}q\text{ out-neighbor}|\text{degree}=d\}f(d)/(1-F(k))\mathrm{d}d}, q\in\{o1,o2\} \\ &= \frac{Pr\{\text{type-}q\text{ out-neighbor}|\text{degree}=d\}f(d)}{\int_k^{\bar{k}} Pr\{\text{type-}q\text{ out-neighbor}|\text{degree}=d\}f(d)\mathrm{d}d}, q\in\{o1,o2\} \end{aligned} \quad \text{(EC.10)}$$

where the condition $d>k$ is imposed upon the original degree distribution for the discussion of out-neighbors (who are born prior to the now-degree-k player in question, thus having larger in-degrees).

[19] The probability that *one* link reaches an in-neighbor of j at the time $i_t(k)$ is $\frac{d_j(i_t(k))}{i_t(k)}$, given there were $i_t(k)$ players in the system and $d_j(i_t(k))$ of them were j's in-neighbors. Since there are o_r such links, the probability that any of these links reaches an in-neighbor of j is approximated by o_r multiplied by the above probability.

[20] Since some in-neighbor of hers has been linked in 1), node j must lie among the $o_r m$ nodes which node i could possibly reach as tier-2 out-neighbors. Then the probability that node j is reached by the neighborhood search with o_n links is $\frac{o_n}{o_r m}$.

[21] Same as (1) of Jackson and Rogers (2007), (EC.9) is not an exact calculation since it ignores the potential overlapping of search ranges. However, it remains a reasonable approximation for large networks ($N >> o_r, o_n$) and o_n is small relative to $o_r m$. See footnote 20 of Jackson and Rogers (2007).

Note then the term $1-F(k)$ is cancelled off. This gives us

$$f_k^{o1}(d) \underbrace{=}_{(EC.10)} \frac{f(d)}{\int_k^{\bar{k}} f(d)\mathrm{d}d} = \frac{f(d)}{1-F(k)} \underbrace{=}_{(1)} \frac{(1+r)(k+rm)^{1+r}}{(d+rm)^{r+2}}. \tag{EC.11}$$

(EC.11) results from the fact that $Pr\{\text{tier-1 out-neighbor}|\text{degree}=d\}$ is independent of d, so that it cancels off from the fraction. Integrating , we have the cdf.

$$F_k^{o1}(d)=1-\left(\frac{k+rm}{d+rm}\right)^{r+1}, \forall d>k.$$

For tier-2 out-neighbors,

$$f_k^{o2}(d) \underbrace{=}_{(EC.10)} \frac{(d-k)f(d)}{\int_k^{\bar{k}}(d-k)f(d)\mathrm{d}d} \underbrace{=}_{(EC.50)} \frac{(d-k)/(d+rm)^{r+2}}{\int_k^{\bar{k}}(d-k)/(d+rm)^{r+2}\mathrm{d}d} = \frac{r(r+1)(d-k)(k+rm)^r}{(d+rm)^{r+2}}.$$

Integrating, we get the cdf.

$$F_k^{o2}(d)=1-\left(\frac{k+rm}{d+rm}\right)^r\left(\frac{r(d-k)}{d+rm}+1\right), \forall d>k.$$

To see that $F_k^{o1}(\cdot), F_k^{o2}(\cdot)$ first-order stochastically increase with m, note that

$$\frac{\partial}{\partial m}\{1-F_k^{o1}(\cdot)\} = \frac{(d-k)r(r+1)}{(d+rm)^2}\left(\frac{k+rm}{d+rm}\right)^r>0,$$
$$\frac{\partial}{\partial m}\{1-F_k^{o2}(\cdot)\} = \frac{(d-k)^2r^2(r+1)}{(d+rm)^3}\left(\frac{k+rm}{d+rm}\right)^{r-1}>0,$$

given $d>k$. To see $F_k(\cdot)$ first-order stochastically increases with r and m, rewrite

$$1-F_k(d)=\frac{(k+rm)\frac{rm}{d+rm}-rm}{k}=\left(1-\frac{d}{d+rm}\right)\frac{k-d}{k}.$$

Then observe that $1-F_k(d)$ increases with r and m given $d<k$. □

Proof of Proposition 2. Out-neighbors. Let us first deal with out-neighbors. Recall that for tier-1 out-neighbors,

$$F_k^{o1}(d) = 1 - \left(\frac{k+rm}{d+rm}\right)^{r+1}, \forall d > k.$$

For tier-2 out-neighbors,

$$F_k^{o2}(d) = 1 - \left(\frac{k+rm}{d+rm}\right)^{r}\left(\frac{r(d-k)}{d+rm}+1\right), \forall d > k.$$

The probability density functions are:

$$\begin{aligned} f_k^{o1}(d) &= \frac{r+1}{d+rm}\left(\frac{k+rm}{d+rm}\right)^{r+1}, \forall d > k, \\ f_k^{o2}(d) &= \frac{r(r+1)(d-k)}{(d+rm)^2}\left(\frac{k+rm}{d+rm}\right)^{r}, \forall d > k. \end{aligned}$$

For the out-neighbors, denote the set of degree-k's tier-1 and tier-2 out-neighbor degree types by N_k^{o1} and N_k^{o2}, respectively. Then we can calculate:

$$\begin{aligned} &\mathbb{E}[\sum_{j\in N_k^o} x(j)|k] \\ &= \mathbb{E}[\sum_{j\in N_k^{o1}} x(j) + \sum_{j\in N_k^{o2}} x(j)|k] \\ &= o_r\mathbb{E}[x(j)|_{j\in N_k^{o1}}k] + o_n\mathbb{E}[x(j)|_{j\in N_k^{o2}}k] \\ &= o_r\int_k^N x(y)f_k^{o1}(y)dy + o_n\int_k^N x(y)f_k^{o2}(y)dy. \end{aligned}$$

where the upper bound of integration is set to be the network size, given it being large enough and thus trivializing the differences from setting it to infinity or to $N-1$. Since the consumption is only determined by in-degree, one can further combine notations and express the above as (using $r = \frac{o_r}{o_n}$):

$$\mathbb{E}[\sum_{j\in N_k^o} x(j)|k] = o_n\int_k^N x(y)\left[rf_k^{o1}(y) + f_k^{o2}(y)\right]dy.$$

Plug in the definitions of $f_k^{o1}(d)$ and $f_k^{o2}(d)$:

$$\begin{aligned} rf_k^{o1}(y) + f_k^{o2}(y) &= r\frac{(r+1)}{y+rm}\left(\frac{k+rm}{y+rm}\right)^{r+1} + \frac{r(r+1)(y-k)}{(y+rm)^2}\left(\frac{k+rm}{y+rm}\right)^{r} \\ &= \frac{r(r+1)}{y+rm}\left(\frac{k+rm}{y+rm}\right)^{r}. \end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\mathbb{E}[\sum_{j\in N_k^o} x(j)|k] \\
&= o_n \int_k^N x(y)\frac{r(r+1)}{y+rm}\left(\frac{k+rm}{y+rm}\right)^r dy \\
&= o_n r(r+1)\left(k+rm\right)^r \int_k^N x(y)\frac{1}{(y+rm)^{r+1}}dy \\
&= rm\left(k+rm\right)^r \int_k^N \frac{x(y)}{(y+rm)^{r+1}}dy,
\end{aligned}$$

Now suppose $x(\cdot)$ is increasing. Notice

$$\begin{aligned}
&\frac{d}{dk}\mathbb{E}[\sum_{j\in N_k^o} x(j)|k] \\
&= rm\left\{\begin{matrix} r\left(k+rm\right)^{r-1}\int_k^N x(y)\frac{1}{(y+rm)^{r+1}}dy \\ -\left(k+rm\right)^r x(k)\frac{1}{(k+rm)^{r+1}}\end{matrix}\right\} \\
&= rm\left\{\begin{matrix} r\left(k+rm\right)^{r-1}\int_k^N x(y)\frac{1}{(y+rm)^{r+1}}dy \\ -\frac{x(k)}{k+rm}\end{matrix}\right\} \\
&= \frac{r}{k+rm}\left\{\begin{matrix} \mathbb{E}[\sum_{j\in N_k^o} x(j)|k] \\ -mx(k)\end{matrix}\right\} \\
&= \frac{r}{k+rm}\mathbb{E}[\sum_{j\in N_k^o} x(j)|k] - \frac{rm}{k+rm}x(k).
\end{aligned} \tag{EC.12}$$

Since one's out-neighbors have higher in-degree than oneself, and that the consumption is assumed increasing in in-degree, each out-neighbor should have higher consumption than does oneself. That is, $\mathbb{E}[\sum_{j\in N_k^o} x(j)|k] > mx(k)$. Elaborated in detail,

$$\begin{aligned}
\mathbb{E}[\sum_{j\in N_k^o} x(j)|k] &= rm\left(k+rm\right)^r \int_k^N x(y)\frac{1}{(y+rm)^{r+1}}dy \\
&\underbrace{>}_{x(\cdot)\text{ increasing}} rm\left(k+rm\right)^r x(k)\int_k^N \frac{1}{(y+rm)^{r+1}}dy \\
&\underbrace{=}_{N\text{ large enough}} mx(k)
\end{aligned}$$

Therefore, $\frac{d}{dk}\mathbb{E}[\sum_{j\in N_k^o} x(j)|k] > 0$ if $x(k)$ increases in k.

In-neighbors. Now we consider the case with in-neighbors.

$$\begin{aligned}
\mathbb{E}[\sum_{j\in N_k^i} x(j)|k] &= \sum_{j\in N_k^i} \mathbb{E}[x(j)|k] \\
&= \sum_{j\in N_k^i} \left\{ x(0) + \int_0^k \frac{dx(y)}{dy}\left[1-F_k(y)\right]dy \right\} \\
&= k\left\{ x(0) + \int_0^k \frac{dx(y)}{dy}\frac{rm(k-y)}{k(y+rm)}dy \right\} \\
&= kx(0) + \int_0^k \frac{dx(y)}{dy}\frac{rm(k-y)}{y+rm}dy \\
&\underbrace{=}_{\text{integration by parts}} kx(0) + 0 - kx(0) - \int_0^k x(y) d\frac{rm(k-y)}{y+rm} \\
&= (k+rm)rm\int_0^k \frac{x(y)}{(y+rm)^2}dy
\end{aligned}$$

It easily follows that $\mathbb{E}[\sum_{j\in N_k^i} x(j)|k]$ is increasing in k. □

Proof of Theorem 1. The proof is composed of three steps. Step (i): We explore the structural properties of the optimization problem. Step (ii): We rewrite the firm objective as a function of consumption only. Step (iii): Using the change of variables, we show that the objective can be optimized by calculus of variations. We also verify that the candidate solution from calculus of variations indeed satisfies the remaining constraints.

Step (i): Structural properties of the optimization problem. As the standard mechanism design approach, we first explore some properties of (4)-(6) that are essential to reducing the firm's problem.

Recall the consumer's payoff when others report the truth

$$\pi_k(\hat{k}, x(\cdot), P(\cdot)) = ax(\hat{k}) - bx^2(\hat{k}) + x(\hat{k})\delta\mathbb{E}[\sum_{j\in N_k^o} x(j)|k] - P(x(\hat{k})).$$

The first-order condition for IC constraints implies:

$$\begin{aligned}
& d\pi_k(\hat{k}, x(\cdot), P(\cdot))/d\hat{k}\Big|_{\hat{k}=k} \\
&= ax'(\hat{k}) - 2bx(\hat{k})x'(\hat{k}) + x'(\hat{k})\delta\mathbb{E}[\sum_{j\in N_k^o} x(j)|k] - P'(x(\hat{k}))x'(\hat{k})\Big|_{\hat{k}=k}
\end{aligned}$$

$$
\begin{aligned}
&= ax'(k) - 2bx(k)x'(k) + \delta x'(k)\mathbb{E}[\sum_{j\in N_k^o} x(j)|k] - P'(x(k))x'(k) \\
&= \left[a - 2bx(k) + \delta\mathbb{E}[\sum_{j\in N_k^o} x(j)|k] - P'(x(k))\right] x'(k) \\
&= 0. \qquad \text{(EC.13)}
\end{aligned}
$$

The local second-order condition for IC constraints is

$$
\begin{aligned}
&\frac{d^2}{d\hat{k}^2}\pi_k(\hat{k}, x(\cdot), P(\cdot))\Big|_{\hat{k}=k} \\
= \quad &\left[-2bx'(\hat{k}) - \frac{d}{d\hat{k}}P'(x(\hat{k}))\right] x'(\hat{k}) + \left[a - 2bx(\hat{k}) + \delta\mathbb{E}[\sum_{j\in N_k^o} x(j)|k] - P'(x(\hat{k}))\right] x''(\hat{k})\Big|_{\hat{k}=k} \\
\underbrace{=}_{\text{(EC.13)}} \quad &\left[-2bx'(k) - \frac{d}{dk}P'(x(k))\right] x'(k) \\
\underbrace{=}_{\text{(EC.13)}} \quad &\left[-\delta\frac{d}{dk}\mathbb{E}[\sum_{j\in N_k^o} x(j)|k]\right] x'(k) \\
< \quad &0 \text{ if } x'(k) > 0 \qquad \text{(EC.14)}
\end{aligned}
$$

Therefore the local concavity for truth reporting requires the *monotonicity condition*, $x'(k) > 0$, which in our problem also leads to $\frac{d}{dk}\mathbb{E}[\sum_{j\in N_k^o} x(j)|k] > 0$ by Proposition 2. (EC.13) and (EC.14) constitute the local IC condition, under which the customer does not attempt to lie locally. We will soon show that the single crossing condition, justified in Proposition 2, extends local IC to global. Define the payoff in the truth telling equilibrium:

$$
V(k) := \pi_k(k, x(\cdot), P(\cdot)) = ax(k) - bx^2(k) + x(k)\delta\mathbb{E}[\sum_{j\in N_k^o} x(j)|k] - P(x(k)), \qquad \text{(EC.15)}
$$

and IR constraints (6) imply that $V(k) \geq 0, \forall k$.

$$
\begin{aligned}
\frac{d}{dk}V(k) \quad &= \quad ax'(k) - 2bx(k)x'(k) + \frac{dx(k)\delta\mathbb{E}[\sum_{j\in N_k^o} x(j)|k]}{dk} - P'(x(k))x'(k) \qquad \text{(EC.16)} \\
&= \quad ax'(k) - 2bx(k)x'(k) + x'(k)\delta\mathbb{E}[\sum_{j\in N_k^o} x(j)|k] + \\
&\qquad x(k)\delta\frac{d}{dk}\mathbb{E}[\sum_{j\in N_k^o} x(j)|k] - P'(x(k))x'(k) \\
&\underbrace{=}_{(EC.13)} \quad x(k)\delta\frac{d}{dk}\mathbb{E}[\sum_{j\in N_k^o} x(j)|k],
\end{aligned}
$$

which is positive if $x'(k)>0$ (which leads to $\frac{d}{dk}\mathbb{E}[\sum_{j\in N_k^o}x(j)|k]>0$ by Proposition 2).

