Proofs and Derivations

EC.1. The Dynamic Choice Model

EC.1.1. Individual customer's sequential decisions.

Given price trajectories and personal preferences, each individual customer solves her own dynamic program (DP) and behaves according to the corresponding optimal policy. The individual customer's DP model consists of a state variable s_t, a control variable a_t, the Markov state transition probability $p(s_{t'}|s_t, a_t)$, the instantaneous utility function $r_t(s_t, a_t)$, and a discount rate $\rho \geqslant 0$. The individual optimal policy is given by $\delta^\star = \arg\max_\delta \mathbb{E}_\delta\big(\int_0^T e^{-\rho t} r_t(s_t, a_t) dt | s_0\big)$, where the expectations $\mathbb{E}_\delta$ are taken over the state-control realizations induced by δ. We introduce the choice set $A_n = \{n, p, i\}$ for non-passholder customers and $A_p = \{n, r, i\}$ for passholders, where p, r, i, n stands for pass purchase, redemption, individual purchase, and do nothing, respectively.

Now we derive the customer's DP by considering what happens in a small time interval $[t, t+h)$ based on the Bernoulli approximation to the Poisson process. Such an approximation disregards $o(h)$ terms and is standard in stochastic models. When doing so, the resulting dynamic equations are effectively in discrete time with the step (time unit) of length h. The paper is focused on the case in which $h \to 0$, and the stated discrete-time dynamic program provides an intuitive justification of the utility evolution process. A decision opportunity arises with probability $\lambda h + o(h)$, the probability of no decision opportunity is given by $1 - \lambda h + o(h)$, and the probability of having more than one opportunity is $o(h)$. If no decision opportunity arises in this interval, the situation is not different from choosing the do-nothing action ($a_t = n$). Therefore, the Bellman equation for a customer at state s_t is

$$V_t(s_t) = \max_{a_t} \Big\{ \lambda h \big[r_t(s_t, a_t) + e^{-\rho h} \mathbb{E}(V_{t+h}(s_{t+h}) | s_t, a_t) \big] + (1 - \lambda h) \big[r_t(s_t, n) + e^{-\rho h} \mathbb{E}(V_{t+h}(s_{t+h}) | s_t, n) \big] + o(h) \Big\}. \tag{EC.1}$$

The individual's state variable, s_t, may involve many factors such as the number of pass credits at one's disposal, demographic attributes, personal preferences, loyalty, and consumption needs. The customer can take all factors into account but not all are observable by the firm. From the firm's perspective, the number of remaining credits (credit balance) is perhaps the easiest factor to observe, sometimes the only one. For example, many passes are sold in the form of anonymous chip cards that only store the credit balance. Some passes track the credit balance by stamping on the pass. Since these passes do not store personal information, the credit balance is the only state variable observable to the firm. Therefore, it is plausible to partition the state variable into two components, namely, $s_t = (x_t, \epsilon_t)$, where $x_t \in \{0, 1, \ldots, \bar{k}\}$ denotes the number of remaining credits, which is *observable* to the firm, and ϵ_t represents the myriad of information observed by the customer but *unobservable* to the firm.

The discount factor ρ is related to the level of strategic behavior: when $\rho \to \infty$, we have $e^{-\rho h} \to 0$ in (EC.1) meaning that the customers disregard future outcomes and solely focus on the present. That is, they are *myopic*. When $\rho \to 0$, we have $e^{-\rho h} \to 1$, in which case future utility is not discounted and impacts the present with full strength. In this case customers are *fully strategic*. For $0 < \rho < \infty$, the customers are *partially strategic*, which is the most realistic case. In general, a lower ρ is associated with more strategic customers.

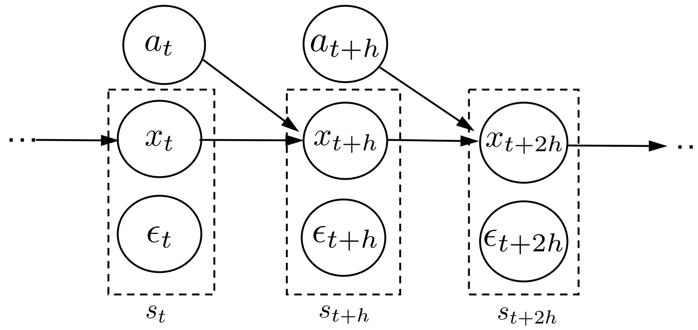
We make two standard assumptions in dynamic discrete choice models: additively separable utility and conditionally independent state components. These assumptions are required to make the problem tractable and amenable to statistical inference.

Assumption EC.1 **(Additively Separable)**. *The instantaneous utility function is additively separable in the observable and unobservable components, namely,* $r_t(s_t, a_t) = u_t(x_t, a_t) + \epsilon_t(a_t)$, *where* $\epsilon_t(a_t)$ *is a random variable representing the unobservable component related to action* a_t.

For non-passholder customers ($x_t = 0$), the choice set is $A_n = \{n, p, i\}$. Let a denote the average value generated by each consumption. Then the observable components are given by $u_t(0, n) = 0$, $u_t(0, p) = a - p_t$, and $u_t(0, i) = a - f_t$. The unobservable components are included in the vector $\epsilon_t = (\epsilon_t(n), \epsilon_t(p), \epsilon_t(i))$. Similarly, for passholders ($x_t = k \geqslant 1$), the choice set is $A_p = \{n, r, i\}$ and the observable components are $u_t(k, n) = 0$, $u_t(k, r) = a$, and $u_t(k, i) = a - f_t$, where $k = 1, \ldots, \bar{k}$. The unobservable component vector is $\epsilon_t = (\epsilon_t(n), \epsilon_t(r), \epsilon_t(i))$. Let $g(\epsilon_t)$ denote the joint density function, and according to the standard practice in discrete choice model, we normalize it so that the marginal mean is zero, i.e., $\mathbb{E}[\epsilon_t(a_t)] = 0$ for all action a_t.

Assumption EC.2 **(Conditional Independence)**. *Conditional on* x_t *and* a_t, *the density for* x_{t+h} *does not depend on the unobservable component* ϵ_t. *Further,* ϵ_t *is independent and identically distributed. This assumption can be summarized as* $p(x_{t+h}, \epsilon_{t+h} | x_t, \epsilon_t, a_t) = p(x_{t+h} | x_t, a_t) g(\epsilon_{t+h})$.

The concept of Assumption EC.2 is illustrated in Figure EC.1. Intuitively, it assumes that ϵ_t is noise superimposed on the main dynamics that are described by $p(x_{t+h} | x_t, a_t)$. Note that each action impacts the credit balance with certainty. For example, if a non-passholder customer purchases the pass, her credit balance jumps from zero to $\bar{k}$ with probability one. If a passholder redeems the kth credit, the credit balance becomes $k - 1$ with certainty, otherwise it remains unchanged. Hence, the conditional distribution $p(x_{t+h} | x_t, a_t)$ is degenerate.



EC1.pdf

Figure EC.1 Structure of dependence implied by the conditional independence assumption.

Under Assumptions EC.1-EC.2, the individual customer's Bellman equation (EC.1) becomes

$$V_t(x_t,\epsilon_t) = \max_{a_t}\Big\{\lambda h\big[u_t(x_t,a_t)+\epsilon_t(a_t)+e^{-\rho h}\mathbb{E}(V_{t+h}(x_{t+h},\epsilon_{t+h})|x_t,a_t)\big] \qquad \text{(EC.2)}$$
$$+(1-\lambda h)\big[u_t(x_t,n)+\epsilon_t(n)+e^{-\rho h}\mathbb{E}(V_{t+h}(x_{t+h},\epsilon_{t+h})|x_t,n)\big]+o(h)\Big\},\ x_t=0,\ldots,\bar{k}.$$

EC.1.2. Expressing the expected utility.

Since ϵ_t is the noise superimposed on the number of credits, we calculate the expected utility of k credits by integrating ϵ_t out, namely,

$$U_t(k) \triangleq \int V_t(k,\epsilon_t)g(\epsilon_t)d\epsilon_t,$$

which is called the *integrated value function* in the dynamic choice literature. For simplicity, we will call it the *expected utility function* in this paper. Intuitively, it is the continuation value of having k credits, just before ϵ_t is revealed. This utility function plays a central role in dynamic choice modelling. Since it captures the long-term utilities, it must be distinguished from the instantaneous utilities.

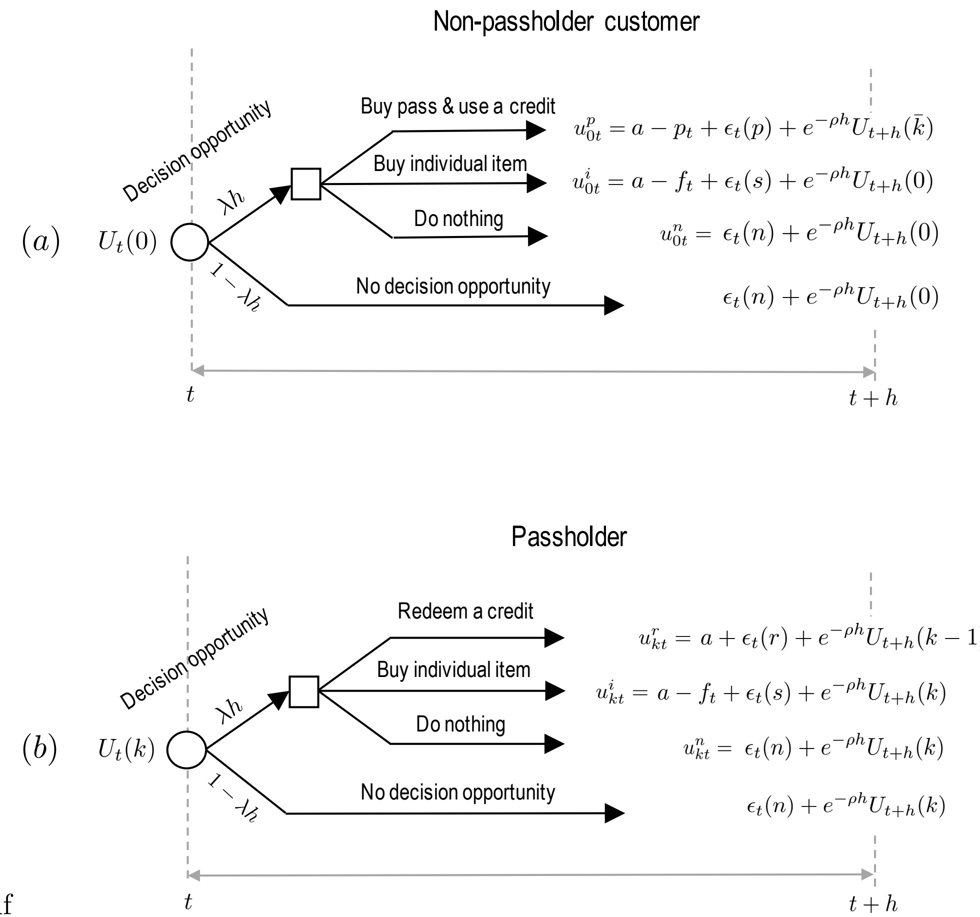
We now begin to derive the system of differential equations that characterize this function. We integrate the unobservable components out from both sides of (EC.2) to obtain the following *integrated Bellman equation* :

$$U_t(k) = \lambda h\int \max_{a_t}\Big\{u_t(k,a_t)+\epsilon_t(a_t)+e^{-\rho h}U_{t+h}(x_{t+h}|k,a_t)\Big\}g(\epsilon_t)d\epsilon_t$$
$$+(1-\lambda h)e^{-\rho h}U_{t+h}(k)+o(h),\quad k=0,\ldots,\bar{k}. \qquad \text{(EC.3)}$$

The derivation of equation (EC.3) is based on two facts (i) unobservable component has zero marginal mean, i.e., $\mathbb{E}[\epsilon_t(n)]=0$, and (ii) the state transition probability distribution $p(x_{t+h}|x_t,a_t)$ is degenerate.

For non-passholder customers ($k=0$), the integrated Bellman equation becomes

$$U_t(0) = \lambda h\int \max\{u^p_{0t},u^i_{0t},u^n_{0t}\}g(\epsilon_t)d\epsilon_t+(1-\lambda h)e^{-\rho h}U_{t+h}(0)+o(h), \qquad \text{(EC.4)}$$



EC2.pdf

Figure EC.2 Illustration of integrated Bellman equations for (a) non-passholder customers and (b) passholders.

where u^p_{0t}, u^i_{0t} and u^n_{0t} are given in Figure EC.2-a. The optimal action is the one that gives the maximum utility. For example, it is optimal to purchase the pass if $u^p_{0t} = \max\left\{u^p_{0t}, u^i_{0t}, u^n_{0t}\right\}$. Therefore, the corresponding purchase probability is given by $\pi^p_{0t} = \Pr(u^p_{0t} = \max\{u^p_{0t}, u^i_{0t}, u^n_{0t}\})$. In the same vein, the integrated Bellman equation for passholders with k credits is

$$U_t(k) = \lambda h \int \max\{u^r_{kt}, u^i_{kt}, u^n_{kt}\} g(\epsilon_t) d\epsilon_t + (1-\lambda h) e^{-\rho h} U_{t+h}(k) + o(h),\ k = 1, \ldots, \bar{k}, \qquad \text{(EC.5)}$$

where u^r_{kt}, u^i_{kt}, u^n_{kt} are shown in Figure EC.2-b. It is optimal to redeem a credit if $u^r_{kt} = \max\left\{u^r_{kt}, u^i_{kt}, u^n_{kt}\right\}$, and hence the redemption probability is $\pi^r_{kt} = \Pr(u^r_{kt} = \max\{u^r_{kt}, u^i_{kt}, u^n_{kt}\})$.

EC.1.3. Expressions for the choice model

We make another standard assumption on the joint distribution of the unobservable components that corresponds to the nested logit model (McFadden 1978).

Assumption EC.3 **(GEV Distribution)**. *The joint density function $g(\epsilon_t)$, or $g(\epsilon_{1t}, \epsilon_{2t}, \epsilon_{3t})$, follows a generalized extreme value (GEV) distribution with the following joint distribution function:*

$$\Pr(\epsilon_{1t} < y_1, \epsilon_{2t} < y_2, \epsilon_{3t} < y_3) = \exp\left\{e^{\gamma_{em}}[-e^{-\frac{y_1}{\mu}} - (e^{-\frac{y_2}{\mu\gamma}} + e^{-\frac{y_3}{\mu\gamma}})^{\gamma}]\right\}, \tag{EC.6}$$

where $\gamma_{em} \approx 0.5772$ is the Euler-Mascheroni constant used for normalization.

The parameter $\mu > 0$ measures the variability in unobserved components, and the parameter $\gamma \in (0,1]$ measures the degree of independence between the idiosyncratic terms ϵ_{2t} (consume individually) and ϵ_{3t} (consume with the pass), which belong to the same nest. A larger γ represents greater independence. For example, $\gamma = 1$ means ϵ_{2t} and ϵ_{3t} are independent, $\gamma \to 0$ means they are perfectly correlated. In reality, γ is usually somewhere in the between: ϵ_{2t} and ϵ_{3t} are generally correlated because both individual items and passes relate to consumption, but the correlation is not perfect.

To simplify the notation, we write $U_t(k)$ as U_{kt} from now on. By taking the limit $\Delta \to 0$ (see Appendix EC.2, the formal derivation follows from Chapter 2.3 of Gīhman and Skorohod (1979)) and making use of the properties of the GEV distribution in (EC.6), the integrated Bellman equations can be written as a system of ordinary differential equations, and the choice probabilities take the form of a nested logit model as shown in Theorem 1.

EC.2. Proof of Theorem 1

Proof: The expressions of choice probabilities are based on Theorem 1 in McFadden (1978). We focus on the derivation of utility expressions. For sake of brevity, we only cover the case for non-passholder customers, but the passholder case can be derived in the same way.

Based on the integrated Bellman equation for non-passholder customers ($k = 0$), we have

$$\begin{aligned}
\dot{U}_t(0) &= \lim_{h\to 0} \frac{U_{t+h}(0) - U_t(0)}{h} \\
&= -\lambda\Big\{\int \max\Big[a - p_t + \epsilon_t(p) + e^{-\rho h}U_{t+h}(\bar{k}), a - f_t + \epsilon_t(s) + e^{-\rho h}U_{t+h}(0), \\
&\qquad \epsilon_t(n) + e^{-\rho h}U_{t+h}(0)\Big] g(\epsilon_t)d\epsilon_t - \mu\gamma_{em} - e^{-\rho h}U_{t+h}(0)\Big\} + \lim_{h\to 0}\frac{1-e^{-\rho h}}{h}U_{t+h}(0) \\
&= -\lambda\Big\{\int \max\Big[a - p_t + \epsilon_t(p) + U_t(\bar{k}), a - f_t + \epsilon_t(s) + U_t(0), \epsilon_t(n) + U_t(0)\Big] g(\epsilon_t)d\epsilon_t - \mu\gamma_{em} - U_t(0)\Big\} + \rho U_t(0).
\end{aligned}$$

The integral on the right-hand side is the expected maximum utility. Based on the corollary of Theorem 1 in McFadden (1978), we can verify that

$$\begin{aligned}
&\int \max\Big[a - p_t + \epsilon_t(p) + U_t(\bar{k}), a - f_t + \epsilon_t(s) + U_t(0), \epsilon_t(n) + U_t(0)\Big] g(\epsilon_t)d\epsilon_t \\
=&\mu \ln\Big\{1 + \exp\Big(\frac{a - f_t}{\mu}\Big) + \exp\Big(\frac{a - p_t + U_t(\bar{k}) - U_t(0)}{\mu}\Big)\Big\} + \mu\gamma_{em} + U_t(0).
\end{aligned}$$

Therefore,

$$\dot{U}_t(0) = -\lambda\mu \ln\left\{1+\exp\left(\frac{a-f_t}{\mu}\right)+\exp\left(\frac{a-p_t+U_t(\bar{k})-U_t(0)}{\mu}\right)\right\}+\rho U_t(0).$$

The equations for $U_t(k)$ can be derived in the same way.

Now we prove the uniqueness of the solution. Note that function on the right-hand sides of (2) are measurable and global Lipschitz continuous with the constant $\lambda+\rho$. Then, by Corollary 2.4.5 of Vinter (2000), the solution to the terminal value problem is unique. □

EC.3. Properties of the expected utilities

To gain a deeper understanding of the integrated Bellman equations, we use the example of $\bar{k}=1$ to highlight some important properties of the expected utilities. First, the expected utility of having one credit is always higher than having no credit:

Corollary EC.1. *When $\bar{k}=1$, we have $U_{1t} \geqslant U_{0t} \geqslant 0$, for $0 \leqslant t \leqslant T$.*

This illustrates the expected utility's dependence on the number of remaining credits. Intuitively, we can understand the monotonicity result by a relaxation argument. The monotonicity in k suggests that the utility of having $k+1$ credits is higher than the utility of having only k credits. For a customer with $k+1$ credits in hand, she can "throw away" one credit and consume the rest as if she had only k credits. Hence, the value of holding $k+1$ credits is at least as high as holding k credits, namely, $U_{(k+1)t} \geqslant U_{kt}$.

Second, the expected utility is not fixed and generally decreases as one approaches the end of horizon:

Corollary EC.2. *When $\rho<\infty$, we have $\dot{U}_{kt} \neq 0$, for $0 \leqslant t \leqslant T$. When $\rho=0$, we have $\dot{U}_{kt}<0$, for $0 \leqslant t \leqslant T$.*

This illustrates the utility's time-dependence. Because one has fewer decision opportunities near the expiration date, the utility of the credit is also lower. We also note that the expected utility of the no-purchase option is not always zero because, by choosing the no-purchase option now, the customer is also waiting for future purchase opportunities that may generate positive utilities (except at the end of the horizon when no future opportunity is available). Third, the credit's marginal utility is higher when customers are more strategic (ρ is smaller):

Corollary EC.3. *When $\bar{k}=1$, let $\{U^i_{kt}, 0 \leqslant t \leqslant T, k=0,1\}$ denote the utility path associated with the discount factor ρ_i. Then, $\rho_1 \geqslant \rho_2$ implies that $U^1_{1t}-U^1_{0t} \leqslant U^2_{1t}-U^2_{0t}$, for $0 \leqslant t \leqslant T$.*

EC.4. Proof of Corollary EC.1

Proof: We first transform the problem using the reversed time index $t' = T - t$. Clearly, all utility and price paths can be written in the reversed time as $U_{kt} = U_{k(T-t')}$, $f_t = f_{T-t'}$, and $p_t = p_{T-t'}$. We define the time-reversed utility path as $V_{kt'} \triangleq U_{k(T-t')}$ for $k = 0, 1$. Similarly, define time-reversed price path as $f'_{t'} \triangleq f_{T-t'}$ and $p'_{t'} \triangleq p_{T-t'}$, respectively.

