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Appendix A: Deferred Proofs from Section 2

Proof of Lemma 1. Fix any adaptive algorithm (which knows the arrival sequence, but not the realizations of the customers' purchase decisions, at the start) and consider its execution on setup $\mathcal{S}$ with arrival sequence $\mathcal{A}$. Let $X_{t,i}^{(j)}$ be the indicator random variable (0 or 1) for the algorithm offering item i at price j to customer t, and $P_{t,i}^{(j)}$ be the indicator random variable for customer t accepting when item i is offered to her at price j. On a given run, the constraints $\sum_{t=1}^{T}\sum_{j=1}^{m_i} P_{t,i}^{(j)} X_{t,i}^{(j)} \le k_i$ and $\sum_{i=1}^{n}\sum_{j=1}^{m_i} X_{t,i}^{(j)} \le 1$ are satisfied. Therefore, they are still satisfied after taking an expectation over all runs, and furthermore we can use independence to show that $\mathbb{E}[P_{t,i}^{(j)} X_{t,i}^{(j)}] = \mathbb{E}[P_{t,i}^{(j)}] \cdot \mathbb{E}[X_{t,i}^{(j)}] = p_{t,i}^{(j)} x_{t,i}^{(j)}$. Therefore, the algorithm must satisfy constraints (5b) and (5c) of the LP. Since its revenue on a given run is $\sum_{t=1}^{T}\sum_{i=1}^{n}\sum_{j=1}^{m_i} P_{t,i}^{(j)} r_i^{(j)} X_{t,i}^{(j)}$, taking an expectation over it yields (5a), completing the proof. $\square$

Proof of Proposition 1. The statement for $\sigma^{(j)}, \ldots, \sigma^{(j)}$ is immediate from the fact that the explicit value of $\sigma^{(j)}$ is $(1-\frac{r^{(j-1)}}{r^{(j)}})(1+\sum_{j'=2}^{m}(1-\frac{r^{(j'-1)}}{r^{(j')}}))^{-1}$, for all $j \in [m]$. To prove the statement for $\alpha^{(1)}, \ldots, \alpha^{(m)}$, we show that the solution to the system of n equations formed by (7) and $\alpha^{(1)} + \ldots \alpha^{(m)} = 1$ is unique and strictly positive.

Let $\gamma^{(j)} = e^{-\alpha^{(j)}}$ for all j. Then the constraint $\alpha^{(1)} + \ldots \alpha^{(m)} = 1$ can be rewritten as $\prod_{j=1}^{m}\gamma^{(j)} = \frac{1}{e}$. Furthermore, we derive from (7) that for all $j > 1$, $\gamma^{(j)} = (1-\frac{r^{(j-1)}}{r^{(j)}})\gamma^{(1)} + \frac{r^{(j-1)}}{r^{(j)}}$. Therefore,

$$\gamma^{(1)} \cdot \prod_{j=2}^{m}\Big((1-\frac{r^{(j-1)}}{r^{(j)}})\gamma^{(1)} + \frac{r^{(j-1)}}{r^{(j)}}\Big) = \frac{1}{e}. \tag{EC.1}$$

Consider the LHS of (EC.1) as a function of $\gamma^{(1)}$ on $[\frac{1}{e}, 1]$. This is a continuous, strictly increasing function which is at most $\frac{1}{e}$ when $\gamma^{(1)} = \frac{1}{e}$ and 1 when $\gamma^{(1)} = 1$. Therefore, there is a unique solution with $\gamma^{(1)} \in [\frac{1}{e}, 1)$, and the resulting value of $\alpha^{(1)}$ is positive. For $j > 1$, since $\gamma^{(j)}$ can also be written as $\gamma^{(1)} + \frac{r^{(j-1)}}{r^{(j)}}(1-\gamma^{(1)})$, it can be seen that $\gamma^{(j)} \in [\frac{1}{e}, 1)$, hence the unique value for $\alpha^{(j)}$ is positive as well. $\square$

Proof of Proposition 2. For the first inequality in (11), observe that $f(x) = \frac{x}{1-e^{-x}}$ is a strictly increasing function on $[0,1]$. Since $\sigma^{(1)} \in (0,1)$, $\frac{\sigma^{(1)}}{1-e^{-\sigma^{(1)}}} < \frac{1}{1-\frac{1}{e}}$, which is the desired result.

For the second inequality in (11), we show $\alpha^{(1)} > \sigma^{(1)}$, by showing that for all $j = 2, \ldots, m$, $\alpha^{(j)}$ is a smaller multiple of $\alpha^{(1)}$ than $\sigma^{(j)}$ is of $\sigma^{(1)}$. This suffices because both the fractions $\alpha^{(1)}, \ldots, \alpha^{(m)}$ and $\sigma^{(1)}, \ldots, \sigma^{(m)}$ must sum to 1. For a given j, we must establish that $\frac{\alpha^{(j)}}{\alpha^{(1)}} < \frac{\sigma^{(j)}}{\sigma^{(1)}}$. By definition, $\frac{\sigma^{(j)}}{\sigma^{(1)}} = 1 - \frac{r^{(j-1)}}{r^{(j)}} = \frac{1-e^{-\alpha^{(j)}}}{1-e^{-\alpha^{(1)}}}$. Therefore, is suffices to show that $\frac{\alpha^{(j)}}{\alpha^{(1)}} < \frac{1-e^{-\alpha^{(j)}}}{1-e^{-\alpha^{(1)}}}$, or $\frac{\alpha^{(j)}}{1-e^{-\alpha^{(j)}}} < \frac{\alpha^{(1)}}{1-e^{-\alpha^{(1)}}}$. This follows from the fact that the function $f(x) = \frac{x}{1-e^{-x}}$ is strictly increasing.

To prove (12), note that $\sigma^{(1)} = (1+\sum_{j=2}^{m}(1-\frac{r^{(j-1)}}{r^{(j)}}))^{-1}$, while $1 + \ln\frac{r^{(m)}}{r^{(1)}} = 1 + \sum_{j=2}^{m}\ln\frac{r^{(j)}}{r^{(j-1)}}$. Therefore, it suffices to show that for any $j = 2, \ldots, m$, $\ln\frac{r^{(j)}}{r^{(j-1)}} > 1 - \frac{r^{(j-1)}}{r^{(j)}}$. Letting $x = \ln\frac{r^{(j-1)}}{r^{(j)}} < 0$, the desired inequality becomes $-x > 1 - e^{x}$, which is immediate.

For (13), we would like to prove that $\alpha < \alpha^{(1)}$. Note that $\alpha^{(1)}$ is the unique solution to

$$\alpha^{(1)} + \sum_{j=2}^{m} \Big[-\ln\big(1-(1-e^{-\alpha^{(1)}})(1-\frac{r^{(j-1)}}{r^{(j)}})\big)\Big] = 1, \tag{EC.2}$$

while α is the unique solution to

$$\alpha + \sum_{j=2}^{m} (1-e^{-\alpha}) \ln \frac{r^{(j)}}{r^{(j-1)}} = 1. \tag{EC.3}$$

The LHS of (EC.2), as a function of $\alpha^{(1)}$, is increasing over $(0,1)$; the same can be said about the LHS of (EC.3) as a function of α. Therefore, it suffices to show that if $\alpha^{(1)} = \alpha = x$, then the LHS of (EC.2) is strictly less than the LHS of (EC.3), for all $x \in (0,1)$.

Let $F = 1-e^{-x}$ and consider any $j > 1$. Let $s = \frac{r^{(j-1)}}{r^{(j)}} \in (0,1)$. It suffices to show that $-\ln(1-F(1-s)) < F \cdot \ln\frac{1}{s}$, which can be rearranged as $\frac{1-s^F}{1-s} > F$. For the final inequality, note that $f(s) = s^F$ is a strictly concave function on $(0,1)$, since $F \in (0,1)$. Therefore, $\frac{1-s^F}{1-s} > F$, because the LHS is the slope of the secant line through (s, s^F) and $(1,1)$, while the RHS is the slope of the tangent line through $(1,1)$. □

Appendix B: Supplement to Section 3

The first subsection contains the deferred proofs from Section 3. In the second subsection, we explain how to optimize the randomized procedure for generating a single value function. In the third subsection, we put together the proof of Theorem 1.