The above result can be reached by the envelope theorem as well, in consideration of the IC constraints:

$$\frac{d}{dk}V(k)=\frac{\partial}{\partial k}\pi_k(\hat{k}^*,x(\cdot),P(\cdot))\Big|_{\hat{k}^*=k}=x(k)\delta\frac{d}{dk}\mathbb{E}[\sum_{j\in N_k^o}x(j)|k].$$

Accordingly, (from Fundamental Theorem of Calculus)

$$V(k)=V(\underline{\mathrm{k}})+\int_{\underline{\mathrm{k}}}^{k}x(u)\delta\frac{d}{du}\mathbb{E}[\sum_{j\in N_u^o}x(j)|u]du,$$

where $\underline{\mathrm{k}}$ is the lowest degree type considered. One can write

$$\begin{aligned}
& V(k)-\pi_k(\hat{k},x(\cdot),P(\cdot)) \\
= \;& V(k)-\left(V(\hat{k})-x(\hat{k})\delta\mathbb{E}[\sum_{j\in N_{\hat{k}}^o}x(j)|\hat{k}]+x(\hat{k})\delta\mathbb{E}[\sum_{j\in N_k^o}x(j)|k]\right) \\
= \;& V(k)-V(\hat{k})+x(\hat{k})\delta\left(\mathbb{E}[\sum_{j\in N_{\hat{k}}^o}x(j)|\hat{k}]-\mathbb{E}[\sum_{j\in N_k^o}x(j)|k]\right) \\
= \;& \int_{\hat{k}}^{k}x(u)\delta\frac{d}{du}\mathbb{E}[\sum_{j\in N_u^o}x(j)|u]du \\
& +x(\hat{k})\delta\left(\mathbb{E}[\sum_{j\in N_{\hat{k}}^o}x(j)|\hat{k}]-\mathbb{E}[\sum_{j\in N_k^o}x(j)|k]\right) \\
\underbrace{=}_{\text{Integration by parts}} \;& (x(k)-x(\hat{k}))\delta\mathbb{E}[\sum_{j\in N_k^o}x(j)|k]-\int_{\hat{k}}^{k}x'(u)\delta\mathbb{E}[\sum_{j\in N_u^o}x(j)|u]du \\
= \;& \int_{\hat{k}}^{k}x'(u)\delta\mathbb{E}[\sum_{j\in N_k^o}x(j)|k]du-\int_{\hat{k}}^{k}x'(u)\delta\mathbb{E}[\sum_{j\in N_u^o}x(j)|u]du \\
\geq \;& 0,
\end{aligned}$$

If $k>\hat{k}$, the above quantity is nonnegative given $x'(\cdot)>0$ (which also means $\mathbb{E}[\sum_{j\in N_k^o}x(j)|k]$ increases in k). If $k<\hat{k}$, rewrite the above as

$$\int_{k}^{\hat{k}}x'(u)\delta\mathbb{E}[\sum_{j\in N_u^o}x(j)|u]du-\int_{k}^{\hat{k}}x'(u)\delta\mathbb{E}[\sum_{j\in N_k^o}x(j)|k]du,$$

which is also nonnegative given $x'(\cdot)>0$ (which also means $\mathbb{E}[\sum_{j\in N_k^o}x(j)|k]$ increases in k). Thus IC is achieved globally. The payment $P(x(k))$ is

$$P(x(k))=ax(k)-bx^2(k)+x(k)\delta\mathbb{E}[\sum_{j\in N_k^o}x(j)|k]-V(\underline{\mathrm{k}})-\int_{\underline{\mathrm{k}}}^{k}x(u)\delta\frac{d}{du}\mathbb{E}[\sum_{j\in N_u^o}x(j)|u]du. \quad \text{(EC.17)}$$

As a result, IC and IR constraints (5)-(6) can reduce to a single monotonicity constraint $x'(\cdot) > 0$.

Step (ii): Rewriting the objective. Now we return to the firm's problem. Its objective can be rewritten as follows:

$$
\begin{aligned}
& N\int_{\underline{k}}^{N}(P(x(k))-cx(k))f(k)dk \qquad \text{(EC.18)}\\
=\ & N\int_{\underline{k}}^{N}\left\{\begin{array}{c}(a-c)x(k)-bx^2(k)+x(k)\delta\mathbb{E}[\sum_{j\in N_k^o}x(j)|k]-V(\underline{k})\\ -\int_{\underline{k}}^{k}x(u)\delta\frac{d}{du}\mathbb{E}[\sum_{j\in N_u^o}x(j)|u]du\end{array}\right\}f(k)dk.
\end{aligned}
$$

Note for any random variable $Y\in[\underline{y},\bar{y}]$ with pdf., cdf. f,F, and any generic function $g(\cdot)$, we have:

$$
\begin{aligned}
\mathbb{E}[g(Y)] &= \int_{\underline{y}}^{\bar{y}} g(y)f(y)dy\\
&= g(\underline{y})+\int_{\underline{y}}^{\bar{y}} g^{'}(y)[1-F(y)]dy.
\end{aligned}
$$

In that way,

$$
\begin{aligned}
&\int_{\underline{k}}^{N}\left\{\int_{\underline{k}}^{k}x(u)\delta\frac{d}{du}\mathbb{E}[\sum_{j\in N_u^o}x(j)|u]du\right\}f(k)dk\\
=&\int_{\underline{k}}^{N}H(k)x(k)\delta\frac{d}{dk}\mathbb{E}[\sum_{j\in N_k^o}x(j)|k]f(k)dk,
\end{aligned}
$$

where in the last equality we recall the definition $H(k)=[1-F(k)]/f(k)$. Given the above, we rewrite the objective function as:

$$
\begin{aligned}
& \pi_0(x(\cdot)) \qquad \text{(EC.19)}\\
=\ & N\int_{\underline{k}}^{N}\left\{\begin{array}{c}(a-c)x(k)-bx^2(k)+x(k)\delta\mathbb{E}[\sum_{j\in N_k^o}x(j)|k]-V(\underline{k})\\ -\int_{\underline{k}}^{k}x(u)\delta\frac{d}{du}\mathbb{E}[\sum_{j\in N_u^o}x(j)|u]du\end{array}\right\}f(k)dk.\\
=\ & N\int_{\underline{k}}^{N}\left\{\begin{array}{c}(a-c)x(k)-bx^2(k)+x(k)\delta\mathbb{E}[\sum_{j\in N_k^o}x(j)|k]-V(\underline{k})\\ -H(k)x(k)\delta\frac{d}{dk}\mathbb{E}[\sum_{j\in N_k^o}x(j)|k]\end{array}\right\}f(k)dk\\
\underbrace{=}_{V(\underline{k})=0\text{ at optimum}}\ & N\int_{\underline{k}}^{N}\left[\begin{array}{c}(a-c)x(k)-bx^2(k)\\ +x(k)\delta\left\{\begin{array}{c}\mathbb{E}[\sum_{j\in N_k^o}x(j)|k]\\ -H(k)\frac{d}{dk}\mathbb{E}[\sum_{j\in N_k^o}x(j)|k]\end{array}\right\}\end{array}\right]f(k)dk
\end{aligned}
$$

The expression in (EC.19) shows that the firm's expected payoff is decreasing in $V(\underline{k})$. Thus, at optimality $V(\underline{k})=0$.

The firm's (transformed) problem is

$$\max_{x(\cdot)} \quad N\int_{\underline{k}}^{N}\left[\begin{array}{c}(a-c)x(k)-bx^2(k)\\ +x(k)\delta\left\{\begin{array}{c}\mathbb{E}[\sum_{j\in N_k^o}x(j)|k]\\ -H(k)\frac{d}{dk}\mathbb{E}[\sum_{j\in N_k^o}x(j)|k]\end{array}\right\}\end{array}\right]f(k)dk \tag{EC.20}$$

$$\text{s.t.} \quad x(\cdot)\text{ is increasing.} \tag{EC.21}$$

For the moment, let us first ignore the constraint (EC.21) and study the firm's objective (EC.20).

Step (iii): Calculus of variations and verification of the candidate solution and its properties. From Proposition 2, the aggregate out-neighbor consumption of a degree-k customer is

$$\mathbb{E}[\sum_{j\in N_k^o}x(j)|k]=rm\,(k+rm)^r\int_k^N\frac{x(y)}{(y+rm)^{r+1}}dy,$$

and

$$\begin{aligned}&\frac{d}{dk}\mathbb{E}[\sum_{j\in N_k^o}x(j)|k]\\ &=\frac{r}{k+rm}\mathbb{E}[\sum_{j\in N_k^o}x(j)|k]-\frac{rm}{k+rm}x(k).\end{aligned} \tag{EC.22}$$

Based on that, we rewrite the firm's transformed objective function $\pi_0(x(\cdot),P(\cdot))$ as

$$\begin{aligned}\pi_0(x(\cdot))&=N\int_{\underline{k}}^{N}\left[\begin{array}{c}(a-c)x(k)-bx^2(k)\\ +x(k)\delta\left\{\begin{array}{c}\mathbb{E}[\sum_{j\in N_k^o}x(j)|k]\\ -H(k)\frac{d}{dk}\mathbb{E}[\sum_{j\in N_k^o}x(j)|k]\end{array}\right\}\end{array}\right]f(k)dk \qquad \text{(EC.23)}\\ &=N\int_{\underline{k}}^{N}\left[\begin{array}{c}(a-c)x(k)-bx^2(k)\\ +x(k)\delta\left\{\begin{array}{c}\mathbb{E}[\sum_{j\in N_k^o}x(j)|k]\\ -H(k)\left(\begin{array}{c}\frac{r}{k+rm}\mathbb{E}[\sum_{j\in N_k^o}x(j)|k]\\ -\frac{rm}{k+rm}x(k)\end{array}\right)\end{array}\right\}\end{array}\right]f(k)dk\end{aligned}$$

$$= N\int_{\underline{k}}^{N}\left[\begin{array}{c}(a-c)x(k)-bx^2(k)\\ +x(k)\delta\left\{\begin{array}{c}\mathbb{E}[\sum_{j\in N_k^o}x(j)|k]\\ -\frac{r}{1+r}\left(\begin{array}{c}\mathbb{E}[\sum_{j\in N_k^o}x(j)|k]\\ -mx(k)\end{array}\right)\end{array}\right\}\end{array}\right]f(k)dk$$

$$= N\int_{\underline{k}}^{N}\left[\begin{array}{c}(a-c)x(k)-bx^2(k)\\ +x(k)\delta\left\{\begin{array}{c}\frac{1}{1+r}\mathbb{E}[\sum_{j\in N_k^o}x(j)|k]\\ -\frac{r}{1+r}\left(-mx(k)\right)\end{array}\right\}\end{array}\right]f(k)dk$$

$$= N\int_{\underline{k}}^{N}\left[\begin{array}{c}(a-c)x(k)-\left(b-\frac{\delta rm}{1+r}\right)x^2(k)\\ +x(k)\delta\left\{\frac{1}{1+r}\mathbb{E}[\sum_{j\in N_k^o}x(j)|k]\right\}\end{array}\right]f(k)dk \tag{EC.24}$$

where we recall that $H(k)=\frac{k+rm}{1+r}$. Since $\frac{\delta}{1+r}>0$, the presence of externality term $\mathbb{E}[\sum_{j\in N_k^o}x(j)|k]$ will positively shift the optimal consumption. That implies a lower bound on optimal consumption, denoted by $\underline{x}(\cdot)$, is the solution to the following objective that ignores the externality.

$$\max_{x(\cdot)}\underline{\pi}_0(x(\cdot))=N\int_{\underline{k}}^{N}\left[(a-c)x(k)-\left(b-\frac{\delta rm}{r+1}\right)x^2(k)\right]f(k)dk.$$

Pointwise maximization of $\underline{\pi}_0(x(\cdot))$ (or by Euler equation approach) yields

$$\underline{x}(k)\equiv\frac{a-c}{2\left(b-\frac{\delta rm}{r+1}\right)},$$

for which the second-order condition holds obviously. It follows $\underline{x}(k)>0$ given Assumptions 1, 2 ($a>c$ and $b>\delta m>\delta\frac{rm}{r+1}$). Thus the nonnegativity of $x(\cdot)$ is guaranteed. Denote $\underline{x}(k)$ by $\underline{x}^o$ for clarity.

