Online Appendix for "Operational Risk Management: A Stochastic Control Framework with Preventive and Corrective Control"

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Appendix A: Proofs

Proof of Theorem 1. Define the Hamilton-Jacobi-Bellman (HJB) equation:

$$\sup_{u\geq 0}\left\{\frac{\partial V}{\partial t}+\left(r(t)-\frac{1}{2}\sigma(t)^2-u\right)\frac{\partial V}{\partial x}+\frac{1}{2}\sigma(t)^2\frac{\partial^2 V}{\partial x^2}\right.$$
$$\left.-G(t,u,\lambda(t))\int_0^\infty[V(t,x)-V(t,x-y)]f(y)dy\right\}=0,$$

with boundary condition $V(T,x)=e^{\beta x}$. We will show that this HJB equation has a classical solution $w(t,x)$ that equals the value function $V^{\mathcal{U}}(t,x)$.

We show that $w(t,x)\in C^{1,2}$ is a classical solution to the HJB equation, when $w(t,x)$ is of the form $w(t,x)=A(t)e^{\beta x}$ and where $C^{1,2}$ denotes continuous differentiability in time and twice continuous differentiability in space. It is easy to see that $A(t)$ has to satisfy the ODE

$$A'(t)+\left[\beta\left(r(t)-\frac{1}{2}\sigma(t)^2-u^*(t)\right)+\frac{1}{2}\sigma(t)^2\beta^2-G(t,u^*(t),\lambda(t))[1-\mathcal{L}(\beta)]\right]A(t)=0,$$

with boundary condition $A(T)=1$, where $u^*(t)$ is defined as

$$u^*(t)=\arg\max_{u\geq 0}\left\{-u\beta-G(t,u,\lambda(t))\left(1-\mathcal{L}(\beta)\right)\right\}.$$

We can solve the ODE for $A(t)$ and get

$$A(t)=\exp\left\{\int_t^T\left[\beta\left(r(s)-\frac{1}{2}\sigma(s)^2-u^*(s)\right)+\frac{1}{2}\sigma(s)^2\beta^2\right.\right.$$
$$\left.\left.-G(s,u^*(s),\lambda(s))[1-\mathcal{L}(\beta)]\right]ds\right\}.$$

We will show later that $u^*(t)$ indeed gives the optimal control. Before we proceed, we notice first that from the definition

$$-u^*(t)\beta - G(t,u^*(t),\lambda(t))(1-\mathcal{L}(\beta)) \geq -G(t,0,\lambda(t))(1-\mathcal{L}(\beta)) = -\lambda(t)(1-\mathcal{L}(\beta)),$$

it follows that

$$u^*(t) \leq \frac{\lambda(t)}{\beta}(1-\mathcal{L}(\beta)).$$

Since $\lambda(t)$ is continuous in t, $u^*(t)$ is bounded on $[0,T]$, and finally since $u^*(t)$ is deterministic, we have verified that $u^*(\cdot)\in\mathcal{U}$. Moreover, since G is continuous differentiable and $\lambda(t)$ is continuous, $u^*(t)$ is continuous, and hence $A(t)$ is continuously differentiable and $w(t,x)$ belongs to $C^{1,2}$. Therefore, $w(t,x)$ is indeed a solution to the HJB equation.

Next, we need to verify that $V^{\mathcal{U}}(t,x)=w(t,x)$. We first show that $w(t,x)\geq V^{\mathcal{U}}(t,x)$. Note that for any $u(t)\in\mathcal{U}$, we have

$$\begin{aligned}\mathbb{E}[e^{\beta X_T}] &= \mathbb{E}[w(T,X_T)] \\ &= w(t,x) + \mathbb{E}\Bigg[\int_t^T \Bigg[\frac{\partial w}{\partial s}(s,X_s) - \left(r(s)-\frac{1}{2}\sigma(s)^2-u(s)\right)\frac{\partial w}{\partial x}(s,X_s) + \frac{1}{2}\sigma(s)^2\frac{\partial^2 w}{\partial x^2}(s,X_s) \\ &\qquad - G(s,u(s),\lambda(s))\int_0^\infty [w(s,X_s)-w(s,X_s-y)]f(y)dy\Bigg]ds\Bigg] \\ &\leq w(t,x). \end{aligned} \tag{1}$$