The following inequality will be useful throughout the paper. For all $j = 2,\dots,m$, (7) says that $1-e^{-\alpha^{(j)}} \le 1-\frac{r^{(j-1)}}{r^{(j)}}$, where we have used the fact that $1-e^{-\alpha^{(1)}} \le 1$. Therefore, for all $j = 2,\dots,m$, we can derive that

$$\frac{r^{(j-1)}}{r^{(j)}} \le e^{-\alpha^{(j)}}. \tag{EC.4}$$

B.1. Deferred Proofs

Proof of Theorem 4. Define $N_{t,i}$ to be the algorithm's value for N_i at the *end* of time t ($N_{0,i}$ is understood to be 0), for all $t \in [T]$ and $i \in [n]$. For all $t \in [T]$, define $R_t = r_{i_t^*}^{(j_t^*)}$ and $Z_t = \tilde{\Phi}_{i_t^*}(\tilde{L}_{i_t^*}^{(j_t^*)}) - \tilde{\Phi}_{i_t^*}(N_{i_t^*}/k_{i_t^*})$ if a sale was made during time t; define $R_t = Z_t = 0$ otherwise.

Consider the solution to the dual LP (19) formed by setting $y_i = \mathbb{E}[\tilde{\Phi}_i(\frac{N_{T,i}}{k_i})]$ for all $i \in [n]$, and $z_t = \mathbb{E}[Z_t]$ for all $t \in [T]$. We claim that this solution is feasible. The non-negativity constraint (19c) can be verified directly from the definitions.

Now, consider constraint (19b) for a fixed $t \in [T], i \in [n], j \in [m_i]$. Given the initializations of $\tilde{L}_i^{(1)},\dots,\tilde{L}_i^{(m_i)},\tilde{\Phi}_i$ and the value of $N_{t-1,i}$, the algorithm will always make a decision during time t

which earns pseudorevenue whose conditional expectation is at least $p^{(j)}_{t,i}(\tilde{\Phi}_i(\tilde{L}^{(j)}_i) - \tilde{\Phi}_i(\frac{N_{t-1,i}}{k_i}))$, by definition (16). Formally,

$$\mathbb{E}[Z_t|\tilde{L}^{(1)}_i,\dots,\tilde{L}^{(m_i)}_i,\tilde{\Phi}_i,N_{t-1,i}] \ge p^{(j)}_{t,i}(\tilde{\Phi}_i(\tilde{L}^{(j)}_i) - \tilde{\Phi}_i(\frac{N_{t-1,i}}{k_i})),$$

for all values of $\tilde{L}^{(1)}_i,\dots,\tilde{L}^{(m_i)}_i,\tilde{\Phi}_i,N_{t-1,i}$. By the tower property of conditional expectation, $z_t = \mathbb{E}[Z_t] \ge \mathbb{E}[p^{(j)}_{t,i}(\tilde{\Phi}_i(\tilde{L}^{(j)}_i) - \tilde{\Phi}_i(\frac{N_{t-1,i}}{k_i}))]$. Meanwhile, y_i has been set to $\mathbb{E}[\tilde{\Phi}_i(\frac{N_{T,i}}{k_i})]$. Since $N_{T,i} \ge N_{t-1,i}$ and $\tilde{\Phi}_i$ is increasing, $y_i \ge \mathbb{E}[\tilde{\Phi}_i(\frac{N_{t-1,i}}{k_i})]$. Therefore, the LHS of (19b), $p^{(j)}_{t,i}y_i + z_t$, is at least $\mathbb{E}[p^{(j)}_{t,i}(\tilde{\Phi}_i(\tilde{L}^{(j)}_i))]$. By (18), this is at least $r^{(j)}_i$, completing the proof of feasibility.

Applying weak duality, we obtain

$$\begin{aligned}\mathsf{OPT}(\mathcal{S},\mathcal{A}) &\le \sum_{i=1}^n k_i\mathbb{E}[\tilde{\Phi}_i(\frac{N_{T,i}}{k_i})] + \sum_{t=1}^T \mathbb{E}[Z_t] \\ &= \sum_{i=1}^n k_i\mathbb{E}\Big[\sum_{t=1}^T(\tilde{\Phi}_i(\frac{N_{t,i}}{k_i}) - \tilde{\Phi}_i(\frac{N_{t-1,i}}{k_i}))\Big] + \sum_{t=1}^T \mathbb{E}[Z_t] \\ &= \sum_{t=1}^T \mathbb{E}\Big[\sum_{i=1}^n k_i(\tilde{\Phi}_i(\frac{N_{t,i}}{k_i}) - \tilde{\Phi}_i(\frac{N_{t-1,i}}{k_i})) + Z_t\Big]. \end{aligned} \tag{EC.5}$$

We now analyze the term inside the expectation,

$$\sum_{i=1}^n k_i(\tilde{\Phi}_i(\frac{N_{t,i}}{k_i}) - \tilde{\Phi}_i(\frac{N_{t-1,i}}{k_i})) + Z_t, \tag{EC.6}$$

for every $t \in [T]$. We would like to argue that it is at most $\frac{R_t}{c}$, on every sample path.

There are two cases. If an item $i = i^*_t$ was sold at price $j = j^*_t$ during time t, then (EC.6) equals

$$k_i(\tilde{\Phi}_i(\frac{N_{t-1,i}+1}{k_i}) - \tilde{\Phi}_i(\frac{N_{t-1,i}}{k_i})) + \tilde{\Phi}_i(\tilde{L}^{(j)}_i) - \tilde{\Phi}_i(\frac{N_{t-1,i}}{k_i}). \tag{EC.7}$$

Indeed, $N_{t,i} = N_{t-1,i}+1$, $N_{t,i} = N_{t-1,i}$ for all $i \ne i$, and $Z_t = \tilde{\Phi}_i(\tilde{L}^{(j)}_i) - \tilde{\Phi}_i(\frac{N_{t-1,i}}{k_i})$ by definition. Furthermore, since Z_t is positive, $N_{t-1,i}$ must by less than $\tilde{L}^{(j)}_i k$. Therefore, we can invoke (17) to get that (EC.7) is at most $r^{(j)}_i/c$, which is equal to $\frac{R_t}{c}$ by definition. In the other case, if no item was sold during time t, then (EC.7) is 0, while $R_t = 0$ too, so (EC.7) is still at most $\frac{R_t}{c}$.

Substituting back into (EC.5), we conclude that $\mathsf{OPT}(\mathcal{S},\mathcal{A}) \le \sum_{t=1}^T \mathbb{E}[\frac{R_t}{c}]$, which is equal to $\frac{1}{c}\mathbb{E}[\mathsf{ALG}(\mathcal{S},\mathcal{A})]$ by definition. This completes the proof of Algorithm 1 having a competitive ratio at least c. $\square$

Proof of Proposition 3. For (21), note that $\mathbb{E}[\tilde{L}^{(j)}] = \frac{\lfloor L^{(j)}k\rfloor+1}{k}(L^{(j)}k - \lfloor L^{(j)}k\rfloor) + \frac{\lfloor L^{(j)}k\rfloor}{k}(1 - (L^{(j)}k - \lfloor L^{(j)}k\rfloor)) = \frac{1}{k}(L^{(j)}k - \lfloor L^{(j)}k\rfloor) + \frac{\lfloor L^{(j)}k\rfloor}{k} = L^{(j)}$.