Define

$$z^o(k):=\int_{k}^{N}\frac{x(y)}{(y+rm)^{r+1}}dy.$$

Given this definition, we have:

$$\mathbb{E}[\sum_{j\in N_k^o}x(j)|k]=rm\,(k+rm)^r\,z^o(k),$$
$$x(k)=-(k+rm)^{r+1}(z^o)'(k).$$

Then one can rewrite the integrand within the objective function $\pi_0(x(\cdot))$, in terms of $z^o(k)$ and its derivative $(z^o)^{'}(k)$:

$$G^o(k,z^o(k),(z^o)^{'}(k)) := \begin{bmatrix} (a-c)x(k)-(b-\frac{\delta rm}{1+r})x^2(k) \\ +x(k)\delta\left\{ \frac{1}{1+r}\mathbb{E}[\sum_{j\in N_k^o} x(j)|k] \right\} \end{bmatrix} f(k)$$

$$= -\begin{bmatrix} (a-c)(k+rm)^{r+1}(z^o)^{'}(k)+(b-\frac{\delta rm}{1+r})\left((k+rm)^{r+1}(z^o)^{'}(k)\right)^2 \\ +(k+rm)^{2r+1}(z^o)^{'}(k)\frac{\delta rm}{r+1}z^o(k) \end{bmatrix} f(k).$$

From (1), one can derive:

$$f(k)=\frac{(1+r)(rm)^{1+r}}{(k+rm)^{2+r}}$$

$$\Rightarrow f^{'}(k)=\frac{-(1+r)(2+r)(rm)^{1+r}}{(k+rm)^{3+r}}=-\frac{r+2}{k+rm}f(k).$$

Taking the partial derivatives of G^o w.r.t. $z^o(k)$ and $(z^o)^{'}(k)$ gives

$$G^o_{z^o(k)}(k,z^o(k),(z^o)^{'}(k)) = -\delta\frac{m}{r+1}r(rm)^{r+1}(r+1)(k+rm)^{r-1}(z^o)^{'}(k)$$

$$G^o_{(z^o)^{'}(k)}(k,z^o(k),(z^o)^{'}(k)) = \frac{1}{k+rm}\begin{Bmatrix} (rm)^{r+1}(r+1)\big[-(a-c)+(k+rm)^r \\ \left(-\delta\frac{m}{r+1}rz(k)+2(k+rm)(-b+\delta\frac{m}{r+1}r)(z^o)^{'}(k)\right)\big] \end{Bmatrix}$$

From the standard argument of calculus of variation, the optimal solution $z^{o*}(\cdot)$ for the firm is the solution to the following *Euler equation*:

$$G^o_{z^o(k)}(k,z^o(k),(z^o)^{'}(k))=\frac{d}{dk}G^o_{(z^o)^{'}(k)}(k,z^o(k),(z^o)^{'}(k)).$$

That is, $z^{o*}(\cdot)$ solves:

$$(a-c)+(k+rm)^r\left[-\frac{\delta rm}{r+1}(r-1)z^o(k)+2(k+rm)\left(-b+\frac{\delta rm}{r+1}\right)\left(r(z^o)^{'}(k)+(k+rm)(z^o)^{''}(k)\right)\right]=0\ \forall k. \tag{EC.25}$$

(EC.25) takes form of a second-order differential equation with regard to $z^o(\cdot)$, whose solution is explicit but cumbersome. Fortunately, we can verify the required monotonicity without recourse to the explicit solution (see below). The second-order condition, also called the Legendre condition, is satisfied under Assumption 2:

$$G^o_{z^{o\prime}(k)z^{o\prime}(k)}(k,z^o(k),z^{o\prime}(k))=-2(rm)^{r+1}(r+1)(k+rm)^r\left(b-\delta\frac{m}{r+1}r\right)\leq 0,\ \forall k.$$

To check for monotonicity, we put Euler equation into x-notation,

$$\begin{aligned}(a-c)-\delta\tfrac{m}{r+1}(r-1)r(k+rm)^r\int_k^N \tfrac{x^*(y)}{(y+rm)^{r+1}}dy \\ +2\left(b-\delta\tfrac{m}{r+1}r\right)(-x^*(k)+(k+rm)x^{*\prime}(k))\end{aligned} = 0,$$

and rearrange it as follows:

$$\begin{aligned}-(a-c)+\delta\tfrac{m}{r+1}(r-1)r(k+rm)^r\int_k^N \tfrac{x^*(y)}{(y+rm)^{r+1}}dy \\ +2\left(b-\delta\tfrac{m}{r+1}r\right)x^*(k)\end{aligned} = 2\left(b-\delta\frac{m}{r+1}r\right)(k+rm)x^{*\prime}(k). \qquad \text{(EC.26)}$$

Under Assumption 2, $b-\delta\frac{m}{r+1}r>0$. Observe that

$$\begin{aligned}2\left(b-\delta\frac{m}{r+1}r\right)x^*(k) &> 2\left(b-\delta\frac{m}{r+1}r\right)\underline{x}^o \\ &= 2\left(b-\delta\frac{m}{r+1}r\right)\frac{a-c}{2\left(b-\frac{\delta rm}{r+1}\right)} \\ &= a-c.\end{aligned}$$

Thus, the right-hand side of (EC.26) is greater than $\delta\frac{m}{r+1}(r-1)r(k+rm)^r\int_k^N \frac{x^*(y)}{(y+rm)^{r+1}}dy$, which suggests $x^{*\prime}(k)$ being positive if $r\geq 1$. Therefore, the monotonicity constraint is satisfied if $r\geq 1$ (Assumption 2).

Lastly, for the ease of presentation, we recollect the Eular equation using more existing notations:

$$\begin{aligned}&\begin{aligned}-(a-c)+\tfrac{\delta rm}{r+1}(r-1)(k+rm)^r\int_k^N \tfrac{x^*(y)}{(y+rm)^{r+1}}dy \\ +2\left(b-\tfrac{\delta rm}{r+1}\right)x^*(k)\end{aligned} = 2\left(b-\frac{\delta rm}{r+1}\right)(k+rm)x^{*\prime}(k) \\ &\Leftrightarrow -(a-c)+\delta\frac{r-1}{r+1}\mathbb{E}[\sum_{j\in N_k^o}x^*(j)|k]+2\left(b-\frac{\delta rm}{r+1}\right)x^*(k) = 2\left(b-\frac{\delta rm}{r+1}\right)(k+rm)x^{*\prime}(k) \\ &\Leftrightarrow -\left(\frac{a-c}{b-\frac{\delta rm}{r+1}}\right)+\frac{\delta(r-1)}{b+br-\delta rm}\mathbb{E}[\sum_{j\in N_k^o}x^*(j)|k]+2x^*(k) = 2(k+rm)x^{*\prime}(k) \\ &\Leftrightarrow \frac{\delta(r-1)}{2(b+br-\delta rm)}\mathbb{E}[\sum_{j\in N_k^o}x^*(j)|k]+x^*(k)-(k+rm)x^{*\prime}(k)-\underline{x}^o = 0\end{aligned}$$

This form of Eular equation is presented in Theorem 1. Plugging $x^*(\cdot)$ into the payment function (EC.17) gives the optimal payment scheme $P^*(\cdot)$ below.[22]

$$P^*(x^*(k)) = ax^*(k)-b(x^*(k))^2+x^*(k)\delta\mathbb{E}[\sum_{j\in N_k^o}x^*(j)|k]-\int_0^k x^*(u)\delta\frac{d}{du}\mathbb{E}[\sum_{j\in N_u^o}x^*(j)|u]du$$

[22] Note the induced consumption below $\underline{\mathrm{k}}$ is zero.

Given the nature of coordination in the consumption game, one may suspect whether a certain pricing scheme can lead to multiple consumption equilibria. To address this issue, we will show that the equilibrium optimal to the firm can be uniquely implemented. Consider a prescribed menu $\{x^*(\cdot), P^*(\cdot)\}$ determined optimally from Theorem 1 and agents report their degrees under this menu. Denote by $\tilde{k}(\cdot)$ the strategy that the focal customer perceives that his neighbors will play (which maps true degree to a reported one). Then let $\tilde{x}(k) := x^*(\tilde{k}(k))$ be the resulting consumption from the pattern of neighbors' misreporting as speculated by the focal customer. Note that $\tilde{x}(k)$ may not be consistent with the desired consumption $x^*(k)$. We will show that any belief $\tilde{x}(\cdot)$ other than $x^*(\cdot)$ will not get implemented by $P^*(\cdot)$. The degree-k customer's payoff when reporting $\hat{k}$ is

$$\pi_k(\hat{k}, \tilde{x}(\cdot), x^*(\cdot), P^*(\cdot)) = ax^*(\hat{k}) - bx^{*2}(\hat{k}) + x^*(\hat{k})\delta\mathbb{E}[\sum_{j\in N_k^o} \tilde{x}(j)|k] - P^*(x^*(\hat{k})) \tag{EC.27}$$

$$\underbrace{=}_{(14)} x^*(\hat{k})\delta\mathbb{E}[\sum_{j\in N_k^o} \tilde{x}(j)|k] - x^*(\hat{k})\delta\mathbb{E}[\sum_{j\in N_{\hat{k}}^o} x^*(j)|\hat{k}] + \int_0^{\hat{k}} x^*(u)\delta\frac{d}{du}\mathbb{E}[\sum_{j\in N_u^o} x^*(j)|u]du \tag{EC.28}$$

$$= x^*(\hat{k})\delta\left(\mathbb{E}[\sum_{j\in N_k^o} \tilde{x}(j)|k] - \mathbb{E}[\sum_{j\in N_{\hat{k}}^o} x^*(j)|\hat{k}]\right) + \int_0^{\hat{k}} x^*(u)\delta\frac{d}{du}\mathbb{E}[\sum_{j\in N_u^o} x^*(j)|u]du \tag{EC.29}$$

Define k_0 such that $\mathbb{E}[\sum_{j\in N_k^o} \tilde{x}(j)|k] = \mathbb{E}[\sum_{j\in N_{k_0}^o} x^*(j)|k_0]$. Note that $k = k_0$ if $\tilde{x}(\cdot)$ coincides with $x^*(\cdot)$ (or truth reporting $\tilde{k}(k) = k$). Thus

$$\pi_k(k_0, \tilde{x}(\cdot), x^*(\cdot), P^*(\cdot)) = x^*(k_0)\delta\left(\mathbb{E}[\sum_{j\in N_k^o} \tilde{x}(j)|k] - \mathbb{E}[\sum_{j\in N_{k_0}^o} x^*(j)|k_0]\right) + \int_0^{k_0} x^*(u)\delta\frac{d}{du}\mathbb{E}[\sum_{j\in N_u^o} x^*(j)|u]du \tag{EC.30}$$

$$= \int_0^{k_0} x^*(u)\delta\frac{d}{du}\mathbb{E}[\sum_{j\in N_u^o} x^*(j)|u]du \tag{EC.31}$$

$$\pi_k(k_0, \tilde{x}(\cdot), x^*(\cdot), P^*(\cdot)) - \pi_k(\hat{k}, \tilde{x}(\cdot), x^*(\cdot), P^*(\cdot)) = \int_{\hat{k}}^{k_0} x^*(u)\delta\frac{d}{du}\mathbb{E}[\sum_{j\in N_u^o} x^*(j)|u]du - x^*(\hat{k})\delta\left(\mathbb{E}[\sum_{j\in N_{k_0}^o} x^*(j)|k_0] - \mathbb{E}[\sum_{j\in N_{\hat{k}}^o} x^*(j)|\hat{k}]\right) \tag{EC.32}$$

$$> 0 \text{ given } x^*(\cdot) \text{ increasing} \tag{EC.33}$$

Therefore under $P^*(\cdot)$, the degree-k player anticipating others to play $\tilde{x}(\cdot)$ will optimally declare k_0. The resulting consumption $x^*(k_0)$ generally does not equal $\tilde{x}(k)$, by definition of k_0, unless $\tilde{x}(\cdot)$

coincides with $x^*(\cdot)$. That means $x^*(\cdot)$ (or truth reporting) will be the *only* equilibrium implemented by the scheme $P^*(\cdot)$. $\square$

Proof of Proposition 3. To study the marginal price in variation with degree, note that incentive compatibility (EC.13) implies

$$P'(x(k)) = a - 2bx(k) + \delta \mathbb{E}[\sum_{j \in N_k^o} x(j)|k] \tag{EC.34}$$

$$\frac{dP'(x(k))}{dk} = -2bx'(k) + \frac{r}{k+rm}\left[\delta \mathbb{E}[\sum_{j \in N_k^o} x(j)|k] - \delta m x(k)\right] \tag{EC.35}$$

Substituting the optimal solution into the above expression,

$$\begin{aligned}\frac{dP'(x^*(k))}{dk} &= -2bx^{*\prime}(k) + \frac{r}{k+rm}\left[\delta \mathbb{E}[\sum_{j \in N_k^o} x^*(j)|k] - \delta m x^*(k)\right] \\ &\underbrace{<}_{(13),\, x^*(\cdot) \geq \underline{x}} -2bx^{*\prime}(k) + \frac{r}{k+rm}\left[\frac{r+1}{r-1} 2\left(b - \frac{\delta rm}{r+1}\right)(k+rm)x^{*\prime}(k) - \delta m x^*(k)\right]\end{aligned}$$

The above quantity is negative if the multiplier of $x^{*\prime}(k)$ is negative, which can reduce to $\frac{b}{b+br-\delta rm} > \frac{r}{r-1}$. In this case, the scheme charges a lower marginal price for higher degree customers at optimum. Then,

$$P''(x^*(k)) = \frac{dP'(x^*(k))}{dk} / x^{*\prime}(k) < 0.$$

which gives rise to a quantity discount menu.