Note that $dU_{kt}/dt = -dU_{k(T-t')}/dt' = -dV_{kt'}/dt'$ and $V_{k0} = 0$, for $k = 0, 1$. Then the state equation for utility can be rewritten as

$$\frac{dV_{1t'}}{dt'} = \lambda\mu \ln\left\{1 + \left[\exp\left(\frac{a - V_{1t'} + V_{0t'}}{\mu\gamma}\right) + \exp\left(\frac{a - f'_{t'}}{\mu\gamma}\right)\right]^{\gamma}\right\} - \rho V_{1t'}, \tag{EC.7}$$

$$\frac{dV_{0t'}}{dt'} = \lambda\mu \ln\left\{1 + \left[\exp\left(\frac{a - p'_{t'} + V_{1t'} - V_{0t'}}{\mu\gamma}\right) + \exp\left(\frac{a - f'_{t'}}{\mu\gamma}\right)\right]^{\gamma}\right\} - \rho V_{0t'}. \tag{EC.8}$$

with initial value $V_{k0} = 0$, for $k = 0, 1$. Define $V'_{0t'} \triangleq e^{\rho t'} V_{0t'}$ and $V'_{1t'} \triangleq e^{\rho t'} V_{1t'}$, then according to (EC.7) and (EC.8), we obtain

$$\frac{dV'_{1t'}}{dt'} = e^{\rho t'} \lambda\mu \ln\left\{1 + \left[\exp\left(\frac{a - e^{-\rho t'} V'_{1t'} + e^{-\rho t'} V'_{0t'}}{\mu\gamma}\right) + \exp\left(\frac{a - f'_{t'}}{\mu\gamma}\right)\right]^{\gamma}\right\}, \tag{EC.9}$$

$$\frac{dV'_{0t'}}{dt'} = e^{\rho t'} \lambda\mu \ln\left\{1 + \left[\exp\left(\frac{a - p'_{t'} + e^{-\rho t'} V'_{1t'} - e^{-\rho t'} V'_{0t'}}{\mu\gamma}\right) + \exp\left(\frac{a - f'_{t'}}{\mu\gamma}\right)\right]^{\gamma}\right\}. \tag{EC.10}$$

It is clear that the log terms on the RHS. of the equations above are nonnegative, the Barrow's formula (Arnol'd 1992) suggests that $V'_{kt'} \geqslant 0$, for $k = 0, 1$, which implies that $V_{kt'} \geqslant 0$ and then $U_{kt} \geqslant 0$, for $k = 0, 1$.

Next, we prove $U_{1t} \geqslant U_{0t}$, for $0 \leqslant t \leqslant T$. Define $\Delta V'_{t'} \triangleq V'_{1t'} - V'_{0t'}$. Subtracting (EC.10) from (EC.9) leads to

$$\frac{d\Delta V'_{t'}}{dt'} = e^{\rho t'} \lambda\mu \ln \frac{1 + \left[\exp\left(\frac{a - e^{-\rho t'} \Delta V'_{t'}}{\mu\gamma}\right) + \exp\left(\frac{a - f'_{t'}}{\mu\gamma}\right)\right]^{\gamma}}{1 + \left[\exp\left(\frac{a - p'_{t'} + e^{-\rho t'} \Delta V'_{t'}}{\mu\gamma}\right) + \exp\left(\frac{a - f'_{t'}}{\mu\gamma}\right)\right]^{\gamma}}.$$

Because the pass price is assumed to be nonnegative, i.e., $p'_{t'} \geqslant 0$, the RHS. of the equation above must also be nonnegative. Again by the Barrow's formula we have $\Delta V'_{t'} \geqslant 0$, for $0 \leqslant t \leqslant T$, which implies that $V'_{1t'} \geqslant V'_{0t'}$, then $V_{1t'} \geqslant V_{0t'}$ and finally $U_{1t} \geqslant U_{0t}$, for $0 \leqslant t \leqslant T$. $\square$

EC.5. Proof of Corollary EC.2

Proof: We prove the corollary by contradiction. Suppose we can find that $\dot{U}_{kt} = 0$ for $0 \leqslant t \leqslant T$. Then we must have $U_{kt} = 0$ for $0 \leqslant t \leqslant T$ because the terminal condition requires that $U_{kT} = 0$. This would further imply

$$\lambda\mu \ln\left\{1 + \left[\exp\left(\frac{a - U_{kt} + U_{(k-1)t}}{\mu\gamma}\right) + \exp\left(\frac{a - f_t}{\mu\gamma}\right)\right]^{\gamma}\right\} = 0, \text{ if } k = 1, \ldots, \bar{k},$$

$$\lambda\mu \ln\left\{1 + \left[\exp\left(\frac{a - p_t + U_{\bar{k}t} - U_{0t}}{\mu\gamma}\right) + \exp\left(\frac{a - f_t}{\mu\gamma}\right)\right]^{\gamma}\right\} = 0, \text{ if } k = 0.$$

Since both of the above equations are impossible for finite prices, we obtain a contradiction. Therefore, it is impossible to have $\dot{U}_{kt}=0$ for all $0\leqslant t\leqslant T$. An exception is the limiting case of $\rho\to\infty$, which represents the myopic-customer scenario with all utility being zero.

When $\rho=0$, the utility state equations (2) become

$$\dot{U}_{kt}=-\lambda\mu\ln\left\{1+\left[\exp\left(\frac{a-U_{kt}+U_{(k-1)t}}{\mu\gamma}\right)+\exp\left(\frac{a-f_t}{\mu\gamma}\right)\right]^{\gamma}\right\},\; k=1,\ldots,\bar{k},$$
$$\dot{U}_{0t}=-\lambda\mu\ln\left\{1+\left[\exp\left(\frac{a-p_t+U_{\bar{k}t}-U_{0t}}{\mu\gamma}\right)+\exp\left(\frac{a-f_t}{\mu\gamma}\right)\right]^{\gamma}\right\}.$$

It is clear that the left-hand sides of above equations are both negative, therefore we have shown that $\dot{U}_{kt}<0$, for $k=0,1,\ldots,\bar{k}$ and $0\leqslant t\leqslant T$. □

EC.6. Proof of Corollary EC.3

Proof: We consider the problem in the reversed time frame $t'=T-t$. Then, similar to the proof of Corollary EC.1, we have $U^i_{kt}=U^i_{k(T-t')}$, for $i=1,2$, $f_t=f_{T-t'}$, and $p_t=p_{T-t'}$. The time-reversed utility paths are defined as $V^i_{kt'}\triangleq U^i_{k(T-t')}$ with initial values $V^i_{k0}=0$, for $k=0,1$ and $i=1,2$. The time-reversed price path are $f'_{t'}\triangleq f_{T-t'}$ and $p'_{t'}\triangleq p_{T-t'}$, respectively.

We define $\Delta V^1_{t'}\triangleq V^1_{1t'}-V^1_{0t'}$ and $\Delta V^2_{t'}\triangleq V^2_{1t'}-V^2_{0t'}$, then by Theorem 1, we have

$$\begin{aligned}\frac{d\Delta V^1_{t'}}{dt'}&=\lambda\mu\ln\frac{1+\left[\exp\left(\frac{a-\Delta V^1_{t'}}{\mu\gamma}\right)+\exp\left(\frac{a-f'_{t'}}{\mu\gamma}\right)\right]^{\gamma}}{1+\left[\exp\left(\frac{a-p'_{t'}+\Delta V^1_{t'}}{\mu\gamma}\right)+\exp\left(\frac{a-f'_{t'}}{\mu\gamma}\right)\right]^{\gamma}}-\rho_1\Delta V^1_{t'}\\&\leqslant\lambda\mu\ln\frac{1+\left[\exp\left(\frac{a-\Delta V^1_{t'}}{\mu\gamma}\right)+\exp\left(\frac{a-f'_{t'}}{\mu\gamma}\right)\right]^{\gamma}}{1+\left[\exp\left(\frac{a-p'_{t'}+\Delta V^1_{t'}}{\mu\gamma}\right)+\exp\left(\frac{a-f'_{t'}}{\mu\gamma}\right)\right]^{\gamma}}-\rho_2\Delta V^1_{t'},\end{aligned}\tag{EC.11}$$

the inequality follows from $\rho_1\geqslant\rho_2$ and $\Delta V^1_{t'}\geqslant 0$ (Corollary EC.1 implies that $\Delta U^1_t\geqslant 0$, which then implies $\Delta V^1_{t'}\geqslant 0$). Next, we will derive an upper-bound solution for $\Delta V^1_{t'}$. We use the RHS of the inequality in (EC.11) to construct a new differential equation as follows

$$\frac{d\Delta V_{t'}}{dt'}=\lambda\mu\ln\frac{1+\left[\exp\left(\frac{a-\Delta V_{t'}}{\mu\gamma}\right)+\exp\left(\frac{a-f'_{t'}}{\mu\gamma}\right)\right]^{\gamma}}{1+\left[\exp\left(\frac{a-p'_{t'}+\Delta V_{t'}}{\mu\gamma}\right)+\exp\left(\frac{a-f'_{t'}}{\mu\gamma}\right)\right]^{\gamma}}-\rho_2\Delta V_{t'},\tag{EC.12}$$

with initial value $\Delta V_0=0$. Then, we must have $\Delta V^1_{t'}\leqslant\Delta V_{t'}$. That is, $\Delta V_{t'}$ is an upper bound for $\Delta V^1_{t'}$. One can further recognize that (EC.12) is actually the differential equation that determines $\Delta V^2_{t'}$. Therefore, $\Delta V_{t'}=\Delta V^2_{t'}$. Now we have $\Delta V^1_{t'}\leqslant\Delta V^2_{t'}$, which suggests that $V^1_{1t'}-V^1_{0t'}\leqslant V^2_{1t'}-V^2_{0t'}$ and finally $U^1_{1t}-U^1_{0t}\leqslant U^2_{1t}-U^2_{0t}$, thereby completing the proof. □

EC.7. The Optimality Conditions

We apply Pontryagin's maximum principle to derive the necessary conditions for the optimal pricing policy (Sethi and Thompson 2000). The control variables are prices, f_t and p_t; the state variables are population distribution over credits, w_{kt}, and the expected utilities, U_{kt}. According to the classical interpretation, the Hamiltonian function combines the direct instantaneous contribution to the objective with the indirect contributions of the state variables:

$$
\begin{aligned}
H =& \lambda(p_t-c)w_{0t}\pi^p_{0t} + \sum_{k=0}^{\bar{k}} \lambda(f_t-c)w_{kt}\pi^s_{kt} - \sum_{k=1}^{\bar{k}} \lambda c w_{kt}\pi^r_{kt} \\
&+ [\lambda\mu\ln(1-\pi^r_{1t}-\pi^s_{1t}) - \lambda\mu\ln(1-\pi^p_{0t}-\pi^s_{0t}) + \rho\Delta U_{1t}]\eta^u_{1t} \\
&+ \sum_{k=2}^{\bar{k}} [\lambda\mu\ln(1-\pi^r_{kt}-\pi^s_{kt}) - \lambda\mu\ln(1-\pi^r_{(k-1)t}-\pi^s_{(k-1)t}) + \rho\Delta U_{kt}]\eta^u_{kt} \\
&+ (\lambda w_{1t}\pi^r_{1t} - \lambda w_{0t}\pi^p_0)\eta^w_{0t} + \sum_{k=1}^{\bar{k}-1} (\lambda w_{(k+1)t}\pi^r_{(k+1)t} - \lambda w_{kt}\pi^r_{kt})\eta^w_{kt} + (\lambda w_{0t}\pi^p_{0t} - \lambda w_{\bar{k}t}\pi^r_{\bar{k}t})\eta^w_{\bar{k}t},
\end{aligned}
$$

where the adjoint variables η^w_{kt} and η^u_{kt} are interpreted as the shadow prices for the population distribution and utility, respectively. The adjoint variables η^w_{kt} satisfy the following adjoint equations:

$$
\begin{aligned}
\dot{\eta}^w_{0t} &= -\frac{\partial H}{\partial w_{0t}} = -\lambda\big\{p_t\pi^p_{0t} + f_t\pi^s_{0t} - \pi^p_{0t}(\eta^w_{0t} - \eta^w_{\bar{k}t}) - (\pi^s_{0t} + \pi^p_{0t})c\big\}, \\
\dot{\eta}^w_{kt} &= -\frac{\partial H}{\partial w_{kt}} = -\lambda\big\{f_t\pi^s_{kt} + \pi^r_{kt}(\eta^w_{(k-1)t} - \eta^w_{kt}) - (\pi^s_{kt} + \pi^r_{kt})c\big\}, \quad k = 1, \ldots, \bar{k}.
\end{aligned}
$$

The terminal values $\eta^w_{kT} = 0, k = 0, \ldots, \bar{k}$ imply that the population distribution has no indirect effect on the objective at the end of the sales horizon. From this point on, we introduce new variables for the credit's marginal shadow price, $\Delta\eta^w_{kt} \triangleq \eta^w_{(k-1)t} - \eta^w_{kt}$, for $k = 1, \ldots, \bar{k}$, and let $\Delta\eta^w_t \triangleq \sum_{k=1}^{\bar{k}} \Delta\eta^w_{kt} = \eta^w_{0t} - \eta^w_{\bar{k}t}$. Then, the new variables satisfy the following equations:

$$
\begin{aligned}
\Delta\dot{\eta}^w_{1t} &= -\lambda\Big\{\pi^p_{0t}(p_t - \sum_{k=1}^{\bar{k}} \Delta\eta^w_{kt} - c) + \pi^s_{0t}(f_t - c) - [\pi^s_{1t}(f_t - c) + \pi^r_{1t}(\Delta\eta^w_{1t} - c)]\Big\}, \qquad \text{(EC.13)} \\
\Delta\dot{\eta}^w_{kt} &= -\lambda\Big\{\pi^s_{(k-1)t}(f_t - c) + \pi^r_{(k-1)t}(\Delta\eta^w_{(k-1)t} - c) - [\pi^s_{kt}(f_t - c) + \pi^r_{kt}(\Delta\eta^w_{kt} - c)]\Big\}, \quad k = 2, \ldots, \bar{k},
\end{aligned}
$$

with terminal values $\Delta\eta^w_{kT} = 0, k = 0, \ldots, \bar{k}$. To make the presentation more concise, let us define

$$
\phi^\gamma_{0t} \triangleq \frac{\gamma(1-\pi^p_{0t}-\pi^s_{0t}) - 1}{\pi^p_{0t} + \pi^s_{0t}}, \quad \phi^\gamma_{kt} \triangleq \frac{\gamma(1-\pi^r_{kt}-\pi^s_{kt}) - 1}{\pi^r_{kt} + \pi^s_{kt}}, \; k = 1, \ldots, \bar{k}.
$$

Then, the adjoint variables η^u_{kt} for marginal utilities satisfy the adjoint equations:

$$
\begin{aligned}
\dot{\eta}^u_{kt} = -\frac{\partial H}{\partial \Delta U_{kt}} =& -\lambda w_{0t}\Big\{\frac{\pi^p_{0t}}{\mu\gamma}(1 + \pi^p_{0t}\phi^\gamma_{0t})(p_t - \sum_{k=1}^{\bar{k}} \Delta\eta^w_{kt} - c) + \frac{\pi^p_{0t}\pi^s_{0t}}{\mu\gamma}\phi^\gamma_{0t}(f_t - c)\Big\} \\
&- \lambda w_{kt}\Big\{-\frac{\pi^r_{kt}\pi^s_{kt}}{\mu\gamma}\phi^\gamma_{kt}(f_t - c) - \frac{\pi^r_{kt}}{\mu\gamma}(1 + \pi^r_{kt}\phi^\gamma_{kt})(\Delta\eta^w_{kt} - c)\Big\} \\
&- \lambda\pi^p_{0t}\eta^u_{1t} - \rho\eta^u_{kt} + \begin{cases} \lambda(\eta^u_{(k+1)t} - \eta^u_{kt})\pi^r_{kt}, & k = 1, \ldots, \bar{k}-1. \\ -\lambda\eta^u_{\bar{k}t}\pi^r_{\bar{k}t}, & k = \bar{k}, \end{cases} \qquad \text{(EC.14)}
\end{aligned}
$$

with initial values $\eta^u_{k0}=0, k=1,\dots,\bar{k}$. Note that the equation for η^u_{kt} has an initial value, which is opposite to the standard adjoint equation with terminal values. With terminal values, a positive adjoint variable represents a positive contribution to the objective. But with an initial value, a negative adjoint variable represents a positive contribution.

One of the first-order conditions for the Hamiltonian is given by $\partial H/\partial p_t=0$, more specifically,

$$\lambda w_{0t}\Big\{-\frac{\pi^p_{0t}}{\mu\gamma}(1+\pi^p_{0t}\phi^\gamma_{0t})(p_t-\sum_{k=1}^{\bar{k}}\Delta\eta^w_{kt}-c)+\pi^p_{0t}-\frac{\pi^p_{0t}\pi^s_{0t}}{\mu\gamma}\phi^\gamma_{0t}(f_t-c)\Big\}-\lambda\eta^u_{1t}\pi^p_{0t}=0. \tag{EC.15}$$

The other first-order condition is given by $\partial H/\partial f_t=0$, namely,

$$\begin{aligned}&\lambda w_{0t}\Big\{-\frac{\pi^p_{0t}\pi^s_{0t}}{\mu\gamma}\phi^\gamma_{0t}(p_t-\sum_{k=1}^{\bar{k}}\Delta\eta^w_{kt}-c)+\pi^s_{0t}-\frac{\pi^s_{0t}}{\mu\gamma}(1+\pi^s_{0t}\phi^\gamma_{0t})(f_t-c)\Big\}\\&+\lambda\sum_{k=1}^{\bar{k}}w_{kt}\Big\{-\frac{\pi^s_{kt}}{\mu\gamma}(1+\pi^s_{kt}\phi^\gamma_{kt})(f_t-c)+\pi^s_{kt}-\frac{\pi^r_{kt}\pi^s_{kt}}{\mu\gamma}\phi^\gamma_{kt}(\Delta\eta^w_{kt}-c)\Big\}\\&-\lambda\eta^u_{1t}\pi^s_{0t}-\lambda\sum_{k=1}^{\bar{k}-1}(\eta^u_{(k+1)t}-\eta^u_{kt})\pi^s_{kt}+\lambda\eta^u_{\bar{k}t}\pi^s_{\bar{k}t}=0.\end{aligned} \tag{EC.16}$$

The conditions above correspond to the maximization of the combined direct and indirect instantaneous contribution to the objective at every point of the optimal price paths.

Lemma EC.1 **(Optimality Conditions)**. *For any optimal controls (i.e., the pass and individual price paths p_t, f_t) and the corresponding state trajectory (i.e., $w^\star_t, \Delta U^\star_t$) satisfying the state equations* (1), (7)*, and their boundary conditions, there exists an adjoint trajectory (i.e., $\eta^{w\star}_{kt}, \eta^{u\star}_{kt}$) satisfying the adjoint equations* (EC.13), (EC.14)*, together with their boundary conditions, such that, at each t, the values of purchase probabilities deliver the global maximum of the Hamiltonian, with the first-order conditions given by* (EC.15) *and* (EC.16).

The conditions stated in Lemma EC.1 provide a characterization of the optimal prices and the corresponding state and adjoint trajectories. It is a boundary-value problem for a large system of nonlinear differential and algebraic equations. Problems of this size are generally prohibitive to solve analytically and almost always require numerical studies to gain deeper understanding.

Our numerical solution to optimal control problem is based on nonlinear programming (NLP). We use the direct transcription formulation (Section 4.5 of Betts (2010)) for the computation, with the trapezoidal method as the Runge-Kutta scheme. The values of the state and control variables are treated as NLP variables, and the differential equations are replaced by a finite set of constraints. The implementation is performed under AMPL with solver Knitro (ver. 11.1.2).

EC.8. Derivation of Optimality Conditions

We provide some details omitted in Section EC.7 as a supplement. After introducing the new adjoint variables $\Delta\eta^w_{kt}$, the Hamiltonian can be rewritten as the following

$$\begin{aligned}H =&\lambda w_{0t}(p_t\pi^p_{0t}+f_t\pi^s_{0t})+\lambda\sum_{k=1}^{\bar{k}} f_t w_{kt}\pi^s_{kt}+[\lambda\mu\ln(1-\pi^r_{1t}-\pi^s_{1t})-\lambda\mu\ln(1-\pi^p_{0t}-\pi^s_{0t})+\rho\Delta U_{1t}]\eta^u_{1t}\\ &+\sum_{k=2}^{\bar{k}}[\lambda\mu\ln(1-\pi^r_{kt}-\pi^s_{kt})-\lambda\mu\ln(1-\pi^r_{(k-1)t}-\pi^s_{(k-1)t})+\rho\Delta U_{kt}]\eta^u_{kt}\\ &+\lambda\sum_{k=1}^{\bar{k}}(w_{kt}\pi^r_{kt}-w_{0t}\pi^p_{0t})\Delta\eta^w_{kt}-[\lambda w_{0t}(\pi^s_{0t}+\pi^p_{0t})+\lambda\sum_{k=1}^{\bar{k}} w_{kt}(\pi^r_{kt}+\pi^s_{kt})]c,\end{aligned}$$

which can be further simplified as

$$\begin{aligned}H =&\lambda w_{0t}\Big[\pi^p_{0t}(p_t-\sum_{k=1}^{\bar{k}}\Delta\eta^w_{kt}-c)+\pi^s_{0t}(f_t-c)\Big]+\lambda\sum_{k=1}^{\bar{k}} w_{kt}\Big[\pi^s_{kt}(f_t-c)+\pi^r_{kt}(\Delta\eta^w_{kt}-c)\Big]\\ &-\lambda\mu\ln(1-\pi^p_{0t}-\pi^s_{0t})\eta^u_{1t}+\sum_{k=1}^{\bar{k}-1}\Big\{-\lambda\mu\ln(1-\pi^r_{kt}-\pi^s_{kt})(\eta^u_{(k+1)t}-\eta^u_{kt})\Big\}\\ &+\lambda\mu\ln(1-\pi^r_{\bar{k}t}-\pi^s_{\bar{k}t})\eta^u_{\bar{k}t}+\rho\sum_{k=1}^{\bar{k}}\Delta U_{kt}\eta^u_{kt}.\end{aligned}$$

The adjoint variables η^u_{kt} satisfy the adjoint equations:

$$\begin{aligned}\dot{\eta}^u_{kt}=-\frac{\partial H}{\partial\Delta U_{kt}}=&-\lambda w_{0t}\Big[\frac{\partial\pi^p_{0t}}{\partial\Delta U_{kt}}(p_t-\sum_{k=1}^{\bar{k}}\Delta\eta^w_{kt}-c)+\frac{\partial\pi^s_{0t}}{\partial\Delta U_{kt}}(f_t-c)\Big]\\ &-\lambda w_{kt}\Big[\frac{\partial\pi^s_{kt}}{\partial\Delta U_{kt}}(f_t-c)+\frac{\partial\pi^r_{kt}}{\partial\Delta U_{kt}}(\Delta\eta^w_{kt}-c)\Big]\\ &+\lambda\mu\frac{\partial\ln(1-\pi^p_{0t}-\pi^s_{0t})}{\partial\Delta U_{kt}}\eta^u_{1t}-\rho\eta^u_{kt}+\begin{cases}-\lambda\mu\frac{\eta^u_{(k+1)t}-\eta^u_{kt}}{1-\pi^r_{kt}-\pi^s_{kt}}\frac{\partial(\pi^s_{kt}+\pi^r_{kt})}{\partial\Delta U_{kt}}, & k=1,\dots,\bar{k}-1.\\ +\lambda\mu\frac{\eta^u_{\bar{k}t}}{1-\pi^r_{\bar{k}t}-\pi^s_{\bar{k}t}}\frac{\partial(\pi^s_{\bar{k}t}+\pi^r_{\bar{k}t})}{\partial\Delta U_{\bar{k}t}}, & k=\bar{k},\end{cases}\end{aligned}$$

with initial values $\eta^u_{k0}=0, k=1,\dots,\bar{k}$.

