Online Appendix

EC.1. Related Literature

In contrast to the voluminous literature on option pricing, optimally investing in options on multiple assets is a far less developed problem, even as empirical work documents large risk premia in options markets (Coval and Shumway 2001, Bakshi and Kapadia 2003, Santa-Clara and Saretto 2009, Schneider and Trojani 2015).

In a complete market, optimal option positions are implied by the condition that marginal utility be proportional to the state-price density. Carr and Madan (2001a) and Carr et al. (2001) show how to compute such optimal payoffs under different beliefs and preferences and one underlying asset. Schneider (2015) links optimal payoffs to the likelihood-ratio swap contract, and shows how to replicate such a contract with a portfolio of options. Guasoni et al. (2011) show how fund managers can use option-writing strategies to create the appearance of outperformance, even in the absence of any ability to predict returns.

In an incomplete market with stochastic investment opportunities and jumps, Liu and Pan (2003) solve in closed form the dynamic portfolio choice problem of an investor trading one stock and one out-of-the-money (OTM) put option on such stock. Eraker (2013) considers combinations of at-the-money (ATM) straddles with OTM calls and puts on the S&P 500 index and finds that they deliver Sharpe ratios close to one. Faias and Santa-Clara (2011) use a simulation approach to find optimal portfolio weights in options on the S&P 500 index and also find significant Sharpe ratios over more than a decade. While these papers focus on options on one underlying asset, Malamud (2014) develops a methodology to find "Greek-efficient" portfolios, identified in terms of higher moments of the underlying assets' returns.

Our duality approach builds on the intuition that dates back to He and Pearson (1991) and Karatzas et al. (1991), whereby portfolio optimization in an incomplete market is equivalent to portfolio optimization in the *least favorable completion* of such a market – an idea which has proven effective also in tackling portfolio performance evaluation (Haugh et al. 2006) and option pricing (Rogers 2002, Haugh and Kogan 2004). In the evaluation of empirical asset pricing models, such a duality arises in relation to distance and discrepancy measures: for example, the minimization of the square-integral distance of Hansen and Jagannathan (1991, 1997) and the Cressie-Read discrepancy family (Almeida and Garcia 2012, 2016) are the dual counterparts of portfolio optimization with mean-variance and HARA utility, respectively.

EC.2. Proof of Theorem 1

Denote by L^2_p the space of (equivalence classes of) Lebesgue measurable functions $f:\mathbb{R}^n\mapsto\mathbb{R}$ such that $\|f\|_p^2 := E[f(X)^2] = \int_{\mathcal{D}} |f(u)|^2 p(u)du < \infty$, which is a Hilbert space with inner product $\langle f, g\rangle_p := \int_{\mathcal{D}} f(u)g(u)p(u)du$.

EC.2.1. Existence and Uniqueness

Proof of Theorem 1 (i) Denote by $\Pi : L^2(\Omega,\mathcal{F},P)\to L^2(\Omega,\mathcal{G},P|_{\mathcal{G}})$ the conditional expectation operator. Recall that $\mathcal{M}\subset L^2(\Omega,\mathcal{F},P)$ consists of those discount factors M for which $m=\Pi(M)$ satisfies

$$\int_{\mathcal{D}_i^c} m(\xi)p(\xi)d\xi_i^c = q_i(\xi_i), \quad 1\le i\le n.$$

Hence, $\mathcal{M}$ is convex, and non-empty by assumption. To check that it is closed, consider a sequence $(M_k)_{k=1}^\infty\subset\mathcal{M}$ converging to some $M_0\in L^2(\Omega,\mathcal{F},P)$. As the sequence M_k converges, the projections $m_k=\Pi(M_k)$ also converge, hence are bounded in $L^2(\Omega,\mathcal{F},P|_{\mathcal{G}})$ norm and thus uniformly integrable. Therefore, the following limit holds in almost sure sense:

$$\int_{\mathcal{D}_i^c} m_0(\xi)p(\xi)d\xi_i^c = \lim_{k\to\infty}\int_{\mathcal{D}_i^c} m_k(\xi)p(\xi)d\xi_i^c = q_i(\xi_i), \quad 1\le i\le n, \tag{EC.1}$$

which proves that $\mathcal{M}$ is closed in $L^2(\Omega,\mathcal{F},P)$. As any non-empty, closed, convex set in a Hilbert space has a unique element of minimum norm (Rudin 1986, Theorem 4.10), the proof is complete.

EC.2.2. Linearity

Proof of Theorem 1 (ii) For any $\psi \in L^\infty(\mathbb{R}^n)$ define

$$\widetilde{\psi}(\xi) := \psi(\xi) - \sum_{i=1}^n \underbrace{\left(\frac{p_i^c(\xi_i^c)}{p(\xi)} \int_{\mathcal{D}_i^c} \psi(\xi)p(\xi)d\xi_i^c\right)}_{=:\widetilde{\psi}_i} + (n-1)\int_{\mathcal{D}} \psi(\eta)p(\eta)d\eta \tag{EC.2}$$

and observe that $\widetilde{\psi} \in \mathcal{N}$, defined as

$$\mathcal{N} := \left\{ \widetilde{\psi} \in L_p^2 \,\middle|\, \int_{\mathcal{D}_i^c} \widetilde{\psi}(\xi)p(\xi)d\xi_i^c \equiv 0, \quad 1 \le i \le n \right\}. \tag{EC.3}$$

Indeed, a repeated application of Fubini's theorem yields

$$\begin{aligned}\int_{\mathcal{D}_i^c} \widetilde{\psi}(\xi)p(\xi)d\xi_i^c &= \int_{\mathcal{D}_i^c} \left(\psi(\xi)p(\xi) - \widetilde{\psi}_i(\xi)p(\xi)\right) d\xi_i^c \\ &\quad - \left(\sum_{j\neq i} \int_{\mathcal{D}_i^c} \widetilde{\psi}_j(\xi)p(\xi)d\xi_i^c - (n-1)p_i(\xi_i)\int_{\mathcal{D}} \psi(\xi)p(\xi)d\xi\right) = 0.\end{aligned}$$