For the out-neighbor model, recall the upper bound of consumption, derived from the variant game, $\bar{x}(k) = \frac{a-c+\delta \mathbb{E}[\sum_{j \in N_k^o} x(j)|k]}{2b}$, or equivalently,

$$\mathbb{E}[\sum_{j \in N_k^o} x(j)|k] > \frac{2bx(k) - (a-c)}{\delta}. \tag{EC.36}$$

In seeking for conditions for quantity premium menu, we obtain

$$\begin{aligned}\frac{dP'(x^*(k))}{dk} &= -2bx^{*\prime}(k) + \frac{r}{k+rm}\left[\delta \mathbb{E}[\sum_{j \in N_k^o} x^*(j)|k] - \delta m x^*(k)\right] \\ &= \frac{1}{k+rm}\left[-2b(k+rm)x^{*\prime}(k) + \delta r\left(\mathbb{E}[\sum_{j \in N_k^o} x^*(j)|k] - m x^*(k)\right)\right]\end{aligned}$$

$$
\begin{aligned}
&\underbrace{=}_{(13)} && \frac{1}{k+rm}\Big[-b\Big(-\Big(\tfrac{a-c}{b-\frac{\delta rm}{r+1}}\Big)+\tfrac{\delta}{b-\frac{\delta rm}{r+1}}\tfrac{r-1}{r+1}\mathbb{E}[\textstyle\sum_{j\in N_k^o}x^*(j)|k]+2x^*(k)\Big)\\
&&& +\delta r\Big(\mathbb{E}[\sum_{j\in N_k^o}x^*(j)|k]-mx^*(k)\Big)\Big]\\
&> && \frac{\delta}{k+rm}\Big[-\Big(\tfrac{b}{b-\frac{\delta rm}{r+1}}\tfrac{r-1}{r+1}\mathbb{E}[\textstyle\sum_{j\in N_k^o}x^*(j)|k]+2\tfrac{b}{\delta}x^*(k)\Big)\\
&&& +\Big(r\mathbb{E}[\sum_{j\in N_k^o}x^*(j)|k]-rmx^*(k)\Big)\Big]\\
&= && \frac{\delta}{k+rm}\left[\left(\left(r-\frac{b}{b-\frac{\delta rm}{r+1}}\frac{r-1}{r+1}\right)\mathbb{E}[\sum_{j\in N_k^o}x^*(j)|k]-\left(rm+\frac{2b}{\delta}\right)x^*(k)\right)\right]\\
&\underbrace{>}_{(EC.36)} && \frac{\delta}{k+rm}\left[\left(\left(r-\frac{b(r-1)}{b+br-\delta rm}\right)\left(\frac{2bx^*(k)-(a-c)}{\delta}\right)-\left(rm+\frac{2b}{\delta}\right)x^*(k)\right)\right]\\
&= && \frac{1}{k+rm}\left[\left(\left(r-\frac{b(r-1)}{b+br-\delta rm}\right)(2bx^*(k)-(a-c))-(\delta rm+2b)\,x^*(k)\right)\right]\\
&\underbrace{>}_{x^*(k)>\underline{x}^o,(EC.37)} && \frac{1}{k+rm}\left[\left(\left(r-\frac{b(r-1)}{b+br-\delta rm}\right)(2b\underline{x}^o-(a-c))-(\delta rm+2b)\,\underline{x}^o\right)\right]\\
&= && \frac{a-c}{k+rm}\left[\frac{-\delta^2r^2m^2(r-1)-2b^2(r+1)^2+b\delta rm(r^2+3)}{2(b+br-\delta rm)^2}\right]\\
&\underbrace{>}_{(EC.38)} && 0
\end{aligned}
$$

where the sufficient conditions include

$$2b(r-1)\left[1-\frac{b}{b+br-\delta rm}\right]-\delta rm > 0, \tag{EC.37}$$

$$-\delta^2r^2m^2(r-1)-2b^2(r+1)^2+b\delta rm(r^2+3) > 0. \tag{EC.38}$$

In this case, the firm charges a higher marginal price for higher degree customers at optimum. This implies quantity premium, since $P''(x^*(k))=\frac{dP'(x^*(k))}{dk}/x^{*\prime}(k)>0$. □

Proof of Corollary 1 Denote by $\Pi^o(k):=P(x(k))-cx(k)$ the profit earned from a single degree-k customer. Therefore we obtain

$$
\begin{aligned}
\frac{d}{dk}\Pi^o(k) &= x'(k)\left(P'(x(k))-c\right)\\
&\underbrace{=}_{(EC.13)} x'(k)\left(a-c-2bx(k)+\delta\mathbb{E}[\sum_{j\in N_k^o}x(j)|k]\right)
\end{aligned}
$$

At optimality, $x^{*\prime}(k) > 0$, and $a - c - 2bx^*(k) + \delta\mathbb{E}[\sum_{j\in N_k^o} x^*(j)|k] > 0$ (as $x^*(k) < \bar{x}(k)$). Thus $\frac{d}{dk}\Pi^{*o}(k) > 0$. That is, the firm grasps higher profit from higher degree customers at optimum.

Comparing the firm's second-best objective to that of the first-best, denoted as $\pi_0^{oFB}(x(\cdot))$,

$$\begin{aligned}
\max_{x(\cdot)} \pi_0^o(x(\cdot)) &= N\int_{\underline{k}}^{N} \begin{bmatrix} (a-c)x(k) - (b - \frac{\delta rm}{1+r})x^2(k) \\ +x(k)\delta\left\{ \frac{1}{1+r}\mathbb{E}[\sum_{j\in N_k^o} x(j)|k] \right\} \end{bmatrix} f(k)dk, \\
&= N\int_{\underline{k}}^{N} \begin{bmatrix} (a-c)x(k) - bx^2(k) \\ +x(k)\delta\mathbb{E}[\sum_{j\in N_k^o} x(j)|k] \\ -x(k)\delta\frac{r}{1+r}\mathbb{E}[\sum_{j\in N_k^o} x(j)|k] + \frac{\delta rm}{1+r}x^2(k) \end{bmatrix} f(k)dk, \\
&= N\int_{\underline{k}}^{N} \begin{bmatrix} (a-c)x(k) - bx^2(k) + x(k)\delta\mathbb{E}[\sum_{j\in N_k^o} x(j)|k] \\ -x(k)\frac{\delta r}{1+r}\left(\mathbb{E}[\sum_{j\in N_k^o} x(j)|k] - mx(k)\right) \end{bmatrix} f(k)dk \\
\max_{x(\cdot)} \pi_0^{oFB}(x(\cdot)) &= N\int_{\underline{k}}^{N} \begin{bmatrix} (a-c)x(k) - bx^2(k) \\ +x(k)\delta\mathbb{E}[\sum_{j\in N_k^o} x(j)|k] \end{bmatrix} f(k)dk
\end{aligned}$$

Since any feasible second-best solution will satisfy $\mathbb{E}[\sum_{j\in N_k^o} x(j)|k] - mx(k) > 0$ (given the consumption must be increasing in degree), the difference between first-best and second-best objectives,

$$x(k)\frac{\delta r}{1+r}\left(\mathbb{E}[\sum_{j\in N_k^o} x(j)|k] - mx(k)\right),$$

consists of $x(k)$ and a positive multiplier. Thus the incentive for consumption in second-best is less than that in first-best, under any feasible choice of $x(\cdot)$ in the second-best case. That suggests downward distortion in consumption in the second-best scenario.[23]

[23] Note that the value of $\underline{k}$ is lower in the first-best profit function than that of the second-best case, as the firm earns more in FB from each degree type-k customers (extracting the information rent), thus selling to more customers down the list of types in FB. Therefore, the downward distortion result still holds in the range of types between $\underline{k}$ of FB and $\underline{k}$ of SB, where the induced consumption is positive for FB and 0 for SB. For types lower than $\underline{k}$ of FB, consumptions in both FB and SB are 0. (As shown before:) In types higher than $\underline{k}$ of SB, consumption in FB dominates that in SB (both nonzero). Hence, downward distortion holds for all degree types.

That $x^*(\cdot)$ increases by k is a direct consequence of being a feasible solution to the firm's problem (4)-(6). Now we show that $x^*(\cdot)$ also increases in m. Recall the firm's profit

$$\pi_0^o(x(\cdot)) = N\int_{\underline{k}}^{N}\left[\begin{array}{c}(a-c)x(k)-\left(b-\frac{\delta rm}{1+r}\right)x^2(k)\\ +x(k)\delta\left\{\frac{1}{1+r}\mathbb{E}[\sum_{j\in N_k^o}x(j)|k]\right\}\end{array}\right]f(k)dk,$$

where the bracketed term

$$(a-c)x(k)-\left(b-\frac{\delta rm}{1+r}\right)x^2(k)+x(k)\delta\left\{\tfrac{1}{1+r}\mathbb{E}[\textstyle\sum_{j\in N_k^o}x(j)|k]\right\} \tag{EC.39}$$

captures the profit gathered from a single degree-k customer under consumption $x(\cdot)$. We start with the hypothesis that $\mathbb{E}[\sum_{j\in N_k^o}x(j)|k]$ increases in m, and will show that this can be reinforced at optimum. Now suppose m increases. Then the disutility coefficient $b-\frac{\delta rm}{1+r}$ in (EC.39) is reduced, and $\mathbb{E}[\sum_{j\in N_k^o}x(j)|k]$ increases (by hypothesis). Therefore, the induced incentive for consumption for the customer in question increases. The optimal consumption level thus increases in m. Given that $x(k)$ increases in k (at optimum) and that increasing m triggers a first-order stochastic increase of the out-neighbor degree distributions F_k^{o1}, F_k^{o2} (Proposition 1), it follows that $\mathbb{E}[\sum_{j\in N_k^o}x(j)|k]$ increases in m,[24] which reinforces the foregoing hypothesis and concludes the proof.

It remains to show that the optimal profit of the firm also increases in m. Given that the coefficient $b-\frac{\delta rm}{1+r}$ declines in m, and $\mathbb{E}[\sum_{j\in N_k^o}x(j)|k]$ increase in m at optimum (as shown above), it follows that the profit earned from one single degree-k player, (EC.39), also increases in m at optimum. Since the degree distribution $F(\cdot)$ first-order stochastically increases in m (c.f. Theorem 7 of Jackson and Rogers (2007)), and (EC.39) is at optimum an increasing function of degree k (c.f. the first bullet point of Corollary 1), we conclude that the firm's overall profit $\pi_0^o(x(\cdot))$ (aggregated over all degree types) increases in m at optimum. □

Proof of Proposition 4. To allow for arbitrary price discrimination, suppose that the firm draws a random number $\eta\in[0,1]$ for each agent from cdf. $Q(\cdot)$, based on which the allocation

[24] Recall that $\mathbb{E}[\sum_{j\in N_k^o}x(j)|k] = o_r\int_k^N x(y)f_k^{o1}(y)dy + o_n\int_k^N x(y)f_k^{o2}(y)dy$, where $o_r=\frac{rm}{r+1}$ and $o_n=\frac{m}{r+1}$ both increase in m.

$\{x(\eta), P(\eta)\}$ is made to that agent. The agent will then decide to accept or reject. Notice that, since the firm does not know who is connected to whom, it cannot enforce any correlation between random numbers of neighbors. So everyone will perceive his neighbor's number as samely randomly distributed according to $Q(\cdot)$. Note that the topological correlation is not explored here, exactly because the pricing does not solicit the degree information. If it does, the firm can then utilize the knowledge of neighbor degree distribution to refine individual allocations.

For any player with number η, the expected total out-neighbor consumption when neighbors accept the contract is $\mathbb{E}[\sum_{j\in N_\eta^o} x(j)|\eta] = m\int_0^1 x(j)q(j)dj$. Furthermore, the expected out-neighbor consumption does not change with one's own number because neighbors' numbers are independent, i.e. $\frac{d}{d\eta}\mathbb{E}[\sum_{j\in N_\eta^o} x(j)|\eta] = 0$. For given $Q(\cdot)$, the firm's optimization resembles a first-best pricing problem, where the agent's type – his number – is observable to the firm.[25] If accepting the firm's offer, the type-η agent's payoff, while others accepting the contract, is given by

$$\pi_\eta(x(\cdot), P(\cdot)) = ax(\eta) - bx^2(\eta) + x(\eta)\delta\mathbb{E}[\sum_{j\in N_\eta^o} x(j)|\eta] - P(\eta),$$

Under first-best, we should have $\pi_\eta(x(\cdot), P(\cdot)) = 0$ (no information rent for the agent). That leads to the pricing scheme is $P^*(\eta) = ax(\eta) - bx^2(\eta) + x(\eta)\delta\mathbb{E}[\sum_{j\in N_\eta^o} x(j)|\eta]$. The firm then obtains the maximum social welfare as follows (where $\mathbb{E}, \mathbb{V}$ respectively represents expectation and variance).

$$\max \pi_0(x, P(\cdot)) = N\mathbb{E}\left\{(a-c)x(\eta) - bx^2(\eta) + x(\eta)\delta\mathbb{E}[\sum_{j\in N_\eta^o} x(j)|\eta]\right\} \quad \text{(EC.40)}$$

$$= N\left\{(a-c)\mathbb{E}[x(\eta)] - b\mathbb{E}[x^2(\eta)] + \mathbb{E}[x(\eta)]\delta m\mathbb{E}[x(j)|_{j\in N_\eta^o}\eta]\right\} \quad \text{(EC.41)}$$

$$= N\left\{(a-c)\mathbb{E}[x(\eta)] - b\mathbb{E}[x^2(\eta)] + \delta m\mathbb{E}[x(\eta)]^2\right\} \quad \text{(EC.42)}$$

$$= N\left\{(a-c)\mathbb{E}[x(\eta)] - b\left(\mathbb{V}[x(\eta)] + \mathbb{E}[x(\eta)]^2\right) + \delta m\mathbb{E}[x(\eta)]^2\right\} \quad \text{(EC.43)}$$

$$= N\left\{(a-c)\mathbb{E}[x(\eta)] - b\mathbb{V}[x(\eta)] - (b-\delta m)\mathbb{E}[x(\eta)]^2\right\} \quad \text{(EC.44)}$$

[25] The decision of $Q(\cdot)$ is cosmetic since η merely serves a proxy for the firm to arbitrarily price-discriminate the customers. As shown in Proposition 4, the induced consumption and optimal pricing are independent of $Q(\cdot)$.