The first equality in (1) holds due to the fact that $A(T)=1$. The second equality in (1) holds due to Dynkin's formula (see, for example, Theorem 1.24 in Øksendal and Sulem (2005)) given that the following technical condition holds

$$\begin{aligned}&\mathbb{E}[|w(T,X_T)|] + \mathbb{E}\Bigg[\int_t^T \Bigg|\frac{\partial w}{\partial s}(s,X_s) - \left(r(s)-\frac{1}{2}\sigma(s)^2-u(s)\right)\frac{\partial w}{\partial x}(s,X_s) + \frac{1}{2}\sigma(s)^2\frac{\partial^2 w}{\partial x^2}(s,X_s) \\ &\qquad - G(s,u(s),\lambda(s))\int_0^\infty [w(s,X_s)-w(s,X_s-y)]f(y)dy\Bigg|ds\Bigg] \\ &+ \mathbb{E}\left[\int_t^T \left(\sigma(s)\frac{\partial w}{\partial x}(s,X_s)\right)^2 ds\right] \\ &\quad + \mathbb{E}\left[\int_t^T G(s,u(s),\lambda(s))\int_0^\infty [w(s,X_s)-w(s,X_s-y)]^2 f(y)dyds\right] < \infty. \end{aligned}$$

By the definition $w(t,x)=A(t)e^{\beta x}$, and since $G(s,u(s),\lambda(s))\leq\lambda(s)$, $A(T)=1$, thus it suffices to show that

$$\begin{aligned}&\mathbb{E}[e^{\beta X_T}] + \left(\max_{t\leq s\leq T}|A'(s)| + \max_{t\leq s\leq T}\frac{1}{2}\sigma(s)^2 A(s)\beta^2\right)\int_t^T \mathbb{E}[e^{\beta X_s}]ds \\ &\quad + \left(\max_{t\leq s\leq T}\left(r(s)+\frac{1}{2}\sigma(s)^2 + \text{ess sup } u(s)\right)A(s)\beta + \max_{t\leq s\leq T}\lambda(s)(1-\mathcal{L}(\beta))\right)\int_t^T \mathbb{E}[e^{\beta X_s}]ds \\ &\quad + \max_{t\leq s\leq T}\sigma(s)^2\beta^2\int_0^T \mathbb{E}[e^{2\beta X_s}]ds + \max_{t\leq s\leq T}\lambda(s)\int_0^\infty [1-e^{-\beta y}]^2 f(y)dy\int_0^T \mathbb{E}[e^{2\beta X_s}]ds < \infty. \end{aligned} \tag{2}$$

For any $u \in \mathcal{U}$ and $\beta \geq 0$,

$$\begin{aligned}\mathbb{E}[e^{\beta X_t}] &\leq \mathbb{E}[e^{\beta(\int_0^t r(s) - \frac{1}{2}\sigma(s)^2)ds + \beta \int_0^t \sigma(s) dB_s}] \\ &= e^{\beta(\int_0^t r(s) - \frac{1}{2}\sigma(s)^2)ds + \frac{1}{2}\beta^2 \int_0^t \sigma(s)^2 ds} \leq e^{t \max_{0\leq t\leq T} |\beta r(t) - \beta \frac{1}{2}\sigma(t)^2 + \frac{1}{2}\beta^2\sigma(t)^2|}.\end{aligned}$$

Therefore, the inequality (2) holds.

Finally, the inequality in (1) holds because

$$\begin{aligned}&\mathbb{E}\bigg[\int_t^T \bigg[\frac{\partial w}{\partial s}(s, X_s) - \left(r(s) - \frac{1}{2}\sigma(s)^2 - u(s)\right)\frac{\partial w}{\partial x}(s, X_s) + \frac{1}{2}\sigma(s)^2 \frac{\partial^2 w}{\partial x^2}(s, X_s) \\ &\qquad - G(s, u(s), \lambda(s)) \int_0^\infty [w(s, X_s) - w(s, X_s - y)] f(y) dy\bigg] ds\bigg] \\ &\leq \mathbb{E}\bigg[\int_t^T \sup_{u\geq 0} \bigg[\frac{\partial w}{\partial s}(s, X_s) - \left(r(s) - \frac{1}{2}\sigma(s)^2 - u\right)\frac{\partial w}{\partial x}(s, X_s) + \frac{1}{2}\sigma(s)^2 \frac{\partial^2 w}{\partial x^2}(s, X_s) \\ &\qquad - G(s, u, \lambda(s)) \int_0^\infty [w(s, X_s) - w(s, X_s - y)] f(y) dy\bigg] ds\bigg] = 0,\end{aligned}$$