Furthermore, as the first and last terms in the definition of $\widetilde{\psi}$ are in L_p^2, it suffices to show that $\widetilde{\psi}_i \in L_p^2$, for $1 \le i \le n$ to conclude that $\widetilde{\psi} \in L_p^2$. Indeed, by Jensen's inequality,

$$\|\widetilde{\psi}_i\|_p^2 = \int_{\mathcal{D}} |\widetilde{\psi}_i(\xi)|^2 p(\xi)d\xi = \int_{\mathcal{D}} \frac{(p_i^c(\xi_i^c))^2}{p(\xi)^2}\left(\int_{\mathcal{D}_i^c} \psi(\xi_i^c,\xi_i)p(\xi_i^c,\xi_i)d\xi_i^c\right)^2 p(\xi)d\xi \tag{EC.4}$$

$$\le \int_{\mathcal{D}} \frac{(p_i^c(\xi_i^c))^2 p_i^2(\xi_i)}{p(\xi)} \left(\int_{\mathcal{D}_i^c} |\psi(\xi_i^c,\xi_i)|^2 \frac{p(\xi_i^c,\xi_i)}{p_i(\xi_i)} d\xi_i^c\right) d\xi \le \|\psi\|_\infty^2 \left\|\frac{p_i\, p_i^c}{p}\right\|_p^2 < \infty. \tag{EC.5}$$

Let $M^* = m^*(X)$ be the solution of Theorem 1 (i). By minimality, for any $\varepsilon > 0$ and $\psi^* \in \mathcal{N}$,

$$\int_{\mathcal{D}} |m^*(\xi) + \varepsilon\psi^*(\xi)|^2 p(\xi)d\xi - \int_{\mathcal{D}} |m^*(\xi)|^2 p(\xi)d\xi \le 0.$$

Dividing by ε and passing to the limit $\varepsilon \to 0$, and observing the same argument holds for both ε and $-\varepsilon$, the first order condition holds, i.e.,

$$\langle m^*, \psi^*\rangle_p := \int_{\mathcal{D}} \psi^*(\xi)m^*(\xi)p(\xi)d\xi = 0. \tag{EC.6}$$

Hence, for any $\psi \in L^\infty(\mathbb{R}^n)$, setting $\psi^* = \widetilde{\psi}$ it follows that

$$\langle m^*, \widetilde{\psi}\rangle_p = \left\langle \overbrace{m^* - \left(\sum_{i=1}^n \int_{\mathcal{D}_i^c} m^*(\xi)p_i^c(\xi_i^c)d\xi_i^c\right) + (n-1)}^{=:\Psi}, \psi \right\rangle_p = 0. \tag{EC.7}$$

Now, set

$$\Phi_i(\xi_i) = n \int_{\mathcal{D}_i^c} m^*(\xi) p_i^c(\xi_i^c) d\xi_i^c - (n-1), \quad 1 \le i \le n.$$

Note that $\Phi_i \in L_p^1$, because

$$\int_{\mathcal{D}} \left| \frac{\Phi_i(\xi_i) + (n-1)}{n} \right| p(\xi) d\xi = \int_{\mathcal{D}} \left| \int_{\mathcal{D}_i^c} m^*(\xi_i, \eta_i^c) p_i^c(\eta_i^c) d\eta_i^c \right| p(\xi_i, \xi_i^c) d\xi_i d\xi_i^c$$
$$\le \int_{\mathcal{D}} |m^*(\xi)| p_i(\xi_i) p_i^c(\xi_i^c) d\xi = \int_{\mathcal{D}} |m^*(\xi)| \sqrt{p(\xi)} \frac{p_i(\xi_i) p_i^c(\xi_i^c)}{\sqrt{p(\xi)}} d\xi \le \|m^*\|_p^2 \left\| \frac{p_i\, p_i^c}{p} \right\|_p^2 < \infty,$$

where the second inequality follows by Cauchy-Schwarz's, and the last one by the integrability assumption and the square integrability of m^*.

Because $m^* \in L_p^2 \subset L_p^1$, it follows that $\Psi \in L_p^1$. As $L^\infty(\mathbb{R})$ is in duality with L_p^1, and (EC.7) holds for all $\psi \in L^\infty(\mathbb{R})$, it follows that $\Psi = 0$ a.s. in p, that is

$$m^*(\xi) = \frac{1}{n} \sum_{i=1}^n \Phi_i(\xi_i) \quad p\text{-a.s.} \tag{EC.8}$$

EC.2.3. Identification

Theorem 1 (i) and (ii) show that a unique minimizing discount factor exists, and its linear structure allows an interpretation as option portfolios. However, the existence proof was not constructive. This section characterizes the solution in terms of a constrained, vector-valued integral equation, which is conveniently solved by discretization.

Proof of Theorem 1 (iii) By the proof of Theorem 1 (ii), the unique minimal discount factor $M^* = m^*(X)$, where $m^* \in L_p^2$ is of the form (EC.8). Therefore, the n constraints in (20) imply the validity of the n equations (21). It remains to establish the constraints (22). Note that

$$\int_{I_i} \Phi_i(\xi_i) p_i(\xi_i) d\xi_i = n \int_{I_i} \left(\int_{\mathcal{D}_i^c} m^*(\xi)\, p_i^c(\xi_i^c) d\xi_i^c \right) p_i(\xi_i) d\xi_i - (n-1)$$
$$= \int_{I_i} \left(\int_{\mathcal{D}_i^c} \left(\sum_{j=1}^n \Phi_j(\xi_j) \right) p_i^c(\xi_i^c) d\xi_i^c \right) p_i(\xi_i) d\xi_i - (n-1) \qquad 1 \le i \le n,$$

which simplifies to a system of n linear equations for $\vartheta_j := \int_{I_i} \Phi_j(\xi_j) p_j(\xi_j) d\xi_j$, $1 \le j \le n$, i.e.,

$$\sum_{j: j \ne i} \vartheta_j = (n-1), \qquad 1 \le i \le n,$$

which has the unique solution $\vartheta_j = 1$ for all $1 \le j \le n$.