Thus the firm at optimum wants to minimize the variance of $x(\eta)$ by providing a menu that contains only a single consumption level x, i.e. $\mathbb{V}[x(\eta)]=0, \mathbb{E}[x(\eta)]=x$. So the firm's problem reduces to

$$\max \pi_0(x,P(\cdot)) = N\left[(a-c)x-(b-\delta m)x^2\right] \tag{EC.45}$$

Provided $b>\delta m$ (Assumption 2), the optimal consumption level $x^*(\eta)=\frac{a-c}{2(b-\delta m)}$; and the optimal payment is $P^*(\eta)=ax^*(\eta)-bx^{*2}(\eta)+x^*(\eta)\delta\mathbb{E}[\sum_{j\in N^o_\eta}x^*(j)|\eta]=\frac{a^2-c^2}{4(b-\delta m)}$. Substituting x^* into (EC.45), the firm's profit at the optimum is $N\frac{(a-c)^2}{4(b-\delta m)}$.

Observe the firm's profit with the DS pricing, (EC.47), and that under NDS, (EC.45),

$$\pi_0(x(\cdot)) \underbrace{=}_{(16)} N\int_{\underline{k}}^{N}\left[\begin{array}{c}(a-c)x(k)-bx^2(k)\\ +x(k)\delta\left\{\begin{array}{c}\mathbb{E}[\sum_{j\in N^o_k}x(j)|k]\\ -H(k)\frac{d}{dk}\mathbb{E}[\sum_{j\in N^o_k}x(j)|k]\end{array}\right\}\end{array}\right]f(k)dk \tag{EC.46}$$

$$\underbrace{=}_{(EC.24)} N\int_{\underline{k}}^{N}\left[\begin{array}{c}(a-c)x(k)-\left(b-\frac{\delta rm}{1+r}\right)x^2(k)\\ +x(k)\delta\left\{\frac{1}{1+r}\mathbb{E}[\sum_{j\in N^o_k}x(j)|k]\right\}\end{array}\right]f(k)dk \tag{EC.47}$$

and note that (EC.47) reduces to (EC.45) if $x(k)\equiv x$ and $k\to 0$ (i.e. all customers served in a homogenous equilibrium). Since the firm is strictly better off with more flexible control of $x(k)$ over all k (as in (EC.47)) than restricting to a single x (as in (EC.45)), [26] the profit earned through NDS pricing is dominated by that under DS pricing. □

Proof of Proposition 5. In case of the Erdös and Rényi graph, $r\to\infty$, $\lim_{r\to\infty}H(k):=\lim_{r\to\infty}\frac{k+rm}{1+r}=m$. We obtain:

$$\begin{aligned}&\lim_{r\to\infty}\mathbb{E}[\sum_{j\in N^o_k}x(j)|k]\\ =&\lim_{r\to\infty}\left\{rm\,(k+rm)^r\int_k^N\frac{x(y)}{(y+rm)^{r+1}}dy\right\}\end{aligned}$$

[26] Note that maximizing (EC.47) will necessarily result in degree-heterogeneous consumption that differs from the solution to (EC.45), provided that the degree distribution and neighbor degree distributions are not symmetric across degrees. Only in the Erdös and Rényi graph where the network externalities are cancelled off, the solutions of DS pricing and NDS pricing will become identical. See Proposition 5.

$$= \int_k^N x(y) \left\{ \lim_{r\to\infty} \frac{rm\,(k+rm)^r}{(y+rm)^{r+1}} \right\} dy$$
$$= \int_k^N x(y) \exp\left(\frac{k-y}{m}\right) dy.$$

Accordingly, its derivative is:

$$\begin{aligned} &\frac{d}{dk}\mathbb{E}[\sum_{j\in N_k^o} x(j)|k]\Big|_{r\to\infty} \\ &= \frac{r}{k+rm}\mathbb{E}[\sum_{j\in N_k^o} x(j)|k]\Big|_{r\to\infty} - \frac{rm}{k+rm}x(k) \\ &= \frac{1}{m}\mathbb{E}[\sum_{j\in N_k^o} x(j)|k]\Big|_{r\to\infty} - x(k). \end{aligned}$$

Recall that in the firm's objective (16); the profit earned from a single degree-k player in the Erdös and Rényi graph can be rewritten as:

$$\begin{aligned} \Pi_o^R(x(\cdot)) &= (a-c)x(k) - bx^2(k) + x(k)\delta \left\{ \begin{array}{c} \mathbb{E}[\sum_{j\in N_k^o} x(j)|k] \\ -H(k)\frac{d}{dk}\mathbb{E}[\sum_{j\in N_k^o} x(j)|k] \end{array} \right\} \\ &= (a-c)x(k) - bx^2(k) + x(k)\delta \left\{ \begin{array}{c} \mathbb{E}[\sum_{j\in N_k^o} x(j)|k] \\ -m\left[\frac{1}{m}\mathbb{E}[\sum_{j\in N_k^o} x(j)|k] - x(k)\right] \end{array} \right\} \\ &= (a-c)x(k) - (b-\delta m)x^2(k). \end{aligned}$$

Notice that the externality term is cancelled off in the deduction. The resulting objective function of the firm becomes identical to (EC.45). The proof follows from that of Proposition 4. □

Proof of Theorem 2. The proof is similar to that of Theorem 1, except for the modification of expected neighbor consumption. In steps (i) and (ii), the analysis is identical and therefore we omit it. We start directly with step (iii). We decompose it into two sub-steps: (a) Calculus of variations and (b) Verification of the candidate solution and its properties.

(a) Calculus of variations. Let

$$z(k) \equiv \int_0^k \frac{x(y)}{(y+rm)^2} dy = \frac{1}{(k+rm)rm}\mathbb{E}[\sum_{j\in N_k^i} x(j)|k].$$

Hence we have

$$\begin{aligned} x(k) &= z'(k)(k+rm)^2, \\ z'(k) &= \frac{x(k)}{(k+rm)^2}, \\ z''(k) &= \frac{x'(k)}{(k+rm)^2} - \frac{2x(k)}{(k+rm)^3}. \end{aligned}$$

Using this definition, we convert the firm's profit (16) as a function of $z(k)$ and $z^{'}(k)$ below.

$$\begin{aligned} &\pi_0(x(\cdot)) \\ &= N\int_{\underline{k}}^{N} \left\{ \begin{array}{c} (a-c)z^{'}(k)(k+rm)^2 - b[z^{'}(k)]^2(k+rm)^4 \\ +z^{'}(k)\,(k+rm)^2\delta \left[\begin{array}{c} rm(k+rm)z(k) \\ -H(k)\frac{d}{dk}\{rm(k+rm)z(k)\} \end{array} \right] \end{array} \right\} f(k)dk \\ &= N\int_{\underline{k}}^{N} \left\{ \begin{array}{c} (a-c)z^{'}(k)(k+rm)^2 - b[z^{'}(k)]^2(k+rm)^4 \\ +z^{'}(k)\,(k+rm)^2\delta \left[\begin{array}{c} rm(k+rm)z(k) \\ -H(k)rmz(k) \\ -H(k)rm(k+rm)z^{'}(k) \end{array} \right] \end{array} \right\} f(k)dk. \end{aligned} \tag{EC.48}$$

Then (EC.48) exhibits the structure of calculus of variations. Let

$$G^i(k,z(k),z^{'}(k)) := \left\{ \begin{array}{c} (a-c)z^{'}(k)(k+rm)^2 - b[z^{'}(k)]^2(k+rm)^4 \\ +z^{'}(k)\,(k+rm)^2\delta \left[\begin{array}{c} rm(k+rm)z(k) - H(k)rmz(k) \\ -H(k)rm(k+rm)z^{'}(k) \end{array} \right] \end{array} \right\} f(k).$$

We obtain its partial derivatives as follows:

$$\begin{aligned} G^i_{z(k)}(k,z(k),z^{'}(k)) &= \{z^{'}(k)\,(k+rm)^2\delta[rm(k+rm) - H(k)rm]\}f(k) \\ G^i_{z'(k)}(k,z(k),z^{'}(k)) &= f(k)\left\{ \begin{array}{c} (a-c)(k+rm)^2 - 2bz^{'}(k)\,(k+rm)^4 \\ +\delta(k+rm)^2 \left[\begin{array}{c} rm(k+rm)z(k) \\ -2H(k)rm(z(k)+(k+rm)z'(k)) \end{array} \right] \end{array} \right\}. \end{aligned}$$

The optimal solution $z^*(\cdot)$ is identified by the first-order condition below (referred as the Euler equation):

$$G^i_{z(k)}(k,z(k),z^{'}(k)) = dG^i_{z'(k)}(k,z(k),z^{'}(k))/dk. \tag{EC.49}$$

Note that (EC.49) takes form of a second-order differential equation with regard to $z(\cdot)$, whose solution is explicit but cumbersome. In our case, we can verify the required monotonicity without writing down the explicit form of the solution (see later Step (b) of the proof).

Recall that

$$\begin{aligned} H(k) &= \frac{1-F(k)}{f(k)} = \frac{k+rm}{1+r} \Rightarrow H^{'}(k) = \frac{1}{r+1}, \\ f(k) &= \frac{(1+r)(rm)^{1+r}}{(k+rm)^{2+r}} \Rightarrow f^{'}(k) = \frac{-(r+1)(r+2)(rm)^{r+1}}{(k+rm)^{r+3}}. \end{aligned} \tag{EC.50}$$

Putting the Euler equation in x-notation, we have

$$\begin{aligned} &(a-c)r(1+r) + \delta m(r-1)r^2(k+rm)\int_0^k \tfrac{x^*(y)}{(y+rm)^2}dy \\ &+2(b+br+\delta rm)(-rx^*(k)+(k+rm)x^{*\prime}(k)) = 0. \end{aligned} \tag{EC.51}$$

The second-order condition for optimality (Legendre condition)

$$G^i_{z'(k)z'(k)}(k, z(k), z^{'}(k)) \le 0, \ \forall k,$$

can be easily shown satisfied. Thus, the Euler equation (EC.49) identifies the maximum point of the firm's objective function.

(b) Verification of the candidate solution and its properties. Now we proceed to show the solution to Euler equation (EC.51) does qualify the monotonic constraint (EC.21). First, we need to derive some lower bound on the induced consumption. As suggested in Proposition 2,

$$\mathbb{E}[\sum_{j\in N_k^i} x(j)|k] = (k+rm)rm\int_0^k \frac{x(y)}{(y+rm)^2}dy.$$

Its derivative is

$$\frac{d}{dk}\mathbb{E}[\sum_{j\in N_k^i} x(j)|k] = rm\left(\int_0^k \frac{x_j(y)}{(y+rm)^2}dy + \frac{x_j(k)}{k+rm}\right) = \frac{\mathbb{E}[\sum_{j\in N_k^i} x(j)|k]}{k+rm} + \frac{rmx(k)}{k+rm}. \tag{EC.52}$$

Rewrite the firm's transformed objective function by substituting $\frac{d}{dk}\mathbb{E}[\sum_{j\in N_k^i} x(j)|k]$ above:

$$\max_{x(\cdot)} \pi_0(x(\cdot))$$

$$= N\int_{\underline{k}}^{N}\left[\begin{array}{c}(a-c)x(k)-bx^2(k)\\ +x(k)\delta\left\{\mathbb{E}[\sum_{j\in N_k^i}x(j)|k]-H(k)\frac{d}{dk}\mathbb{E}[\sum_{j\in N_k^i}x(j)|k]\right\}\end{array}\right]f(k)dk$$

$$= N\int_{\underline{k}}^{N}\left[\begin{array}{c}(a-c)x(k)-bx^2(k)\\ +x(k)\delta\left\{\mathbb{E}[\sum_{j\in N_k^i}x(j)|k]-H(k)\left(\frac{\mathbb{E}[\sum_{j\in N_k^i}x(j)|k]}{k+rm}+\frac{rmx(k)}{k+rm}\right)\right\}\end{array}\right]f(k)dk$$

$$= N\int_{\underline{k}}^{N}\left[\begin{array}{c}(a-c)x(k)-bx^2(k)\\ +x(k)\delta\left\{\mathbb{E}[\sum_{j\in N_k^i}x(j)|k]\left(1-\frac{H(k)}{k+rm}\right)-H(k)\left(\frac{rm}{k+rm}\right)x(k)\right\}\end{array}\right]f(k)dk$$

$$= N\int_{\underline{k}}^{N}\left[\begin{array}{c}(a-c)x(k)-bx^2(k)\\ +x(k)\delta\left\{\mathbb{E}[\sum_{j\in N_k^i}x(j)|k]\left(1-\frac{1}{1+r}\right)-H(k)\left(\frac{rm}{k+rm}\right)x(k)\right\}\end{array}\right]f(k)dk.$$

Since $1-\frac{1}{1+r}>0$, the presence of externality term $\mathbb{E}[\sum_{j\in N_k^i}x(j)|k]$ will positively shift the optimal consumption. That implies a lower bound on optimal consumption, denoted by $\underline{x}(\cdot)$, is the solution to the following objective that ignores the externality.

$$\max_{x(\cdot)}\underline{\pi}_0(x(\cdot))=N\int_{\underline{k}}^{N}\left[\begin{array}{c}(a-c)x(k)-bx^2(k)\\ +x^2(k)\delta\left\{-H(k)\left(\frac{rm}{k+rm}\right)\right\}\end{array}\right]f(k)dk$$

Pointwise maximization of $\underline{\pi}_0(x(\cdot))$ (or by Euler equation approach equivalently) yields

$$\underline{x}(k)\equiv\frac{a-c}{2\left(b+\delta H(k)\frac{rm}{k+rm}\right)}=\frac{a-c}{2\left(b+\delta\frac{rm}{1+r}\right)}=\frac{(a-c)(1+r)}{2(b+br+\delta rm)},\tag{EC.53}$$

for which the second-order condition holds obviously. $\underline{x}(k)>0$ given $a>c$. Thus the nonnegativity of $x(\cdot)$ is guaranteed. Denote $\underline{x}(k)$ by $\underline{x}^i$ for clarity.