The first-order conditions for the Hamiltonian are

$$\frac{\partial H}{\partial p_t}=\lambda w_{0t}\Big\{\frac{\partial\pi^p_{0t}}{\partial p_t}(p_t-\sum_{k=1}^{\bar{k}}\Delta\eta^w_{kt}-c)+\pi^p_{0t}+\frac{\partial\pi^s_{0t}}{\partial p_t}(f_t-c)\Big\}+\lambda\mu\frac{\eta^u_{1t}}{1-\pi^p_{0t}-\pi^s_{0t}}\Big(\frac{\partial\pi^p_{0t}}{\partial p_t}+\frac{\partial\pi^s_{0t}}{\partial p_t}\Big)=0,$$

$$\begin{aligned}\frac{\partial H}{\partial f_t}=&\lambda w_{0t}\Big\{\frac{\partial\pi^p_{0t}}{\partial f_t}(p_t-\sum_{k=1}^{\bar{k}}\Delta\eta^w_{kt}-c)+\pi^s_{0t}+\frac{\partial\pi^s_{0t}}{\partial f_t}(f_t-c)\Big\}+\lambda\sum_{k=1}^{\bar{k}} w_{kt}\Big[\frac{\partial\pi^s_{kt}}{\partial f_t}(f_t-c)+\pi^s_{kt}+\frac{\partial\pi^r_{kt}}{\partial f_t}(\Delta\eta^w_{kt}-c)\Big]\\ &+\lambda\mu\frac{\eta^u_{1t}}{1-\pi^p_{0t}-\pi^s_{0t}}\Big(\frac{\partial\pi^p_{0t}}{\partial f_t}+\frac{\partial\pi^s_{0t}}{\partial f_t}\Big)+\lambda\mu\sum_{k=1}^{\bar{k}-1}\frac{\eta^u_{(k+1)t}-\eta^u_{kt}}{1-\pi^r_{kt}-\pi^s_{kt}}\Big(\frac{\partial\pi^r_{kt}}{\partial f_t}+\frac{\partial\pi^s_{kt}}{\partial f_t}\Big)\\ &-\lambda\mu\frac{\eta^u_{\bar{k}t}}{1-\pi^r_{\bar{k}t}-\pi^s_{\bar{k}t}}\Big(\frac{\partial\pi^r_{\bar{k}t}}{\partial f_t}+\frac{\partial\pi^s_{\bar{k}t}}{\partial f_t}\Big)=0.\end{aligned}$$

The full expression of the derivatives are

$$\begin{aligned}
-\frac{\partial \pi^p_{0t}}{\partial \Delta U_{kt}} = \frac{\partial \pi^p_{0t}}{\partial p_t} &= -\frac{1}{\mu\gamma}\pi^p_{0t}\Big[1+(\gamma-1)\frac{\pi^p_{0t}}{\pi^p_{0t}+\pi^s_{0t}} - \gamma\pi^p_{0t}\Big],\\
\frac{\partial \pi^s_{0t}}{\partial f_t} &= -\frac{1}{\mu\gamma}\pi^s_{0t}\Big[1+(\gamma-1)\frac{\pi^s_{0t}}{\pi^p_{0t}+\pi^s_{0t}} - \gamma\pi^s_{0t}\Big],\\
-\frac{\partial \pi^s_{0t}}{\partial \Delta U_{kt}} = \frac{\partial \pi^s_{0t}}{\partial p_t} = \frac{\partial \pi^p_{0t}}{\partial f_t} &= -\frac{1}{\mu\gamma}\pi^p_{0t}\pi^s_{0t}\Big[\frac{\gamma(1-\pi^p_{0t}-\pi^s_{0t})-1}{\pi^p_{0t}+\pi^s_{0t}}\Big],\\
-\Big(\frac{\partial \pi^p_{0t}}{\partial p_t}+\frac{\partial \pi^p_{0t}}{\partial f_t}\Big) = -\Big(\frac{\partial \pi^p_{0t}}{\partial p_t}+\frac{\partial \pi^s_{0t}}{\partial p_t}\Big) &= \frac{1}{\mu}\pi^p_{0t}(1-\pi^p_{0t}-\pi^s_{0t}),\\
-\Big(\frac{\partial \pi^s_{0t}}{\partial p_t}+\frac{\partial \pi^s_{0t}}{\partial f_t}\Big) = -\Big(\frac{\partial \pi^p_{0t}}{\partial f_t}+\frac{\partial \pi^s_{0t}}{\partial f_t}\Big) &= \frac{1}{\mu}\pi^s_{0t}(1-\pi^p_{0t}-\pi^s_{0t}),
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial \pi^r_{kt}}{\partial \Delta U_{kt}} &= -\frac{1}{\mu\gamma}\pi^r_{kt}\Big[1+(\gamma-1)\frac{\pi^r_{kt}}{\pi^r_{kt}+\pi^s_{kt}} - \gamma\pi^r_{kt}\Big],\\
\frac{\partial \pi^s_{kt}}{\partial f_t} &= -\frac{1}{\mu\gamma}\pi^s_{kt}\Big[1+(\gamma-1)\frac{\pi^s_{kt}}{\pi^r_{kt}+\pi^s_{kt}} - \gamma\pi^s_{kt}\Big],\\
\frac{\partial \pi^s_{kt}}{\partial \Delta U_{kt}} = \frac{\partial \pi^r_{kt}}{\partial f_t} &= -\frac{1}{\mu\gamma}\pi^r_{kt}\pi^s_{kt}\Big[\frac{\gamma(1-\pi^r_{kt}-\pi^s_{kt})-1}{\pi^r_{kt}+\pi^s_{kt}}\Big],\\
\frac{\partial \pi^r_{kt}}{\partial f_t}+\frac{\partial \pi^s_{kt}}{\partial f_t} &= -\frac{1}{\mu}\pi^s_{kt}(1-\pi^r_{kt}-\pi^s_{kt}),\\
\frac{\partial \pi^r_{kt}}{\partial \Delta U_{kt}}+\frac{\partial \pi^s_{kt}}{\partial \Delta U_{kt}} &= -\frac{1}{\mu}\pi^r_{kt}(1-\pi^r_{kt}-\pi^s_{kt}).
\end{aligned}$$

By substituting the derivatives back to the adjoint equations for η^u_{kt}, we obtain (EC.14). Substituting the derivatives to the first-order conditions gives (EC.15) and (EC.16).

EC.9. Turnpike Equations

The turnpike customer-flow equations are obtained by setting the derivatives in (1) to zeros:

$$0 = \lambda\bar{w}_{(k+1)}\bar{\pi}^r_{(k+1)} - \lambda\bar{w}_k\bar{\pi}^r_k, \quad k=1,\dots,\bar{k}-1, \tag{EC.17}$$

$$0 = \lambda\bar{w}_0\bar{\pi}^p_0 - \lambda\bar{w}_{\bar{k}}\bar{\pi}^r_{\bar{k}}, \tag{EC.18}$$

$$0 = \lambda\bar{w}_1\bar{\pi}^r_1 - \lambda\bar{w}_0\bar{\pi}^p_0. \tag{EC.19}$$

The turnpike marginal utility equations are obtained by setting the derivatives in (7) to zeros:

$$\lambda\mu[\ln(1-\bar{\pi}^r_{(k-1)}-\bar{\pi}^s_{(k-1)}) - \ln(1-\bar{\pi}^r_k-\bar{\pi}^s_k)] = \rho\Delta\bar{U}_k,\ k=2,\dots,\bar{k}, \tag{EC.20}$$

$$\lambda\mu[\ln(1-\bar{\pi}^p_0-\bar{\pi}^s_0) - \ln(1-\bar{\pi}^r_1-\bar{\pi}^s_1)] = \rho\Delta\bar{U}_1. \tag{EC.21}$$

By setting the derivatives in the adjoint equations (EC.13) to zeros, we obtain the following turnpike adjoint equations for customer flows:

$$\bar{\pi}^p_0(\bar{p}-\sum_{k=1}^{\bar{k}}\Delta\bar{\eta}^w_k - c) + \bar{\pi}^s_0(\bar{f}-c) = \bar{\pi}^s_k(\bar{f}-c) + \bar{\pi}^r_k(\Delta\bar{\eta}^w_k - c), \quad k=1,\dots,\bar{k}. \tag{EC.22}$$

The steady states of the adjoint equations for utility (EC.14) are given by

$$
\begin{aligned}
0 = &-\lambda\bar{w}_0\bar{\pi}_0^p\frac{1}{\mu\gamma}\Big\{\bar{p}-\sum_{k=1}^{\bar{k}}\Delta\bar{\eta}_k^w - c + \frac{\gamma(1-\bar{\pi}_0^p-\bar{\pi}_0^s)-1}{\bar{\pi}_0^p+\bar{\pi}_0^s}\Big[\bar{\pi}_0^p(\bar{p}-\sum_{k=1}^{\bar{k}}\Delta\bar{\eta}_k^w - c)+\bar{\pi}_0^s(\bar{f}-c)\Big]\Big\}\\
&+\lambda\bar{w}_k\bar{\pi}_k^r\frac{1}{\mu\gamma}\Big\{\Delta\bar{\eta}_k^w - c + \frac{\gamma(1-\bar{\pi}_k^r-\bar{\pi}_k^s)-1}{\bar{\pi}_k^r+\bar{\pi}_k^s}\Big[\bar{\pi}_k^s(\bar{f}-c)+\bar{\pi}_k^r(\Delta\bar{\eta}_k^w-c)\Big]\Big\}\\
&-\lambda\bar{\pi}_0^p\bar{\eta}_1^u - \rho\bar{\eta}_k^u + \begin{cases}\lambda(\bar{\eta}_{(k+1)}^u-\bar{\eta}_k^u)\bar{\pi}_k^r, & k=1,\dots,\bar{k}-1,\\ -\lambda\bar{\eta}_{\bar{k}}^u\bar{\pi}_{\bar{k}}^r, & k=\bar{k}.\end{cases}
\end{aligned}
\tag{EC.23}
$$

One of the turnpike first-order conditions is given by the steady state of (EC.15):

$$
\bar{p}-\sum_{k=1}^{\bar{k}}\Delta\bar{\eta}_k^w - c - \mu\gamma + \frac{\gamma(1-\bar{\pi}_0^p-\bar{\pi}_0^s)-1}{\bar{\pi}_0^p+\bar{\pi}_0^s}\Big[\bar{\pi}_0^p(\bar{p}-\sum_{k=1}^{\bar{k}}\Delta\bar{\eta}_k^w-c)+\bar{\pi}_0^s(\bar{f}-c)\Big]+\mu\gamma\frac{\bar{\eta}_1^u}{\bar{w}_0}=0. \tag{EC.24}
$$

The other turnpike first-order condition is given by the steady state of (EC.16):

$$
\begin{aligned}
&\lambda\frac{1}{\mu\gamma}\bar{w}_0\bar{\pi}_0^s\Big\{\bar{f}-c-\mu\gamma+\frac{\gamma(1-\bar{\pi}_0^p-\bar{\pi}_0^s)-1}{\bar{\pi}_0^p+\bar{\pi}_0^s}\Big[\bar{\pi}_0^p(\bar{p}-\sum_{k=1}^{\bar{k}}\Delta\bar{\eta}_k^w-c)+\bar{\pi}_0^s(\bar{f}-c)\Big]\Big\}\\
&+\lambda\frac{1}{\mu\gamma}\sum_{k=1}^{\bar{k}}\bar{w}_k\bar{\pi}_k^s\Big\{\bar{f}-c-\mu\gamma+\frac{\gamma(1-\bar{\pi}_k^r-\bar{\pi}_k^s)-1}{\bar{\pi}_k^r+\bar{\pi}_k^s}\Big[\bar{\pi}_k^s(\bar{f}-c)+\bar{\pi}_k^r(\Delta\bar{\eta}_k^w-c)\Big]\Big\}\\
=&-\lambda\bar{\eta}_1^u\bar{\pi}_0^s-\lambda\sum_{k=1}^{\bar{k}-1}(\bar{\eta}_{(k+1)}^u-\bar{\eta}_k^u)\bar{\pi}_k^s+\lambda\bar{\eta}_{\bar{k}}^u\bar{\pi}_{\bar{k}}^s.
\end{aligned}
\tag{EC.25}
$$

To complete the system of equations, we also need the following turnpike choice probabilities:

$$
\bar{\pi}_0^p=\frac{\exp\left(\frac{a-\bar{p}+\sum_{k=1}^{\bar{k}}\Delta\bar{U}_k}{\mu\gamma}\right)\left[\exp\left(\frac{a-\bar{p}+\sum_{k=1}^{\bar{k}}\Delta\bar{U}_k}{\mu\gamma}\right)+\exp\left(\frac{a-\bar{f}}{\mu\gamma}\right)\right]^{\gamma-1}}{1+\left[\exp\left(\frac{a-\bar{p}+\sum_{k=1}^{\bar{k}}\Delta\bar{U}_k}{\mu\gamma}\right)+\exp\left(\frac{a-\bar{f}}{\mu\gamma}\right)\right]^{\gamma}}, \tag{EC.26}
$$

$$
\bar{\pi}_0^s=\frac{\exp\left(\frac{a-\bar{f}}{\mu\gamma}\right)\left[\exp\left(\frac{a-\bar{p}+\sum_{k=1}^{\bar{k}}\Delta\bar{U}_k}{\mu\gamma}\right)+\exp\left(\frac{a-\bar{f}}{\mu\gamma}\right)\right]^{\gamma-1}}{1+\left[\exp\left(\frac{a-\bar{p}+\sum_{k=1}^{\bar{k}}\Delta\bar{U}_k}{\mu\gamma}\right)+\exp\left(\frac{a-\bar{f}}{\mu\gamma}\right)\right]^{\gamma}}, \tag{EC.27}
$$

$$
\bar{\pi}_k^r=\frac{\exp\left(\frac{a-\Delta\bar{U}_k}{\mu\gamma}\right)\left[\exp\left(\frac{a-\Delta\bar{U}_k}{\mu\gamma}\right)+\exp\left(\frac{a-\bar{f}}{\mu\gamma}\right)\right]^{\gamma-1}}{1+\left[\exp\left(\frac{a-\Delta\bar{U}_k}{\mu\gamma}\right)+\exp\left(\frac{a-\bar{f}}{\mu\gamma}\right)\right]^{\gamma}}, \tag{EC.28}
$$

$$
\bar{\pi}_k^s=\frac{\exp\left(\frac{a-\bar{f}}{\mu\gamma}\right)\left[\exp\left(\frac{a-\Delta\bar{U}_k}{\mu\gamma}\right)+\exp\left(\frac{a-\bar{f}}{\mu\gamma}\right)\right]^{\gamma-1}}{1+\left[\exp\left(\frac{a-\Delta\bar{U}_k}{\mu\gamma}\right)+\exp\left(\frac{a-\bar{f}}{\mu\gamma}\right)\right]^{\gamma}}. \tag{EC.29}
$$

These turnpike choice probabilities are derived from (3), (4), (5), (6) by replacing the utility paths with the corresponding turnpike marginal utilities. The equations (EC.17) -(EC.29) constitute a closed system that fully characterizes the turnpike.

EC.10. Proof of Lemma 1

Proof: Consider the full expression of the turnpike utility equation (EC.20)

$$\lambda\mu \ln \frac{1+\left[\exp\left(\frac{a-\Delta\bar{U}_k}{\mu\gamma}\right)+\exp\left(\frac{a-\bar{f}}{\mu\gamma}\right)\right]^\gamma}{1+\left[\exp\left(\frac{a-\Delta\bar{U}_{k-1}}{\mu\gamma}\right)+\exp\left(\frac{a-\bar{f}}{\mu\gamma}\right)\right]^\gamma}=\rho\Delta\bar{U}_k,\;\; k=2,\dots,\bar{k}, \tag{EC.30}$$

$$\lambda\mu \ln \frac{1+\left[\exp\left(\frac{a-\Delta\bar{U}_1}{\mu\gamma}\right)+\exp\left(\frac{a-\bar{f}}{\mu\gamma}\right)\right]^\gamma}{1+\left[\exp\left(\frac{a-\bar{p}+\sum_{k=1}^{\bar{k}}\Delta\bar{U}_k}{\mu\gamma}\right)+\exp\left(\frac{a-\bar{f}}{\mu\gamma}\right)\right]^\gamma}=\rho\Delta\bar{U}_1. \tag{EC.31}$$

Note that (EC.30) has a special structure that leads to the following equivalence for all $k=1,\dots,\bar{k}$:

$$\Delta\bar{U}_k > \Delta\bar{U}_{k-1} \Leftrightarrow \Delta\bar{U}_k < 0,$$
$$\Delta\bar{U}_k = \Delta\bar{U}_{k-1} \Leftrightarrow \Delta\bar{U}_k = 0,$$
$$\Delta\bar{U}_k < \Delta\bar{U}_{k-1} \Leftrightarrow \Delta\bar{U}_k > 0.$$

which implies that, for all k, if $\Delta\bar{U}_k<0$, then $\Delta\bar{U}_{k-1}<0$ because $0>\Delta\bar{U}_k>\Delta\bar{U}_{k-1}$. By the same argument, if $\Delta\bar{U}_k=0$, then $\Delta\bar{U}_{k-1}=0$. If $\Delta\bar{U}_k>0$, then $\Delta\bar{U}_{k-1}>0$ since $0<\Delta\bar{U}_k<\Delta\bar{U}_{k-1}$. In summary, the signs of $\Delta\bar{U}_k$'s must be the *same*.

Next, we will prove the lemma by contradiction. Note that signs of $\Delta\bar{U}_k$ are the same for all k, we suppose that $\Delta\bar{U}_k<0, k=1,\dots,\bar{k}$. Note further that (EC.31) suggests that

$$\Delta\bar{U}_1 > \bar{p}-\sum_{k=1}^{\bar{k}}\Delta\bar{U}_k \Leftrightarrow \Delta\bar{U}_1<0.$$

Therefore, we have $0>\Delta\bar{U}_1>\bar{p}-\sum_{k=1}^{\bar{k}}\Delta\bar{U}_k$, which contradicts with $\bar{p}\geqslant 0$ and $\Delta\bar{U}_k<0, k=1,\dots,\bar{k}$. Thus, $\Delta\bar{U}_k\geqslant 0, k=1,\dots,\bar{k}$. □

EC.11. Proof of Proposition 1

Proof: According to (3), (4), (5), (6), one can verify that the following equations hold:

$$\left(\frac{\bar{\pi}_0^p+\bar{\pi}_0^s}{1-\bar{\pi}_0^p-\bar{\pi}_0^s}\right)^{\frac{1}{\gamma}}\frac{\bar{\pi}_0^s}{\bar{\pi}_0^p+\bar{\pi}_0^s}=\left(\frac{\bar{\pi}_k^r+\bar{\pi}_k^s}{1-\bar{\pi}_k^r-\bar{\pi}_k^s}\right)^{\frac{1}{\gamma}}\frac{\bar{\pi}_k^s}{\bar{\pi}_k^r+\bar{\pi}_k^s},\quad k=1,\dots,\bar{k} \tag{EC.32}$$

Since $\Delta\bar{U}_k\geqslant 0$ by Lemma 1, the turnpike equations (EC.20) implies that

$$\begin{aligned}&\bar{\pi}_1^r+\bar{\pi}_1^s\geqslant\bar{\pi}_0^p+\bar{\pi}_0^s\\&\bar{\pi}_k^r+\bar{\pi}_k^s\geqslant\bar{\pi}_{(k-1)}^r+\bar{\pi}_{(k-1)}^s, k=2,\dots,\bar{k}\end{aligned} \tag{EC.33}$$

We observe that the R.H.S. of (EC.32) is invariant w.r.t. k, we have

$$\frac{\bar{\pi}_{(k-1)}^s}{\bar{\pi}_k^s}=\left(\frac{1-\bar{\pi}_{(k-1)}^p-\bar{\pi}_{(k-1)}^s}{1-\bar{\pi}_k^r-\bar{\pi}_k^s}\right)^{\frac{1}{\gamma}}\left(\frac{\bar{\pi}_k^r+\bar{\pi}_k^s}{\bar{\pi}_{(k-1)}^r+\bar{\pi}_{(k-1)}^s}\right)^{\frac{1}{\gamma}-1}\geqslant 1,\quad k=2,\dots,\bar{k} \tag{EC.34}$$

in which the inequality follows from (EC.33) and $0<\gamma\leqslant 1$. Using similar technique, we have

$$\frac{\bar{\pi}_0^s}{\bar{\pi}_1^s}=\left(\frac{1-\bar{\pi}_0^p-\bar{\pi}_0^s}{1-\bar{\pi}_k^r-\bar{\pi}_k^s}\right)^{\frac{1}{\gamma}}\left(\frac{\bar{\pi}_k^r+\bar{\pi}_k^s}{\bar{\pi}_0^r+\bar{\pi}_0^s}\right)^{\frac{1}{\gamma}-1}\geqslant 1. \tag{EC.35}$$