For (22), note that $|(\tilde{L}^{(j)} - \tilde{L}^{(j')}) - (L^{(j)} - L^{(j')})| = |(\tilde{L}^{(j)} - L^{(j)}) - (\tilde{L}^{(j')} - L^{(j')})|$. We will prove that $(\tilde{L}^{(j)} - L^{(j)}) - (\tilde{L}^{(j')} - L^{(j')}) \le \frac{1}{k}$; the inequality that $(\tilde{L}^{(j)} - L^{(j)}) - (\tilde{L}^{(j')} - L^{(j')}) \ge -\frac{1}{k}$ follows by symmetry. The maximum value of $k\tilde{L}^{(j)}$ is $\lfloor kL^{(j)}\rfloor+1$ while the minimum value of $k\tilde{L}^{(j')}$ is $\lfloor kL^{(j')}\rfloor$, hence the result is immediate unless $(\lfloor kL^{(j)}\rfloor+1)-kL^{(j)}+kL^{(j')}-\lfloor kL^{(j')}\rfloor>1$, i.e. $kL^{(j')}-\lfloor kL^{(j')}\rfloor>kL^{(j)}-\lfloor kL^{(j)}\rfloor$. However, in this case, if $k\tilde{L}^{(j)}=\lfloor kL^{(j)}\rfloor+1$, then $W<kL^{(j)}-\lfloor kL^{(j)}\rfloor<kL^{(j')}-\lfloor kL^{(j')}\rfloor$ and hence $\tilde{L}^{(j')}$ is rounded up as well. Similarly, if $\tilde{L}^{(j')}$ is rounded down, then $\tilde{L}^{(j)}$ must be rounded down as well. If $\tilde{L}^{(j)}$ and $\tilde{L}^{(j')}$ are rounded in the same direction, then (iii) holds. □

Proof of Theorem 5. First we prove (18), the claim that $\mathbb{E}[\tilde{\Phi}(\tilde{L}^{(j)})]\ge r^{(j)}$, inductively. Clearly $\mathbb{E}[\Phi(\tilde{L}^{(0)})]\ge r^{(0)}=0$. Now consider $j\in[m]$ and suppose we have established (18) for the $j-1$ case. We can compare expression (20) with $q=\tilde{L}^{(j)}$ and $q=\tilde{L}^{(j-1)}$ to obtain $\tilde{\Phi}(\tilde{L}^{(j)})=\tilde{\Phi}(\tilde{L}^{(j)})+(r^{(j)}-r^{(j-1)})\frac{\exp(\tilde{L}^{(j)}-\tilde{L}^{(j-1)})-1}{\exp(\alpha^{(j)})-1}$. Therefore,

$$\begin{aligned}\mathbb{E}[\tilde{\Phi}(\tilde{L}^{(j)})]&\ge r^{(j-1)}+(r^{(j)}-r^{(j-1)})\frac{\mathbb{E}[\exp(\tilde{L}^{(j)}-\tilde{L}^{(j-1)})]-1}{\exp(\alpha^{(j)})-1}\\&\ge r^{(j-1)}+(r^{(j)}-r^{(j-1)})\frac{\exp(\mathbb{E}[\tilde{L}^{(j)}-\tilde{L}^{(j-1)}])-1}{\exp(\alpha^{(j)})-1}\\&=r^{(j-1)}+(r^{(j)}-r^{(j-1)})\frac{\exp(\alpha^{(j)})-1}{\exp(\alpha^{(j)})-1}\end{aligned}$$

where the first inequality uses the induction hypothesis, and the second inequality uses Jensen's inequality (the exponential function exp is convex). The equality follows from (21) and the definition that $\alpha^{(j)}=L^{(j)}-L^{(j-1)}$, completing the induction.

Now we prove (17) for an arbitrary $j\in[m]$ and $N\in\{0,\dots,\tilde{L}^{(j)}k-1\}$. Let $q=\frac{N}{k}$ and $\ell=\tilde{\ell}(q)$. Note that $1\le\ell\le j$, and $\tilde{L}^{(\ell-1)}\le q<\tilde{L}^{(\ell)}$. Substituting $q=\frac{N}{k}$ into the LHS of (17), we get $k\big(\tilde{\Phi}(q+\frac{1}{k})-\tilde{\Phi}(q)\big)+\tilde{\Phi}(\tilde{L}^{(j)})-\tilde{\Phi}(q)$. Adding and subtracting $\tilde{\Phi}(\tilde{L}^{(\ell)})$ and rearranging, we get

$$k\big(\tilde{\Phi}(q+\frac{1}{k})-\tilde{\Phi}(q)\big)+\tilde{\Phi}(\tilde{L}^{(\ell)})-\tilde{\Phi}(q)+\tilde{\Phi}(\tilde{L}^{(j)})-\tilde{\Phi}(\tilde{L}^{(\ell)}). \tag{EC.8}$$

The following upper bound can be derived for expression (EC.8):

$$\begin{aligned}&k\big(\tilde{\Phi}(q+\frac{1}{k})-\tilde{\Phi}(q)\big)+\tilde{\Phi}(\tilde{L}^{(\ell)})-\tilde{\Phi}(q)+\tilde{\Phi}(\tilde{L}^{(j)})-\tilde{\Phi}(\tilde{L}^{(\ell)})\\&=(r^{(\ell)}-r^{(\ell-1)})\frac{e^{q+1/k-\tilde{L}^{(\ell-1)}}(k-(k+1)e^{-1/k})+e^{\tilde{L}^{(\ell)}-\tilde{L}^{(\ell-1)}}}{e^{\alpha^{(\ell)}}-1}+\sum_{\ell'=\ell+1}^{j}(r^{(\ell')}-r^{(\ell'-1)})\frac{e^{\tilde{L}^{(\ell')}-\tilde{L}^{(\ell'-1)}}-1}{e^{\alpha^{(\ell')}}-1}\\&\le(r^{(\ell)}-r^{(\ell-1)})\frac{e^{\tilde{L}^{(\ell)}-\tilde{L}^{(\ell-1)}}(k-(k+1)e^{-1/k})+e^{\tilde{L}^{(\ell)}-\tilde{L}^{(\ell-1)}}}{e^{\alpha^{(\ell)}}-1}+\sum_{\ell'=\ell+1}^{j}(r^{(\ell')}-r^{(\ell'-1)})\frac{e^{\tilde{L}^{(\ell')}-\tilde{L}^{(\ell'-1)}}-1}{e^{\alpha^{(\ell')}}-1}\\&=(r^{(\ell)}-r^{(\ell-1)})\frac{e^{\tilde{L}^{(\ell)}-\tilde{L}^{(\ell-1)}-\alpha^{(\ell)}}(1+k)(1-e^{-1/k})}{1-e^{-\alpha^{(\ell)}}}+\sum_{\ell'=\ell+1}^{j}(r^{(\ell')}-r^{(\ell'-1)})\frac{e^{\tilde{L}^{(\ell')}-\tilde{L}^{(\ell'-1)}-\alpha^{(\ell')}}-e^{-\alpha^{(\ell')}}}{1-e^{-\alpha^{(\ell')}}}.\end{aligned} \tag{EC.9}$$

The inequality holds because $k-(1+k)e^{-1/k}>0$ for all $k\in\mathbb{N}$, and q is at most $\tilde{L}^{(\ell)}-1/k$.

It suffices to show that expression (EC.9) is bounded from above by

$$r^{(j)}\frac{(1+k)(e^{1/k}-1)}{1-e^{-\alpha^{(1)}}}. \tag{EC.10}$$

To assist in this task, we would like to establish the following for all $\ell' = \ell+1,\dots,j$ and $\ell'' \in \{\ell,\dots,\ell'-1\}$:

$$\begin{aligned}&(r^{(\ell'-1)}-r^{(\ell'-2)})\frac{e^{\tilde{L}^{(\ell'-1)}-\tilde{L}^{(\ell''-1)}-L^{(\ell'-1)}+L^{(\ell''-1)}}(1+k)(1-e^{-1/k})}{1-e^{-\alpha^{(\ell'-1)}}}+(r^{(\ell')}-r^{(\ell'-1)})\frac{e^{\tilde{L}^{(\ell')}-\tilde{L}^{(\ell'-1)}-\alpha^{(\ell')}}-e^{-\alpha^{(\ell')}}}{1-e^{-\alpha^{(\ell')}}}\\ \le&(r^{(\ell')}-r^{(\ell'-1)})\frac{e^{\max\{\tilde{L}^{(\ell')}-\tilde{L}^{(\ell''-1)}-L^{(\ell')}+L^{(\ell''-1)},\tilde{L}^{(\ell')}-\tilde{L}^{(\ell'-1)}-L^{(\ell')}+L^{(\ell'-1)}\}}(1+k)(1-e^{-1/k})}{1-e^{-\alpha^{(\ell')}}}.\end{aligned} \tag{EC.11}$$