EC.2.4. Performance

Proof of Theorem 1 (iv) The efficient frontier is spanned by excess returns of the form $a(M^* - E^Q[M^*])$, with $a < 0$ (cf. note 1). Because $M^* \in \mathcal{R}$ and $M^* \in \mathcal{M}$, $E[(M^*)^2]$ represents the price of M^* and thus the sum of the prices of $\Phi_i(X_i)$ divided by n. As a result, because the risk-neutral density for options on X_i is q_i,

$$E[(M^*)^2] = \frac{1}{n}\sum_{i=1}^{n}\int_{I_i}\Phi_i(\xi_i)q_i(\xi)d\xi_i,$$

and, as $E[M^*] = 1$, the maximal Sharpe ratio is

$$\frac{|E[M^*] - E^Q[M^*]|}{\sqrt{\mathrm{Var}^Q(M^*)}} = \sqrt{E^Q[M^*] - 1} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\int_{I_i}\Phi_i(\xi_i)q_i(\xi_i)d\xi_i - 1}.$$

EC.2.5. Regularity

The above arguments prove that, $\mathcal{M} \neq \varnothing$, then a unique minimal discount factor $M^* = m^*(X)$ exists, but they are silent on the regularity of the map $x \mapsto m^*(x)$. This section shows that, if the physical and risk-neutral densities are regular, so are the asset-specific payoffs Φ_i and hence the optimal payoff m^*.

Proof of Theorem 1 (v) As all $(p_i)_{1\le i\le n}$ are strictly positive, the integral equation for Φ_i is equivalent to

$$\Phi_i(\xi_i) = \frac{nq_i(\xi_i)}{p_i(\xi_i)} - \sum_{j\neq i}\frac{1}{p_i(\xi_i)}\int_{\mathcal{D}_i^c}\Phi_j(\xi_j)p(\xi)d\xi_i^c. \tag{EC.9}$$

By assumption, $q_i(\xi_i)$ is continuously differentiable. It suffices to show that the same regularity holds for the functions

$$g_{i,j} : \xi_i \mapsto \int_{\mathcal{D}_i^c} g(\xi_j)p(\xi)d\xi_i^c, \qquad \text{for all } j \neq i.$$

where the integrand $g \in L_p^2$. Once this is shown, then the claim follows by setting $g = 1$ to obtain the regularity of the marginal densities p_i, and setting $g = \Phi_j$, $j \neq i$, to obtain the regularity of the sum in (EC.9). Note that, by assumption, there exists $\alpha \in (1/2, 1]$ such that locally in ξ_i, the integrand

$$|g(\xi_j)\partial_{\xi_i}^{\beta}p(\xi)| \le C|g(\xi_j)|(p_i^c)^{\alpha}$$

admits an upper bound, independent of ξ_i and integrable, in view of the Cauchy-Schwarz inequality,

$$\int_{\mathcal{D}_i^c} |g(\xi_j)|(p_i^c)^\alpha d\xi_i^c \leq \int_{\mathcal{D}_i^c} |g(\xi_j)|^2 p_i^c d\xi_i^c \times \int_{\mathcal{D}_i^c} p_i^c(\xi_i^c)^{2\alpha-1} d\xi_i^c < \infty.$$

Hence, by dominated convergence and the continuity of the integrand, also $g_{i,j}$ is k times continuously differentiable.

Remark EC.1. Trading options on a larger set of underlying assets generates at least the same Sharpe ratio as trading options on a smaller set of assets because any option strategy on the smaller set retains the same performance in the larger market. The Sharpe ratio in the larger market is strictly larger if and only if the options in the larger market are not priced correctly by the minimal SDF in the smaller market.

To see this point, denote by M_k^* and M_{k+1}^* the minimal SDFs in the markets with the first k and $k+1$ assets, respectively. Because M_{k+1}^* must satisfy one extra marginal constraint, its second moment is minimal in a smaller class, hence greater or equal than that of M_k^*, i.e., $E[(M_{k+1}^*)^2] \geq E[(M_k^*)^2]$. By Theorem 1, such second moments represent the squared maximal Sharpe ratios in the respective markets. If they are equal, then $M_{k+1}^* = M_k^*$ by the uniqueness of the minimal SDF, as M_{k+1}^* is (trivially) an SDF for the first k assets. Vice versa, if $E[(M_{k+1}^*)^2] > E[(M_k^*)^2]$, then denote by $N = M_{k+1}^* - M_k^*$, which is a payoff involving all $k+1$ assets, and has price $E[NM_{k+1}^*]$. Note also that $E[NM_k^*] = 0$ because M_k^* is minimal among all SDF that price the first k assets. Thus, it follows that

$$\begin{aligned} E[NM_{k+1}^*] - E[NM_k^*] =& E[N^2] = E[N(M_{k+1}^* - M_k^*)] = E[NM_{k+1}^*] \\ =& E[(M_{k+1}^*)^2] - E[M_{k+1}^* M_k^*] = E[(M_{k+1}^*)^2] - E[(M_k^*)^2] > 0, \end{aligned}$$

which means that M_k^* prices N incorrectly.

EC.3. Proofs of Statements in Section EC.4

Proof of Proposition EC.1 Rearrange the two integral equations (12)–(13) as

$$\frac{\Phi_X(x)}{2} = -\frac{1}{2}\int_{I_Y} \Phi_Y(y)\frac{p(x,y)}{p_X(x)}dy + \frac{q_X(x)}{p_X(x)}, \quad \frac{\Phi_Y(y)}{2} = -\frac{1}{2}\int_{I_X} \Phi_X(x)\frac{p(x,y)}{p_Y(y)}dx + \frac{q_Y(y)}{p_Y(y)}$$

and note that

$$m^*(x,y) = \frac{\Phi_X(x)+\Phi_Y(y)}{2} = -\frac{1}{2}\int_{I_Y}\Phi_Y(y)\frac{p(x,y)}{p_X(x)}dy + \frac{q_X(x)}{p_X(x)} - \frac{1}{2}\int_{I_X}\Phi_X(x)\frac{p(x,y)}{p_Y(y)}dx + \frac{q_Y(y)}{p_Y(y)}$$
$$\geq -\frac{\gamma}{2}\int_{I_Y}\Phi_Y(y)p_Y(y)dy - \frac{\gamma}{2}\int_{I_X}\Phi_X(x)p_X(x)dx + \alpha/2+\beta/2 = -\gamma+\alpha/2+\beta/2 > 0, \quad \text{a.e.}$$

as the last equality follows from the conditions $\int \Phi_Y(y)p_Y(y)dy = \int \Phi_X(x)p_X(x)dx = 1$.