Thus, we have proved that $\bar{\pi}_0^s\geqslant\bar{\pi}_1^s\geqslant\cdots\geqslant\bar{\pi}_{(k-1)}^s\geqslant\bar{\pi}_k^s\geqslant\cdots\geqslant\bar{\pi}_{\bar{k}}^s$, which together with (EC.33) lead to $\bar{\pi}_0^p\leqslant\bar{\pi}_1^r+\bar{\pi}_1^s-\bar{\pi}_0^s\leqslant\bar{\pi}_1^r$ and $\bar{\pi}_{(k-1)}^r\leqslant\bar{\pi}_k^r+\bar{\pi}_k^s-\bar{\pi}_{(k-1)}^s\leqslant\bar{\pi}_k^r$ for $k=2,\ldots,\bar{k}$. Therefore, we have also established the ordering $\bar{\pi}_0^p\leqslant\bar{\pi}_1^r\leqslant\cdots\leqslant\bar{\pi}_{(k-1)}^r\leqslant\bar{\pi}_k^r\leqslant\cdots\leqslant\bar{\pi}_{\bar{k}}^r$. □

EC.12. Proof of Proposition 2

Proof: Consider the redemption probability at the turnpike

$$\bar{\pi}_k^r=\frac{\exp\left(\frac{a-\Delta\bar{U}_k}{\mu\gamma}\right)\left[\exp\left(\frac{a-\Delta\bar{U}_k}{\mu\gamma}\right)+\exp\left(\frac{a-\bar{f}}{\mu\gamma}\right)\right]^{\gamma-1}}{1+\left[\exp\left(\frac{a-\Delta\bar{U}_k}{\mu\gamma}\right)+\exp\left(\frac{a-\bar{f}}{\mu\gamma}\right)\right]^{\gamma}}.$$

Note that the RHS of the above equation is decreasing in $\Delta\bar{U}_k$ because

$$\frac{\partial\bar{\pi}_k^r}{\partial\Delta\bar{U}_k}=\frac{1}{\mu}\frac{\bar{\pi}_k^r}{\bar{\pi}_k^r+\bar{\pi}_k^s}\left[-(1-\bar{\pi}_k^r-\bar{\pi}_k^s)\bar{\pi}_k^r-\frac{\bar{\pi}_k^s}{\gamma}\right]<0.$$

Suppose that $\Delta\bar{U}_{(k+1)}>\Delta\bar{U}_k$, then $\bar{\pi}_{(k+1)}^r<\bar{\pi}_k^r$, which contradicts with Proposition 1. Therefore, we must have $\Delta\bar{U}_{(k+1)}\leqslant\Delta\bar{U}_k$, thereby proving the first part of the proposition. To prove the second part, note that

$$\lambda\mu\ln\frac{1+\left[\exp\left(\frac{a-\Delta\bar{U}_1}{\mu\gamma}\right)+\exp\left(\frac{a-\bar{f}}{\mu\gamma}\right)\right]^{\gamma}}{1+\left[\exp\left(\frac{a-\bar{p}+\sum_{k=1}^{\bar{k}}\Delta\bar{U}_k}{\mu\gamma}\right)+\exp\left(\frac{a-\bar{f}}{\mu\gamma}\right)\right]^{\gamma}}=\rho\Delta\bar{U}_1, \tag{EC.36}$$

and $\Delta\bar{U}_1>0$ by Lemma 1. Therefore, we must have $a-\Delta\bar{U}_1>a-\bar{p}+\sum_{k=1}^{\bar{k}}\Delta\bar{U}_k$, which implies the second part of the proposition. □

EC.13. Proof of Proposition 3

Proof: The proposition follows from the turnpike equation (EC.22).

EC.14. Proof of Proposition 4

Proof: When customers are fully strategic, i.e., $\rho=0$, the turnpike equations for utilities become

$$\lambda\mu[\ln(1-\bar{\pi}_{(k-1)}^r-\bar{\pi}_{(k-1)}^s)-\ln(1-\bar{\pi}_k^r-\bar{\pi}_k^s)]=0,\ k=2,\ldots,\bar{k},$$

$$\lambda\mu[\ln(1-\bar{\pi}_0^p-\bar{\pi}_0^s)-\ln(1-\bar{\pi}_1^r-\bar{\pi}_1^s)]=0,$$

which imply that $\bar{\pi}_0^p + \bar{\pi}_0^s = \bar{\pi}_k^r + \bar{\pi}_k^s$, for $k = 1, \ldots, \bar{k}$. Note further that the choice model (3), (4), (5), (6) give rise to the following equations:

$$\Big(\frac{\bar{\pi}_0^p + \bar{\pi}_0^s}{1 - \bar{\pi}_0^p - \bar{\pi}_0^s}\Big)^{\frac{1}{\gamma}} \frac{\bar{\pi}_0^s}{\bar{\pi}_0^p + \bar{\pi}_0^s} = \Big(\frac{\bar{\pi}_k^r + \bar{\pi}_k^s}{1 - \bar{\pi}_k^r - \bar{\pi}_k^s}\Big)^{\frac{1}{\gamma}} \frac{\bar{\pi}_k^s}{\bar{\pi}_k^r + \bar{\pi}_k^s}, \quad k = 1, \ldots, \bar{k}$$

Therefore, we have $\bar{\pi}_0^s = \bar{\pi}_k^s$, for $k = 1, \ldots, \bar{k}$, which then imply that $\bar{\pi}_0^p = \bar{\pi}_k^r$, for $k = 1, \ldots, \bar{k}$. For simplicity, we will denote $\bar{\pi}^s \triangleq \bar{\pi}_k^s$, for $k = 0, \ldots, \bar{k}$ and $\bar{\pi}^p \triangleq \bar{\pi}_0^p = \bar{\pi}_k^r$, for $k = 1, \ldots, \bar{k}$. It then follows from (EC.22) that $\Delta\bar{\eta}_k^w$ is invariant w.r.t. k (hence we can denote it as $\Delta\bar{\eta}^w$) and

$$\bar{p} = (1 + \bar{k})\Delta\bar{\eta}^w. \tag{EC.37}$$

Note that the probability ratio of pass to individual purchase takes the following form:

$$\frac{\bar{\pi}^p}{\bar{\pi}^s} = \exp\Big(\frac{-\bar{p} + \sum_{k=1}^{\bar{k}} \Delta\bar{U}_k + \bar{f}}{\mu\gamma}\Big). \tag{EC.38}$$

We now combine (EC.22) and (EC.25) to obtain the following result:

$$\bar{f} - c - \mu\gamma + \frac{\gamma(1 - \bar{\pi}^p - \bar{\pi}^s) - 1}{\bar{\pi}^p + \bar{\pi}^s}\alpha = 0. \tag{EC.39}$$

The combination of (EC.22), (EC.24) and (EC.37) leads to

$$\Delta\bar{\eta}^w - c - \mu\gamma + \frac{\gamma(1 - \bar{\pi}^p - \bar{\pi}^s) - 1}{\bar{\pi}^p + \bar{\pi}^s}\alpha = -\mu\gamma\frac{\bar{\eta}_1^u}{\bar{w}_0}. \tag{EC.40}$$

From (EC.23) and (EC.37) we obtain

$$0 = -\lambda\bar{\eta}_1^u + \begin{cases} \lambda(\bar{\eta}_{(k+1)}^u - \bar{\eta}_k^u), & k = 1, \ldots, \bar{k} - 1. \\ -\lambda\bar{\eta}_{\bar{k}}^u, & k = \bar{k}, \end{cases} \tag{EC.41}$$

which implies that $\bar{\eta}_1^u = \bar{\eta}_2^u - \bar{\eta}_1^u = \bar{\eta}_3^u - \bar{\eta}_2^u = \cdots = \bar{\eta}_{\bar{k}}^u - \bar{\eta}_{\bar{k}-1}^u = -\bar{\eta}_{\bar{k}}^u$. That is, $\bar{\eta}_{\bar{k}}^u = k\bar{\eta}_1^u$ and $\bar{\eta}_{\bar{k}}^u = -\bar{\eta}_1^u$. Therefore, $\bar{\eta}_1^u = \bar{\eta}_k^u = 0$ for $k = 2, \ldots, \bar{k}$. Substitute this result back to (EC.40) we obtain

$$\Delta\bar{\eta}^w - c - \mu\gamma + \frac{\gamma(1 - \bar{\pi}^p - \bar{\pi}^s) - 1}{\bar{\pi}^p + \bar{\pi}^s}\alpha = 0. \tag{EC.42}$$

It is obvious from (EC.39) and (EC.42) that $\bar{f} = \Delta\bar{\eta}^w$. Then, by (EC.37) we obtain $\bar{p} = (1 + \bar{k})\bar{f}$.

According to the expressions of choice probabilities and the fact that $\bar{\pi}_0^p + \bar{\pi}_0^s = \bar{\pi}_k^r + \bar{\pi}_k^s$, for $k = 1, \ldots, \bar{k}$, it is easy to see that

$$\bar{p} = \Delta\bar{U}_k + \sum_{k=1}^{\bar{k}} \Delta\bar{U}_k,$$

for all $k = 1, \ldots, \bar{k}$. This suggests that $\Delta\bar{U}_k$ is invariant w.r.t. k and hence can be denoted as $\Delta\bar{U}$. Since we have just shown that $\bar{p} = (1 + \bar{k})\bar{f}$, it is easy to see that $\bar{f} = \Delta\bar{U}$. So, $\bar{p} = (1 + \bar{k})\bar{f} = (1 + \bar{k})\Delta\bar{U}$. Substitute it back to (EC.38), we get $\bar{\pi}_0^s = \bar{\pi}_0^p = \bar{\pi}_k^s = \bar{\pi}_k^r$, for $k = 1, \ldots, \bar{k}$. All choice probabilities are identical. For convenience, we denote all of them as $\bar{\pi}$. An immediate consequence of identical choice probabilities is that $\bar{w}_0 = \bar{w}_k = 1/(1+\bar{k})$, for $k = 1, \ldots, \bar{k}$. Note in (EC.37) that $\bar{p} = (1+\bar{k})\Delta\bar{\eta}^w$. Hence, we have $\Delta\bar{\eta}^w = \Delta\bar{U} = \bar{f}$.

Substituting $\bar{p} = \bar{f} + \bar{k}\Delta\bar{\eta}^w$, $\bar{\pi}_0^s = \bar{\pi}_0^p = \bar{\pi}_k^s = \bar{\pi}_k^r = \bar{\pi}$, and $\bar{\eta}_1^u = 0$ into the equation (EC.24), we obtain

$$(\bar{f} - c)(1 - 2\bar{\pi}) = \frac{\bar{f} - c}{1 + 2^\gamma \exp\left(\frac{a - \bar{f}}{\mu}\right)} = \mu. \tag{EC.43}$$

The solution to the above equation is the optimal turnpike individual price:

$$\bar{f} = \mu + c + \mu\mathbb{W}\left[2^\gamma \exp\left(\frac{a - \mu - c}{\mu}\right)\right],$$

where $\mathbb{W}$ stands for the Lambert-W function. The optimal turnpike price for the pass is given by

$$\bar{p} = (1+\bar{k})\left\{\mu + c + \mu\mathbb{W}\left[2^\gamma \exp\left(\frac{a - \mu - c}{\mu}\right)\right]\right\}.$$

The choice probabilities are given by

$$\bar{\pi}_0^s = \bar{\pi}_0^p = \bar{\pi}_k^s = \bar{\pi}_k^r = \bar{\pi} = \frac{\mathbb{W}\left[2^\gamma \exp\left(\frac{a-\mu-c}{\mu}\right)\right]}{2 + 2\mathbb{W}\left[2^\gamma \exp\left(\frac{a-\mu-c}{\mu}\right)\right]}, \quad k = 1, \ldots, \bar{k}.$$

The full solution is complete. □

EC.15. Proof of Proposition 5

Proof: We first derive the turnpike equations for myopic customers. When customers are myopic ($\rho = \infty$), we have $\bar{U}_0 = \bar{U}_k = 0$, for $k = 1, \ldots, \bar{k}$. The choice probability model (3), (4), (5), (6) imply that $\bar{\pi}_k^s = \bar{\pi}^s$ and $\bar{\pi}_k^r = \bar{\pi}^r$, for $k = 1, \ldots, \bar{k}$. More specifically,

$$\bar{\pi}_0^p = \frac{\exp\left(\frac{a-\bar{p}}{\mu\gamma}\right)\left[\exp\left(\frac{a-\bar{p}}{\mu\gamma}\right) + \exp\left(\frac{a-\bar{f}}{\mu\gamma}\right)\right]^{\gamma-1}}{1 + \left[\exp\left(\frac{a-\bar{p}}{\mu\gamma}\right) + \exp\left(\frac{a-\bar{f}}{\mu\gamma}\right)\right]^\gamma}, \tag{EC.44}$$

$$\bar{\pi}_0^s = \frac{\exp\left(\frac{a-\bar{f}}{\mu\gamma}\right)\left[\exp\left(\frac{a-\bar{p}}{\mu\gamma}\right) + \exp\left(\frac{a-\bar{f}}{\mu\gamma}\right)\right]^{\gamma-1}}{1 + \left[\exp\left(\frac{a-\bar{p}}{\mu\gamma}\right) + \exp\left(\frac{a-\bar{f}}{\mu\gamma}\right)\right]^\gamma} \tag{EC.45}$$

$$\bar{\pi}^r = \frac{\exp\left(\frac{a}{\mu\gamma}\right)\left[\exp\left(\frac{a}{\mu\gamma}\right) + \exp\left(\frac{a-\bar{f}}{\mu\gamma}\right)\right]^{\gamma-1}}{1 + \left[\exp\left(\frac{a}{\mu\gamma}\right) + \exp\left(\frac{a-\bar{f}}{\mu\gamma}\right)\right]^\gamma}, \tag{EC.46}$$

$$\bar{\pi}^s = \frac{\exp\left(\frac{a-\bar{f}}{\mu\gamma}\right)\left[\exp\left(\frac{a}{\mu\gamma}\right) + \exp\left(\frac{a-\bar{f}}{\mu\gamma}\right)\right]^{\gamma-1}}{1 + \left[\exp\left(\frac{a}{\mu\gamma}\right) + \exp\left(\frac{a-\bar{f}}{\mu\gamma}\right)\right]^\gamma}. \tag{EC.47}$$

Therefore, by (EC.17) we obtain $\bar{w}_k = \bar{w}$, for $k = 1, \ldots, \bar{k}$. Further, the turnpike adjoint equation (EC.22) implies that $\Delta\bar{\eta}^w_k = \Delta\bar{\eta}^w$. So (EC.22) can be rewritten as

$$\bar{\pi}^p_0(\bar{p} - \bar{k}\Delta\bar{\eta}^w - c) + \bar{\pi}^s_0(\bar{f} - c) = \bar{\pi}^s(\bar{f} - c) + \bar{\pi}^r(\Delta\bar{\eta}^w - c), \tag{EC.48}$$

Note that price is positive ($\bar{p} > 0$), the expressions for the choice probabilities also suggest that $\bar{\pi}^s_0 > \bar{\pi}^s$, $\bar{\pi}^p_0 < \bar{\pi}^r$ and

$$\bar{\pi}^s_0 + \bar{\pi}^p_0 < \bar{\pi}^s + \bar{\pi}^r. \tag{EC.49}$$

There is no utility equation for myopic customers. The first order condition for the Hamiltonian are

$$\bar{p} - \bar{k}\Delta\bar{\eta}^w - c - \mu\gamma + \frac{\gamma(1 - \bar{\pi}^p_0 - \bar{\pi}^s_0) - 1}{\bar{\pi}^p_0 + \bar{\pi}^s_0}\Big[\bar{\pi}^p_0(\bar{p} - \bar{k}\Delta\bar{\eta}^w - c) + \bar{\pi}^s_0(\bar{f} - c)\Big] = 0, \tag{EC.50}$$

and

$$\begin{aligned} 0 =& \bar{w}_0\bar{\pi}^s_0\Big\{\bar{f} - c - \mu\gamma + \frac{\gamma(1 - \bar{\pi}^p_0 - \bar{\pi}^s_0) - 1}{\bar{\pi}^p_0 + \bar{\pi}^s_0}\Big[\bar{\pi}^p_0(\bar{p} - \bar{k}\Delta\bar{\eta}^w - c) + \bar{\pi}^s_0(\bar{f} - c)\Big]\Big\} \\ &+ (1 - \bar{w}_0)\bar{\pi}^s\Big\{\bar{f} - c - \mu\gamma + \frac{\gamma(1 - \bar{\pi}^r - \bar{\pi}^s) - 1}{\bar{\pi}^r + \bar{\pi}^s}\Big[\bar{\pi}^s(\bar{f} - c) + \bar{\pi}^r(\Delta\bar{\eta}^w - c)\Big]\Big\}. \end{aligned} \tag{EC.51}$$

Now we start proving the proposition. This is essentially solving the myopic turnpike equation (EC.44) -(EC.51) for the case of $\gamma = 1$. First, define

$$\alpha \triangleq \bar{\pi}^p_0(\bar{p} - \bar{k}\Delta\bar{\eta}^w - c) + \bar{\pi}^s_0(\bar{f} - c) = \bar{\pi}^s(\bar{f} - c) + \bar{\pi}^r(\Delta\bar{\eta}^w - c). \tag{EC.52}$$

When $\gamma = 1$, we have $(\gamma(1 - \bar{\pi}^p_0 - \bar{\pi}^s_0) - 1)/(\bar{\pi}^p_0 + \bar{\pi}^s_0) = -1$, then the first-order condition (EC.50) can be reduced to

$$\bar{p} - \bar{k}\Delta\bar{\eta}^w = c + \mu + \alpha. \tag{EC.53}$$

The other first-order condition (EC.51) becomes

$$0 = \bar{w}_0\bar{\pi}^s_0(\bar{f} - c - \mu - \alpha) + (1 - \bar{w}_0)\bar{\pi}^s(\bar{f} - c - \mu - \alpha) \tag{EC.54}$$

Because $\bar{w}_0\bar{\pi}^s_0 + (1 - \bar{w}_0)\bar{\pi}^s > 0$, (EC.54) suggests that

$$\bar{f} = c + \mu + \alpha \tag{EC.55}$$

Combining (EC.53) and (EC.55), we get $\bar{p} = \bar{f} + \bar{k}\Delta\bar{\eta}^w$. Then, substitute (EC.53) and (EC.55) into (EC.52), we have

$$\alpha \triangleq (\bar{\pi}^p_0 + \bar{\pi}^s_0)(\mu + \alpha) = \bar{\pi}^s(\mu + \alpha) + \bar{\pi}^r(\Delta\bar{\eta}^w - c),$$

from which we obtain

$$\alpha = \frac{(\bar{\pi}_0^p + \bar{\pi}_0^s)\mu}{1-\bar{\pi}_0^p-\bar{\pi}_0^s},$$
$$\Delta\bar{\eta}^w = \frac{(1-\bar{\pi}^s)(\bar{\pi}_0^p+\bar{\pi}_0^s)\mu}{\bar{\pi}^r(1-\bar{\pi}_0^p-\bar{\pi}_0^s)} - \mu\frac{\bar{\pi}^s}{\bar{\pi}^r} + c = \mu\Big(e^{-\frac{\bar{p}}{\mu}} + e^{\frac{a-\bar{p}}{\mu}} + e^{\frac{a-\bar{f}}{\mu}}\Big) + c > 0. \tag{EC.56}$$

Because we have just shown that $\bar{p} = \bar{f} + \bar{k}\Delta\bar{\eta}^w$, it is obvious that $\bar{p} > \bar{f}$ and

$$\frac{\bar{p}-\bar{f}}{\bar{k}} = \mu\Big(e^{-\frac{\bar{p}}{\mu}} + e^{\frac{a-\bar{p}}{\mu}} + e^{\frac{a-\bar{f}}{\mu}}\Big) + c \tag{EC.57}$$

Substitute $\bar{p} = \bar{f} + \bar{k}\Delta\bar{\eta}^w$ into (EC.50), we obtain $(1-\bar{\pi}_0^p-\bar{\pi}_0^s)(\bar{f}-c) = \mu$, or

$$\bar{f} = \mu\Big(1 + e^{\frac{a-\bar{p}}{\mu}} + e^{\frac{a-\bar{f}}{\mu}}\Big) + c. \tag{EC.58}$$

Subtracting (EC.57) from (EC.58), we get

$$\bar{p} = (1+\bar{k})\bar{f} - \mu\bar{k}[1-\exp(-\bar{p}/\mu)]. \tag{EC.59}$$

Thus we have obtained two equations, (EC.58) and (EC.59), that fully specify $\bar{f}$ and $\bar{p}$. Since $\bar{p} > 0$, it is clear that $\bar{p} < (1+\bar{k})\bar{f}$. Because $\bar{p} > 0$, we have $\exp(-\bar{p}/\mu) < 1$, then by (EC.56) and (EC.58) we have $\Delta\bar{\eta}^w < \bar{f}$.

By Proposition 5, we can express $\bar{p}$ in terms of $\bar{f}$ as the following

$$\bar{p} = \mu\mathbb{W}\Big[\bar{k}\exp\Big(-\frac{\bar{k}(\bar{f}-\mu)+\bar{f}}{\mu}\Big)\Big] + \bar{k}(\bar{f}-\mu) + \bar{f}, \tag{EC.60}$$

where $\mathbb{W}$ is the Lambert-W function. Note that $\bar{f} > \mu$, again by Proposition 5, then it can be verified by differentiation that the RHS of (EC.60) is increasing in both $\bar{k}$ and $\bar{f}$.

Next, consider the second equation given in Proposition 5,

$$\bar{f} = \mu\big(1 + \exp[(a-\bar{p})/\mu] + \exp[(a-\bar{f})/\mu]\big) + c.$$

The LHS of the above equation is clearly increasing in $\bar{f}$. By (EC.60), we also know that $\bar{p}$ is increasing in both $\bar{k}$ and $\bar{f}$, so the RHS of the above equation is decreasing in both $\bar{k}$ and $\bar{f}$. Hence, the solution to the above equation is unique and it is decreasing in $\bar{k}$. However, the RHS of the above equation is increasing in both a and c, its solution therefore is also increasing in a and c.