But $\frac{r^{(\ell'-1)}-r^{(\ell'-2)}}{1-e^{-\alpha^{(\ell'-1)}}} = \frac{r^{(\ell')}-r^{(\ell'-1)}}{1-e^{-\alpha^{(\ell')}}}\cdot\frac{r^{(\ell'-1)}}{r^{(\ell')}}$ due to the definition of α in (7), and $\frac{r^{(\ell'-1)}}{r^{(\ell')}} \le e^{-\alpha^{(\ell')}}$ due to (EC.4). Substituting back into inequality (EC.11), it suffices to prove

$$\begin{aligned}&e^{\tilde{L}^{(\ell'-1)}-\tilde{L}^{(\ell''-1)}-L^{(\ell')}+L^{(\ell''-1)}}(1+k)(1-e^{-1/k})+e^{\tilde{L}^{(\ell')}-\tilde{L}^{(\ell'-1)}-\alpha^{(\ell')}}-e^{-\alpha^{(\ell')}}\\ \le&e^{\max\{\tilde{L}^{(\ell')}-\tilde{L}^{(\ell''-1)}-L^{(\ell')}+L^{(\ell''-1)},\tilde{L}^{(\ell')}-\tilde{L}^{(\ell'-1)}-L^{(\ell')}+L^{(\ell'-1)}\}}(1+k)(1-e^{-1/k})\end{aligned}$$

where we have used Definition 1 to rewrite the first exponent. Now,

$$e^{\tilde{L}^{(\ell'-1)}-\tilde{L}^{(\ell''-1)}-L^{(\ell')}+L^{(\ell''-1)}}(k-(1+k)e^{-1/k})\le e^{\tilde{L}^{(\ell')}-\tilde{L}^{(\ell''-1)}-L^{(\ell')}+L^{(\ell''-1)}}(k-(1+k)1-e^{-1/k}),$$

since $k-(1+k)e^{-1/k}>0$ and $\tilde{L}^{(\ell'-1)}\le\tilde{L}^{(\ell')}$. Thus it remains to prove that

$$e^{\tilde{L}^{(\ell'-1)}-\tilde{L}^{(\ell''-1)}-L^{(\ell')}+L^{(\ell''-1)}}+e^{\tilde{L}^{(\ell')}-\tilde{L}^{(\ell'-1)}-\alpha^{(\ell')}}-e^{-\alpha^{(\ell')}}\le e^{\max\{\tilde{L}^{(\ell')}-\tilde{L}^{(\ell''-1)}-L^{(\ell')}+L^{(\ell''-1)},\tilde{L}^{(\ell')}-\tilde{L}^{(\ell'-1)}-L^{(\ell')}+L^{(\ell'-1)}\}}. \tag{EC.12}$$

We consider two cases. First suppose $\tilde{L}^{(\ell')}-\tilde{L}^{(\ell''-1)}-L^{(\ell')}+L^{(\ell''-1)}\le\tilde{L}^{(\ell')}-\tilde{L}^{(\ell'-1)}-L^{(\ell')}+L^{(\ell'-1)}$, i.e. $\tilde{L}^{(\ell'-1)}-\tilde{L}^{(\ell''-1)}\le L^{(\ell'-1)}-L^{(\ell''-1)}$. Then the LHS of (EC.12) equals $e^{-\alpha^{(\ell')}}+e^{\tilde{L}^{(\ell')}-\tilde{L}^{(\ell'-1)}-\alpha^{(\ell')}}-e^{-\alpha^{(\ell')}}=e^{\tilde{L}^{(\ell')}-\tilde{L}^{(\ell'-1)}-\alpha^{(\ell')}}$, which equals the RHS of (EC.12) by the assumption that $\tilde{L}^{(\ell'-1)}-\tilde{L}^{(\ell''-1)}\le L^{(\ell'-1)}-L^{(\ell''-1)}$. In the second case, suppose $\tilde{L}^{(\ell')}-\tilde{L}^{(\ell''-1)}-L^{(\ell')}+L^{(\ell''-1)}>\tilde{L}^{(\ell')}-\tilde{L}^{(\ell'-1)}-L^{(\ell')}+L^{(\ell'-1)}$, i.e. $\tilde{L}^{(\ell'-1)}-\tilde{L}^{(\ell''-1)}>L^{(\ell'-1)}-L^{(\ell''-1)}$. Then inequality (EC.12) can be rearranged as

$$e^{-\alpha^{(\ell')}}(e^{\tilde{L}^{(\ell'-1)}-\tilde{L}^{(\ell''-1)}-L^{(\ell'-1)}+L^{(\ell''-1)}}-1)(e^{\tilde{L}^{(\ell')}-\tilde{L}^{(\ell'-1)}}-1)\ge 0.$$

The first bracket is positive by the assumption that $\tilde{L}^{(\ell'-1)}-\tilde{L}^{(\ell''-1)}>L^{(\ell'-1)}-L^{(\ell''-1)}$ and the second bracket is non-negative since $\tilde{L}^{(\ell'-1)}\le\tilde{L}^{(\ell')}$. This finishes the proof of (EC.12), and hence (EC.11).

Equipped with (EC.11), we return the task of proving that expression (EC.9) is at most expression (EC.10). If we inductively apply inequality (EC.11) to expression (EC.9) for $\ell' = \ell+1,\dots,j$ (when $\ell' = \ell+1$, $\ell'' = \ell$; when $\ell' = \ell+2$, $\ell'' = \ell$ if we arrived at case two during iteration $\ell+1$ and $\ell'' = \ell+1$ otherwise,...), we conclude that expression (EC.9) is bounded from above by

$$(r^{(j)} - r^{(j-1)})\frac{e^{\tilde{L}^{(j)} - \tilde{L}^{(\ell''-1)} - L^{(j)} + L^{(\ell''-1)}}(1+k)(1-e^{-1/k})}{1-e^{-\alpha^{(j)}}}$$

for some $\ell'' \in \{\ell,\dots,j\}$. The fact that $1 - e^{-\alpha^{(1)}} = \frac{r^{(j)}}{r^{(j)} - r^{(j-1)}}(1-e^{-\alpha^{(j)}})$, due to (7), and the fact that $(\tilde{L}^{(j)} - \tilde{L}^{(\ell''-1)}) - (L^{(j)} - L^{(\ell''-1)}) \le 1/k$, due to (22), complete the proof of expression (EC.9) being at most expression (EC.10), and thus the proof of Theorem 5 for general m.

Finally, when $m=1$, $\alpha^{(1)} = 1$. In the above proof, since j and ℓ are always 1, (EC.9) can be replaced by $r^{(1)} \cdot \frac{(1)(1+k)(1-e^{-1/k})}{1-e^{-1}}$, where we have used the fact that $L^{(1)} = k$ always. This is immediately at most $\frac{r^{(j)}}{c}$, for the improved value of $c = \frac{1-e^{-\alpha^{(1)}}}{(1+k)(1-e^{-1/k})}$, completing the proof of Theorem 5 in its entirety. □

B.2. Optimizing the Randomized Procedure

We can explicitly formulate the optimization problem over randomized procedures for a single item with starting inventory k and m prices $r^{(1)},\dots,r^{(m)}$. Using the "balls in bins" counting argument, the number of configurations satisfying (14) is $D := \binom{k+m-1}{m-1}$.