Proof of Proposition EC.2 Let $M \in \mathcal{M}_+$. By non-negativity of M,

$$E[M\hat{m}(X,Y)] \geq E[M(\Phi_X(X)+\Phi_Y(Y))/2] = E[\hat{m}(X,Y)(\Phi_X(X)+\Phi_Y(Y))/2] = E[\hat{m}^2(X,Y)],$$

where, for the first equality, note that both $\hat{m}(X,Y)$ and M price the individual options on each underlying asset. It follows that

$$E[(M-\hat{m}(X,Y))\hat{m}(X,Y)] \geq 0,$$

and therefore

$$E[M^2] = E[\hat{m}^2(X,Y)] + 2E[(M-\hat{m}(X,Y))\hat{m}(X,Y)] + E[(M-\hat{m}(X,Y))^2] \geq E[\hat{m}^2(X,Y)].$$

To prove the second part, note that $M \in \mathcal{M}_+ \subset \mathcal{M}$, whence $\mathcal{M} \neq \varnothing$. Thus, Theorem 1 implies that a minimal discount factor $M^* \in \mathcal{M}$ uniquely exists and is of the "linear" form $(\Phi_X^*(X)+\Phi_Y^*(Y))/2$. Hence $\hat{m}(X,Y) \neq M^*$, unless $P[\Phi_X(X)+\Phi_Y(Y) \geq 0] = 1$.

Proof of Theorem EC.1 The uniqueness of Φ^ε follows as in the case of Theorem 1 (ii). Furthermore, direct substitution confirms that $f = (f_X, f_Y)$ solves the unperturbed system of integral equations, (9)-(11). Because f_X, $f_Y \in L_p^2$ by assumption, it follows that $\hat{M} \in \mathcal{M}$. By Theorem 1, the unique minimal SDF M^* exists and has the form $M^* = \frac{\Phi_X(X)+\Phi_Y(Y)}{2}$, where $\Phi = (\Phi_X, \Phi_Y)$ solves the unperturbed system of integral equations, (9)-(11). Again, by Theorem 1, the solution to these equations is unique, whence $f_X = \Phi_X$, $f_Y = \Phi_Y$ almost everywhere. Strict positivity follows because $\hat{M} = (1-\varepsilon)M_\varepsilon^* + \varepsilon \geq \varepsilon > 0$.

EC.4. No Arbitrage and Positive SDFs

The appeal of the minimal SDF obtained in Theorem 1 is that it is the average of separate functions of the underlying assets, and that it is identified by a system of linear equations. The drawback of such simplicity is that such SDF is not guaranteed to be strictly positive, potentially leading to arbitrage opportunities if used to price contingent claims with nonlinear dependence on multiple assets.

This section discusses two approaches to obtain a strictly positive SDF: (a) formulating sufficient conditions for the minimal discount factor in Theorem 1 to be strictly positive; and (b) adding a positivity constraint to the optimization problem and obtain a strictly positive SDF by perturbation. To ease notation, the bulk of this section concentrates on two underlying assets, the multivariate setting being analogous.

EC.4.1. Criteria for Positivity

First, as the SDF in Theorem 1 separates in the cross section, its strict positivity is straightforward to check, because

$$\inf_{\xi} m^*(\xi) = \inf_{\xi} \frac{1}{n}\sum_{i=1}^{n} \Phi_i(\xi_i) = \frac{1}{n}\sum_{i=1}^{n} \inf_{\xi_i} \Phi_i(\xi_i). \tag{EC.10}$$

Thus, the strict positivity of the SDF is equivalent to the strict positivity of the sum of the infima of its additive components.

Equation (EC.10) is sufficient to determine whether the infimum of the SDF is positive from the solution of the integral equations. The following, more restrictive, criterion guarantees the same conclusion a priori, i.e., without solving the equations.

Proposition EC.1. *Let the assumptions of Theorem 1 hold, and assume that for some $1 \le \gamma < (\alpha+\beta)/2$, with $0 \le \alpha, \beta \le 2$*

$$\frac{q_X(x)}{p_X(x)} \ge \alpha/2, \quad \frac{q_Y(y)}{p_Y(y)} \ge \beta/2, \quad \text{and} \quad \frac{p(x,y)}{p_X(x)p_Y(y)} \le \gamma \quad \text{for a.e. } x, y. \tag{EC.11}$$

Then $m^*(x,y) > 0$ almost everywhere.

The interpretation of this result is that when marginal risk premia are large enough and the assets are not too dependent, absence of arbitrage follows. (Note that independence fulfills the assumption with $\gamma = 1$ and risk neutrality with $\alpha = \beta = 2$.) In general, the condition requires that marginal state price densities, which represent the price of Arrow-Debreu securities concentrated in a small interval, are uniformly bounded from below, which means that no state of nature is "too cheap" – risk premia are high. Vice versa, the second part of the condition is equivalent to $P(Y \in dy|X \in dx) = p(x,y)/p_X(x) \le \gamma p_Y(y) = \gamma P(Y \in dy)$, which means that information on the value of X cannot increase the density of Y by a factor higher than γ, i.e. dependence is not too strong. (Of course, the same reasoning applies if X and Y are exchanged.)

EC.4.2. The Hansen-Jagannathan approach

To obtain positive discount factors, Hansen and Jagannathan (1991) propose to consider SDFs obtained as positive parts of portfolios, which are interpreted as call options on several assets. As they acknowledge, a limit of this approach is that such SDFs may not be unique and may also vanish with positive probability, thereby leading to potential arbitrage opportunities (if used to price claims on both assets). In the present setting, this observation is made precise with the following statement:

PROPOSITION EC.2. *Under the assumptions of Theorem 1. Any SDF of the form* $\hat{m}(X,Y) = \frac{1}{2}\left(\Phi_X(X) + \Phi_Y(Y)\right)_+$ *is minimal in* $\mathcal{M}_+ := \{m \in \mathcal{M} \mid m \ge 0\} \subseteq \mathcal{M}$. *Only if* $\Phi_X(X) + \Phi_Y(Y) \ge 0$ *a.s. such an SDF is minimal in* $\mathcal{M}$ *and closes the duality gap.*

We describe a perturbation argument that yields a strictly positive SDF starting from an SDF that is merely positive. This is the case, for example, for the minimal SDF M_+ in the constrained set $\mathcal{M}_+$. (Indeed, M_+ exists by the same argument as in the proof of Theorem 1. If it were strictly positive, then the positivity constraint would not be binding, and therefore – by uniqueness – it would coincide with the unconstrained M^* in Theorem 1. Thus, whenever positivity is binding,