When $\gamma \to 0$ and $\bar{k} = 1$, the closed-form solution comes from Section EC.18. Since $\rho \to \infty$, we have $\Delta\bar{U}_1 \to 0$, therefore (EC.68) becomes

$$\Delta\bar{\eta}_1^w = \mu\frac{1+\exp(\frac{a}{\mu})}{\exp(\frac{a}{\mu})}\mathbb{W}\big[\exp\big(\frac{a-c-\mu}{\mu}\big)\big] + c.$$

Because $\bar{f} = c + \mu + \mu\mathbb{W}(\exp[(a-c-\mu)/\mu])$ and $\bar{p} = \Delta\bar{\eta}_1^w + \bar{f}$, we have

$$\begin{aligned}\bar{p} &= c + \mu + \mu\mathbb{W}(\exp[(a-c-\mu)/\mu]) + \mu\frac{1+\exp(\frac{a}{\mu})}{\exp(\frac{a}{\mu})}\mathbb{W}\big[\exp\big(\frac{a-c-\mu}{\mu}\big)\big] + c \\ &= 2c + \mu + \mu[2+\exp(-a/\mu)]\mathbb{W}(\exp[(a-c-\mu)/\mu]).\end{aligned}$$

Note further that $\bar{p} - 2\bar{f} = \mu\{e^{-a/\mu}\mathbb{W}(\exp[(a-c-\mu)/\mu]) - 1\}$. Because

$$e^{-a/\mu}\mathbb{W}(\exp[(a-c-\mu)/\mu]) < \exp(-a/\mu)\mathbb{W}[\exp(a/\mu)] < 1,$$

where the last inequality follows by definition, therefore $\bar{p} - 2\bar{f} < 0$, completing the proof. □

EC.16. Proof of Proposition 6

Proof: By Proposition 4, $\bar{f}_0$ is the solution to the following equation when $\gamma = 1$:

$$\bar{f} - c = \mu\big[1 + 2\exp\big(\frac{a-\bar{f}}{\mu}\big)\big]. \tag{EC.61}$$

By Proposition 5, $\bar{f}_\infty$ is the solution to the following equation

$$\bar{f} - c = \mu\Big[1 + \exp\big(\frac{a-\bar{p}}{\mu}\big) + \exp\big(\frac{a-\bar{f}}{\mu}\big)\Big]. \tag{EC.62}$$

Because (EC.60) and Proposition 5 suggests that $\bar{p}_\infty > \bar{f}_\infty$, one can easily see that the RHS of (EC.62) is less than the RHS of (EC.61). Therefore, $\bar{f}_\infty < \bar{f}_0$, for the case of $\gamma = 1$. Note further that $\bar{p}_0 = (1+\bar{k})\bar{f}_0$ and

$$\bar{p}_\infty = (1+\bar{k})\bar{f}_\infty - \mu\bar{k}[1-\exp(-\bar{p}_\infty/\mu)] < (1+\bar{k})\bar{f}_\infty < (1+\bar{k})\bar{f}_0 = \bar{p}_0.$$

Thus, we have proved part 1 of the proposition.

Now we prove part 2. By Proposition 4 and Proposition 5, the expressions of all choice probabilities, when $\gamma = 1$, are

$$\bar{\pi}_0^p = \frac{\exp\left(\frac{a-\bar{f}_0}{\mu}\right)}{1+\exp\left(\frac{a-\bar{f}_0}{\mu}\right)+\exp\left(\frac{a-\bar{f}_0}{\mu}\right)}, \qquad \bar{\pi}_\infty^p = \frac{\exp\left(\frac{a-\bar{p}_\infty}{\mu}\right)}{1+\exp\left(\frac{a-\bar{f}_\infty}{\mu}\right)+\exp\left(\frac{a-\bar{p}_\infty}{\mu}\right)},$$

$$\bar{\pi}_0^s = \frac{\exp\left(\frac{a-\bar{f}_0}{\mu}\right)}{1+\exp\left(\frac{a-\bar{f}_0}{\mu}\right)+\exp\left(\frac{a-\bar{f}_0}{\mu}\right)}, \qquad \bar{\pi}_\infty^s = \frac{\exp\left(\frac{a-\bar{f}_\infty}{\mu}\right)}{1+\exp\left(\frac{a-\bar{f}_\infty}{\mu}\right)+\exp\left(\frac{a-\bar{p}_\infty}{\mu}\right)},$$

$$\bar{\pi}_{k,0}^s = \frac{\exp\left(\frac{a-\bar{f}_0}{\mu}\right)}{1+\exp\left(\frac{a-\bar{f}_0}{\mu}\right)+\exp\left(\frac{a-\bar{f}_0}{\mu}\right)}, \qquad \bar{\pi}_{k,\infty}^s = \frac{\exp\left(\frac{a-\bar{f}_\infty}{\mu}\right)}{1+\exp\left(\frac{a-\bar{f}_\infty}{\mu}\right)+\exp\left(\frac{a}{\mu}\right)},$$

$$\bar{\pi}_{k,0}^r = \frac{\exp\left(\frac{a-\bar{f}_0}{\mu}\right)}{1+\exp\left(\frac{a-\bar{f}_0}{\mu}\right)+\exp\left(\frac{a-\bar{f}_0}{\mu}\right)}, \qquad \bar{\pi}_{k,\infty}^r = \frac{\exp\left(\frac{a}{\mu}\right)}{1+\exp\left(\frac{a-\bar{f}_\infty}{\mu}\right)+\exp\left(\frac{a}{\mu}\right)}.$$

Note further that

$$\bar{f}_0 - c = \mu\Big\{1 + \exp\big(\frac{a-\bar{f}_0}{\mu}\big) + \exp\big(\frac{a-\bar{f}_0}{\mu}\big)\Big\}$$
$$\bar{f}_\infty - c = \mu\Big\{1 + \exp\big(\frac{a-\bar{f}_\infty}{\mu}\big) + \exp\big(\frac{a-\bar{p}_\infty}{\mu}\big)\Big\}.$$

Because $\bar{f}_0 > \bar{f}_\infty$, we have $\exp\big(\frac{a-\bar{f}_0}{\mu}\big) < \exp\big(\frac{a-\bar{f}_\infty}{\mu}\big)$ and $1 + \exp\big(\frac{a-\bar{f}_0}{\mu}\big) + \exp\big(\frac{a-\bar{f}_0}{\mu}\big) > 1 + \exp\big(\frac{a-\bar{f}_\infty}{\mu}\big) + \exp\big(\frac{a-\bar{p}_\infty}{\mu}\big)$. Then, it is clear that $\bar{\pi}^s_0 < \bar{\pi}^s_\infty$.

Since $1 + \exp\big(\frac{a-\bar{f}_0}{\mu}\big) + \exp\big(\frac{a-\bar{f}_0}{\mu}\big) = 1/(1-\bar{\pi}^p_0-\bar{\pi}^s_0)$ and $1 + \exp\big(\frac{a-\bar{f}_\infty}{\mu}\big) + \exp\big(\frac{a-\bar{p}_\infty}{\mu}\big) = 1/(1-\bar{\pi}^p_\infty-\bar{\pi}^s_\infty)$, we have $\pi^p_0 + \bar{\pi}^s_0 > \bar{\pi}^p_\infty + \bar{\pi}^s_\infty$. Note further that $\pi^p_0 = \bar{\pi}^s_0$ and $\bar{\pi}^p_\infty < \bar{\pi}^s_\infty$ (Proposition 5 suggests that $\bar{p}_\infty > \bar{f}_\infty$). Hence, $\pi^p_0 = \frac{1}{2}(\pi^p_0 + \bar{\pi}^s_0) > \frac{1}{2}(\bar{\pi}^p_\infty + \bar{\pi}^s_\infty) > \bar{\pi}^p_\infty$. Because $\exp\big(\frac{a-\bar{f}_0}{\mu}\big) < \exp\big(\frac{a}{\mu}\big)$ and $1 + \exp\big(\frac{a-\bar{f}_0}{\mu}\big) + \exp\big(\frac{a-\bar{f}_0}{\mu}\big) > 1 + \exp\big(\frac{a-\bar{f}_\infty}{\mu}\big) + \exp\big(\frac{a-\bar{p}_\infty}{\mu}\big)$, we have $\bar{\pi}^r_{k,0} < \bar{\pi}^r_{k,\infty}$. □

EC.17. Relation between the choice probabilities and marginal utilities

We first derive the partial derivatives of choice probabilities w.r.t. marginal utilities as follows:

$$\partial\bar{\pi}^p_0/\partial\Delta\bar{U}_k = \bar{\pi}^p_0[(1-\bar{\pi}^p_0-\bar{\pi}^s_0)\bar{\pi}^p_0 + \bar{\pi}^s_0/\gamma]/[\mu(\bar{\pi}^p_0+\bar{\pi}^s_0)] > 0,$$

$$\partial\bar{\pi}^s_0/\partial\Delta\bar{U}_k = \bar{\pi}^p_0[(1-\bar{\pi}^p_0-\bar{\pi}^s_0)\bar{\pi}^s_0 - \bar{\pi}^s_0/\gamma]/[\mu(\bar{\pi}^p_0+\bar{\pi}^s_0)] < 0,$$

$$\partial\bar{\pi}^r_k/\partial\Delta\bar{U}_k = -\bar{\pi}^r_k[(1-\bar{\pi}^r_k-\bar{\pi}^s_k)\bar{\pi}^r_k + \bar{\pi}^s_k/\gamma]/[\mu(\bar{\pi}^r_k+\bar{\pi}^s_k)] < 0,$$

$$\partial\bar{\pi}^s_k/\partial\Delta\bar{U}_k = \bar{\pi}^r_k[\bar{\pi}^s_k/\gamma - (1-\bar{\pi}^r_k-\bar{\pi}^s_k)\bar{\pi}^s_k]/[\mu(\bar{\pi}^r_k+\bar{\pi}^s_k)] > 0.$$

For a non-passholder customer, a pass will be more appealing if the credit marginal value $\Delta\bar{U}_k$ increases. This is because $\partial\bar{\pi}^p_0/\partial\Delta\bar{U}_k > 0$. As a result, she will be less likely to make individual purchases, as evidenced by $\partial\bar{\pi}^s_0/\partial\Delta\bar{U}_k < 0$. This means that some of the individual demand will be shifted toward passes. However, the situation is drastically different for a passholder with $k>0$ remaining credits. Since a larger $\Delta\bar{U}_k$ suggests that the credit k has a higher future value, she will hesitate to redeem it and tend to make individual purchases instead, as reflected by $\partial\bar{\pi}^r_k/\partial\Delta\bar{U}_k < 0$, and $\partial\bar{\pi}^s_k/\partial\Delta\bar{U}_k > 0$. In this situation, the demand shifts in the opposite direction: from passes to individual items. We can see that marginal utilities can have a counteracting impact on the demand, depending on whether the customers have passes.

EC.18. The limiting case of $\gamma \to 0$ (for $\bar{k} = 1$)

We first consider the case of fully strategic customers ($\rho = 0$), it is obvious from Proposition 4 that, as $\gamma \to 0$, the turnpike has closed form solution:

1. $\bar{f} = \mu + c + \mu\mathbb{W}\big[\exp\big(\frac{a-\mu-c}{\mu}\big)\big]$, $\bar{p} = (1+\bar{k})\bar{f}$.
2. $\bar{\pi}^s_0 = \bar{\pi}^p_0 = \bar{\pi}^s_k = \bar{\pi}^r_k = \mathbb{W}\big[\exp\big(\frac{a-\mu-c}{\mu}\big)\big]/\big(2+2\mathbb{W}\big[\exp\big(\frac{a-\mu-c}{\mu}\big)\big]\big)$, $\quad k = 1,\ldots,\bar{k}$.

3. $\bar{w}_0 = \bar{w}_k = 1/(1+\bar{k}),\ k = 1,\dots,\bar{k}$.
4. $\Delta\bar{U}_k = \Delta\bar{\eta}^w_k = \bar{f}, \bar{\eta}^u_k = 0, \quad k = 1,\dots,\bar{k}$.

where $\mathbb{W}$ stands for the Lambert-W function.

Next, we consider the case of partially strategic customers ($\rho > 0$). When $\gamma \to 0$ and $\rho > 0$, we observe from (EC.26)-(EC.29) that the following four cases may emerge: Case I: $\bar{\pi}^p_0 \to 0, \bar{\pi}^s_0 \to \exp(\frac{a-\bar{f}}{\mu})/[1+\exp(\frac{a-\bar{f}}{\mu})],\ \bar{\pi}^s_1 \to 0, \bar{\pi}^r_1 \to \exp(\frac{a-\Delta\bar{U}_1}{\mu})/[1+\exp(\frac{a-\Delta\bar{U}_1}{\mu})]$. Case II: $\bar{\pi}^p_0 \to 0, \bar{\pi}^s_0 \to \exp(\frac{a-\bar{f}}{\mu})/[1+\exp(\frac{a-\bar{f}}{\mu})],\ \bar{\pi}^s_1 \to \exp(\frac{a-\bar{f}}{\mu})/[1+\exp(\frac{a-\bar{f}}{\mu})], \bar{\pi}^r_1 \to 0$. Case III: $\bar{\pi}^p_0 \to \exp(\frac{a-\bar{p}+\Delta\bar{U}_1}{\mu})/[1+\exp(\frac{a-\bar{p}+\Delta\bar{U}_1}{\mu})], \bar{\pi}^s_0 \to 0,\ \bar{\pi}^s_1 \to 0, \bar{\pi}^r_1 \to \exp(\frac{a-\Delta\bar{U}_1}{\mu})/[1+\exp(\frac{a-\Delta\bar{U}_1}{\mu})]$. Case IV: $\bar{\pi}^p_0 \to \exp(\frac{a-\bar{p}+\Delta\bar{U}_1}{\mu})/[1+\exp(\frac{a-\bar{p}+\Delta\bar{U}_1}{\mu})], \bar{\pi}^s_0 \to 0,\ \bar{\pi}^s_1 \to \exp(\frac{a-\bar{f}}{\mu})/[1+\exp(\frac{a-\bar{f}}{\mu})], \bar{\pi}^r_1 \to 0$. We first rule out Case II and Case IV from the optimality, and then show that Case III has a lower turnpike profit rate than Case I. In Case II, note that $\bar{\pi}^s_0 = \bar{\pi}^s_1$, then by (EC.20) we have $\Delta\bar{U}_1 = 0$, which contradicts $\Delta\bar{U}_1 > 0$. Therefore, Case II cannot be optimal. Case IV implies that $\bar{p} - \Delta\bar{U}_1 < \bar{f}$ and $\bar{f} < \Delta\bar{U}_1$, which lead to $\bar{p} > 2\Delta\bar{U}_1$, contradicting Proposition 2. Therefore, Case IV also cannot be optimal. We now compare the profit rate of Case I and Case III.

$$
\begin{aligned}
\text{Profit rate of Case III} &\triangleq \max_{\bar{p}} \lambda\bar{w}_0\bar{\pi}^p_0(\bar{p}-c) - \lambda\bar{w}_1\bar{\pi}^r_1 c = \max_{\bar{p}} \lambda\bar{w}_0\bar{\pi}^p_0(\bar{p}-2c)\\
&= \max_{\bar{p}} \lambda\frac{\bar{p}-2c}{\frac{1}{\bar{\pi}^r_1}+\frac{1}{\bar{\pi}^p_0}} = \max_{\bar{p}} \lambda\frac{\bar{p}-2c}{2+\exp(\frac{\Delta\bar{U}_1-a}{\mu})+\exp(\frac{p-a-\Delta\bar{U}_1}{\mu})}\\
&\leqslant \max_{p} \lambda\frac{\bar{p}-2c}{2+\exp(\frac{\bar{p}-2a}{2\mu})} = \max_{\frac{\bar{p}}{2}} \lambda\frac{\frac{\bar{p}}{2}-c}{1+\exp(\frac{\frac{\bar{p}}{2}-a}{\mu})}\\
&= \max_{\bar{f}} \lambda\frac{\bar{f}-c}{1+\exp(\frac{\bar{f}-a}{\mu})} \triangleq \text{Profit rate of Case I},
\end{aligned}
$$

in which we used the inequality of arithmetic and geometric means (AM-GM). Therefore, (EC.24) leads to

$$\bar{p} - \Delta\bar{\eta}^w_1 - \bar{f} \to 0, \tag{EC.63}$$

and (EC.25) leads to

$$\mu - (1-\bar{\pi}^s_0)(\bar{f}-c) = \mu\frac{\bar{\eta}^u_1}{\bar{w}_0}. \tag{EC.64}$$

Since individual purchase is the only source of revenue in this limiting case, the first-order condition implies $\mu = (1-\bar{\pi}^s_0)(\bar{f}-c)$, namely,

$$\mu = \frac{\bar{f}-c}{1+\exp(\frac{a-\bar{f}}{\mu})}, \tag{EC.65}$$

from which we get $\bar{f} = c + \mu + \mu\mathbb{W}(\exp[(a-c-\mu)/\mu])$. Combine (EC.64) and (EC.65), one gets $\bar{\eta}_1^u = 0$. In addition, (EC.20) leads to

$$\lambda\mu \ln\Big(\frac{1+\exp(\frac{a-\Delta\bar{U}_1}{\mu})}{1+\exp(\frac{a-\bar{f}}{\mu})}\Big) = \rho\Delta\bar{U}_1, \tag{EC.66}$$

which can be combine with (EC.65) to give

$$\lambda\mu \ln\Big(\frac{1+\exp(\frac{a-\Delta\bar{U}_1}{\mu})}{1+\mathbb{W}(\exp[(a-c-\mu)/\mu])}\Big) = \rho\Delta\bar{U}_1, \tag{EC.67}$$

from which one can solve for $\Delta\bar{U}_1$. (EC.22) becomes

$$\bar{\pi}_0^s(\bar{f}-c) = \bar{\pi}_1^r(\Delta\bar{\eta}_1^w - c),$$

which can be combined with (EC.65) to yield:

$$\mu\mathbb{W}\big[\exp\big(\frac{a-c-\mu}{\mu}\big)\big] = \frac{\exp(\frac{a-\Delta\bar{U}_1}{\mu})}{1+\exp(\frac{a-\Delta\bar{U}_1}{\mu})}(\Delta\bar{\eta}_1^w - c),$$

$$\Delta\bar{\eta}_1^w = \mu\frac{1+\exp(\frac{a-\Delta\bar{U}_1}{\mu})}{\exp(\frac{a-\Delta\bar{U}_1}{\mu})}\mathbb{W}\big[\exp\big(\frac{a-c-\mu}{\mu}\big)\big] + c. \tag{EC.68}$$

EC.19. Proof of Proposition 7

Proof: We first show that for $\gamma = 1$, the turnpike revenue rate per capita is the same between passholders and non-pass holders when $\rho = 0$ or $\rho = \infty$. Clearly, when $\rho = 0$, we have $\bar{p} = \bar{f} + \sum_{k=1}^{\bar{k}} \Delta\bar{\eta}_k^w$ from Proposition 4. Similarly, Proposition 5 suggests that $\bar{p} = \bar{f} + \sum_{k=1}^{\bar{k}} \Delta\bar{\eta}_k^w$ also holds for $\rho = \infty$. For both cases, there is no EPP, namely, $e_{pp} = 0$. So the turnpike revenue rate per capita is homogeneous across the customer population. By Theorem 4, we therefore obtain

$$\bar{R} = \lambda(\bar{f}-c)(\bar{\pi}_0^s + \bar{\pi}_0^p). \tag{EC.69}$$

When selling to myopic customers ($\rho = \infty$) with $\gamma = 1$, the equation (EC.58) becomes $(1-\bar{\pi}_0^p - \bar{\pi}_0^s)(\bar{f}-c) = \mu$, then by (EC.69) we have $\bar{R} = \lambda(\bar{f}-c)(\bar{\pi}_0^s + \bar{\pi}_0^p) = \lambda(\bar{f}-c-\mu)$. Similarly, when selling to fully strategic customers ($\rho = 0$), the equation (EC.43) also implies $(1-\bar{\pi}_0^p - \bar{\pi}_0^s)(\bar{f}-c) = \mu$, and hence the profit expression $\bar{R} = \lambda(\bar{f}-c)(\bar{\pi}_0^s + \bar{\pi}_0^p) = \lambda(\bar{f}-c-\mu)$ also holds for fully strategic customers. Therefore, using the notation defined in the theorem, we have $\bar{R}_0 - \bar{R}_\infty = \lambda(\bar{f}_0 - \bar{f}_\infty)$ for the case of $\gamma = 1$.