We refer to these configurations in an arbitrary order using the index $d \in [D]$, where we let ρ_d denote the probability of choosing configuration d, $f_d(\cdot)$ denote the value function for d, and $L_d^{(j)}$ denote the value of $\tilde{L}^{(j)}$ under configuration d for all $j = 0,\dots,m$. The optimization problem of satisfying (17)–(18) with a maximal value of c can be formulated as follows:

$$\tilde{F} := \sup c \qquad \text{(EC.13a)}$$

$$k\Big(f_d\big(\frac{N+1}{k}\big) - f_d\big(\frac{N}{k}\big)\Big) + f_d(L_d^{(j)}) - f_d\big(\frac{N}{k}\big) \le \frac{r^{(j)}}{c} \qquad d \in [D], j \in [m], 0 \le N \le kL_d^{(j)} - 1 \qquad \text{(EC.13b)}$$

$$f_d(1) \ge \dots \ge f_d\big(\frac{1}{k}\big) \ge f_d(0) = 0 \qquad d \in [D] \qquad \text{(EC.13c)}$$

$$\sum_{d=1}^{D} \rho_d f_d(L_d^{(j)}) \ge r^{(j)} \qquad j \in [m] \qquad \text{(EC.13d)}$$

$$\sum_{d=1}^{D} \rho_d = 1 \qquad \text{(EC.13e)}$$

$$f_d(0), f_d(\frac{1}{k}), \ldots, f_d(1) \in \mathbb{R}; \rho_d \geq 0 \qquad d \in [D] \qquad \text{(EC.13f)}$$

Constraint (EC.13b) corresponds to (17), constraint (EC.13d) corresponds to (18), while constraint (EC.13c) enforces the definition of a value function in (15). We let $\tilde{F}$ denote the optimal objective value of (EC.13). Unfortunately, it is difficult to solve (EC.13) exactly, since the number of configurations D is exponential in the number of prices m, and constraint (EC.13d) is non-linear.

Nonetheless, (EC.13) is useful at determining the best competitive ratio which could be established *using our analysis*. We know that the randomized procedure from Definition 3 (based on Φ) is an optimal solution to (EC.13) as $k \to \infty$, since it achieves the optimal competitive ratio possible.

We can also solve (EC.13) exactly when $k = 1$, in which case $D = m$, where we will let $d \in [D]$ denote the configuration with $\tilde{L}^{(0)} = \ldots = \tilde{L}^{(d-1)} = 0$ and $\tilde{L}^{(d)} = \ldots = \tilde{L}^{(m)} = 1$. (EC.13b) reduces to $2f_d(1) \leq \frac{r^{(j)}}{c}$, and needs to hold for $d \in [D]$, $j \geq d$ (for $j < d$, $kL_d^{(j)} - 1 = -1$). However, clearly only the constraint with $j = d$ is binding. As a result, (EC.13b) corresponds to m constraints. (EC.13d) corresponds to m constraints of the form $\sum_{d=1}^{j} \rho_d f_d(1) \geq r^{(j)}$, for $j \in [m]$.

Not counting $f_d(0)$, which must be set to 0, there are $2m+1$ variables: $\{f_d(1), \rho_d : d \in [D]\}$ and c. Consider the system of equations obtained in these $2m+1$ variables by setting (EC.13b), (EC.13d), and (EC.13e) to equality. It can be checked that the unique solution is

$$f_d(1) = \frac{r^{(d)}}{\sigma^{(1)}}, \forall d \in [D]; \rho_d = \sigma^{(d)}, \forall d \in [D]; c = \frac{\sigma^{(1)}}{2} \qquad \text{(EC.14)}$$

with $\sigma^{(1)}, \ldots, \sigma^{(m)}$ defined from $r^{(1)}, \ldots, r^{(m)}$ according to (8). Furthermore, this solution is both feasible, satisfying the non-negativity constraints in (EC.13c) and (EC.13f), and optimal. Therefore, the value of $\tilde{F}$ is $\frac{\sigma^{(1)}}{2}$.

B.3. Proof of Theorem 1

Now we put together the proof of Theorem 1. For all items $i \in [n]$, $\tilde{F}_i$ is defined to be the optimal objective value of (EC.13), with $k = k_i$, $m = m_i$, and $r^{(1)} = r_i^{(1)}, \ldots, r^{(m)} = r_i^{(m_i)}$. Consider Algorithm 1, where for all i, the randomized procedure used to initialize $\tilde{\Phi}_i$ is an optimal solution to

(EC.13) achieving the objective value of $\tilde{F}_i$. For all i, (17)–(18) is satisfied as long as $c \le \tilde{F}_i$. Therefore, the maximum value of c satisfying the conditions of Theorem 4 is $\min_i \tilde{F}_i$. By Theorem 4, this algorithm achieves a competitive ratio of $\min_i \tilde{F}_i$.

To establish bounds (i)–(iii) from Theorem 1, for all i, we need to find a feasible randomized procedure with an objective value in (EC.13) equal to the bound. For bounds (i) and (iii), this is established directly by the randomized procedure from Definition 3, via the statement of Theorem 5. For bound (ii), we need to split the k_i units of item i into k_i disparate items. For each single-unit item, its value function in Algorithm 1 is initialized according to the randomized procedure described by (EC.14). This yields a value of $\frac{\sigma_i^{(1)}}{2}$, completing the proof of Theorem 1.

Appendix C: Deferred Proofs from Section 4

Proof of Lemma 2. Since the algorithm was willing to sell item i at price j, it must be the case that $W_i < L_i^{(j)}$. Let ℓ denote $\ell_i(W_i)$, which is at most j. We ignore measure-zero events and assume that $W_i \neq L^{(\ell-1)}$. We can rearrange Z_t as

$$\begin{aligned} & r_i^{(j)} - r_i^{(\ell)} + r_i^{(\ell)} - \left(r_i^{(\ell-1)} + (r_i^{(\ell)} - r_i^{(\ell-1)}) \frac{\exp(W_i - L_i^{(\ell-1)}) - 1}{\exp(\alpha_i^{(\ell)}) - 1} \right) \\ & = r_i^{(j)} - r_i^{(\ell)} + (r_i^{(\ell)} - r_i^{(\ell-1)}) \frac{\exp(\alpha_i^{(\ell)}) - \exp(W_i - L_i^{(\ell-1)})}{\exp(\alpha_i^{(\ell)}) - 1}. \end{aligned}$$

Adding $Y_i = \Phi'_{\mathcal{P}_i}(W_i) = (r_i^{(\ell)} - r_i^{(\ell-1)}) \frac{\exp(W_i - L_i^{(\ell-1)})}{\exp(\alpha_i^{(\ell)}) - 1}$ to this expression, we get $r_i^{(j)} - r_i^{(\ell)} + \frac{r_i^{(\ell)} - r_i^{(\ell-1)}}{1 - \exp(-\alpha_i^{(\ell)})}$, which can be re-written as $r_i^{(j)} - r_i^{(\ell)} + \frac{r_i^{(\ell)}}{1 - \exp(-\alpha_i^{(1)})}$ due to (7). The result follows immediately. $\square$

Proof of Lemma 3. It suffices to show that constraint (19b) holds for all $t \in [T]$ and $i \in [n]$. Since $p_{t,i}^{(j)} \in \{0, 1\}$ and the constraint clearly holds when $p_{t,i}^{(j)} = 0$, it suffices to show that $\mathbb{E}[Y_i + Z_t] \ge r_i^{(j_{t,i})}$, where $j_{t,i} \neq 0$. We will let $j = j_{t,i}$ for brevity.

Fix the realization of $W_{i'}$ for all $i' \neq i$, and consider the run of the algorithm on a modified setup with item i removed. Having fixed the values of $W_{i'}$, such a run is deterministic. Let Z^{crit} denote the pseudorevenue earned on this run during time t, possibly 0. $\Phi_{\mathcal{P}_i}$ maps $[0, L_i^{(j)}]$ to $[0, r_i^{(j)}]$ bijectively, so we can set W^{crit} to be the value in $[0, L_i^{(j)}]$ for which $\Phi_{\mathcal{P}_i}(W^{\mathsf{crit}}) = \max\{r_i^{(j)} - Z^{\mathsf{crit}}, 0\}$.

We now consider the run of the algorithm on the full setup with item i, which is dependent on the realization of W_i. The following two claims from Devanur et al. (2013) generalize to our multi-price setting.