Now note the Euler equation

$$(a-c)r(1+r)+\delta m(r-1)r^2(k+rm)\int_0^k\frac{x^*(y)}{(y+rm)^2}dy+2(b+br+\delta rm)(-rx^*(k)+(k+rm)x^{*\prime}(k))=0$$

implies

$$\begin{aligned}&(k+rm)x^{*\prime}(k)\\ =\;&rx^*(k)-\frac{(a-c)r(1+r)}{2(b+br+\delta rm)}-\frac{\delta m(r-1)r^2(k+rm)}{2(b+br+\delta rm)}\int_0^k\frac{x^*(y)}{(y+rm)^2}dy\end{aligned}$$

$$
\begin{aligned}
&\underbrace{=}_{(EC.53)} r(x^*(k)-\underline{x}^i)+\frac{\delta m(1-r)r^2(k+rm)}{2(b+br+\delta rm)}\int_0^k \frac{x^*(y)}{(y+rm)^2}dy \\
&> \frac{\delta m(1-r)r^2(k+rm)}{2(b+br+\delta rm)}\int_0^k \frac{x^*(y)}{(y+rm)^2}dy \\
&> 0 \text{ if } r<1
\end{aligned}
$$

This suggests that the monotonicity constraint, $x^{*\prime}(\cdot)>0$, is satisfied if $r<1$ (Assumption 3).

For the ease of presentation, we rewrite Eular equation using more existing notations:

$$
\begin{aligned}
(k+rm)x^{*\prime}(k) &= r(x^*(k)-\underline{x}^i)+\frac{\delta m(1-r)r^2(k+rm)}{2(b+br+\delta rm)}\int_0^k \frac{x^*(y)}{(y+rm)^2}dy \\
&= r(x^*(k)-\underline{x}^i)+\frac{\delta(1-r)r}{2(b+br+\delta rm)}\mathbb{E}[\sum_{j\in N_k^i} x^*(j)|k], \qquad \text{(EC.54)}
\end{aligned}
$$

which then gives us the optimality condition presented in Theorem 2:

$$
\frac{\delta(1-r)r}{2(b+br+\delta rm)}\mathbb{E}[\sum_{j\in N_k^i} x^*(j)|k]+r(x^*(k)-\underline{x}^i)-(k+rm)x^{*\prime}(k)=0.
$$

Finally, one can substitute $x^*(\cdot)$ back to (EC.17) (and note the consumption of degree types below $\underline{k}$ is zero) to obtain the payment function $P^*(\cdot)$ at optimum. Similar to Theorem 1, it can also be shown that the equilibrium $x^*(\cdot)$ can be uniquely implemented under $P^*(\cdot)$. □

Proof of Proposition 6. The first-order condition of IC constraints (EC.13) implies that, at optimum,

$$
\begin{aligned}
P'(x(k)) &= a-2bx(k)+\delta\mathbb{E}[\sum_{j\in N_k^i} x(j)|k], \\
\frac{dP'(x(k))}{dk} &= -2bx'(k)+\delta\frac{d\mathbb{E}[\sum_{j\in N_k^i} x(j)|k]}{dk} \\
&\underbrace{=}_{(EC.52)} -2bx'(k)+\delta\left(\frac{\mathbb{E}[\sum_{j\in N_k^i} x(j)|k]}{k+rm}+\frac{rmx(k)}{k+rm}\right)
\end{aligned}
$$

Substituted into the derivative of incentive-compatible marginal price,

$$
\begin{aligned}
\frac{dP'(x^*(k))}{dk} &= -2bx^{*\prime}(k)+\delta\left(\frac{\mathbb{E}[\sum_{j\in N_k^i} x^*(j)|k]}{k+rm}+\frac{rmx^*(k)}{k+rm}\right) \qquad \text{(EC.55)} \\
&= \frac{1}{k+rm}\left(-2b(k+rm)x^{*\prime}(k)+\delta\left(\mathbb{E}[\sum_{j\in N_k^i} x^*(j)|k]+rmx^*(k)\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&\underbrace{=}_{(19)} \frac{1}{k+rm}\left[-2b\begin{pmatrix} r(x^*(k)-\underline{x}^i) \\ +\frac{\delta(1-r)r}{2(b+br+\delta rm)}\mathbb{E}[\sum_{j\in N_k^i} x^*(j)|k] \end{pmatrix} + \delta\left(\mathbb{E}[\sum_{j\in N_k^i} x^*(j)|k] + rmx^*(k)\right)\right] \\
&= \frac{1}{k+rm}\begin{bmatrix} \delta rmx^*(k) - 2br(x^*(k)-\underline{x}^i) \\ +\left(\delta - \frac{b\delta(1-r)r}{b+br+\delta rm}\right)\mathbb{E}[\sum_{j\in N_k^i} x^*(j)|k] \end{bmatrix} \\
&> \frac{1}{k+rm}\begin{bmatrix} \delta rmx^*(k) - 2br(\bar{x}(k)-\underline{x}^i) \\ +\left(\delta - \frac{b\delta(1-r)r}{b+br+\delta rm}\right)\mathbb{E}[\sum_{j\in N_k^i} x^*(j)|k] \end{bmatrix} \\
&= \frac{1}{k+rm}\begin{bmatrix} \delta rmx^*(k) + 2br\underline{x}^i - r(a-c+\delta\mathbb{E}[\sum_{j\in N_k^i} x^*(j)|k]) \\ +\left(\delta - \frac{b\delta(1-r)r}{b+br+\delta rm}\right)\mathbb{E}[\sum_{j\in N_k^i} x^*(j)|k] \end{bmatrix} \\
&= \frac{1}{k+rm}\begin{bmatrix} \delta rmx^*(k) + 2br\underline{x}^i - r(a-c) \\ +\left(\delta(1-r) - \frac{b\delta(1-r)r}{b+br+\delta rm}\right)\mathbb{E}[\sum_{j\in N_k^i} x^*(j)|k] \end{bmatrix} \\
&> \frac{1}{k+rm}\begin{bmatrix} (\delta rm + 2br)\underline{x}^i - r(a-c) \\ +\left(\delta(1-r) - \frac{b\delta(1-r)r}{b+br+\delta rm}\right)\mathbb{E}[\sum_{j\in N_k^i} x^*(j)|k] \end{bmatrix} \\
&\underbrace{>}_{r<1} 0
\end{aligned}
\tag{EC.56}
$$

Notice $\frac{br}{b+br+\delta rm} < 1$, and that $r<1$ implies $\frac{2r\delta m}{1+r} < \delta m$, which then leads to $\underline{x}^i > \frac{a-c}{2b+\delta m}$. Altogether it gives (EC.56). This indicates the optimal payment scheme charges higher marginal price at optimum for higher degree customers. In this case we also conclude

$$P''(x^*(k)) = \frac{dP'(x^*(k))}{dk}/x^{*\prime}(k) > 0.$$

which gives rise to a *quantity premium* menu.

Next we will show that the firm, at optimality, reaps more profit from higher degree customers. The profit grasped from a degree-k consumer is given by

$$\Pi(k) := P(x(k)) - cx(k).$$

Note

$$
\begin{aligned}
\frac{d}{dk}P(x(k)) &= P'(x(k))x'(k) \\
&= \left(a - 2bx(k) + \delta\mathbb{E}[\sum_{j\in N_k^i} x(j)|k]\right)x'(k)
\end{aligned}
$$

So

$$\frac{d}{dk}\Pi(k) = x'(k)\left(a - c - 2bx(k) + \delta\mathbb{E}[\sum_{j\in N_k^i} x(j)|k]\right)$$

At optimality, $x^{*\prime}(k) > 0$, and $a - c - 2bx^*(k) + \delta\mathbb{E}[\sum_{j\in N_k^i} x^*(j)|k] > 0$ (as $x^*(k) < \bar{x}(k)$). Thus $\frac{d}{dk}\Pi^*(k) > 0$. □

Proof of Corollary 2. We first show the induced consumption is downward shifted compared to the consumption if the firm had complete information on customer degrees (first-best scenario). To be brief, the firm's first-best objective is

$$\max_{x(\cdot)} \pi_0^{FB}(x(\cdot)) = N\int_{\underline{k}}^{N}\begin{bmatrix}(a-c)x(k) - bx^2(k)\\ +x(k)\delta\left\{\mathbb{E}[\sum_{j\in N_k^i} x(j)|k]\right\}\end{bmatrix} f(k)dk, \tag{EC.57}$$

while recall the second-best firm's objective as

$$\begin{aligned}\max_{x(\cdot)} \pi_0(x(\cdot)) &= N\int_{\underline{k}}^{N}\begin{bmatrix}(a-c)x(k) - bx^2(k)\\ +x(k)\delta\left\{\mathbb{E}[\sum_{j\in N_k^i} x(j)|k] - H(k)\frac{d}{dk}\mathbb{E}[\sum_{j\in N_k^i} x(j)|k]\right\}\end{bmatrix} f(k)dk\\ &= N\int_{\underline{k}}^{N}\begin{bmatrix}(a-c)x(k) - bx^2(k)\\ +x(k)\delta\left\{\mathbb{E}[\sum_{j\in N_k^i} x(j)|k] - \frac{1}{1+r}\left(\mathbb{E}[\sum_{j\in N_k^i} x(j)|k] + rmx(k)\right)\right\}\end{bmatrix} f(k)dk\\ &= N\int_{\underline{k}}^{N}\begin{bmatrix}(a-c)x(k) - \left(b+\frac{\delta rm}{1+r}\right)x^2(k)\\ +x(k)\delta\left\{\mathbb{E}[\sum_{j\in N_k^i} x(j)|k]\left(1-\frac{1}{1+r}\right)\right\}\end{bmatrix} f(k)dk. \end{aligned} \tag{EC.58}$$

Compared to that of first-best (EC.57), the linear benefit of consumption is discounted in the second-best objective (EC.58) since $1 - \frac{1}{1+r} < 1$, while the quadratic disutility term is strengthened. Thus the resulting consumption $x^*(k)$ in second-best should be lower than that in first-best. In other words, the firm faces a downward distortion in the consumption when its information regarding the social network is incomplete.[27] Lastly, note that the monotonicity of $x^*(k)$ with regard to k follows from its feasibility to the firm's problem (4)-(6). □

Proof of Proposition 7. The derivation of $f_k^t(\cdot)$, i.e. the pdf of neighbor degree distribution for a degree-k player (he) at time t involves the following steps:

[27] By the same argument as in footnote 23, it can be shown the downward distortion holds for the whole range of degree types, although the first-best and second-best profit functions possess different values of $\underline{k}$.

Under mean field approximation, the preferential attachment gives rise to $k_i(t) = M(t/i_t(k))^{1/2}$ (Barabási and Albert 1999), where $i_t(k)$ is the birthdate of the focal player. Thereby we have $i_t(k) = M^2t/k^2$. Observe:

(i) At his birthdate, the focal player already had M out of the k neighbors that he has now. If a now-degree-d player (referred as player j) was of degree d' at the moment $i_t(k)$, we must have $d' = dM/k$. [28]

Therefore, the probability that player j became a neighbor to the focal player is $\frac{Md'}{2Mi_t(k)} = \frac{d'}{2i_t(k)} = \frac{dM/k}{2i_t(k)} = \frac{dMk}{2M^2t} = \frac{dk}{2Mt}$.[29] Applying Bayes' theorem,

$$\begin{aligned}
f_k^t(d) &= \frac{Pr\{\text{neighbor}|\text{degree}=d\}f(d|d>k)}{\int_k^{\bar{k}} Pr\{\text{neighbor}|\text{degree}=d\}f(d|d>k)\mathrm{d}d} \\
&= \frac{Pr\{\text{neighbor}|\text{degree}=d\}f(d)}{\int_k^{\bar{k}} Pr\{\text{neighbor}|\text{degree}=d\}f(d)\mathrm{d}d} \\
&= \frac{\frac{dk}{2Mt}f(d)}{\int_k^{\bar{k}} \frac{dk}{2Mt}f(d)\mathrm{d}d} \\
&= \frac{df(d)}{\int_k^{\bar{k}} df(d)\mathrm{d}d} \\
&= \frac{1/d^2}{\int_k^{\bar{k}} 1/d^2\mathrm{d}d} \\
&= \frac{1/d^2}{d^{-1}|_N^k} \\
&\underbrace{=}_{N\to\infty} \frac{k}{d^2}
\end{aligned} \tag{EC.59}$$

which is time-invariant. Redenote $f_k^t(d)$ by $f_k^o(d)$, and we have

$$\begin{aligned}
F_k^o(d) &= \int_k^d f_k^o(y)dy \\
&= 1 - \frac{k}{d}
\end{aligned} \tag{EC.60}$$

One can verify the legitimacy of $F_k^o(\cdot)$ on (k,∞) by $F_k^o(k) = 0, F_k^o(\infty) = 1$.

[28] which can be obtained by solving the simultaneous equations $k_j(t) = M(t/j)^{1/2} = d$ and $k_j(i_t(k)) = d'$.

[29] Note that the total degrees of all players at the moment is $2Mi_t(k)$, since each player brings in M links which raise $2M$ degrees systemwise. Thus the probability j is reached by any of the M links under preferential attachment is $\frac{Md'}{2Mi_t(k)}$.