$P(M_+ = 0) > 0$.) Then, a strictly positive SDF may be obtained through the following perturbation argument. For $\varepsilon \in (0,1)$ define

$$q_X^\varepsilon := \frac{1}{1-\varepsilon}(q_X - \varepsilon p_X), \quad q_Y^\varepsilon := \frac{1}{1-\varepsilon}(q_Y - \varepsilon p_Y),$$

and consider the perturbed integral equations with the usual constraint

$$\frac{1}{2} f_X(x) p_X(x) + \frac{1}{2}\int_{I_Y} f_Y(y) p(x,y) dy = q_X^\varepsilon(x), \tag{EC.12}$$

$$\frac{1}{2} f_Y(y) p_Y(y) + \frac{1}{2}\int_{I_X} f_X(x) p(x,y) dx = q_Y^\varepsilon(y), \tag{EC.13}$$

$$\int_{I_X} f_X(x) p_X(x) dx = \int_{I_Y} f_Y(y) p_Y(y) dy = 1. \tag{EC.14}$$

Theorem EC.1. *Under the assumptions of Theorem 1, and assume that for some $\varepsilon > 0$ the integral equations* (EC.12)-(EC.14) *have a solution $\Phi^\varepsilon := (\Phi_X^\varepsilon, \Phi_Y^\varepsilon) \subset L_p^2$ such that $\Phi_X^\varepsilon, \Phi_Y^\varepsilon \geq 0$. Then Φ^ε is the unique solution, and the SDF defined as*

$$\hat{M} = (1-\varepsilon)\frac{\Phi_X^\varepsilon(X) + \Phi_Y^\varepsilon(Y)}{2} + \varepsilon$$

is greater than $\varepsilon > 0$ and satisfies Theorem 1 (i)-(iv).

EC.5. Variance Gamma Model

The Variance Gamma model specifies the asset price as $X_t = X_0 e^{\omega t + Z_t(\sigma,\nu,\theta)}$, where the random variable $Z_t(\sigma,\nu,\theta)$ is identified by the characteristic function

$$E[e^{iuZ_t}] = (1 - i\theta\nu u + \frac{\sigma^2}{2}u^2\nu)^{-t/\nu}, \quad u \in \mathbb{R} \tag{EC.15}$$

hence has mean $E[Z_t(\sigma,\nu,\theta)] = \theta t$ and variance $\mathrm{Var}(Z_t(\sigma,\nu,\theta)) = (\theta^2\nu + \sigma^2)t$. Furthermore, this distribution corresponds to the marginal law of a Levy process where the jump size has the density (Madan et al. 1998)

$$k_Z(x) = \frac{e^{\theta x/\sigma^2}}{\nu|x|} e^{-\frac{\sqrt{\frac{2}{\nu} + \frac{\theta^2}{\sigma^2}}}{\sigma}|x|}. \tag{EC.16}$$

Henceforth, for the sake of simplicity set $\theta = 0$, which means that positive and negative jumps of the same magnitude are equally likely and that the variance of the log price at time t is simply $\sigma^2 t$.

Such an assumption abstracts from the asymmetry of the volatility "smile" observed in practice, while focusing on the tension between physical and risk-neutral volatility as the main determinant of option positions.

A full specification of the model consists of the joint law of assets' returns under the physical probability P and their marginal distributions under the risk-neutral probability Q. The risk-neutral marginal under Q of the i-th asset's return is a Variance Gamma law with parameters $(\sigma_i^Q)_{i=X,Y}$, for a fixed shape parameter ν, assumed common to all assets to keep the volatility parameter as the main determinant of option prices. In addition, the risk-neutrality condition requires that $\omega_i^Q = -\frac{1}{\nu}\log(1-\nu(\sigma_i^Q)^2/2)$, completing the specification of risk-neutral marginals.

The joint physical law is described through the separate specification of physical marginals and their copula. The P marginals are also assumed of the form (EC.15), with different variance parameters $(\sigma_i^P)_{i=X,Y}$ but with the same shape parameter ν used for the risk-neutral dynamics. Furthermore, assume that assets do not carry risk-premia even under the physical measure, whence $\omega_i^P = -\frac{1}{\nu}\log(1-\nu(\sigma_i^P)^2/2)$. Note that such an assumption is not dictated by absence of arbitrage: instead, its purpose is to ensure that all the demand for options in the model is driven by option risk-premia rather than by the motive to gain exposure to the asset's risk premium through options. Put differently, such a condition removes assets' demand from options' demand.

Finally, the dependence among the returns follows a bivariate t-copula with parameters d (the degrees of freedom in the t distribution) and correlation ϱ. Thus, the joint distribution is

$$P(X \le x, Y \le y) = T_{\varrho,d}(T^{-1}(F_X(x)), T^{-1}(F_Y(y))), \tag{EC.17}$$

where $T_{\varrho,d}$ is distribution function of the standard t-copula with parameters ϱ, d, while T is the distribution function of the t-distribution with d degrees of freedom, and F_X, F_Y are the distribution functions of the Variance Gamma marginals X and Y.

EC.5.1. Performance and the Limits of Naïve Optimization

Theorem 1 implies that, in general, the optimal asset-specific option payoff depends on the risk-neutral densities of all assets. Yet, a natural question is whether assets' interdependence (the dependence of each payoff on the risk-neutral densities of other assets) is a second-order effect, which perhaps could be bypassed by a two-stage optimization approach that treats options on each asset in isolation, as follows. First, find the optimal option payoff $\Psi_i(X_i)$ on each asset X_i, as if options on other assets did not exist. Second, construct a portfolio of the form $\sum_{i=1}^n w_i \Psi_i(X_i)$, by choosing the weights w_i that maximize the global Sharpe ratio.

While such a divide-and-conquer approach is intuitively appealing, its performance is significantly worse than the solution to the joint optimization problem formulated in the paper: assets' interdependence is a first-order effect. To see this point clearly, consider the situation described in Figure 1: as options on the asset Y have zero risk-premia, the optimal payoff of options in Y is identically zero. Thus, the two-stage optimization procedure trivially yields the optimal payoff of options on X, while all hedging gains – a prominent feature of global optimization – are lost.

The broader message of this example is that, because the asset-specific option payoffs are entirely determined by their internal risk-return tradeoffs, they are ill-suited for hedging, which is an intrinsically global problem. In addition, two-stage optimization performs most poorly precisely for the cheapest hedging instruments, that is when the risk-premium to be sacrificed in hedging is lowest (zero, in the above example).