1. Dominance: if $W_i \in [0, W^{\mathsf{crit}})$, then in the run with item i, item i gets matched.

Proof: Since $W^{\mathsf{crit}} > W_i$ and $W_i \geq 0$, $W^{\mathsf{crit}} > 0$. Therefore, $\Phi_{\mathcal{P}_i}(W^{\mathsf{crit}}) > 0$. Thus $\Phi_{\mathcal{P}_i}(W^{\mathsf{crit}}) = r_i^{(j)} - Z^{\mathsf{crit}}$ (as opposed to $\Phi_{\mathcal{P}_i}(W^{\mathsf{crit}}) = 0$), and moreover since $W_i < W^{\mathsf{crit}}$ and $\Phi_{\mathcal{P}_i}$ is strictly increasing, $\Phi_{\mathcal{P}_i}(W_i) < r_i^{(j)} - Z^{\mathsf{crit}}$. This implies $r_i^{(j)} - \Phi_{\mathcal{P}_i}(W_i) > \max\{Z^{\mathsf{crit}}, 0\}$, since $Z^{\mathsf{crit}} \geq 0$. Thus on the run with item i, either i is already matched before time t, or it is matched to customer t.

2. Monotonicity: $Z_t \geq Z^{\mathsf{crit}}$ (regardless of the realization of W_i).

Proof: fix the realization of W_i. We compare two deterministic runs of the algorithm: one with item i, and one without. We can inductively establish over $t = 0, \ldots, T$ that at the end of time t, the set of unmatched items in the run with i is a superset of that in the run without i. Therefore, in the run with i, since the algorithm is maximizing pseudorevenue over a superset of items, its pseudorevenue Z_t can be no less than Z^{crit}.

Now, conditioned on the realizations of $W_{i'}$ for $i' \neq i$, which determines the values of Z^{crit} and W^{crit}, we have $Z_t \geq Z^{\mathsf{crit}}$ (by Monotonicity) and in turn $Z^{\mathsf{crit}} \geq r_i^{(j)} - \Phi_{\mathcal{P}_i}(W^{\mathsf{crit}})$ (by the definition of W^{crit}). Meanwhile, as long as i gets matched, Y_i gets set to $\Phi'_{\mathcal{P}_i}(W_i)$, so by Dominance, $\mathbb{E}[Y_i | \{W_{i'} : i' \neq i\}] \geq \int_0^{W^{\mathsf{crit}}} \Phi'_{\mathcal{P}_i}(w) dw = \Phi_{\mathcal{P}_i}(W^{\mathsf{crit}}) - \Phi_{\mathcal{P}_i}(0) = \Phi_{\mathcal{P}_i}(W^{\mathsf{crit}})$. Therefore, $\mathbb{E}[Y_i + Z_t | \{W_{i'} : i' \neq i\}] \geq r_i^{(j)}$. The proof follows from the tower property of conditional expectation. □

Appendix D: Deferred Proofs from Section 5

Proof of Proposition 4. The unique solution to the system (24) is obtained inductively over $j = 2, \ldots, m$ by setting $B_j = \frac{r^{(j-1)} e^{-\alpha^{(j-1)}}}{r^{(j)} e^{-\alpha^{(j)}}} B_{j-1}$. By (EC.4), $\frac{r^{(j-1)}}{r^{(j)}} \leq e^{-\alpha^{(j)}}$, hence $B_j \leq e^{-\alpha^{(j-1)}} B_{j-1}$. But $\alpha^{(j-1)} > 0$ by Proposition 1, completing the proof that $B_j < B_{j-1}$ for $j = 2, \ldots, m$. The fact that $0 < B_m$ is immediate. □

Proof of Lemma 4. Consider the execution of an online algorithm with this randomized arrival sequence. For all $i \in [n]$ and group of customers $t \in [n]$, let $Q_{t,i}$ denote the number of group-t customers to which item π_i is sold, which is a random variable with respect to the random permutation π as well as any randomness in the algorithm. Let $q_{t,i} = \mathbb{E}[Q_{t,i}]$.

Clearly if $i < t$, then $Q_{t,i} = 0$, because group-t customers have no interest in item π_i. Otherwise, for any $i, i' \geq t$, we argue that $q_{t,i} = q_{t,i'}$. This is because while group t is arriving, the online algorithm cannot distinguish between items π_i and $\pi_{i'}$, hence any items it allocates are equally likely to be item π_i and item $\pi_{i'}$. Therefore, we let q_t denote the value of $q_{t,i}$ for $i \geq t$.

Now, consider item π_n. Since it only has k units of inventory, we know that $\sum_{t=1}^{n} Q_{t,n} \leq k$ on every sample path. Using the linearity of expectation, we get that

$$\sum_{t=1}^{n} q_t \leq k. \tag{EC.15}$$

Furthermore, for a $t \in [n]$, on every sample path, $\sum_{i=t}^{n} Q_{t,i} \leq k$, since there are only k customers in group t. Therefore, $(n+1-t)q_t \leq k$, or

$$q_t \leq \frac{k}{n+1-t}. \tag{EC.16}$$

For this proof, let $M_j = \sum_{j'=1}^{j} \beta_{j'}$, for all $j = 0, \ldots, m$. For all $j \in [m]$, let $\lambda_j = \frac{1}{k} \sum_{t=M_{j-1}n+1}^{M_j n} q_t$. Substituting into (EC.15), we get the constraint that $\sum_{j=1}^{m} \lambda_j \leq 1$. For any $j \in [m-1]$, summing inequality (EC.16) for $t = M_{j-1}n+1, \ldots, M_j n$ yields $\lambda_j \leq \ln \frac{B_j}{B_{j+1}}$, since $n \to \infty$, and $B_j = 1 - M_{j-1}$, $B_{j+1} = 1 - M_j$ by definition. It is also clear from definition that $\lambda_j \geq 0$ for all $j \in [m]$.

Finally, the total expected revenue is

$$\sum_{j=1}^{m} r^{(j)} \sum_{t=M_{j-1}n+1}^{M_j n} q_t(n+1-t), \tag{EC.17}$$

since for each group t there are $n+1-t$ items for each of which q_t copies are sold in expectation. Consider any $j \in [m]$. Since $\sum_{t=M_{j-1}n+1}^{M_j n} q_t = \lambda_j k$ by definition, $\sum_{t=M_{j-1}n+1}^{M_j n} q_t(n+1-t)$ is maximized by setting q_t to its upper bound in (EC.16) for $t = M_{j-1}n+1, M_{j-1}n+2, \ldots$ until the capacity of $\lambda_j k$ is reached. Since $n \to \infty$, we can simply compute the value of t for which

$$\frac{k}{n - M_{j-1}n} + \ldots + \frac{k}{n-t} = \lambda_j k, \tag{EC.18}$$

with $t \in [M_{j-1}n, M_j n]$. Letting $t = (M_{j-1} + y\beta_j)n$ with $y \in [0,1]$, and using the definition of B_j, (EC.18) becomes $\ln \frac{B_j}{B_j - y\beta_j} = \lambda_j$, or $y\beta_j = B_j(1 - e^{-\lambda_j})$. Therefore,

$$\sum_{t=M_{j-1}n+1}^{M_j n} q_t(n+1-t) \leq \sum_{t=M_{j-1}n+1}^{(M_{j-1}+B_j(1-e^{-\lambda_j}))n} \frac{k}{n+1-t} \cdot (n+1-t)$$

$$= B_j(1 - e^{-\lambda_j})nk$$

Substituting into (EC.17), we get that the expected revenue of the online algorithm is at most (26), where $\sum_{j=1}^m \lambda_j \le 1$, $\lambda_j \le \ln \frac{B_j}{B_{j+1}}$ for $j \in [m-1]$, and $\lambda_j \ge 0$ for $j \in [m]$, completing the proof.

□

Proof of Lemma 5. We use backward induction over $j = m, \ldots, 1$. When $j = m$, (28) becomes $nkr^{(m)}B_m(1 - \exp(-\tau))$, since $A_m = \alpha^{(m)}$ by definition. Meanwhile, (27) is maximized by setting $\lambda_m = \tau$, resulting in the same expression and establishing the base case.