(ii) After his birth, the focal player i got his $k-M$ neighbors over time. The probability of a now-degree-d player being his neighbor is calculated as follows. At date $i_t(d)=M^2t/d^2$, suppose the focal player i's degree was k'. Then it must be that $k'=kM/d$. [30]

The now-degree-d player reaches the focal player w.p. $M\frac{k'}{2Mi_t(d)}=\frac{k'}{2i_t(d)}=\frac{kM/d}{2i_t(d)}=\frac{kM/d}{2M^2t/d^2}=\frac{kd}{2Mt}$. Applying Bayes' theorem,

$$
\begin{aligned}
f_k^t(d) &= \frac{Pr\{\text{neighbor}|\text{degree}=d\}f(d|d<k)}{\int_M^k Pr\{\text{neighbor}|\text{degree}=d\}f(d|d<k)\mathrm{d}d} \\
&= \frac{Pr\{\text{neighbor}|\text{degree}=d\}f(d)}{\int_M^k Pr\{\text{neighbor}|\text{degree}=d\}f(d)\mathrm{d}d} \\
&= \frac{\frac{dk}{2Mt}f(d)}{\int_M^k \frac{dk}{2Mt}f(d)\mathrm{d}d} \\
&= \frac{df(d)}{\int_M^k df(d)\mathrm{d}d} \\
&= \frac{1/d^2}{\int_M^k 1/d^2\mathrm{d}d} \\
&= \frac{1/d^2}{d^{-1}|_k^M} \\
&= \frac{1/d^2}{1/M-1/k} \\
&= \frac{kM}{d^2(k-M)},
\end{aligned}
\tag{EC.61}
$$

which is time-invariant. Redenote $f_k^t(d)$ by $f_k^i(d)$.

$$
\begin{aligned}
F_k^i(d) &= \int_M^d f_k^i(y)dy \\
&= \frac{k(d-M)}{d(k-M)}
\end{aligned}
\tag{EC.62}
$$

One can verify that $F_k^i(M)=0$ and $F_k^i(k)=1$, so that $F_k^i(\cdot)$ is a legitimate distribution on $[M,k)$. Also notice that $1-F_k^o(d)>1-F_k^i(d)\,\forall d$, which indicates that one's out-neighbors have a stochastically higher degree distribution than in-neighbors do. Note that $1-F_k^i(d)$ increases with M. That means one's in-neighbor's degree distribution first-order stochastically increases in M. □

[30] which can be obtained by solving the simultaneous equations $k=M(t/i)^{1/2}$ and $k'=M(i_t(d)/i)^{1/2}$.

Proof of Proposition 8. We have

$$\begin{aligned}
\mathbb{E}[\sum_{j\in N_k} x(j)|k] &= \lambda M\int_k^N x(d)f_k^o(d)\mathrm{d}d+(1-\lambda)(k-M)\int_M^k x(d)f_k^i(d)\mathrm{d}d \\
&= \lambda M\int_k^N x(d)\frac{k}{d^2}\mathrm{d}d+(1-\lambda)(k-M)\int_M^k x(d)\frac{kM}{d^2(k-M)}\mathrm{d}d \\
&= \lambda kM\int_k^N \frac{x(d)}{d^2}\mathrm{d}d+(1-\lambda)kM\int_M^k \frac{x(d)}{d^2}\mathrm{d}d \\
&= kM\left(\lambda\int_k^N \frac{x(d)}{d^2}\mathrm{d}d+(1-\lambda)\int_M^k \frac{x(d)}{d^2}\mathrm{d}d\right) \qquad \text{(EC.63)}
\end{aligned}$$

In order for $\mathbb{E}[\sum_{j\in N_k} x(j)|k]$ increasing in k, it suffices to have

$$\begin{aligned}
\frac{d}{dk}\left\{kM\int_k^N \frac{x(d)}{d^2}\mathrm{d}d\right\} &= M\left(\int_k^N \frac{x(d)}{d^2}\mathrm{d}d-\frac{x(k)}{k}\right) \\
&= M\left(-\int_k^N x(d)\mathrm{d}\frac{1}{d}-\frac{x(k)}{k}\right) \\
&\underbrace{=}_{\text{Integration by part}} M\left(\int_k^N \frac{1}{d}x'(d)\mathrm{d}d-\frac{x(d)}{d}\Big|_k^N-\frac{x(k)}{k}\right) \\
&\underbrace{=}_{N\to\infty} M\int_k^N \frac{1}{d}x'(d)\mathrm{d}d \\
&> 0 \ \text{ if } x'(\cdot)>0
\end{aligned}$$

Therefore, the single crossing condition is met if the consumption strategy is increasing in degree. □

Proof of Theorem 3. The proof is analogous to that of Theorem 1, and we recombine it into two steps. Step (i): We explore the structural properties of the optimization problem. Step (ii): We rewrite the firm objective as a function of consumption only. Using the change of variables, we show that the objective can be optimized by calculus of variations. We then verify that the candidate solution from calculus of variations indeed satisfies the remaining constraints.

Step (i): Structural properties of the optimization problem. As the standard mechanism design approach, we first reduce the constraints of incentive compatibility and individual rationality by single crossing property. Recall the payoff function of a degree-k customer reporting $\hat{k}$ when others report their own types is

$$\pi_k(\hat{k},x(\cdot),P(\cdot)) = ax(\hat{k})-bx^2(\hat{k})+x(\hat{k})\delta\mathbb{E}[\sum_{j\in N_k} x(j)|k]-P(x(\hat{k})),$$

The first-order condition for IC constraints implies:

$$\begin{aligned}
& d\pi_k(\hat{k}, x(\cdot), P(\cdot))/d\hat{k}\Big|_{\hat{k}=k} \\
&= ax'(\hat{k}) - 2bx(\hat{k})x'(\hat{k}) + x'(\hat{k})\delta\mathbb{E}[\sum_{j\in N_k} x(j)|k] - P'(x(\hat{k}))x'(\hat{k})\Big|_{\hat{k}=k} \\
&= \left[a - 2bx(\hat{k}) + \delta\mathbb{E}[\sum_{j\in N_k} x(j)|k] - P'(x(\hat{k}))\right] x'(\hat{k})\Big|_{\hat{k}=k} \\
&= \left[a - 2bx(k) + \delta\mathbb{E}[\sum_{j\in N_k} x(j)|k] - P'(x(k))\right] x'(k) \\
&= 0.
\end{aligned} \tag{EC.64}$$

So we have $a - 2bx(k) + \delta\mathbb{E}[\sum_{j\in N_k} x(j)|k] = P'(x(k))$, which implies

$$\frac{d}{dk}P'(x(k)) = -2bx'(k) + \delta\frac{d}{dk}\mathbb{E}[\sum_{j\in N_k} x(j)|k] \tag{EC.65}$$

The local second-order condition for IC constraints is

$$\begin{aligned}
& \frac{d^2}{d\hat{k}^2}\pi_k(\hat{k}, x(\cdot), P(\cdot))\Big|_{\hat{k}=k} \\
= & \left[-2bx'(\hat{k}) - \frac{d}{d\hat{k}}P'(x(\hat{k}))\right] x'(\hat{k}) + \left[a - 2bx(\hat{k}) + \delta\mathbb{E}[\sum_{j\in N_k} x(j)|k] - P'(x(\hat{k}))\right] x''(\hat{k})\Big|_{\hat{k}=k} \\
\underbrace{=}_{\text{(EC.64)}} & \left[-2bx'(k) - \frac{d}{dk}P'(x(k))\right] x'(k) \\
\underbrace{=}_{\text{(EC.65)}} & \left[-\delta\frac{d}{dk}\mathbb{E}[\sum_{j\in N_k} x(j)|k]\right] x'(k) \\
< & \;\; 0 \text{ if } x'(k) > 0
\end{aligned} \tag{EC.66}$$

Therefore the local concavity for truth reporting requires the monotonicity condition, $x'(k) > 0$, which in our problem also leads to $\frac{d}{dk}\mathbb{E}[\sum_{j\in N_k} x(j)|k] > 0$ by Proposition 8. (EC.64) and (EC.66) constitute the local IC condition, under which the customer does not attempt to lie locally. We will soon show that the single crossing condition, justified in Proposition 8, extends local IC to global. Define the payoff in the truth telling equilibrium:

$$V(k) := \pi_k(k, x(\cdot), P(\cdot)) = ax(k) - bx^2(k) + x(k)\delta\mathbb{E}[\sum_{j\in N_k} x(j)|k] - P(x(k)), \tag{EC.67}$$

and IR constraints (6) imply that $V(k) \geq 0, \forall k$.

$$\begin{aligned}\frac{d}{dk}V(k) &= ax'(k) - 2bx(k)x'(k) + \frac{dx(k)\delta\mathbb{E}[\sum_{j\in N_k} x(j)|k]}{dk} - P'(x(k))x'(k) \\ &= ax'(k) - 2bx(k)x'(k) + x'(k)\delta\mathbb{E}[\sum_{j\in N_k} x(j)|k] + \\ &\quad x(k)\delta\frac{d}{dk}\mathbb{E}[\sum_{j\in N_k} x(j)|k] - P'(x(k))x'(k) \\ &\underbrace{=}_{(EC.64)} x(k)\delta\frac{d}{dk}\mathbb{E}[\sum_{j\in N_k} x(j)|k],\end{aligned}$$

The above result can be reached by the envelope theorem as well, in consideration of the IC constraints:

$$\frac{d}{dk}V(k) = \frac{\partial}{\partial k}\pi_k(\hat{k}^*, x(\cdot), P(\cdot))\Big|_{\hat{k}^*=k} = x(k)\delta\frac{d}{dk}\mathbb{E}[\sum_{j\in N_k} x(j)|k].$$

Accordingly, (from Fundamental Theorem of Calculus)

$$V(k) = V(\underline{\mathrm{k}}) + \int_{\underline{\mathrm{k}}}^{k} x(u)\delta\frac{d}{du}\mathbb{E}[\sum_{j\in N_u} x(j)|u]du,$$

where $\underline{\mathrm{k}}$ is the lowest degree type considered. One can write

$$\begin{aligned}& V(k) - \pi_k(\hat{k}, x(\cdot), P(\cdot)) \\ =\ & V(k) - \left(V(\hat{k}) - x(\hat{k})\delta\mathbb{E}[\sum_{j\in N_{\hat{k}}} x(j)|\hat{k}] + x(\hat{k})\delta\mathbb{E}[\sum_{j\in N_k} x(j)|k]\right) \\ =\ & V(k) - V(\hat{k}) + x(\hat{k})\delta\left(\mathbb{E}[\sum_{j\in N_{\hat{k}}} x(j)|\hat{k}] - \mathbb{E}[\sum_{j\in N_k} x(j)|k]\right) \\ =\ & \int_{\hat{k}}^{k} x(u)\delta\frac{d}{du}\mathbb{E}[\sum_{j\in N_u} x(j)|u]du \\ & +x(\hat{k})\delta\left(\mathbb{E}[\sum_{j\in N_{\hat{k}}} x(j)|\hat{k}] - \mathbb{E}[\sum_{j\in N_k} x(j)|k]\right) \\ \underbrace{=}_{\text{Integration by parts}}\ & x(k)\delta\mathbb{E}[\sum_{j\in N_k} x(j)|k] - \int_{\hat{k}}^{k}\mathbb{E}[\sum_{j\in N_u} x(j)|u]\delta x'(u)du \\ & +x(\hat{k})\delta\left(-\mathbb{E}[\sum_{j\in N_k} x(j)|k]\right) \\ =\ & (x(k) - x(\hat{k}))\delta\mathbb{E}[\sum_{j\in N_k} x(j)|k] - \int_{\hat{k}}^{k} x'(u)\delta\mathbb{E}[\sum_{j\in N_u} x(j)|u]du\end{aligned}$$

$$
\begin{aligned}
&= \int_{\hat{k}}^{k} x'(u)\delta\mathbb{E}[\sum_{j\in N_k} x(j)|k]du - \int_{\hat{k}}^{k} x'(u)\delta\mathbb{E}[\sum_{j\in N_u} x(j)|u]du \\
&\geq 0,
\end{aligned}
$$

if $\hat{k} < k$, since $x'(\cdot) > 0$ (which also means $\mathbb{E}[\sum_{j\in N_k} x(j)|k]$ increases in k). When $\hat{k} > k$, simply rewrite the above as

$$
\begin{aligned}
& V(k) - \pi_k(\hat{k}, x(\cdot), P(\cdot)) \\
&= \int_{k}^{\hat{k}} x'(u)\delta\mathbb{E}[\sum_{j\in N_u} x(j)|u]du - \int_{k}^{\hat{k}} x'(u)\delta\mathbb{E}[\sum_{j\in N_k} x(j)|k]du \\
&\geq 0,
\end{aligned}
$$

where the last inequality follows from $x'(\cdot) > 0$ (which also means $\mathbb{E}[\sum_{j\in N_k} x(j)|k]$ increases in k). Thus IC is achieved globally. Since $V(\underline{k}) = 0$, the payment $P(x(k))$ is

$$
P(x(k)) = ax(k) - bx^2(k) + x(k)\delta\mathbb{E}[\sum_{j\in N_k} x(j)|k] - \int_{\underline{k}}^{k} x(u)\delta\frac{d}{du}\mathbb{E}[\sum_{j\in N_u} x(j)|u]du. \tag{EC.68}
$$

As a result, IC and IR constraints (5)-(6) can reduce to a single monotonicity constraint $x'(\cdot) > 0$.

Step (ii): **Rewriting the firm's objective**. The firm's problem can be transformed as

$$
\begin{aligned}
\max_{x(\cdot)} \quad & N\int_{\underline{k}}^{N} \left[\begin{array}{c} (a-c)x(k) - bx^2(k) \\ +x(k)\delta \left\{ \begin{array}{c} \mathbb{E}[\sum_{j\in N_k} x(j)|k] \\ -H(k)\frac{d}{dk}\mathbb{E}[\sum_{j\in N_k} x(j)|k] \end{array} \right\} \end{array} \right] f(k)dk \\
\text{s.t.} \quad & x(\cdot) \text{ is increasing.}
\end{aligned} \tag{EC.69}
$$

Observe

$$
\begin{aligned}
\frac{d}{dk}\mathbb{E}[\sum_{j\in N_k} x(j)|k] &= \frac{\mathbb{E}[\sum_{j\in N_k} x(j)|k]}{k} + kM\left(-\lambda\frac{x(k)}{k^2} + (1-\lambda)\frac{x(k)}{k^2}\right) \\
&= \frac{\mathbb{E}[\sum_{j\in N_k} x(j)|k]}{k} + M(1-2\lambda)\frac{x(k)}{k} \\
\frac{d^2}{dk^2}\mathbb{E}[\sum_{j\in N_k} x(j)|k] &= \frac{M(1-2\lambda)}{k}x'(k)
\end{aligned} \tag{EC.70}
$$

Therefore

$$
k\frac{d}{dk}\mathbb{E}[\sum_{j\in N_k} x(j)|k] - \mathbb{E}[\sum_{j\in N_k} x(j)|k] = M(1-2\lambda)x(k) \tag{EC.71}
$$

$$
x(k) = \frac{1}{M(1-2\lambda)}\left(k\frac{d}{dk}\mathbb{E}[\sum_{j\in N_k} x(j)|k] - \mathbb{E}[\sum_{j\in N_k} x(j)|k]\right) \tag{EC.72}
$$

Let

$$G := \left[\begin{array}{c} (a-c)x(k) - bx^2(k) \\ +x(k)\delta \left\{ \begin{array}{c} \mathbb{E}[\sum_{j\in N_k} x(j)|k] \\ -H(k)\frac{d}{dk}\mathbb{E}[\sum_{j\in N_k} x(j)|k] \end{array} \right\} \end{array} \right] f(k) \tag{EC.73}$$

and $z(k) := \mathbb{E}[\sum_{j\in N_k} x(j)|k]$. Thus G can be expressed as a function of $z(k)$ and $z'(k)$, which makes the transformed objective (EC.69) readily solvable by calculus of variation.