Finally, because Proposition 4 suggests that $\bar{f}_0$ is independent of $\bar{k}$ and Proposition 5 suggests that $\bar{f}_\infty$ is decreasing in $\bar{k}$, it is clear that $\bar{R}_0 - \bar{R}_\infty$ is increasing in $\bar{k}$. □

EC.20. Proof of Theorem 2

Proof: When $c=0$, consider the turnpike revenue rate of an individual myopic customer *without* pass, it has the following expression:

$$\begin{aligned}\bar{Q}_\infty(\bar{p},\bar{f}) &= \bar{p}\bar{\pi}_0^p+\bar{f}\bar{\pi}_0^s\\ &= \bar{p}\frac{\exp\left(\frac{a-\bar{p}}{\mu\gamma}\right)\left[\exp\left(\frac{a-\bar{p}}{\mu\gamma}\right)+\exp\left(\frac{a-\bar{f}}{\mu\gamma}\right)\right]^{\gamma-1}}{1+\left[\exp\left(\frac{a-\bar{p}}{\mu\gamma}\right)+\exp\left(\frac{a-\bar{f}}{\mu\gamma}\right)\right]^{\gamma}}+\bar{f}\frac{\exp\left(\frac{a-\bar{f}}{\mu\gamma}\right)\left[\exp\left(\frac{a-\bar{p}}{\mu\gamma}\right)+\exp\left(\frac{a-\bar{f}}{\mu\gamma}\right)\right]^{\gamma-1}}{1+\left[\exp\left(\frac{a-\bar{p}}{\mu\gamma}\right)+\exp\left(\frac{a-\bar{f}}{\mu\gamma}\right)\right]^{\gamma}}.\end{aligned}$$

We will show that $\bar{Q}_\infty(\bar{p},\bar{f})\leqslant \mu\mathbb{W}\left[2^\gamma\exp\left(\frac{a-\mu}{\mu}\right)\right]$. To begin with, note that according to (3) - (6) the prices $\bar{p}$ and $\bar{f}$ can be expressed in terms of the choice probabilities as the following:

$$\begin{aligned}\bar{f} &= a+\mu\gamma\left[\ln(1-\bar{\pi}_0^p-\bar{\pi}_0^s)-\ln\bar{\pi}_0^s\right]+\mu(1-\gamma)\left[\ln(1-\bar{\pi}_0^p-\bar{\pi}_0^s)-\ln(\bar{\pi}_0^p+\bar{\pi}_0^s)\right],\\ \bar{p} &= a+\mu\gamma\left[\ln(1-\bar{\pi}_0^p-\bar{\pi}_0^s)-\ln\bar{\pi}_0^p\right]+\mu(1-\gamma)\left[\ln(1-\bar{\pi}_0^p-\bar{\pi}_0^s)-\ln(\bar{\pi}_0^p+\bar{\pi}_0^s)\right].\end{aligned}\tag{EC.70}$$

Therefore, $\bar{Q}_\infty(\bar{p},\bar{f})$ can also be expressed as a function of the choice probabilities:

$$\begin{aligned}&\bar{Q}_\infty(\bar{p},\bar{f})=\bar{Q}'_\infty(\bar{\pi}_0^p,\bar{\pi}_0^s)\\ =&\left\{a+\mu\gamma\left[\ln(1-\bar{\pi}_0^p-\bar{\pi}_0^s)-\ln\bar{\pi}_0^p\right]+\mu(1-\gamma)\left[\ln(1-\bar{\pi}_0^p-\bar{\pi}_0^s)-\ln(\bar{\pi}_0^p+\bar{\pi}_0^s)\right]\right\}\bar{\pi}_0^p\\ &+\left\{a+\mu\gamma\left[\ln(1-\bar{\pi}_0^p-\bar{\pi}_0^s)-\ln\bar{\pi}_0^s\right]+\mu(1-\gamma)\left[\ln(1-\bar{\pi}_0^p-\bar{\pi}_0^s)-\ln(\bar{\pi}_0^p+\bar{\pi}_0^s)\right]\right\}\bar{\pi}_0^s.\end{aligned}\tag{EC.71}$$

Next, we show $\bar{Q}'_\infty(\bar{\pi}_0^p,\bar{\pi}_0^s)$ is jointly concave in $\bar{\pi}_0^p,\bar{\pi}_0^s$. The Hessian of $\bar{Q}'_\infty(\bar{\pi}_0^p,\bar{\pi}_0^s)$, denoted as $\mathcal{Q}$ for brevity, can be calculated as the following

$$\begin{aligned}\mathcal{Q}(1,1)&=\frac{\partial^2\bar{Q}'_\infty(\bar{\pi}_0^p,\bar{\pi}_0^s)}{\partial(\bar{\pi}_0^p)^2}=-\frac{\bar{\pi}_0^p\bar{\pi}_0^s+(1-\bar{\pi}_0^s)^2}{(1-\bar{\pi}_0^p-\bar{\pi}_0^s)^2\bar{\pi}_0^p}<0,\\ \mathcal{Q}(2,2)&=\frac{\partial^2\bar{Q}'_\infty(\bar{\pi}_0^p,\bar{\pi}_0^s)}{\partial(\bar{\pi}_0^s)^2}=-\frac{\bar{\pi}_0^p\bar{\pi}_0^s+(1-\bar{\pi}_0^p)^2}{(1-\bar{\pi}_0^p-\bar{\pi}_0^s)^2\bar{\pi}_0^s}<0,\\ \mathcal{Q}(1,2)=\mathcal{Q}(2,1)&=\frac{\partial^2\bar{Q}'_\infty(\bar{\pi}_0^p,\bar{\pi}_0^s)}{\partial\bar{\pi}_0^p\partial\bar{\pi}_0^s}=-\frac{(2-\bar{\pi}_0^p-\bar{\pi}_0^s)}{(1-\bar{\pi}_0^p-\bar{\pi}_0^s)^2}<0,\end{aligned}$$

Because $\mathcal{Q}(1,1)<0$, and the determinant of $\mathcal{Q}$ is positive, i.e.,

$$\det(\mathcal{Q})=\mathcal{Q}(1,1)\times\mathcal{Q}(2,2)-\mathcal{Q}(1,2)\times\mathcal{Q}(2,1)=\frac{1}{\bar{\pi}_0^p\bar{\pi}_0^s(1-\bar{\pi}_0^p-\bar{\pi}_0^s)^2}>0,$$

so the Hessian $\mathcal{Q}$ is negative definite and hence $\bar{Q}'_\infty(\bar{\pi}_0^p,\bar{\pi}_0^s)$ is indeed jointly concave in $\bar{\pi}_0^p,\bar{\pi}_0^s$. We can then find the global maximum of $\bar{Q}'_\infty(\bar{\pi}_0^p,\bar{\pi}_0^s)$ using the first-order conditions:

$$\frac{\partial\bar{Q}'_\infty(\bar{\pi}_0^p,\bar{\pi}_0^s)}{\partial\bar{\pi}_0^p}\Big|_{(\bar{\pi}_0^p=\bar{\pi}_0^{*p},\bar{\pi}_0^s=\bar{\pi}_0^{*s})}=0,\quad \frac{\partial\bar{Q}'_\infty(\bar{\pi}_0^p,\bar{\pi}_0^s)}{\partial\bar{\pi}_0^s}\Big|_{(\bar{\pi}_0^p=\bar{\pi}_0^{*p},\bar{\pi}_0^s=\bar{\pi}_0^{*s})}=0,$$

where $(\bar{\pi}_0^{*p}, \bar{\pi}_0^{*s})$ is the global maximizer such that $\bar{Q}'_\infty(\bar{\pi}_0^p, \bar{\pi}_0^s) \leqslant \bar{Q}'_\infty(\bar{\pi}_0^{*p}, \bar{\pi}_0^{*s})$. After some algebra, the first-order conditions imply

$$(1-\bar{\pi}_0^{*p}-\bar{\pi}_0^{*s})\Big[a+\mu\ln(1-\bar{\pi}_0^{*p}-\bar{\pi}_0^{*s})-\mu\gamma\ln(\bar{\pi}_0^{*p})-\mu(1-\gamma)\ln(\bar{\pi}_0^{*p}+\bar{\pi}_0^{*s})\Big]=\mu, \tag{EC.72}$$

$$(1-\bar{\pi}_0^{*p}-\bar{\pi}_0^{*s})\Big[a+\mu\ln(1-\bar{\pi}_0^{*p}-\bar{\pi}_0^{*s})-\mu\gamma\ln(\bar{\pi}_0^{*s})-\mu(1-\gamma)\ln(\bar{\pi}_0^{*p}+\bar{\pi}_0^{*s})\Big]=\mu. \tag{EC.73}$$

Taking the difference between the above two equations gives $\bar{\pi}_0^{*p}=\bar{\pi}_0^{*s}$. Let f^* and p^* be the prices corresponding to $\bar{\pi}_0^{*p}, \bar{\pi}_0^{*s}$. Note further that, by (EC.70), the term in the square bracket of (EC.72) is equal to f^*, and the term in the square bracket of (EC.73) is equal to p^*. Clearly, $f^*=p^*$. Therefore, (EC.72) can be rewritten in terms of f^* as $f^*=\mu\Big[1+2^\gamma\exp\big(\frac{a-f^*}{\mu}\big)\Big]$. Solving this equation, we obtain $f^*=\mu+\mu\mathbb{W}\Big[2^\gamma\exp\Big(\frac{a-\mu}{\mu}\Big)\Big]$, where $\mathbb{W}$ is the Lambert-W function. Substituting the expression of f^* back to $\bar{Q}'_\infty(\bar{\pi}_0^{*p}, \bar{\pi}_0^{*s})$, we obtain $\bar{Q}'_\infty(\bar{\pi}_0^p, \bar{\pi}_0^s) \leqslant \bar{Q}'_\infty(\bar{\pi}_0^{*p}, \bar{\pi}_0^{*s}) = \mu\mathbb{W}\Big[2^\gamma\exp\Big(\frac{a-\mu}{\mu}\Big)\Big]$. Because $\bar{Q}_\infty(\bar{p},\bar{f})=\bar{Q}'_\infty(\bar{\pi}_0^p, \bar{\pi}_0^s)$, they have the same upper bound, therefore,

$$\bar{Q}_\infty(\bar{p},\bar{f}) \leqslant \mu\mathbb{W}\Big[2^\gamma\exp\Big(\frac{a-\mu}{\mu}\Big)\Big]. \tag{EC.74}$$

Now we have obtained the upper bound for $\bar{Q}_\infty(\bar{p},\bar{f})$, namely, the turnpike revenue rate of an individual myopic customer without pass.

Next, we consider the turnpike revenue rate of an individual myopic customer *with* the pass. It has the following expression:

$$\begin{aligned}\bar{V}_\infty(\bar{f}) = \bar{f}\bar{\pi}^s = \bar{f}\,&\frac{\exp\Big(\frac{a-\bar{f}}{\mu\gamma}\Big)\Big[\exp\Big(\frac{a}{\mu\gamma}\Big)+\exp\Big(\frac{a-\bar{f}}{\mu\gamma}\Big)\Big]^{\gamma-1}}{1+\Big[\exp\Big(\frac{a}{\mu\gamma}\Big)+\exp\Big(\frac{a-\bar{f}}{\mu\gamma}\Big)\Big]^{\gamma}}\\ \leqslant \bar{f}\,&\frac{\exp\Big(\frac{a-\bar{f}}{\mu\gamma}\Big)\Big[\exp\Big(\frac{a-\bar{f}}{\mu\gamma}\Big)+\exp\Big(\frac{a-\bar{f}}{\mu\gamma}\Big)\Big]^{\gamma-1}}{1+\Big[\exp\Big(\frac{a-\bar{f}}{\mu\gamma}\Big)+\exp\Big(\frac{a-\bar{f}}{\mu\gamma}\Big)\Big]^{\gamma}} = \frac{1}{2}\bar{f}\frac{2^\gamma\exp\Big(\frac{a-\bar{f}}{\mu}\Big)}{1+2^\gamma\exp\Big(\frac{a-\bar{f}}{\mu}\Big)}\end{aligned}$$

where the inequality follows from $\bar{f}>0$ and $0<\gamma\leqslant 1$. Note that, by restricting $p=f$, the function $\bar{Q}_\infty(p,f)$ becomes

$$\bar{Q}_\infty(f,f) = f\frac{2^\gamma\exp\Big(\frac{a-f}{\mu}\Big)}{1+2^\gamma\exp\Big(\frac{a-f}{\mu}\Big)}.$$

Note that the unrestricted function $\bar{Q}_\infty(\bar{p},\bar{f})$ is bounded from above by $\mu\mathbb{W}\Big[2^\gamma\exp\Big(\frac{a-\mu}{\mu}\Big)\Big]$ following (EC.74), therefore the restricted function $\bar{Q}_\infty(f,f)$ is also bounded from above by $\mu\mathbb{W}\Big[2^\gamma\exp\Big(\frac{a-\mu}{\mu}\Big)\Big]$. Note further that $\bar{V}_\infty(\bar{f})=\bar{Q}_\infty(f,f)/2$, therefore we have

$$\bar{V}_\infty(\bar{f}) < \mu\mathbb{W}\Big[2^\gamma\exp\Big(\frac{a-\mu}{\mu}\Big)\Big].$$

Now we have obtained the upper bound for $\bar{V}_\infty(\bar{f})$, namely, the turnpike revenue rate of an individual myopic customer with pass. It is important to note that this is the same upper bound for $\bar{Q}_\infty(\bar{p},\bar{f})$.

Finally, we consider the total turnpike revenue rate as follows

$$\begin{aligned}\bar{R}_\infty &= \lambda\bar{w}_0(\bar{p}\bar{\pi}_0^p + \bar{f}\bar{\pi}_0^s) + \lambda\bar{k}\bar{w}\bar{f}\bar{\pi}^s = \lambda\bar{w}_0\bar{Q}_\infty(\bar{p},\bar{f}) + \lambda(1-\bar{w}_0)\bar{V}_\infty(\bar{f}) \\ &\leqslant \lambda\bar{w}_0\mu\mathbb{W}\Big[2^\gamma\exp\Big(\frac{a-\mu}{\mu}\Big)\Big] + \lambda(1-\bar{w}_0)\mu\mathbb{W}\Big[2^\gamma\exp\Big(\frac{a-\mu}{\mu}\Big)\Big] \\ &= \lambda\mu\mathbb{W}\Big[2^\gamma\exp\Big(\frac{a-\mu}{\mu}\Big)\Big] = \bar{R}_0.\end{aligned}$$

When $\gamma\to 0$, (EC.65) leads to $\bar{f} = \mu + \mu\mathbb{W}\big[\exp(\frac{a-\mu-c}{\mu})\big]$. Since the individual purchase is the only source of revenue in this limiting case, it can be easily verified that $\bar{R}_\infty = \bar{R}_0$, thereby completing the proof. □

EC.21. Proof of Proposition 8

Proof: Suppose $\bar{p} > \bar{f} + \Delta\bar{\eta}_1^w$, then the turnpike equation (EC.22) suggests that

$$\bar{\pi}_1^s(\bar{f}-c) + \bar{\pi}_1^r(\Delta\bar{\eta}_1^w - c) = \bar{\pi}_0^p(\bar{p}-\Delta\bar{\eta}_1^w - c) + \bar{\pi}_0^s(\bar{f}-c) > \bar{\pi}_0^p(\bar{f}-c) + \bar{\pi}_0^s(\bar{f}-c), \tag{EC.75}$$

which then implies

$$\bar{r}_1 \triangleq \bar{\pi}_1^s(\bar{f}-c) + \bar{\pi}_1^r(\bar{p}-\bar{f}-c) > \bar{\pi}_1^s(\bar{f}-c) + \bar{\pi}_1^r(\Delta\bar{\eta}_1^w - c) > (\bar{\pi}_0^p + \bar{\pi}_0^s)(\bar{f}-c) = \bar{r}_0, \tag{EC.76}$$

where the first inequality follows directly from $\bar{p} > \bar{f} + \Delta\bar{\eta}_1^w$, the second is from (EC.75).

Now suppose $\bar{p} < \bar{f} + \Delta\bar{\eta}_1^w$, by the turnpike equation (EC.22) we have

$$\bar{\pi}_1^s(\bar{f}-c) + \bar{\pi}_1^r(\Delta\bar{\eta}_1^w - c) = \bar{\pi}_0^p(\bar{p}-\Delta\bar{\eta}_1^w - c) + \bar{\pi}_0^s(\bar{f}-c) < \bar{\pi}_0^p(\bar{f}-c) + \bar{\pi}_0^s(\bar{f}-c), \tag{EC.77}$$

which leads to

$$\bar{r}_1 \triangleq \bar{\pi}_1^s(\bar{f}-c) + \bar{\pi}_1^r(\bar{p}-\bar{f}-c) < \bar{\pi}_1^s(\bar{f}-c) + \bar{\pi}_1^r(\Delta\bar{\eta}_1^w - c) < (\bar{\pi}_0^p + \bar{\pi}_0^s)(\bar{f}-c) = \bar{r}_0. \tag{EC.78}$$

Similarly, the first inequality follows from $\bar{p} < \bar{f} + \Delta\bar{\eta}_1^w$ and the second is by (EC.77). Using the same procedure, it is straightforward to show that $\bar{p} = \bar{f} + \Delta\bar{\eta}_1^w$ leads to $\bar{r}_1 = \bar{r}_0$. □

EC.22. Proof of Theorem 3

Proof: We prove each part of the theorem separately.

EC.22.1. The proof of part 1-2

When $\rho = 0$, by Proposition 4 we have $\bar{p} = \bar{f} + \Delta\bar{\eta}_1^w$. Therefore it is obvious that $e_{pp} = 0$. When $\gamma\to 0, \rho = 0$, it is obvious from Proposition 4 that $e_{pp} = 0$. When $\gamma\to 0, \rho > 0$, we substitute the solutions in EC.18 to (EC.24) and obtain $\bar{p} - \Delta\bar{\eta}_1^w - \bar{f} \to 0$, which implies $e_{pp}\to 0$.

EC.22.2. The proof of part 3

In the proof of Proposition 5, we have already shown that, as $\rho \to \infty$, we have $\bar{p} \to \bar{f} + \Delta\bar{\eta}^w$ for $\gamma = 1$, which implies $e_{pp} \to 0$. Also, we have just shown $e_{pp} \to 0$ as $\gamma \to 0$. It remains to show that $e_{pp} > 0$ or $\bar{p} > \bar{f} + \Delta\bar{\eta}^w$ for $0 < \gamma < 1$.

We prove this by contradiction. Suppose that $\bar{p} \leqslant \bar{f} + \Delta\bar{\eta}^w$ for $0 < \gamma < 1$. then (EC.48) implies that

$$(\bar{\pi}_0^p + \bar{\pi}_0^s)(\bar{f} - c) \geqslant \bar{\pi}^s(\bar{f} - c) + \bar{\pi}^r(\Delta\bar{\eta}^w - c), \tag{EC.79}$$

Substitute $\bar{p} \leqslant \bar{f} + \Delta\bar{\eta}^w$ into (EC.50), we obtain

$$(1 - \bar{\pi}_0^p - \bar{\pi}_0^s)(\bar{f} - c) \geqslant \mu, \tag{EC.80}$$

Substituting $\bar{p} \leqslant \bar{f} + \Delta\bar{\eta}^w$ into (EC.51) yields

$$0 \geqslant \bar{f} - c - \mu\gamma + \frac{\gamma(1 - \bar{\pi}^r - \bar{\pi}^s) - 1}{\bar{\pi}^r + \bar{\pi}^s}\Big[\bar{\pi}^s(\bar{f} - c) + \bar{\pi}^r(\Delta\bar{\eta}^w - c)\Big].$$

We combine this inequality with (EC.79) and (EC.80) to get the following inequality

$$(1 - \gamma)(\bar{\pi}_0^p + \bar{\pi}_0^s) \geqslant (1 - \gamma)(\bar{\pi}^r + \bar{\pi}^s).$$

When $0 < \gamma < 1$, the above inequality implies $\bar{\pi}_0^p + \bar{\pi}_0^s \geqslant \bar{\pi}^r + \bar{\pi}^s$, which contradicts with (EC.49), namely, $\bar{\pi}_0^s + \bar{\pi}_0^p < \bar{\pi}^s + \bar{\pi}^r$. Therefore, we must have $\bar{p} > \bar{f} + \Delta\bar{\eta}^w$, for $0 < \gamma < 1$. Hence, $e_{pp} > 0$, for $0 < \gamma < 1$ and $\rho \to \infty$.