Now suppose $j < m$ and that we have already established the lemma in the $j + 1$ case. If we set $\lambda_j = \lambda$, for some $\lambda \in [0, \tau]$, then the maximum value of (27) subject to $\lambda_{j+1}, \ldots, \lambda_m \ge 0$ and $\lambda_{j+1} + \ldots + \lambda_m \le \tau - \lambda$ is, by the inductive hypothesis,

$$r^{(j)}B_j(1 - \exp(-\lambda))nk + nk \sum_{\ell=j+1}^{m} r^{(\ell)}B_\ell \Big(1 - \exp\big(-\alpha^{(\ell)} + \frac{A_{j+1} - (\tau - \lambda)}{m - (j+1) + 1}\big)\Big). \tag{EC.19}$$

Consider this expression as a function of λ. The derivative is

$$r^{(j)}B_j \exp(-\lambda)nk + nk \sum_{\ell=j+1}^{m} r^{(\ell)}B_\ell \cdot \frac{-1}{m-j} \cdot \exp\big(-\alpha^{(\ell)} + \frac{A_{j+1} - (\tau - \lambda)}{m - j}\big) \tag{EC.20}$$

and the second derivative is clearly negative, so the function is concave. Therefore, it is maximized by setting the derivative to 0. By definition (24), $r^{(\ell)}B_\ell e^{-\alpha^{(\ell)}}$ is identical for all $\ell = j+1, \ldots, m$, and equal to $r^{(j)}B_j e^{-\alpha^{(j)}}$. Thus setting (EC.20) to 0 implies:

$$\exp(\alpha^{(j)} - \lambda) = \frac{1}{m-j} \sum_{\ell=j+1}^{m} \exp\big(\frac{A_{j+1} - (\tau - \lambda)}{m - j}\big)$$

$$\alpha^{(j)} - \lambda = \frac{A_{j+1} - (\tau - \lambda)}{m - j}.$$

Rearranging and using the definition that $A_{j+1} = A_j - \alpha^{(j)}$, we get $\lambda = \alpha^{(j)} - \frac{A_j - \tau}{m-j+1}$. Substituting this value of λ into (EC.19), the expression $\frac{A_{j+1} - (\tau - \lambda)}{m-(j+1)+1}$ is equal to $\frac{A_j - \tau}{m-j+1}$, hence (EC.19) is equal to (28), completing the induction and the proof of the lemma. □

Appendix E: Deriving the Multi-price Value Function $\Phi_{\mathcal{P}}$

In this section we explain how we optimized the value function $\Phi_{\mathcal{P}}$ for a given price set $\mathcal{P}$, leading to the system of equations in (7), and the functional form in (9). In Appendix E.1, we use the same method to derive the optimal value function when the price of an item can take any value in the continuum $[r^{\min}, r^{\max}]$.

Consider constraints (17)–(18) in Theorem 5 for a single item with $k \to \infty$. Let $w = \frac{N}{k}$, and we deterministically set $\tilde{\Phi}$ to some Φ. The goal is to solve for the Φ which maximizes the value of F.

Observe that

$$\lim_{k\to\infty} k(\Phi(\frac{N+1}{k}) - \Phi(\frac{N}{k})) = \lim_{k\to\infty} \frac{\Phi(w+1/k) - \Phi(w)}{1/k},$$

which is equal to the derivative of Φ as w, by definition (Φ will end up not being differentiable on a discrete set of measure 0, which can be ignored). Therefore, (17) is equivalent to

$$\Phi'(w) - \Phi(w) \le r^{(j)}(\frac{1}{F} - 1), \tag{EC.21}$$

and needs to hold for all $j \in [m], w \in [0, L^{(j)}]$. For a fixed $w \in (L^{(j-1)}, L^{(j)})$, (EC.21) needs to hold for all $j' = j, \ldots, m$, but is clearly binding when $j' = j$. Therefore, it suffices to fix a $j \in [m]$ and consider (EC.21) when $w \in (L^{(j-1)}, L^{(j)})$.

We should point out that this simplification via the "binding" argument is not possible for a finite k and random $\tilde{\Phi}$, because then (EC.21) becomes $\tilde{\Phi}'(w) - \tilde{\Phi}(w) \le \frac{r^{(j)}}{F} - \tilde{\Phi}(L^{(j)})$, and the RHS in fact may not be increasing in j. This is why we resort to first solving for Φ when $k \to \infty$ and then defining $\tilde{\Phi}$ as a random perturbation of Φ.

If we set (EC.21) to equality for some $j \in [m]$ and all $w \in (L^{(j-1)}, L^{(j)})$, and solve the differential equation, we get that $\Phi(w)$ must be of the form $Ce^w - r^{(j)}(\frac{1}{F} - 1)$ on $(L^{(j-1)}, L^{(j)})$. Setting $\Phi(L^{(j-1)}) = r^{(j-1)}$ and $\Phi(L^{(j)}) = r^{(j)}$, we obtain

$$C = \frac{r^{(j)} - r^{(j-1)}}{e^{L^{(j)}} - e^{L^{(j-1)}}};$$

$$F = \frac{1}{1 - \frac{r^{(j-1)}}{r^{(j)}}} \cdot (1 - e^{-\alpha^{(j)}}). \tag{EC.22}$$

The RHS of (EC.22) is the largest value of F which allows (EC.21) to hold on segment j. It is dependent on $\alpha^{(j)}$, which is equal to $L^{(j)} - L^{(j-1)}$, the length of segment j. For (EC.21) to hold on all segments $j \in [m]$, F must be set to $\min_j \frac{1}{1-r^{(j-1)}/r^{(j)}} \cdot (1 - e^{-\alpha^{(j)}})$.

Therefore, we would like to choose segment lengths $\alpha^{(1)}, \ldots, \alpha^{(m)}$ summing to 1 to maximize the minimum $\frac{1}{1-r^{(j-1)}/r^{(j)}} \cdot (1 - e^{-\alpha^{(j)}})$, which is accomplished by setting $\frac{1}{1-r^{(j-1)}/r^{(j)}} \cdot (1 - e^{-\alpha^{(j)}})$ equal for all $j \in [m]$. This yields the system of equations (7), and Proposition 1. The resulting value of F is equal to $1 - e^{-\alpha^{(1)}}$, since $r^{(0)} = 0$. The resulting value of C, when substituted into the equation for $\Phi(w)$ on each segment $(L^{(j-1)}, L^{(j)})$, yields (9).

The derivation of Φ we just completed, starting from condition (EC.21), comes from our analysis of Multi-price Balance. We note that the exact same inequality (EC.21) can also be derived from our analysis of Multi-price Ranking, which shows that the same value function should be used for both algorithms.

E.1. Continuum of Feasible Prices

Let the feasible price set for the item be $[r^{\min}, r^{\max}]$, where $0 < r^{\min} < r^{\max}$. Using the same "binding" argument, it suffices to maximize the value of F for which the following can hold:

$$\Phi'(w) - \Phi(w) \le r^{\min}(\frac{1}{F} - 1), \qquad w \in (0, \alpha); \tag{EC.23}$$

$$\Phi'(w) - \frac{\Phi(w)}{F} \le 0, \qquad w \in (\alpha, 1). \tag{EC.24}$$

Φ must also satisfy $\Phi(0) = 0, \Phi(\alpha) = r^{\min}, \Phi(1) = r^{\max}$, while $\alpha \in (0,1)$ is an arbitrary "booking limit" for the lowest price of $r^{\min}$.

We know from before that under the optimal solution to (EC.23), the value of F can be at most $1 - e^{-\alpha}$. Solving the differential equation where (EC.24) is set to equality, $\Phi(w)$ must take the form $Ce^{w/F}$ on $(\alpha, 1)$. Substituting $\Phi(\alpha) = r^{\min}$ and $\Phi(1) = r^{\max}$ yields

$$C = (r^{\min})^{\frac{1}{1-\alpha}} (r^{\max})^{-\frac{\alpha}{1-\alpha}};$$

$$F = \frac{1-\alpha}{\ln \frac{r^{\max}}{r^{\min}}}.$$

Therefore, the value of F is also bounded from above by $\frac{1-\alpha}{\ln(r^{\max}/r^{\min})}$. F is maximized by setting $\frac{1-\alpha}{\ln(r^{\max}/r^{\min})}$ equal to the other upper bound of $1-e^{-\alpha}$; the value at which equality is achieved is then the competitive ratio.