Substituted with (EC.72), (EC.73) becomes a function of $G(k, z(k), z'(k))$. The Eular equation

$$G_{z(k)}(k, z(k), (z)^{'}(k)) = \frac{d}{dk} G_{(z)^{'}(k)}(k, z(k), (z)^{'}(k)).$$

yields a surprisingly simple formula (in x-terms):

$$x^*(k) - kx^{*\prime}(k) = \frac{a-c}{2b + \delta(1-2\lambda)M}. \tag{EC.74}$$

This means that the induced consumption is linear in degree, i.e.

$$x^*(k) = \theta k + \frac{a-c}{2b + \delta(1-2\lambda)M}, \tag{EC.75}$$

where θ is a positive real number (to satisfy the monotonicity constraint $x'(\cdot) > 0$). Also note that Assumption 1 implies $2b > \delta M$, which ensures the non-negativity of the consumption regardless of $\lambda \in [0,1]$ and k.

The second order condition, known as Legendre condition, is satisfied:

$$G_{z'(k)z'(k)}(k, z(k), z'(k)) = -\frac{2(2b + \delta(1-2\lambda)M)}{k(1-2\lambda)^2} \le 0, \; \forall k, z(k).$$

Plugging $x^*(\cdot)$ into the payment function (EC.68) gives the optimal payment scheme $P^*(\cdot)$. Similar to Theorems 1 and 2, it can be shown that $x^*(\cdot)$ can be uniquely implemented by $P^*(\cdot)$. □

Proof of Proposition 9. Recall that

$$\begin{aligned} P^{*\prime\prime}(x^*(k)) &= \frac{\mathrm{d}P'(x^*(k))}{\mathrm{d}k} / x^{*\prime}(k) \\ &= -2b + \delta \frac{\mathrm{d}}{\mathrm{d}k} \mathbb{E}[\sum_{j\in N_k} x^*(j)|k] / x^{*\prime}(k). \end{aligned} \tag{EC.76}$$

Substituted with $\frac{d}{dk}\mathbb{E}[\sum_{j\in N_k} x(j)|k] = \frac{\mathbb{E}[\sum_{j\in N_k} x(j)|k]}{k} + M(1-2\lambda)\frac{x(k)}{k} = \frac{1}{k}\left(\mathbb{E}[\sum_{j\in N_k} x(j)|k] + M(1-2\lambda)x(k)\right)$ and $x^{*\prime}(k)=\theta$, it yields

$$\begin{aligned}
P^{*\prime\prime}(x^*(k)) &= -2b+\delta\frac{1}{k\theta}\left(\mathbb{E}[\sum_{j\in N_k} x^*(j)|k]+M(1-2\lambda)x^*(k)\right)\\
&= -2b+\frac{1}{k\theta}\left(\delta\mathbb{E}[\sum_{j\in N_k} x^*(j)|k]+\delta M(1-2\lambda)x^*(k)\right)\\
&> \frac{1}{k\theta}\left(2b(x^*(k)-k\theta)-(a-c)+\delta M(1-2\lambda)x^*(k)\right)\\
&\underbrace{=}_{(25)} \frac{1}{k\theta}\left(2b\left(\frac{a-c}{2b+\delta(1-2\lambda)M}\right)-(a-c)+\delta M(1-2\lambda)x^*(k)\right)\\
&\underbrace{=}_{(25)} \frac{\delta(1-2\lambda)M}{k\theta}\left(\left(\frac{-(a-c)}{2b+\delta(1-2\lambda)M}\right)+x^*(k)\right)\\
&= \delta(1-2\lambda)M
\end{aligned}$$

The inequality ">" above stems from the fact that $x^*(k) < \frac{1}{2b}\{a-c+\delta\mathbb{E}[\sum_{j\in N_k} x^*(j)|k]\} \Leftrightarrow \delta\mathbb{E}[\sum_{j\in N_k} x^*(j)|k] > 2bx^*(k)-(a-c)$. Therefore, we can conclude that, if $\lambda<1/2$, $P^{*\prime\prime}(x^*(k))>0$ (quantity premium). Since $x^{*\prime}(\cdot)>0$, we also have $\frac{\mathrm{d}P^{*\prime}(x^*(k))}{\mathrm{d}k}>0$.

When $\lambda>1/2$,

$$\begin{aligned}
P^{*\prime\prime}(x^*(k)) &= -2b+\delta\frac{1}{k\theta}\left(\mathbb{E}[\sum_{j\in N_k} x^*(j)|k]+M(1-2\lambda)x^*(k)\right)\\
&< -2b+\delta M(1-2\lambda)+\frac{1}{k\theta}\left(\delta\mathbb{E}[\sum_{j\in N_k} x^*(j)|k]\right)\\
&< -2b+\delta M(1-2\lambda)+\frac{1}{\theta}\delta x^*(\bar{k})\\
&\underbrace{=}_{(25)} -2b+\delta M(1-2\lambda)+\delta\left(\bar{k}+\frac{a-c}{\theta(2b+\delta(1-2\lambda)M)}\right)\\
&= -2b+\delta\bar{k}+\delta M(1-2\lambda)+\delta\frac{a-c}{\theta(2b+\delta(1-2\lambda)M)}
\end{aligned}$$

Given Assumption 1, $P^{*\prime\prime}(x^*(k))<0$ (quantity discount) holds so long as $M(1-2\lambda)+\frac{a-c}{\theta(2b+\delta(1-2\lambda)M)}<0$. Since $x'(k)>0$, we also have in this case $\frac{\mathrm{d}P^{*\prime}(x^*(k))}{\mathrm{d}k}<0$. □

Proof of Corollary 3. Recall that of first best,

$$G^{FB} := \begin{bmatrix} (a-c)x(k)-bx^2(k) \\ +x(k)\delta\left\{\mathbb{E}[\sum_{j\in N_k} x(j)|k]\right\}\end{bmatrix} f(k). \tag{EC.77}$$

and G defined in (EC.73), so that

$$G^{FB} - G = \left[x(k)\delta H(k) \frac{d}{dk} \mathbb{E}[\sum_{j\in N_k} x(j)|k] \right] f(k)$$
$$= x(k)\delta(1-F(k)) \frac{d}{dk} \mathbb{E}[\sum_{j\in N_k} x(j)|k]. \quad \text{(EC.78)}$$

For the expression of $G^{FB} - G$, the coefficient for $x(k)$ is positive given that $\mathbb{E}[\sum_{j\in N_k} x(j)|k]$ increases in k (for any feasible $x'(\cdot) > 0$). Thus the incentive for consumption is lower in second best case than that in first best scenario. That implies downward distortion of consumption in the second best case.

To see that the profit earned from degree-k customer increases in k, recall

$$P(x(k)) = ax(k) - bx^2(k) + x(k)\delta \mathbb{E}[\sum_{j\in N_k} x(j)|k] - \int_{\underline{k}}^{k} x(u)\delta \frac{d}{du}\mathbb{E}[\sum_{j\in N_u} x(j)|u]du$$
$$\frac{d}{dk}P(x(k)) = \left(a - 2bx(k) + \delta\mathbb{E}[\sum_{j\in N_k} x(j)|k] \right) x'(k)$$

Therefore for $\Pi(k) := P(x(k)) - cx(k)$,

$$\Pi'(k) = \frac{d}{dk}P(x(k)) - cx'(k)$$
$$= \left(a - c - 2bx(k) + \delta\mathbb{E}[\sum_{j\in N_k} x(j)|k] \right) x'(k)$$

Notice that $x^{*\prime}(k) > 0$ and $a - c - 2bx^*(k) + \delta\mathbb{E}[\sum_{j\in N_k} x^*(j)|k] > 0$ (since $x^*(k) < \bar{x}(k)$). Thus $\Pi^{*\prime}(k) > 0$.

That $x^*(k)$ increases in k directly follows from the monotonicity constraint. Next we show how the consumption and the profit change with the network density. Since the average degree in the network is $2M$, the parameter M captures the density of network. Recall

$$x^*(k) = \theta k + \frac{a-c}{2b + \delta(1-2\lambda)M}, \quad \text{(EC.79)}$$

If $1 - 2\lambda > 0$, the induced consumption $x^*(k)$ declines in M. If $1 - 2\lambda < 0$, $x^*(k)$ increases in M.

To show that the firm's maximum profit increases with M if $1 - 2\lambda < 0$, we need more notations. Let $\tilde{x}(k) := \theta k$; and note $\tilde{x}(k)$ is independent of M and $x^*(k) = \tilde{x}(k) + \frac{a-c}{2b+\delta(1-2\lambda)M}$. Define $\xi(k) :=$

$kM\frac{a-c}{2b+\delta(1-2\lambda)M}\left(\lambda\int_k^N\frac{1}{d^2}dd+(1-\lambda)\int_M^k\frac{1}{d^2}dd\right)=\frac{a-c}{2b+\delta(1-2\lambda)M}\left[(2\lambda-1)M+(1-\lambda)k\right]$, which increases in M given $1-2\lambda<0$. It follows that $\mathbb{E}[\sum_{j\in N_k}x^*(j)|k]=\mathbb{E}[\sum_{j\in N_k}\tilde{x}(j)|k]+\xi(k)$. Observe that

$$\begin{aligned}\frac{d}{dM}\mathbb{E}[\sum_{j\in N_k}\tilde{x}(j)|k] &= \frac{\mathbb{E}[\sum_{j\in N_k}\tilde{x}(j)|k]}{M}-k(1-\lambda)\frac{\tilde{x}(M)}{M}\\ &= \frac{k}{M}\left[\mathbb{E}[\tilde{x}(j)|_{j\in N_k}k]-(1-\lambda)\tilde{x}(M)\right]\\ &> 0,\ \text{given } \tilde{x}(\cdot) \text{ increasing and } M \text{ the lowest degree}\end{aligned}$$

Thus, $\frac{d}{dM}\mathbb{E}[\sum_{j\in N_k}x^*(j)|k]=\frac{d}{dM}\mathbb{E}[\sum_{j\in N_k}\tilde{x}(j)|k]+\frac{d}{dM}\xi(k)>0$. Also recall that

$$\begin{aligned}\delta\mathbb{E}[\sum_{j\in N_k}x^*(j)|k] &> 2bx^*(k)-(a-c)\\ &= 2b\left(\theta k+\frac{a-c}{2b+\delta(1-2\lambda)M}\right)-(a-c)\\ &> 2b\theta k \text{ since } 1-2\lambda<0 \end{aligned}\tag{EC.80}$$

$$\begin{aligned}\Pi_0(k) &= (a-c)x(k)-bx^2(k)+x(k)\delta\left\{\begin{array}{c}\mathbb{E}[\sum_{j\in N_k}x(j)|k]\\ -H(k)\frac{d}{dk}\mathbb{E}[\sum_{j\in N_k}x(j)|k]\end{array}\right\}\\ &= (a-c)x(k)-bx^2(k)+x(k)\delta\left\{\begin{array}{c}\mathbb{E}[\sum_{j\in N_k}x(j)|k]\\ -\frac{1}{2}\left(\mathbb{E}[\sum_{j\in N_k}x(j)|k]+M(1-2\lambda)x(k)\right)\end{array}\right\}\\ &= (a-c)x(k)-bx^2(k)+\frac{1}{2}x(k)\delta\left(\mathbb{E}[\sum_{j\in N_k}x(j)|k]-M(1-2\lambda)x(k)\right)\end{aligned}$$

At optimum,

$$\begin{aligned}\Pi_0^*(k) &= (a-c)x^*(k)+x^*(k)\left[\frac{1}{2}\delta\mathbb{E}[\sum_{j\in N_k}x^*(j)|k]-\left(b+\frac{1}{2}\delta M(1-2\lambda)\right)x^*(k)\right]\\ &\underbrace{=}_{(EC.79)} x^*(k)\left[\frac{1}{2}\delta\mathbb{E}[\sum_{j\in N_k}x^*(j)|k]-\left(b+\frac{1}{2}\delta M(1-2\lambda)\right)\theta k+\frac{1}{2}(a-c)\right]\\ &= \frac{1}{2}x^*(k)\left[\delta\mathbb{E}[\sum_{j\in N_k}x^*(j)|k]-(2b+\delta M(1-2\lambda))\,\theta k+(a-c)\right]\end{aligned}$$

Note that both $x^*(k)$ and the bracketed term $\delta\mathbb{E}[\sum_{j\in N_k}x^*(j)|k]-(2b+\delta M(1-2\lambda))\,\theta k+(a-c)$ are positive (given (EC.80)) and increase in M. Hence, $\Pi_0^*(k)$ increases in M. Furthermore, since

$F(\cdot)$ first-order stochastically increases with M and $\Pi_0^*(k)$ increases in k (as shown above), the firm's overall profit rises with the network density M at optimum. □

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