EC.22.3. The proof of part 4

We now consider the case of $\gamma = 1$, the steady-state adjoint equations for utility is

$$0 = -\lambda\bar{w}_0\bar{\pi}_0^p\frac{1}{\mu}(\bar{p} - \Delta\bar{\eta}_1^w - c - \alpha) + \lambda\bar{w}_1\bar{\pi}_1^r\frac{1}{\mu}(\Delta\bar{\eta}_1^w - c - \alpha) - \lambda\bar{\pi}_0^p\bar{\eta}_1^u - \rho\bar{\eta}_1^u - \lambda\bar{\eta}_1^u\bar{\pi}_1^r. \tag{EC.81}$$

The turnpike first-order conditions (EC.24) can be written as:

$$\mu\frac{\bar{\eta}_1^u}{\bar{w}_0} = -(\bar{p} - \Delta\bar{\eta}_1^w - c - \mu - \alpha). \tag{EC.82}$$

Substitute (EC.82) into (EC.81) and note that $\bar{w}_0\bar{\pi}_0^p = \bar{w}_1\bar{\pi}_1^r$, we obtain

$$\mu\frac{\bar{\eta}_1^u}{\bar{w}_0} = \lambda\frac{\bar{\pi}_0^p}{\rho + \lambda\bar{\pi}_1^r}(\Delta\bar{\eta}_1^w - c - \mu - \alpha). \tag{EC.83}$$

The other turnpike first-order condition (EC.25) becomes:

$$\frac{\lambda}{\mu}(\bar{w}_0\bar{\pi}_0^s + \bar{w}_1\bar{\pi}_1^s)(\bar{f} - c - \mu - \alpha) = \lambda\bar{\eta}_1^u(\bar{\pi}_1^s - \bar{\pi}_0^s).$$

Since $\bar{w}_1\bar{\pi}_1^r = \bar{w}_0\bar{\pi}_0^p$, we divide both sides of the above equation by $\bar{w}_0$ to obtain

$$\frac{\lambda}{\mu}(\bar{\pi}_0^s + \frac{\bar{\pi}_0^p}{\bar{\pi}_1^r}\bar{\pi}_1^s)(\bar{f} - c - \mu - \alpha) = \lambda\frac{\bar{\eta}_1^u}{\bar{w}_0}(\bar{\pi}_1^s - \bar{\pi}_0^s). \tag{EC.84}$$

To simplify notations, we introduce the new variables $x \triangleq \bar{p} - \Delta\bar{\eta}_1^w - c, y \triangleq \bar{f} - c, z \triangleq \Delta\bar{\eta}_1^w - c$ and let $x' \triangleq x - \mu - \alpha$, $y' \triangleq y - \mu - \alpha$, $z' \triangleq z - \mu - \alpha$. Then, by (EC.82) and (EC.83):

$$-(\rho + \lambda\bar{\pi}_1^r)x' = \lambda\bar{\pi}_0^p z'. \tag{EC.85}$$

Substitute (EC.82) into (EC.84), we have

$$(\bar{\pi}_0^s + \frac{\bar{\pi}_0^p}{\bar{\pi}_1^r}\bar{\pi}_1^s)y' = (\bar{\pi}_0^s - \bar{\pi}_1^s)x'. \tag{EC.86}$$

Finally, since $\alpha = \bar{\pi}_0^p x + \bar{\pi}_0^s y = \bar{\pi}_1^s y + \bar{\pi}_1^r z$, we have $\bar{\pi}_0^p(x' + \mu + \alpha) + \bar{\pi}_0^s(y' + \mu + \alpha) = \bar{\pi}_1^s(y' + \mu + \alpha) + \bar{\pi}_1^r(z' + \mu + \alpha) = \alpha$. Therefore,

$$\frac{\bar{\pi}_0^p(x' + \mu) + \bar{\pi}_0^s(y' + \mu)}{1 - \bar{\pi}_0^p - \bar{\pi}_0^s} = \frac{\bar{\pi}_1^r(z' + \mu) + \bar{\pi}_1^s(y' + \mu)}{1 - \bar{\pi}_1^r - \bar{\pi}_1^s}. \tag{EC.87}$$

Note that (EC.85), (EC.86), (EC.87) constitutes a closed system for x', y', z' given all choice probabilities. Solving this system, we obtain

$$x' = \bar{\pi}_0^p C, \quad y' = \frac{\bar{\pi}_0^p\bar{\pi}_1^r(\bar{\pi}_0^s - \bar{\pi}_1^s)}{\bar{\pi}_1^r\bar{\pi}_0^s + \bar{\pi}_0^p\bar{\pi}_1^s}C, \quad z' = -(\bar{\pi}_1^r + \frac{\rho}{\lambda})C,$$

where

$$C = \mu\frac{(\bar{\pi}_1^r + \bar{\pi}_1^s) - (\bar{\pi}_0^p + \bar{\pi}_0^s)}{(\bar{\pi}_0^p)^2(1 - \bar{\pi}_1^r - \bar{\pi}_1^s) + \bar{\pi}_1^r(\bar{\pi}_1^r + \frac{\rho}{\lambda})(1 - \bar{\pi}_0^p - \bar{\pi}_0^s)}.$$

According to Proposition 1, we know that $(\bar{\pi}_1^r + \bar{\pi}_1^s) - (\bar{\pi}_0^p + \bar{\pi}_0^s) \geqslant 0$ and $\bar{\pi}_0^s \geqslant \bar{\pi}_1^s$. Then, it is straightforward to show that $C \geqslant 0$ and $x' \geqslant 0, y' \geqslant 0, z' \leqslant 0$. More specifically, the equalities are obtained if $\rho = 0$ or $\rho \to \infty$. So, when $0 < \rho < \infty$, we have $C > 0$.

Furthermore,

$$e_{pp} = x' - y' = \frac{\bar{\pi}_0^p\bar{\pi}_1^s(\bar{\pi}_0^p + \bar{\pi}_1^r)}{\bar{\pi}_1^r\bar{\pi}_0^s + \bar{\pi}_0^p\bar{\pi}_1^s}C \geqslant 0.$$

When customers are fully strategic, i.e., $\rho = 0$, by Proposition 4 we have $\bar{\pi}_1^r + \bar{\pi}_1^s = \bar{\pi}_0^p + \bar{\pi}_0^s$, which implies $C = 0$ and hence $e_{pp} = 0$. When customers are myopic, namely, as $\rho \to \infty$, it is clear that $C \to 0$ and hence $e_{pp} \to 0$. $e_{pp} > 0$ only occurs when $0 < \rho < \infty$ because $C > 0$. □

EC.23. Proof of Proposition 9

Part 1 of the proposition follows directly from Proposition 4. Here we focus on the second part.

For the case of $\gamma = 1$, the proof is based on the proof of Theorem 3. Recall that

$$x' - y' = \frac{\bar{\pi}_0^p \bar{\pi}_1^s (\bar{\pi}_0^p + \bar{\pi}_1^r)}{\bar{\pi}_1^r \bar{\pi}_0^s + \bar{\pi}_0^p \bar{\pi}_1^s} C \geqslant 0,$$

and we have shown that the equality holds if $\rho = 0$ or $\rho \to \infty$. So, when $0 < \rho < \infty$, we have $x' > y'$, which implies that $\bar{p} - \Delta\bar{\eta}_1^w > \bar{f}$. Similarly, we have

$$x' + z' - 2y' = \Big(\bar{\pi}_0^p - (\bar{\pi}_1^r + \frac{\rho}{\lambda}) - 2\frac{\bar{\pi}_0^p \bar{\pi}_1^r (\bar{\pi}_0^s - \bar{\pi}_1^s)}{\bar{\pi}_1^r \bar{\pi}_0^s + \bar{\pi}_0^p \bar{\pi}_1^s}\Big) C = -\frac{\frac{\rho}{\lambda}(\bar{\pi}_1^r \bar{\pi}_0^s + \bar{\pi}_0^p \bar{\pi}_1^s) + (\bar{\pi}_1^r - \bar{\pi}_0^p)(\bar{\pi}_1^r \bar{\pi}_0^s - \bar{\pi}_0^p \bar{\pi}_1^s)}{\bar{\pi}_1^r \bar{\pi}_0^s + \bar{\pi}_0^p \bar{\pi}_1^s} C.$$

Because $\bar{\pi}_1^r - \bar{\pi}_0^p > 0$ and $\bar{\pi}_0^s > \bar{\pi}_1^s$ following from Proposition 1, the RHS of above equation is negative and hence we obtain $x' + z' < 2y'$, which then implies that $x + z < 2y$ and $\bar{p} - \Delta\bar{\eta}_1^w - c + \Delta\bar{\eta}_1^w - c < 2\bar{f} - 2c$. That is, $\bar{p} < 2\bar{f}$.

Now we turn to the case of $\gamma \to 0$. Since $\bar{p} = \Delta\bar{\eta}_1^w + \bar{f}$, proving $\bar{p} < 2\bar{f}$ is equivalent to proving $\Delta\bar{\eta}_1^w < \bar{f}$. According to (EC.68) and $\bar{f} = c + \mu + \mu\mathbb{W}(\exp[(a - c - \mu)/\mu])$, we obtain

$$\Delta\bar{\eta}_1^w - \bar{f} = \mu \frac{\mathbb{W}\big[\exp\big(\frac{a-c-\mu}{\mu}\big)\big]}{\exp\big(\frac{a-\Delta\bar{U}_1}{\mu}\big)} - \mu. \tag{EC.88}$$

By Lemma 1, $\Delta\bar{U}_1 \geqslant 0$. Therefore, (EC.67) implies that $\frac{\mathbb{W}\big[\exp\big(\frac{a-c-\mu}{\mu}\big)\big]}{\exp\big(\frac{a-\Delta\bar{U}_1}{\mu}\big)} < 1$. Hence, it is obvious that $\Delta\bar{\eta}_1^w < \bar{f}$, and $\bar{p} < 2\bar{f}$.

EC.24. Proof of Theorem 4

Proof: The proof is based on the turnpike equations. We first prove the second equality in the theorem. By definition,

$$\begin{aligned}\bar{R} &\triangleq \lambda\bar{w}_0[\bar{\pi}_0^p(\bar{p} - c) + \bar{\pi}_0^s(\bar{f} - c)] + \lambda\sum_{k=1}^{\bar{k}} \bar{w}_k \bar{\pi}_k^s(\bar{f} - c) - \lambda\sum_{k=1}^{\bar{k}} \bar{w}_k \bar{\pi}_k^r c \\ &= \lambda\bar{w}_0[\bar{\pi}_0^p(\bar{p} - \bar{f} + \bar{f} - c) + \bar{\pi}_0^s(\bar{f} - c)] + \lambda\sum_{k=1}^{\bar{k}} \bar{w}_k[\bar{\pi}_k^s(\bar{f} - c) - \bar{\pi}_k^r c].\end{aligned}$$

Notice that the pass purchase and redemption are balanced in the steady state, namely, $\bar{w}_0\bar{\pi}_0^p = \bar{w}_k\bar{\pi}_k^r$, for all $k = 1, \dots, \bar{k}$. We can then express $\bar{R}$ as follows:

$$\begin{aligned}\bar{R} &= \lambda\bar{w}_0[\bar{\pi}_0^p(\bar{f} - c) + \bar{\pi}_0^s(\bar{f} - c)] + \lambda\bar{w}_0\bar{\pi}_0^p(\bar{p} - \bar{f}) + \lambda\sum_{k=1}^{\bar{k}} \bar{w}_k[\bar{\pi}_k^s(\bar{f} - c) - \bar{\pi}_k^r c] \\ &= \lambda\bar{w}_0[\bar{\pi}_0^p(\bar{f} - c) + \bar{\pi}_0^s(\bar{f} - c)] + \lambda\frac{\sum_{k=1}^{\bar{k}} \bar{w}_k\bar{\pi}_k^r}{\bar{k}}(\bar{p} - \bar{f}) + \lambda\sum_{k=1}^{\bar{k}} \bar{w}_k[\bar{\pi}_k^s(\bar{f} - c) - \bar{\pi}_k^r c] \\ &= \lambda\bar{w}_0(\bar{\pi}_0^p + \bar{\pi}_0^s)(\bar{f} - c) + \lambda\sum_{k=1}^{\bar{k}} \bar{w}_k\Big[\bar{\pi}_k^r\Big(\frac{\bar{p} - \bar{f}}{\bar{k}} - c\Big) + \bar{\pi}_k^s(\bar{f} - c)\Big].\end{aligned}$$

Now we prove the last equality in the theorem. By the definition of $\bar{R}$ and $\bar{p} = \bar{f} + \sum_{k=1}^{\bar{k}} \Delta\bar{\eta}_k^w + e_{pp}$ we have

$$\begin{aligned}
\bar{R} &\triangleq \lambda\bar{w}_0[\bar{\pi}_0^p(\bar{p}-c) + \bar{\pi}_0^s(\bar{f}-c)] + \lambda\sum_{k=1}^{\bar{k}} \bar{w}_k\bar{\pi}_k^s(\bar{f}-c) - \lambda\sum_{k=1}^{\bar{k}} \bar{w}_k\bar{\pi}_k^r c \\
&= \lambda\bar{w}_0\Big[(\bar{f}-c)\bar{\pi}_0^s + \Big(\bar{f}-c+\sum_{k=1}^{\bar{k}}\Delta\bar{\eta}_k^w + e_{pp}\Big)\bar{\pi}_0^p\Big] + \lambda\sum_{k=1}^{\bar{k}} \bar{w}_k\bar{\pi}_k^s(\bar{f}-c) - \lambda\sum_{k=1}^{\bar{k}} \bar{w}_k\bar{\pi}_k^r c \\
&= \lambda\bar{w}_0(\bar{f}-c)(\bar{\pi}_0^s+\bar{\pi}_0^p) + \lambda\bar{w}_0\bar{\pi}_0^p\sum_{k=1}^{\bar{k}}\Delta\bar{\eta}_k^w + \lambda\bar{w}_0\bar{\pi}_0^p e_{pp} + \lambda\sum_{k=1}^{\bar{k}} \bar{w}_k\bar{\pi}_k^s(\bar{f}-c) - \lambda\sum_{k=1}^{\bar{k}} \bar{w}_k\bar{\pi}_k^r c
\end{aligned}$$

Noting again that $\bar{w}_0\bar{\pi}_0^p = \bar{w}_k\bar{\pi}_k^r$, for all $k = 1, \ldots, \bar{k}$, we have

$$\begin{aligned}
\bar{R} &= \lambda\bar{w}_0(\bar{f}-c)(\bar{\pi}_0^s+\bar{\pi}_0^p) + \lambda\sum_{k=1}^{\bar{k}} \bar{w}_k\bar{\pi}_k^r(\Delta\bar{\eta}_k^w - c) + \lambda\sum_{k=1}^{\bar{k}} \bar{w}_k\pi_k^s(\bar{f}-c) + \lambda\bar{w}_0\bar{\pi}_0^p e_{pp} \\
&= \lambda\bar{w}_0(\bar{f}-c)(\bar{\pi}_0^s+\bar{\pi}_0^p) + \lambda\sum_{k=1}^{\bar{k}} \bar{w}_k[\bar{\pi}_k^r(\Delta\bar{\eta}_k^w - c) + \bar{\pi}_k^s(\bar{f}-c)] + \lambda\bar{w}_0\bar{\pi}_0^p e_{pp}.
\end{aligned}$$

By (EC.22) the above equation can be rewritten as

$$\bar{R} = \lambda\bar{w}_0(\bar{f}-c)(\bar{\pi}_0^s+\bar{\pi}_0^p) + \lambda\sum_{k=1}^{\bar{k}} \bar{w}_k[\bar{\pi}_0^p(\bar{p} - \sum_{k=1}^{\bar{k}}\Delta\bar{\eta}_k^w - c) + \bar{\pi}_0^s(\bar{f}-c)] + \lambda\bar{w}_0\bar{\pi}_0^p e_{pp}.$$

Using the definition of e_{pp} again, namely, $e_{pp} \triangleq \bar{p} - \bar{f} - \sum_{k=1}^{\bar{k}} \Delta\bar{\eta}_k^w$, we have

$$\begin{aligned}
\bar{R} &= \lambda\bar{w}_0(\bar{f}-c)(\bar{\pi}_0^s+\bar{\pi}_0^p) + \lambda\sum_{k=1}^{\bar{k}} \bar{w}_k[\bar{\pi}_0^p(\bar{f}-c+e_{pp}) + \bar{\pi}_0^s(\bar{f}-c)] + \lambda\bar{w}_0\bar{\pi}_0^p e_{pp} \\
&= \lambda\bar{w}_0(\bar{f}-c)(\bar{\pi}_0^s+\bar{\pi}_0^p) + \lambda\sum_{k=1}^{\bar{k}} \bar{w}_k[(\bar{\pi}_0^p+\bar{\pi}_0^s)(\bar{f}-c) + \bar{\pi}_0^p e_{pp}] + \lambda\bar{w}_0\bar{\pi}_0^p e_{pp} \\
&= \lambda\Big(\sum_{k=1}^{\bar{k}}\bar{w}_k + \bar{w}_0\Big)(\bar{f}-c)(\bar{\pi}_0^s+\bar{\pi}_0^p) + \lambda\Big(\sum_{k=1}^{\bar{k}}\bar{w}_k + \bar{w}_0\Big)\bar{\pi}_0^p e_{pp} \\
&= \lambda(\bar{f}-c)(\bar{\pi}_0^s+\bar{\pi}_0^p) + \lambda\bar{\pi}_0^p e_{pp},
\end{aligned}$$

where the last equality is obvious because $\bar{w}_0 + \sum_{k=1}^{\bar{k}} \bar{w}_k = 1$. □

EC.25. Proof of Proposition 10

Part 1

Proof: When $\rho = 0$, the turnpike profit rate is

$$\begin{aligned}
\bar{R}_0 &\triangleq \lambda\bar{w}_0[(\bar{f}-c)\bar{\pi}_0^s + (\bar{p}-c)\bar{\pi}_0^p] + \lambda\sum_{k=1}^{\bar{k}} \bar{w}_k(\bar{f}-c)\bar{\pi}_k^s - \lambda\sum_{k=1}^{\bar{k}} \bar{w}_k\bar{\pi}_k^r c \\
&= \lambda\frac{1}{1+\bar{k}}\{(\bar{f}-c)\bar{\pi} + [(1+\bar{k})\bar{f}-c]\bar{\pi}\} + \lambda\bar{k}\frac{1}{1+\bar{k}}(\bar{f}-c)\bar{\pi} - \lambda\bar{k}\frac{1}{1+\bar{k}}\bar{\pi}c
\end{aligned}$$

$$= 2\lambda(\bar{f} - c)\bar{\pi}$$
$$= \lambda\Big\{\mu + \mu\mathbb{W}\Big[2^\gamma \exp\Big(\frac{a-\mu-c}{\mu}\Big)\Big]\Big\}\frac{\mathbb{W}\Big[2^\gamma \exp\Big(\frac{a-\mu-c}{\mu}\Big)\Big]}{1+\mathbb{W}\Big[2^\gamma \exp\Big(\frac{a-\mu-c}{\mu}\Big)\Big]},$$

in which the second equality follows from $\bar{p} = (1+\bar{k})\bar{f}$, $\bar{\pi}_0^s = \bar{\pi}_0^p = \bar{\pi}_k^s = \bar{\pi}_k^r = \bar{\pi}$ and $\bar{w}_0 = \bar{w}_k = 1/(1+\bar{k})$, for $k = 1, \ldots, \bar{k}$, as shown in Proposition 4. The last inequality follows from $\bar{\pi} = \mathbb{W}\big[2^\gamma \exp\big(\frac{a-\mu-c}{\mu}\big)\big]/\big(2+2\mathbb{W}\big[2^\gamma \exp\big(\frac{a-\mu-c}{\mu}\big)\big]\big)$ and $\bar{f} = \mu + c + \mu\mathbb{W}\big[2^\gamma \exp\big(\frac{a-\mu-c}{\mu}\big)\big]$, also obtained in Proposition 4. □

Part 2

Proof: When $\gamma = 1$, Proposition 5 suggests that $\bar{p} = \bar{f} + \sum_{k=1}^{\bar{k}} \Delta\bar{\eta}_k^w$ for $\rho = \infty$. Combine this result with Theorem 4, we have

$$\bar{R} = \lambda(\bar{f} - c)(\bar{\pi}_0^s + \bar{\pi}_0^p). \tag{EC.89}$$

When $\rho = \infty$ and $\gamma = 1$, the equation (EC.58) becomes $(1 - \bar{\pi}_0^p - \bar{\pi}_0^s)(\bar{f} - c) = \mu$, then by (EC.89) we have

$$\bar{R} = \lambda(\bar{f} - c)(\bar{\pi}_0^s + \bar{\pi}_0^p) = \lambda(\bar{f} - c - \mu), \tag{EC.90}$$

in which $\bar{f}$ is the solution to the following system of equations (by Proposition 5):

$$\bar{f} = \mu\big(1 + \exp[(a-\bar{p})/\mu] + \exp[(a-\bar{f})/\mu]\big) + c. \tag{EC.91}$$
$$\bar{p} = (1+\bar{k})\bar{f} - \bar{k}\mu[1 - \exp(-\bar{p}/\mu)]. \tag{EC.92}$$

By (EC.92) we can write $\bar{p}$ in terms of $\bar{f}$ as the following

$$\bar{p} = \mu\mathbb{W}\Big[\bar{k}\exp\Big(-\frac{\bar{k}(\bar{f}-\mu)+\bar{f}}{\mu}\Big)\Big] + \bar{k}(\bar{f}-\mu) + \bar{f}, \tag{EC.93}$$

where $\mathbb{W}$ is the Lambert-W function. Note that $\bar{f} > \mu$, again by Proposition 5, then it can be verified by differentiation that the RHS of (EC.93) is increasing in both $\bar{k}$ and $\bar{f}$.

Now, consider (EC.91), the LHS of the equation is clearly increasing in $\bar{f}$. We have shown that $\bar{p}$ is increasing in both $\bar{k}$ and $\bar{f}$, so the RHS of (EC.91) is decreasing in both $\bar{k}$ and $\bar{f}$. Hence, the solution to (EC.91), $\bar{f}$, is unique and it is decreasing in $\bar{k}$. Combine this with (EC.90), it is cleat that the profit rate $\bar{R}$ is decreasing in $\bar{k}$ and the maximum profit is achieved when $\bar{k} = 1$. □

EC.26. The Application of Turnpike Policy in Stochastic Environment

In this section, we investigate the asymptotic behaviour in the corresponding full fledged discrete-event stochastic system. We found in the numerical experiments that the turnpike solution is asymptotically optimal in the full fledged stochastic system. For the purposes of this investigation, we consider a stochastic system with a finite number of discrete customers and track their individual states and transitions between these states. From a computational standpoint, it is necessary to consider a discrete-time approximation to the continuous-time Markov jump process that drives these transitions. It is also easier to follow the logic of the equations in a discrete-time setting. Thus, for the purposes of the stochastic market dynamics model, we view time as discrete with a step of length h and approximate all transition probabilities by h times the intensity of transitions in the continuous-time model.