following from the HJB equation. Taking the supremum over $u(t) \in \mathcal{U}$ in (1), we obtain $V^{\mathcal{U}}(t,x) \leq w(t,x)$. Denoting $X_t^* = \log V_t^*$, with V_T^* being the asset value process with u^*, we have $V^{\mathcal{U}}(t,x) \geq \mathbb{E}[e^{\beta X_T^*}]$. Then by Dynkin's formula (we can check a similar technical condition as in (1)), we have

$$\begin{aligned}\mathbb{E}[e^{\beta X_T^*}] = e^{\beta x} + \int_t^T \bigg(&\left(r(s) - \frac{1}{2}\sigma(s)^2 - u^*(s)\right)\beta + \frac{1}{2}\sigma(s)^2\beta^2 \\ &- G(s, u^*(s), \lambda(s)) \left[1 - \mathcal{L}(\beta)\right]\bigg) \mathbb{E} e^{\beta X_s^*} ds,\end{aligned}$$

which implies that $\mathbb{E}[e^{\beta X_T^*}] = A(t) e^{\beta x} = w(t,x)$.

Hence we conclude that $V^{\mathcal{U}}(t,x) = w(t,x)$, and the optimal strategy $u^*(t)$ is given by

$$u^*(t) = \arg\max_{u\geq 0} \left\{-u\beta - G(t, u, \lambda(t)) \left(1 - \mathcal{L}(\beta)\right)\right\}.$$

and the value function is given by

$$\begin{aligned}V^{\mathcal{U}}(t,x) = e^{\beta x} \exp\bigg\{\int_t^T \bigg[&\beta\left(r(s) - \frac{1}{2}\sigma(s)^2 - u^*(s)\right) + \frac{1}{2}\sigma(s)^2\beta^2 \\ &- G(s, u^*(s), \lambda(s))[1 - \mathcal{L}(\beta)]\bigg] ds\bigg\}.\end{aligned}$$

Now we can solve the optimization problem for $u^*(t)$. At optimality, if $u^*(t) > 0$, we have

$$\frac{\partial}{\partial u} G(t, u^*(t), \lambda(t)) = -\frac{\beta}{(1 - \mathcal{L}(\beta))}.$$

Recall that H is the inverse function of $\partial G(t, u, \lambda(t))/\partial u$, i.e., $\partial G(t, H(t, x, \lambda(t)), \lambda(t))/\partial u = x$. We conclude that the optimal investment strategy $u^*(t)$ for the optimization problem is a threshold strategy with $u^*(t) = H(t, -\beta/(1 - \mathcal{L}(\beta)), \lambda(t))$ when $H(t, -\beta/(1 - \mathcal{L}(\beta)), \lambda(t)) > 0$, and $u^*(t) = 0$

otherwise. This completes the proof. □

Proof of Proposition 1. If $G(t,u(t),\lambda(t))$ is convex in $u(t)$, then $-G(t,u(t),\lambda(t))$ is concave in $u(t)$ and hence $\partial(-G(t,u(t),\lambda(t)))/\partial u(t)$ is decreasing in u. Note that

$$\frac{\partial}{\partial\beta}\frac{1-e^{-\beta y}}{\beta}=\frac{\beta y e^{-\beta y}-1+e^{-\beta y}}{\beta^2},$$

and the function $xe^{-x}-1+e^{-x}<0$ for any $x>0$ and we have therefore that $\beta/(1-\mathcal{L}(\beta))$ increases in β. Hence we conclude that if $G(t,u(t),\lambda(t))$ is convex in $u(t)$, then $u^*(t)$ decreases in β. □

Proof of Proposition 2. Note here that $G(t,u(t),\lambda(t))$ decreases in $u(t)$, and therefore $\partial G(t,u(t),\lambda(t))/\partial u(t)<0$. Hence, if EF_1^t increases in $\lambda(t)$, then $\partial G(t,u(t),\lambda(t))/\partial u(t)$ actually decreases in $\lambda(t)$.

We first prove the results when EF_1^t decreases in $\lambda(t)$. Recall that

$$\frac{\partial}{\partial u}[G(t,u^*,\lambda)]=-\beta/(1-\mathcal{L}(\beta)).$$

First, $G(t,u(t),\lambda(t))$ is convex in $u(t)$ and the loss reduction rate $\partial G(t,u(t),\lambda(t))/\partial u(t)$ increases in $\lambda(t)$. Assume $\lambda_1<\lambda_2$. If $u_1^*\le u_2^*$, then

$$\frac{\partial}{\partial u}[G(t,u_2^*,\lambda_2)]\ge\frac{\partial}{\partial u}[G(t,u_1^*,\lambda_2)]>\frac{\partial}{\partial u}[G(t,u_1^*,\lambda_1)],$$

which leads to a contradiction since $\partial G(t,u_2^*,\lambda_2)/\partial u=\partial G(t,u_1^*,\lambda_1)/\partial u$. Therefore, we have $u_1^*>u_2^*$.