Letting $R=\ln(r^{\max}/r^{\min})$, the solution to $\frac{1-\alpha}{R}=1-e^{-\alpha}$ can be written as $W(Re^{R-1})-R+1$, where W is the Lambert-W function, the inverse function to $f(x)=xe^x$ for $x\in\mathbb{R}_{\geq 0}$. Indeed, when $\alpha=W(Re^{R-1})$, the following can be derived:

$$\begin{aligned}\frac{1-\alpha}{R}&=1-e^{-\alpha}\\ Re^{-\alpha}&=\alpha+R-1\\ Re^{R-1}&=(\alpha+R-1)e^{\alpha+R-1}\\ W(Re^{R-1})&=\alpha+R-1\end{aligned}$$

Substituting $\alpha=W(\ln(r^{\max}/r^{\min})e^{\ln(r^{\max}/r^{\min})-1})-\ln(r^{\max}/r^{\min})+1$ into the formula for C, and using the fact that $\Phi(w)=Ce^{w/F}$, we get

$$\Phi(w)=(r^{\min})^{\frac{1-w}{1-\alpha}}(r^{\max})^{\frac{w-\alpha}{1-\alpha}}, \qquad w\in[\alpha,1].$$

Meanwhile, the earlier derivation implies that

$$\Phi(w)=r^{\min}\cdot\frac{e^w-1}{e^\alpha-1}, \qquad w\in[0,\alpha].$$

It can be checked that indeed $\Phi(0)=0$, $\Phi(\alpha)=r^{\min}$ (Φ is continuous at $w=\alpha$), and $\Phi(1)=r^{\max}$. Furthermore, unlike the case of discrete prices, it can be checked that Φ is also differentiable at $w=\alpha$ (on $[\alpha,1]$, use the form that $\Phi(w)=Ce^{w/F}$, hence $\Phi'(\alpha)=\frac{\Phi(\alpha)}{F}$).

Appendix F: Supplement to Numerical Experiments

We provide additional details about our choice estimation. We define 8 customer types, one for each combination of the 3 following binary features.

1. Group: whether the customer indicated a party size greater than 1.
2. CRO: whether the customer booked using the Central Reservation Office, as opposed to the hotel's website or a Global Distribution System (for details on these terms, see Bodea et al. (2009)).

Table EC.1 MNL choice models for the 8 customer types. The suffix "L" on a room type means lower fare, while the suffix "H" on a room type means higher fare.

Customer Type				MNL Mean Utilities								
Group?	CRO?	VIP?	Share	KingL	QueenL	SuiteL	2DoubleL	KingH	QueenH	SuiteH	2DoubleH	NoBuy
			0.16	-0.36	-1.22	-2.56	-1.04	0	-0.23	-2.25	-1.8	0
		✓	0.03	-0.82	-1.98	-2.16	-2.09	0	-1.02	-1.45	-1.82	0
	✓		0.28	-1.67	$-\infty$	-3.78	-2.71	0	-1.33	-1.8	-1.58	0
	✓	✓	0.09	-2.13	$-\infty$	-3.38	-3.76	0	-2.12	-1	-1.59	0
✓			0.19	-0.54	-0.97	-2.26	0	-0.91	-1.47	-2.78	-1.41	0
✓		✓	0.04	-0.09	-0.82	-0.95	-0.14	0	-1.35	-1.07	-0.51	0
✓	✓		0.18	-0.93	$-\infty$	-2.56	-0.76	0	-1.66	-1.41	-0.27	0
✓	✓	✓	0.03	-1.39	$-\infty$	-2.16	-1.8	0	-2.45	-0.61	-0.28	0

3. VIP: whether the customer had any kind of VIP status.

We did not use features such as: whether the booking date is a weekend, whether the check-in date is a weekend, the length of stay, or the number of days in advance booked. Such features did not result in a more predictive model.

We estimate the mean MNL utilities for each of the 8 products separately for each customer type. The results are displayed in Table EC.1. The total share of each customer type (out of all the transactions) is also displayed. We should point out that it is possible for a customer to choose the higher fare for a room, even if the lower fare was also offered. This is because the higher fares are often packaged with additional offers, such as airline services, city attractions, in-room services, etc.

We have shifted the mean utilities so that for each customer type, the weights of both the no-purchase option, and the most-preferred purchase option, is equal to 0. (We synthetically set the weight of the no-purchase option because it is not possible to estimate from the data.) The large weights on the no-purchase options ensure that the revenue-maximizing assortments tend to include both the low and high fares.

In the setting with greater fare differentiation (Subsection 7.5), the high prices of the King, Queen, Suite, and Two-double rooms are adjusted to \$614, \$608, \$768, \$612, respectively (twice the lower fares). The mean utility of the no-purchase option is increased by 2 for every customer type, to ensure that the revenue-maximizing assortments still include both the low and high fares.

Figure EC.1 Distribution of arrivals over the days before check-in, formed by aggregating all transactions.

F.1. Details on the Forecasting Bid-price Algorithms

To forecast the remaining number of customers, we assume that we know the average number of customers interested in each occupancy date (1340), as well as the overall trend for how far in advance customers book, which is plotted in Figure EC.1. As an example of how to use these numbers, consider the occupancy date March 31st. At the start, we forecast there to be 1340 arrivals. However, suppose by March 6th, 500 customers have arrived. Since we know from Figure EC.1 that roughly 50% of the total population interested in March 31st will have already booked by March 6th (25 days in advance), we expect there to only be 500 customers remaining.

To forecast the breakdown of remaining customers by type, we assume that we know the aggregate distribution of customer type over all occupancy dates. For example, from Appendix F, we know that 28% of all customers are of Type 3. Then we would estimate $28\% \times 500 = 140$ of the 500 remaining customers to be of Type 3. Alternatively, one can try to learn the specific distribution of customers interested in March 31st. Suppose that only 100, or 20%, of the 500 bookings made before March 6th came from customers of Type 3. Then we would instead estimate $20\% \times 500 = 100$ of the 500 remaining customers to be of Type 3.

To use the forecasted information, algorithms incorporate it into the LP (30), and set the *bid price* of each item i equal to the shadow price of constraint i in (30b). These algorithms then offer each customer t the assortment S (from the available items) maximizing $\sum_{(i,j)\in S} p^{(j)}_{t,i}(S)\big(r^{(j)}_i - \lambda_i\big)$.

We clarify the exact way in which the forecasted information is incorporated into the LP. Let there be A customer types, indexed by $a=1,\ldots,A$. We use $p^{(j)}_{a,i}(S)$ to denote the probability of a customer of type a choosing product (i,j) from assortment S. Suppose that when we want to re-solve the LP (30), the forecasted number of remaining customers of type a is N_a, for all $a\in[A]$, and the remaining inventory of item i is K_i, for all $i\in[n]$. We can formulate the following LP, which is a modification of (30):

$$\max\sum_{a=1}^{A}\sum_{S}x_a(S)\sum_{(i,j)\in S}r^{(j)}_i p^{(j)}_{a,i}(S)$$

$$\sum_{a=1}^{A}\sum_{S}x_a(S)\sum_{j:(i,j)\in S}p^{(j)}_{a,i}(S)\le k_i \qquad i\in[n]$$

$$\sum_{S}x_a(S)=N_a \qquad a\in[A]$$

$$x_a(S)\ge 0 \qquad a\in[A], S\subseteq\{(i,j):i\in[n], j\in[m_i]\}$$

We have set $T=\sum_{a=1}^{A}N_a$ and $|\{t:\text{type of customer } t \text{ is } a\}|=N_a$; note that the ordering of remaining customers is inconsequential for the LP.

Although this LP has an exponential number of variables, we can easily solve it using column generation (e.g., see Liu and Van Ryzin (2008)). Fix an optimal primal solution $(x^*_a(S):a\in[A],S\subseteq\{(i,j):i\in[n],j\in[m_i]\})$ and an optimal dual solution $(y^*_i:i\in[n]),(z^*_a:a\in[A])$. The bid-price algorithm sets the bid price of each item i equal to y^*_i.

We should point out that for every bid-price algorithm based on dual variables, there is a corresponding *random assignment* algorithm based on primal variables. Such an algorithm would, for each customer type a, offer each assortment S with probability proportional to $x^*_a(S)$. We have confirmed that these algorithms perform similarly in the simulations. We compare with the bid-price algorithms instead of the random assignment algorithms because they follow a form more similar to our Multi-price Balance algorithm.