Assuming that customer preferences are still fully determined by the number of credits they hold, it is sufficient to track the total number of customers with each number of remaining credits. Let n_k represent the number of customers with k credits. The total number of customers is N, therefore, $\sum_{k=0}^{\bar{k}} n_k = N$. The stochastic system state is a vector $\mathbf{n} = (n_0, \ldots, n_{\bar{k}})$. We extend our earlier notation for customer utilities and the pricing policy to depend on $\mathbf{n}$. Given a state-dependent pricing policy $f_t(\mathbf{n}), p_t(\mathbf{n})$, the system of difference equations governing customer utilities in each state assumes the form which is in a direct correspondence with differential equations (2) in Theorem 1:

$$\begin{aligned}\frac{U_{k(t+h)}(\mathbf{n}) - U_{kt}(\mathbf{n})}{h} &= -\lambda\mu\ln\Big\{1 + \Big[\exp\Big(\frac{a + U_{(k-1)(t+h)}(\mathbf{n} - \mathbf{e}_k + \mathbf{e}_{k-1}) - U_{k(t+h)}(\mathbf{n})}{\mu\gamma}\Big) \\ &\quad +\exp\Big(\frac{a - f_t(\mathbf{n})}{\mu\gamma}\Big)\Big]^{\gamma}\Big\} + \rho U_{k(t+h)}(\mathbf{n}),\ k = 1, \ldots, \bar{k},\ t \in \{0, h, \ldots, T-h\} \qquad \text{(EC.94)}\\ \frac{U_{0(t+h)}(\mathbf{n}) - U_{0t}(\mathbf{n})}{h} &= -\lambda\mu\ln\Big\{1 + \Big[\exp\Big(\frac{a - p_t(\mathbf{n}) + U_{\bar{k}(t+h)}(\mathbf{n} - \mathbf{e}_0 + \mathbf{e}_{\bar{k}}) - U_{0(t+h)}(\mathbf{n})}{\mu\gamma}\Big) \\ &\quad +\exp\Big(\frac{a - f_t(\mathbf{n})}{\mu\gamma}\Big)\Big]^{\gamma}\Big\} + \rho U_{0(t+h)}(\mathbf{n}),\ t \in \{0, h, \ldots, T-h\}, \qquad \text{(EC.95)}\end{aligned}$$

with the boundary conditions

$$U_{kT}(\mathbf{n}) = 0,\ k = 1, \ldots, \bar{k},\ \text{for all } \mathbf{n}. \qquad \text{(EC.96)}$$

In turn, the seller's problem of finding the optimal pricing policy $f_t(\mathbf{n}), p_t(\mathbf{n})$ can be viewed as an optimal control of a continuous-time Markov jump process where events correspond to sales of passes, credit redemptions, and sales of the individual items. For each individual customer, the intensities of these events are a combination of the shopping opportunity intensity and the respective choice probabilities in which we add an explicit dependence on the state $\mathbf{n}$ policy f, p: $\lambda\pi^p_{0t}(\mathbf{n}, f, p)$, $\lambda\pi^r_{kt}(\mathbf{n}, f, p)$, and $\lambda\pi^s_{kt}(\mathbf{n}, f, p)$, $k = 0, \ldots, \bar{k}$. For the market as a whole in state $\mathbf{n}$, the intensities of these events aggregate across customers to $\lambda n_0\pi^p_{0t}(\mathbf{n}, f, p)$, $\lambda n_k\pi^r_{kt}(\mathbf{n}, f, p)$, $\lambda n_k\pi^s_{kt}(\mathbf{n}, f, p)$. In a discrete-time approximation, the probabilities of events are given by intensities multiplied by h. Moreover, at time t, the customers make their decision based on the expected utilities at time $t+h$, e.g.,

$$\pi^p_{0t}(\mathbf{n},f,p) = \exp\Big(\frac{a-p_t(\mathbf{n})+U_{\bar{k}(t+h)}(\mathbf{n}-\mathbf{e}_0+\mathbf{e}_{\bar{k}})-U_{0(t+h)}(\mathbf{n})}{\mu\gamma}\Big) \\ \times\Big[\exp\Big(\frac{a-p_t(\mathbf{n})+U_{\bar{k}(t+h)}(\mathbf{n}-\mathbf{e}_0+\mathbf{e}_{\bar{k}})-U_{0(t+h)}(\mathbf{n})}{\mu\gamma}\Big)+\exp\Big(\frac{a-f_t(\mathbf{n})}{\mu\gamma}\Big)\Big]^{\gamma-1} \\ \Big/\Big\{1+\Big[\exp\Big(\frac{a-p_t(\mathbf{n})+U_{\bar{k}(t+h)}(\mathbf{n}-\mathbf{e}_0+\mathbf{e}_{\bar{k}})-U_{0(t+h)}(\mathbf{n})}{\mu\gamma}\Big)+\exp\Big(\frac{a-f_t(\mathbf{n})}{\mu\gamma}\Big)\Big]^{\gamma}\Big\},$$

which generalizes equation (3). Other probabilities take a similar form directly generalizing equations (4)-(6).

The optimal policy is characterized by the Hamilton-Jacobi-Bellman equations for the sellers profit-to-go function $R_t(\mathbf{n})$:

$$R_t(\mathbf{n}) = \max_{f,p}\Big\{h\lambda n_0\pi^p_{0t}(\mathbf{n},f,p)\big(p-c+R_{t+h}(\mathbf{n}+\mathbf{e}_{\bar{k}}-\mathbf{e}_0)\big) \\ +\sum_{k=1}^{\bar{k}}h\lambda n_k\pi^r_{kt}(\mathbf{n},f,p)\big(-c+R_{t+h}(\mathbf{n}+\mathbf{e}_{k-1}-\mathbf{e}_k)\big)+\sum_{k=0}^{\bar{k}}h\lambda n_k\pi^s_{kt}(\mathbf{n},f,p)\big(f-c+R_{t+h}(\mathbf{n})\big) \\ +\Big(1-h\lambda n_0\pi^p_{0t}(\mathbf{n},f,p)-\sum_{k=1}^{\bar{k}}h\lambda n_k\pi^r_{kt}(\mathbf{n},f,p)-\sum_{k=0}^{\bar{k}}h\lambda n_k\pi^s_{kt}(\mathbf{n},f,p)\Big)R_{t+h}(\mathbf{n})\Big\}, \quad \text{(EC.97)}$$

with the boundary conditions

$$R_T(\mathbf{n}) = 0, \text{ for all } \mathbf{n}. \quad \text{(EC.98)}$$

In these equations, we account for all possible events in the Markov process describing sales tickets, passes, and redemption of passes. Only pass-related events in the sales process cause transitions in the state of the market, i.e., a change in $\mathbf{n}$. Given $\mathbf{n}$ in period t, the optimization problem can also be written more compactly as

$$\max_{f,p}\Big\{n_0\pi^p_{0t}(\mathbf{n},f,p)\big(p-c+R_{t+h}(\mathbf{n}+\mathbf{e}_{\bar{k}}-\mathbf{e}_0)-R_{t+h}(\mathbf{n})\big) \\ +\sum_{k=1}^{\bar{k}}n_k\pi^r_{kt}(\mathbf{n},f,p)\big(-c+R_{t+h}(\mathbf{n}+\mathbf{e}_{k-1}-\mathbf{e}_k)-R_{t+h}(\mathbf{n})\big)+\sum_{k=0}^{\bar{k}}n_k\pi^s_{kt}(\mathbf{n},f,p)\big(f-c\big)\Big\}.$$

The computational procedure for finding the optimal policy in discrete time proceeds as follows. In decreasing time order, $t=T-h,\dots,0$, we first find the optimal policy $f_t(\mathbf{n}), p_t(\mathbf{n})$ attaining the maximum in equation (EC.97) for each $\mathbf{n}$ and then we use this policy to compute profit-to-go $R_t(\mathbf{n})$ using (EC.97) as well as expected utilities $U_{tk}(\mathbf{n})$ using (EC.95)-(EC.95). Since, when solving for optimal policy at time t, the expected utilities and profit-to-go at $t+h$ are already computed, the necessary inputs for the optimization problem are all defined. The equations for the profit-to-go of

EC3.pdf

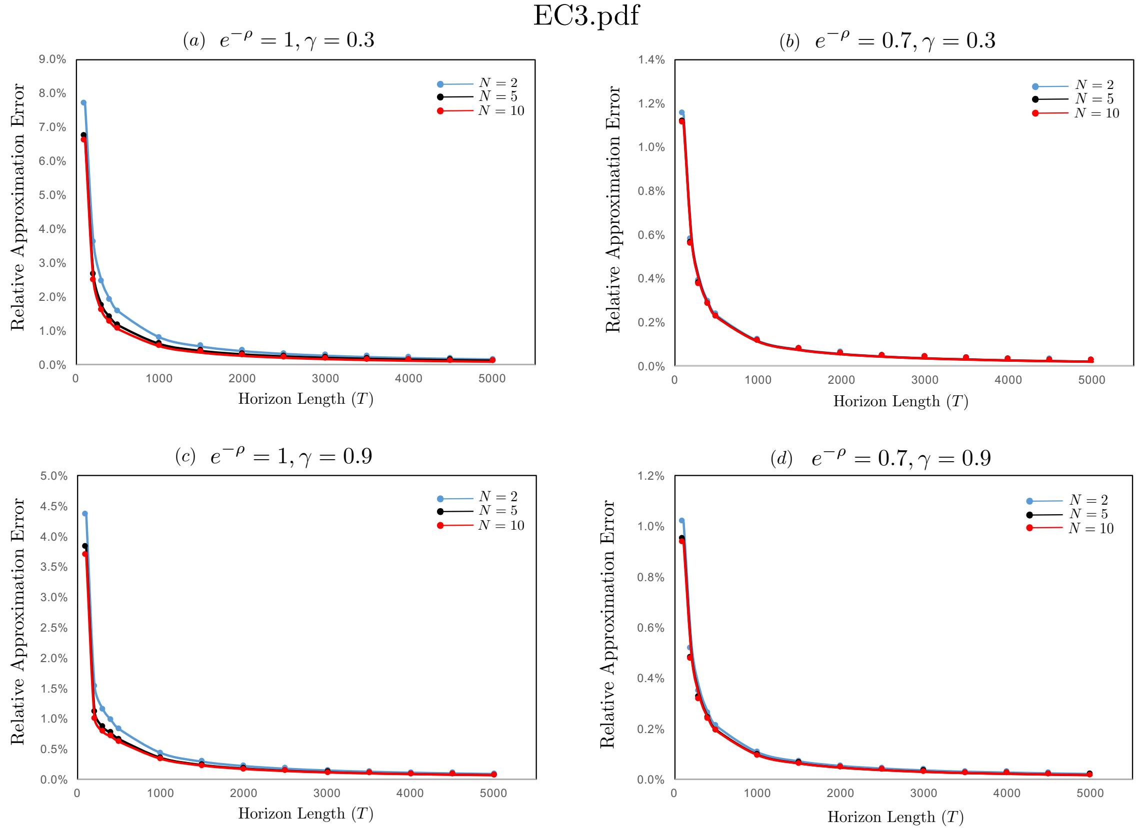


Figure EC.3 Using the turnpike prices in the full fledged stochastic systems.

the fixed price policy are similar and are obtained by substituting the fixed prices into equations similar to (EC.95)-(EC.95) and (EC.97) (for the latter, the maximization step would be dropped).

We have computed the optimal profit as well as the profit resulting from using the turnpike prices (called "Approx. Profit" in the table) for several combinations of the inputs. The relative difference between them is called the relative approximation error. In particular, we fixed $a = 2, \mu = 1, \bar{k} = 1, \lambda = 0.04, c = 0.1$ and varied $e^{-\rho} \in \{0.7, 1\}, \gamma \in \{0.3, 0.9\}, N \in \{2, 5, 10\}$. Time horizon was varied up to $T = 5000$. The results are shown in Figure EC.3 and Table EC.1 ($\bar{R}$ denotes the turnpike profit rate). There is indeed convergence of the turnpike-based fixed-price heuristic to optimality as T increases, and the convergence is taking place faster for larger N.

It is particularly interesting that the number of customers N did not have to be large to observe asymptotic behaviour. In other words, the system is very close to the asymptotic behavior when there are only a few customers. This may appear surprising because the deterministic fluid model is obtained by considering an infinite number of customers. We speculate that the quick convergence to asymptotic behavior is due to the pass purchase and redemption cycles. When the horizon is long, a single customer goes through this cycle for many times, exhibiting a behavior as if many customers are in the system.

Table EC.1 The turnpike prices approximation in full fledged stochastic systems (Figure EC.3-a).

Parameters : $(a,\mu,\bar{k},c,\lambda,e^{-\rho},\gamma)=(2,1,1,0.1,0.04,1,0.3)$ Turnpike prices: $\bar{f}=2.15469, \bar{p}=4.30939$					
	Horizon T	Optimal Profit	Approx. Profit	Approx. Error	Turnpike $\bar{R}\times T$
	100	4.41931	4.07949	7.69%	4.218
$N=2$	200	8.77588	8.45797	3.62%	8.437
	300	13.0488	12.7261	2.47%	12.656
	400	17.2882	16.9543	1.93%	16.875
	500	21.5147	21.1746	1.58%	21.094
	1000	42.6131	42.2688	0.80%	42.188
	1500	63.7070	63.3626	0.54%	63.282
	2000	84.8009	84.4565	0.40%	84.376
	2500	105.895	105.550	0.32%	105.470
	3000	126.989	126.644	0.27%	126.564
	3500	148.083	147.738	0.23%	147.658
	4000	169.176	168.832	0.20%	168.752
	4500	190.270	189.926	0.18%	189.846
	100	4.37245	4.07797	6.73%	4.218
$N=5$	200	8.68865	8.45647	2.67%	8.437
	300	12.9534	12.7256	1.76%	12.656
	400	17.1942	16.9541	1.39%	16.875
	500	21.4237	21.1745	1.16%	21.094
	1000	42.5277	42.2687	0.60%	42.188
	1500	63.6218	63.3625	0.40%	63.282
	2000	84.7157	84.4564	0.30%	84.376
	2500	105.810	105.550	0.24%	105.470
	3000	126.903	126.644	0.20%	126.564
	3500	147.997	147.738	0.17%	147.658
	4000	169.091	168.832	0.15%	168.752
	4500	190.185	189.926	0.13%	189.846
	100	4.36631	4.07748	6.61%	4.219
$N=10$	200	8.67137	8.45598	2.48%	8.437
	300	12.9322	12.7254	1.59%	12.656
	400	17.1725	16.954	1.27%	16.875
	500	21.4025	21.1744	1.06%	21.094
	1000	42.5086	42.2686	0.56%	42.188
	1500	63.6030	63.3625	0.37%	63.282
	2000	84.6969	84.4564	0.28%	84.376
	2500	105.791	105.550	0.22%	105.470
	3000	126.885	126.644	0.18%	126.564
	3500	147.979	147.738	0.16%	147.658
	4000	169.072	168.832	0.14%	168.752
	4500	190.166	189.926	0.12%	189.846

EC.27. Model for Dynamical $\bar{k}$.

Here, we focus on the single change: $\bar{k}$ changes from $\bar{k}_0$ to $\bar{k}_1$ at time τ. The case with multiple change points can be formulated in a similar way. We describe the formulation for the case with $k_0 > k_1$, namely, the number of credits decreases over time.

Before $\bar{k}$ decreases, namely, when $t \leqslant \tau$, the customer population evolves according to the following differential equations

$$\dot{w}_{kt} = \lambda w_{(k+1)t}\pi^r_{(k+1)t} - \lambda w_{kt}\pi^r_{kt}, \quad k=1,\dots,\bar{k}_0-1,$$

$$
\begin{aligned}
\dot{w}_{\bar{k}_0 t} &= \lambda w_{0t}\pi^p_{0t} - \lambda w_{\bar{k}_0 t}\pi^r_{\bar{k}_0 t},\\
\dot{w}_{0t} &= \lambda w_{1t}\pi^r_{1t} - \lambda w_{0t}\pi^p_{0},
\end{aligned}
$$

with initial condition $w_{00}=1$ and $w_{k0}=0$, for $k=1,\ldots,\bar{k}_0$. After $\bar{k}$ decreasing from $\bar{k}_0$ to $\bar{k}_1$, namely, when $t>\tau$, the state dynamics becomes more complex and is described as follows

$$
\begin{aligned}
\dot{w}_{\bar{k}_0 t} &= -\lambda w_{\bar{k}_0 t}\pi^r_{\bar{k}_0 t},\\
\dot{w}_{kt} &= \lambda w_{(k+1)t}\pi^r_{(k+1)t} - \lambda w_{kt}\pi^r_{kt}, \qquad k=\bar{k}_1+1,\ldots,\bar{k}_0-1,\\
\dot{w}_{\bar{k}_1 t} &= \lambda w_{(\bar{k}_1+1)t}\pi^r_{(\bar{k}_1+1)t} + \lambda w_{0t}\pi^p_{0t} - \lambda w_{\bar{k}_1 t}\pi^r_{\bar{k}_1 t},\\
\dot{w}_{kt} &= \lambda w_{(k+1)t}\pi^r_{(k+1)t} - \lambda w_{kt}\pi^r_{kt}, \qquad k=1,\ldots,\bar{k}_1-1,\\
\dot{w}_{0t} &= \lambda w_{1t}\pi^r_{1t} - \lambda w_{0t}\pi^p_{0}.
\end{aligned}
$$

Since the seller stops offering more than $\bar{k}_1+1$ credits after τ, the corresponding passholder population diminishes over time.

The customer utility equations are characterized by the following equations

$$
\begin{aligned}
\dot{U}_{kt} &= -\lambda\mu\ln\left\{1+\left[\exp\left(\frac{a-U_{kt}+U_{(k-1)t}}{\mu\gamma}\right)+\exp\left(\frac{a-f_t}{\mu\gamma}\right)\right]^{\gamma}\right\}+\rho U_{kt},\ k=1,\ldots,\bar{k}_0,\\
\dot{U}_{0t} &= \begin{cases} -\lambda\mu\ln\left\{1+\left[\exp\left(\frac{a-p_t+U_{\bar{k}_0 t}-U_{0t}}{\mu\gamma}\right)+\exp\left(\frac{a-f_t}{\mu\gamma}\right)\right]^{\gamma}\right\}+\rho U_{0t}, & t\leqslant\tau,\\ -\lambda\mu\ln\left\{1+\left[\exp\left(\frac{a-p_t+U_{\bar{k}_1 t}-U_{0t}}{\mu\gamma}\right)+\exp\left(\frac{a-f_t}{\mu\gamma}\right)\right]^{\gamma}\right\}+\rho U_{0t}, & t>\tau. \end{cases}
\end{aligned}
$$

with terminal conditions $U_{kT}=0$. Note that the dynamic $\bar{k}$ has a direct impact on the expression of $\dot{U}_{0t}$, through which it also impacts other $\dot{U}_{kt}$'s.

The choice probabilities are modified as follows

$$
\pi^p_{0t} = \begin{cases} \dfrac{\exp\left(\frac{a-p_t+U_{\bar{k}_0 t}}{\mu\gamma}\right)\left[\exp\left(\frac{a-p_t+U_{\bar{k}_0 t}}{\mu\gamma}\right)+\exp\left(\frac{a-f_t+U_{0t}}{\mu\gamma}\right)\right]^{\gamma-1}}{\exp\left(\frac{U_{0t}}{\mu}\right)+\left[\exp\left(\frac{a-p_t+U_{\bar{k}_0 t}}{\mu\gamma}\right)+\exp\left(\frac{a-f_t+U_{0t}}{\mu\gamma}\right)\right]^{\gamma}}, & t\leqslant\tau,\\[2ex] \dfrac{\exp\left(\frac{a-p_t+U_{\bar{k}_1 t}}{\mu\gamma}\right)\left[\exp\left(\frac{a-p_t+U_{\bar{k}_1 t}}{\mu\gamma}\right)+\exp\left(\frac{a-f_t+U_{0t}}{\mu\gamma}\right)\right]^{\gamma-1}}{\exp\left(\frac{U_{0t}}{\mu}\right)+\left[\exp\left(\frac{a-p_t+U_{\bar{k}_1 t}}{\mu\gamma}\right)+\exp\left(\frac{a-f_t+U_{0t}}{\mu\gamma}\right)\right]^{\gamma}}, & t>\tau, \end{cases}
$$

$$
\pi^s_{0t} = \begin{cases} \dfrac{\exp\left(\frac{a-f_t+U_{0t}}{\mu\gamma}\right)\left[\exp\left(\frac{a-p_t+U_{\bar{k}_0 t}}{\mu\gamma}\right)+\exp\left(\frac{a-f_t+U_{0t}}{\mu\gamma}\right)\right]^{\gamma-1}}{\exp\left(\frac{U_{0t}}{\mu}\right)+\left[\exp\left(\frac{a-p_t+U_{\bar{k}_0 t}}{\mu\gamma}\right)+\exp\left(\frac{a-f_t+U_{0t}}{\mu\gamma}\right)\right]^{\gamma}}, & t\leqslant\tau,\\[2ex] \dfrac{\exp\left(\frac{a-p_t+U_{\bar{k}_1 t}}{\mu\gamma}\right)\left[\exp\left(\frac{a-p_t+U_{\bar{k}_1 t}}{\mu\gamma}\right)+\exp\left(\frac{a-f_t+U_{0t}}{\mu\gamma}\right)\right]^{\gamma-1}}{\exp\left(\frac{U_{0t}}{\mu}\right)+\left[\exp\left(\frac{a-p_t+U_{\bar{k}_1 t}}{\mu\gamma}\right)+\exp\left(\frac{a-f_t+U_{0t}}{\mu\gamma}\right)\right]^{\gamma}}, & t>\tau, \end{cases}
$$

$$
\pi^r_{kt} = \frac{\exp\left(\frac{a+U_{(k-1)t}}{\mu\gamma}\right)\left[\exp\left(\frac{a+U_{(k-1)t}}{\mu\gamma}\right)+\exp\left(\frac{a-f_t+U_{kt}}{\mu\gamma}\right)\right]^{\gamma-1}}{\exp\left(\frac{U_{kt}}{\mu}\right)+\left[\exp\left(\frac{a+U_{(k-1)t}}{\mu\gamma}\right)+\exp\left(\frac{a-f_t+U_{kt}}{\mu\gamma}\right)\right]^{\gamma}}, \quad k=1,\ldots,\bar{k}_0
$$

$$\pi^s_{kt} = \frac{\exp\left(\frac{a-f_t+U_{kt}}{\mu\gamma}\right)\left[\exp\left(\frac{a+U_{(k-1)t}}{\mu\gamma}\right)+\exp\left(\frac{a-f_t+U_{kt}}{\mu\gamma}\right)\right]^{\gamma-1}}{\exp\left(\frac{U_{kt}}{\mu}\right)+\left[\exp\left(\frac{a+U_{(k-1)t}}{\mu\gamma}\right)+\exp\left(\frac{a-f_t+U_{kt}}{\mu\gamma}\right)\right]^{\gamma}}, \quad k=1,\ldots,\bar{k}_0.$$

The objective is

$$\max_{\{p_t,f_t\}} \int_0^T \big[\underbrace{\lambda(p_t-c)w_{0t}\pi^p_{0t}}_{\text{pass sales profit}} + \underbrace{\sum_{k=0}^{\bar{k}}\lambda(f_t-c)w_{kt}\pi^s_{kt}}_{\text{individual sales profit}} - \underbrace{\sum_{k=1}^{\bar{k}}\lambda c w_{kt}\pi^r_{kt}}_{\text{redemption cost}}\big]dt.$$