Figure EC.1 displays the performance of optimal option strategies against correlation, and shows how high (positive or negative) correlation lifts the Sharpe ratio. This effect is present when options on both asset carry risk premia (right panel) and when options on the second asset have no risk premium (left panel), hence are merely hedging instruments. For low correlation, the performance of the optimal portfolio of Theorem 1 (black line) is marginally better than an equally weighted combination of the asset-specific portfolios (green), which is optimal with independent assets (Example 2). Indeed, low correlation implies that hedging opportunities are limited and assets' interdependence insignificant. As correlation becomes stronger, the situation reverses: the performance of the global optimizer and the zero-correlation portfolio diverge quickly. When options on both assets have risk premia (right panel), the two-stage optimizer (blue line) performs better than the zero-correlation portfolio, but still significantly worse than the global optimizer.

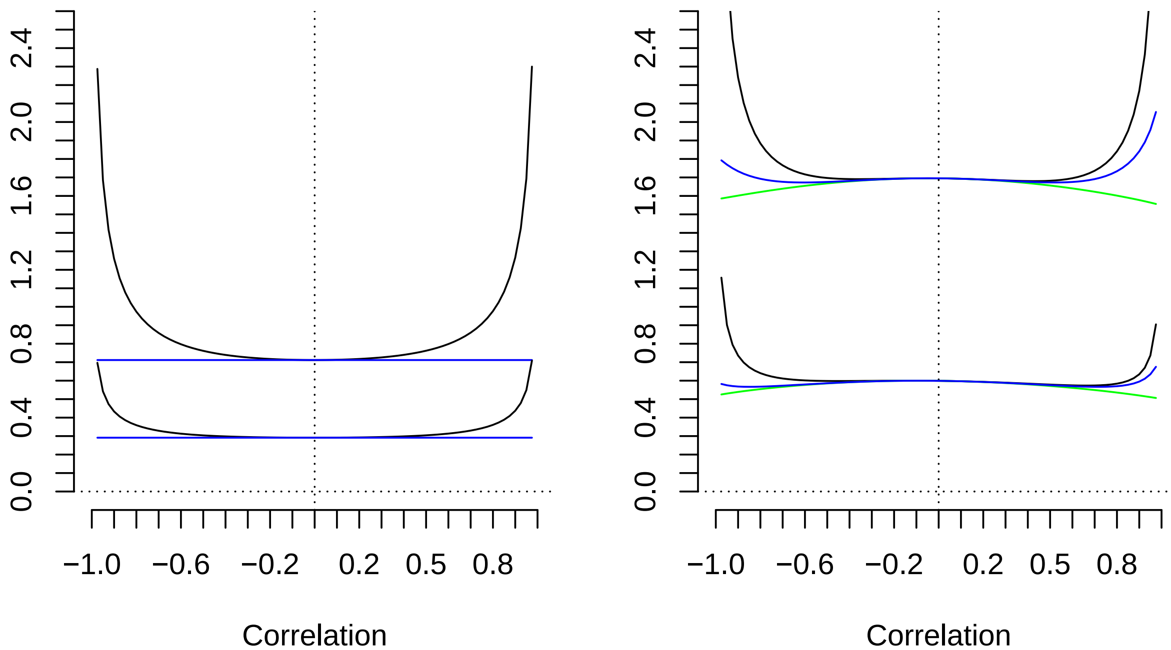

Figure EC.1 Sharpe Ratios (vertical axis) against correlation (horizontal) corresponding to Figure 1, with annual (lower) and monthly (upper) horizon, for the optimal portfolio (black), the optimal combination of asset-specific portfolios for given correlation between the assets' return (blue), and the combination of asset-specific portfolios for zero correlation (green).

Importantly, keeping physical and risk-neutral volatilities constant, the optimal performance rises dramatically as the trading frequency increases from annual to monthly, more than doubling the Sharpe ratio across a range of typical positive correlations, with or without risk premia in the second asset. This phenomenon may seem counterintuitive at first, as the processes considered have independent, identically distributed returns, which hint at a constant Sharpe ratio. Upon closer inspection, however, the Sharpe ratio here does not result from exposure to the assets themselves (assets' risk premia are assumed to be zero), but rather from exposure to the nonlinearity in option payoffs (Gamma, in traders' jargon).

A lower trading frequency (i.e., a longer investment period) reduces an investor's ability to gain nonlinear exposure because, even if a portfolio of options is optimal at the beginning of the period, once the asset price moves in either direction, most of the options become either firmly in the

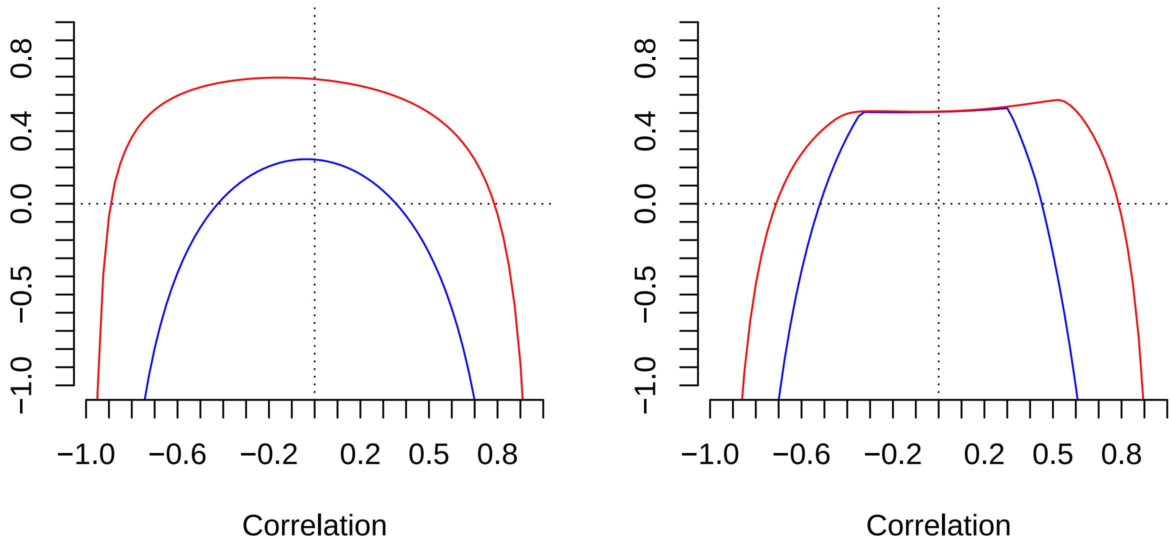

Figure EC.2 Minimum value of the minimal SDF (vertical axis) against correlation (horizontal) corresponding to Figure 1, with monthly (lower, blue) and annual (upper, red) horizon.

money or firmly out of the money, thereby losing much of their nonlinearity for the rest of the period. In addition, once the asset price moves, the option portfolio gains nonzero exposure to the asset itself (i.e., Delta), which has zero risk premium, thereby adding idiosyncratic risk that reduces the Sharpe ratio. Vice versa, with a higher trading frequency the investor can reset the option payoff after typically smaller variations in asset prices, restoring the rewarded nonlinear exposure that generates return while neutralizing linear exposure (Delta hedging) that only adds unrewarded risk – ultimately boosting the Sharpe ratio.

This phenomenon arises for two reasons: First, as the option maturity grows, the second moment of the state-price density increases at varying rates, and the trading frequency scales the average rate for a given maturity to the unit interval. (The ideal trading frequency depends on the model and the parameters considered.) Second, as the trading frequency increases, so does the hedging frequency (with nonzero correlation), reducing risk even further.

The overall message is that access to options on several assets significantly increases the overall performance of an option portfolio, even when some of these options are unattractive *per se*, because they play a critical role as hedging instruments. Performance gains are especially significant at short horizons and for options on highly correlated assets.

EC.5.2. SDF Positivity and Arbitrage

To understand when the SDF implied by Theorem 1 is consistent with the absence of arbitrage, Figure EC.2 examines the minimal value of the SDF in equation (EC.10) as asset correlation varies. The SDF is positive when correlation is weak: for the left panel, corresponding to Figure 1, the absence of arbitrage follows from the absence of risk premia on the second asset ($\frac{q_Y}{p_Y} = 1$). Indeed, the joint specification in (EC.17) implies that zero correlation corresponds to independence, and in this case the minimal SDF reduces to $m^*(x,y) = \frac{q_X(x)}{p_X(x)} + \frac{q_Y(y)}{p_Y(y)} - 1 = \frac{q_X(x)}{p_X(x)} > 0$.

As correlation becomes stronger, the minimum value of the SDF drops below zero first with an annual horizon and eventually even with a monthly horizon for near-perfect correlation. To understand this phenomenon, consider the limit case of perfect (positive or negative) correlation: in this case, the assets' returns are linked by an affine transformation, which in turn identifies one marginal (e.g., Y) in terms of the other (e.g., X). However, if the pre-specified risk-neutral marginal of Y is not compatible with any such transformation (and in general, it is not), then an arbitrage opportunity arises by selling a claim on Y for the price implied by its risk-neutral marginal, while hedging it perfectly for the price implied by the risk neutral marginal of X (or vice versa, depending on which price is higher). Put differently, while two marginals can always be joined through independence, they may not be compatible with an arbitrarily high correlation value. (This is true even for lognormals, as pointed out by Embrechts et al. (2002).) When the prescribed marginals are incompatible with the prescribed correlation, any kernel that joins them cannot be strictly positive, otherwise it would be a joint probability law.

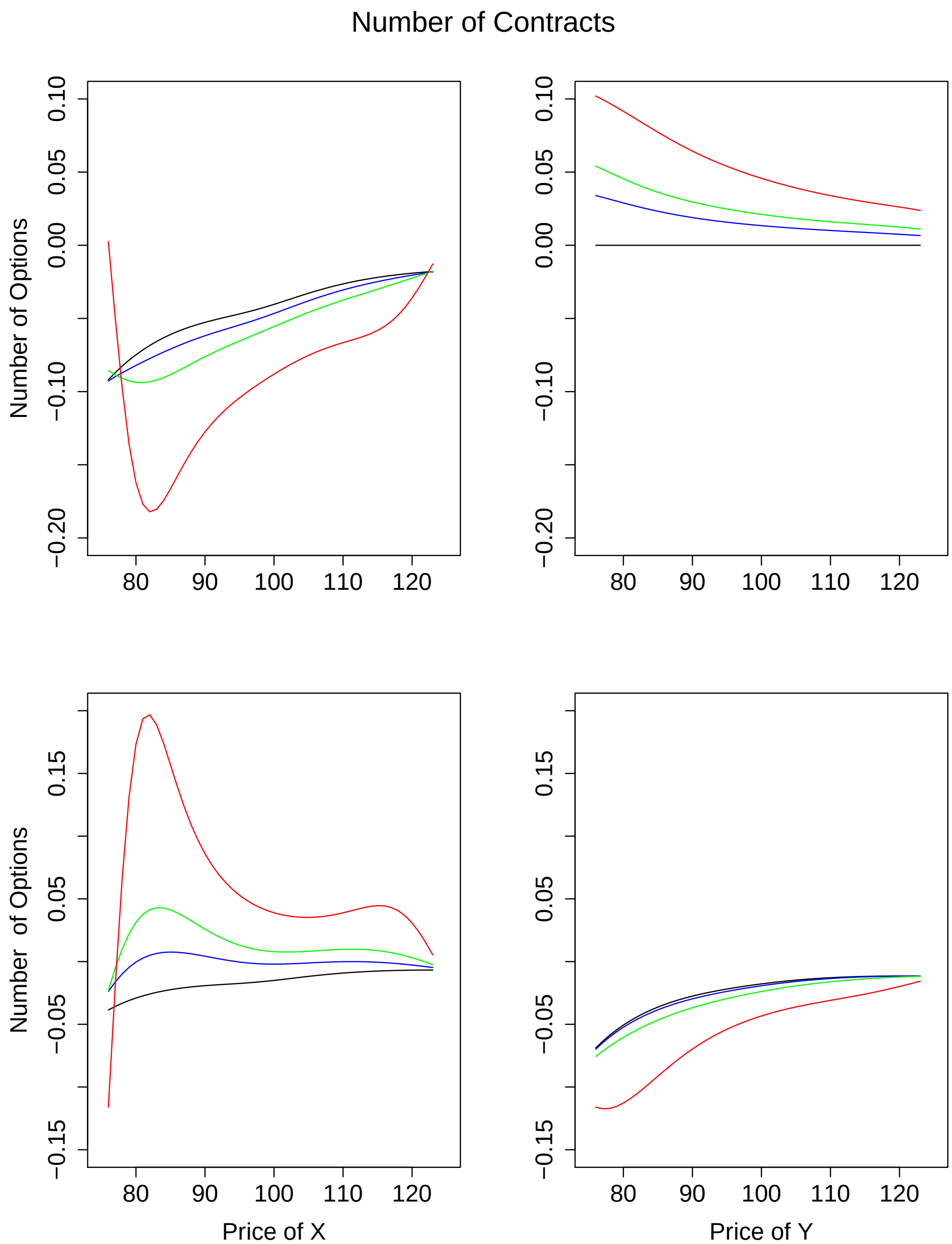

Figure EC.3 Number of contracts (vertical axis) for each strike price (horizontal) in the first (left) and the second (right) asset, corresponding to the parameters of Figure 1. The number of contracts is the same for calls and puts, interchangeably, in view of put-call parity.

EC.5.3. Number of Contracts

A legitimate concern for the implementation of the trading strategies implied by the main result is whether the number of contracts that is required for each strike is actually available in the market.

Figure EC.3 displays the number of contracts (determined up to an arbitrary multiplicative factor, as for optimal payoffs) corresponding to the parameter values in Figure 1. (Note that the number of contracts for a fixed strike is the same for calls and puts, in view of put-call parity.) In most settings of interest, and consistent with the qualitative features of optimal payoffs, the number of contracts moderately increases in absolute value as the strike declines, both for the short positions designed to generate returns, and for their hedging long positions.

The exception to this rule of thumb is the extreme configuration that combines two highly correlated assets (red line, corresponding to 90% correlation) with sharply different risk premia for option prices (in the bottom panels $\sigma_X^Q - \sigma_X^P = 5\%$ while $\sigma_Y^Q - \sigma_Y^P = 15\%$): in such unrealistic scenario, the number of contracts would be hump-shaped, with the number of contracts peaking around 15% below the money, less than one standard deviation from the initial asset price, in the annual horizon considered.

EC.6. Conclusion

We have introduced a method to compute the combination of options, each of them written on one of many assets with several available strikes, as to maximize the Sharpe ratio of the option portfolio. The method identifies optimal payoffs as solutions to a system of integral equations that, under appropriate discretizations of physical and risk-neutral probabilities, reduces to a matrix equation. In a concrete model with correlated assets, optimal payoffs display significant interactions, with the global optimal option portfolio departing significantly from optimal asset-specific payoffs, and sometimes even entailing reverse positions.

This paper focuses on investors who trade options on several underlying assets with a common expiration. An important future development is to extend our analysis to include multiple expirations. The framework in this paper may also be adapted to investigate the Ross (2015) recovery of the physical measure from option prices, subject to additional information on either preferences or market moments, in the spirit of Schneider (2015) and Schneider and Trojani (2018).

References

Almeida C, Garcia R (2012) Assessing misspecified asset pricing models with empirical likelihood estimators. *Journal of Econometrics* 170(2):519–537.

Almeida C, Garcia R (2016) Economic implications of nonlinear pricing kernels. *Management Science* 63(10):3361–3380.

Bakshi G, Kapadia N (2003) Delta-hedged gains and the negative market volatility risk premium. *Review of Financial Studies* 16(2):527–566.

Carr P, Jin X, Madan D (2001) Optimal investment in derivative securities. *Finance and Stochastics* 5(1):33–59.

Carr P, Madan D (2001a) Optimal positioning in derivative securities. *Quantitative Finance* 1:19–37.

Cherubini U, Luciano E (2002) Bivariate option pricing with copulas. *Applied Mathematical Finance* 9(2):69–85.

Coval JD, Shumway T (2001) Expected option returns. *J. of Finance* 56(3):983–1009.

Daniele P, Giuffrè S, Idone G, Maugeri A (2007) Infinite dimensional duality and applications. *Mathematische Annalen* 339(1):221–239.

Donato MB (2011) The infinite dimensional lagrange multiplier rule for convex optimization problems. *Journal of Functional Analysis* 261(8):2083–2093.

Embrechts P, McNeil A, Straumann D (2002) Correlation and dependence in risk management: properties and pitfalls. *Risk management: value at risk and beyond* 1:176–223.

Eraker B (2013) The performance of model based option trading strategies. *Review of Derivatives Research* 16(1):1–23.

Faias J, Santa-Clara P (2011) Optimal option portfolio strategies URL `"http://ssrn.com/abstract=1569380"`.

Guasoni P, Huberman G, Wang Z (2011) Performance maximization of actively managed funds. *Journal of Financial Economics* 101(3):574–595.

Guasoni P, Robertson S (2012) Portfolios and risk premia for the long run. *The Annals of Applied Probability* 22(1):239–284.

Hansen LP, Jagannathan R (1997) Assessing specification errors in stochastic discount factor models. *The Journal of Finance* 52(2):557–590.

Haugh MB, Kogan L (2004) Pricing American options: a duality approach. *Operations Research* 52(2):258–270.

Haugh MB, Kogan L, Wang J (2006) Evaluating portfolio policies: A duality approach. *Operations Research* 54(3):405–418.

He H, Pearson ND (1991) Consumption and portfolio policies with incomplete markets and short-sale constraints: The infinite dimensional case. *Journal of Economic Theory* 54(2):259–304.

Karatzas I, Lehoczky JP, Shreve SE, Xu GL (1991) Martingale and duality methods for utility maximization in an incomplete market. *SIAM Journal on Control and optimization* 29(3):702–730.

Li DX (2000) On default correlation: A copula function approach. *The Journal of Fixed Income* 9(4):43–54.

Liu J, Pan J (2003) Dynamic derivative strategies. *Journal of Financial Economics* 69(3):401–430.

Madan DB, Carr PP, Chang EC (1998) The variance gamma process and option pricing. *European Finance Review* 2(1):79–105.

Madan DB, Seneta E (1990) The variance gamma (VG) model for share market returns. *Journal of Business* 511–524.

Malamud S (2014) Portfolio Selection with Options and Transaction Costs. *Swiss Finance Working Paper Series* .

Rogers LC (2002) Monte Carlo valuation of American options. *Mathematical Finance* 12(3):271–286.

Ross S (2015) The recovery theorem. *The Journal of Finance* 70(2):615–648.

Rudin W (1986) *Real and complex analysis (3rd)* (New York: McGraw-Hill Inc).

Santa-Clara P, Saretto A (2009) Option strategies: Good deals and margin calls. *Journal of Financial Markets* 12(3):391–417.

Schneider P (2015) Generalized risk premia. *Journal of Financial Economics* 116(3):487–504.

Schneider P, Trojani F (2015) Fear trading, available on SSRN (ID: 1994454).