Next we consider the results when EF_1^t increases in $\lambda(t)$. Consider the case $G(t,u(t),\lambda(t))$ convex in $u(t)$ and the loss reduction rate $\partial G(t,u(t),\lambda(t))/\partial u(t)$ decreasing in $\lambda(t)$. Assume $\lambda_1<\lambda_2$. If $u_1^*\ge u_2^*$, then

$$\frac{\partial}{\partial u}[G(t,u_2^*,\lambda_2)]\le\frac{\partial}{\partial u}[G(t,u_1^*,\lambda_2)]<\frac{\partial}{\partial u}[G(t,u_1^*,\lambda_1)],$$

which leads to a contradiction since $\partial G(t,u_2^*,\lambda_2)/\partial u=\partial G(t,u_1^*,\lambda_1)/\partial u$. Therefore, we have $u_1^*<u_2^*$. □

Proof of Proposition 3. Recall that

$$\frac{\partial}{\partial u}[-G(t,u^*,\lambda)]=\frac{\beta}{1-\mathbb{E}[e^{-\beta Y}]}.$$

If $Y_1\ge_{cx}Y_2$, then $\mathbb{E}[e^{-\beta Y_1}]\ge\mathbb{E}[e^{-\beta Y_2}]$ so that

$$\frac{\partial}{\partial u}[-G(t,u_1^*,\lambda)]=\frac{\beta}{1-\mathbb{E}[e^{-\beta Y_1}]}\ge\frac{\beta}{1-\mathbb{E}[e^{-\beta Y_2}]}=\frac{\partial}{\partial u}[-G(t,u_2^*,\lambda)].$$

Since G is convex in u, $-G$ must be concave in u, and $\partial[-G]/\partial u$ is thus decreasing in u. This implies that $u_1^* \leq u_2^*$. □

Proof of Theorem 2. The proof is similar to the proof of Theorem 1, and is therefore omitted here. □

Proof of Theorem 3. Define the Hamilton-Jacobi-Bellman (HJB) equation:

$$\sup_{v\geq 0}\left\{\frac{\partial V}{\partial t}+\left(r(t)-\frac{1}{2}\sigma(t)^2-v\right)\frac{\partial V}{\partial x}+\frac{1}{2}\sigma(t)^2\frac{\partial^2 V}{\partial x^2}\right.$$
$$\left.-\lambda(t)\int_0^\infty [V(t,x)-V(t,x-K(t,v,y))]f(y)dy\right\}=0,$$

with boundary condition $V(T,x)=e^{\beta x}$.

We will show that this HJB equation has a classical solution $w(t,x)$ which equals to the value function $V^{\mathcal{U}}(t,x)$.

We show that $w(t,x)\in C^{1,2}$ is a classical solution to the HJB equation, where $C^{1,2}$ denotes continuous differentiability in time and twice continuous differentiability in space, where $w(t,x)$ is of the form $w(t,x)=A(t)e^{\beta x}$. It is easy to see that $A(t)$ has to satisfy the ODE

$$A'(t)+\left[\beta\left(r(t)-\frac{1}{2}\sigma(t)^2-v^*(t)\right)+\frac{1}{2}\sigma(t)^2\beta^2-\lambda(t)\left(1-\int_0^\infty e^{-\beta K(t,v^*(t),y)}f(y)dy\right)\right]A(t)=0$$

with boundary condition $A(T)=1$, where $v^*(t)$ is defined as

$$v^*(t)=\arg\max_{v\geq 0}\left\{-v\beta+\lambda(t)\int_0^\infty e^{-\beta K(t,v,y)}f(y)dy\right\}.$$

We can solve this ODE and get

$$A(t)=\exp\left\{\int_t^T\left[\beta\left(r(s)-\frac{1}{2}\sigma(s)^2-v^*(s)\right)+\frac{1}{2}\sigma(s)^2\beta^2-\lambda(s)\left(1-\int_0^\infty e^{-\beta K(s,v^*(s),y)}f(y)dy\right)\right]ds\right\}.$$

We will show later that $v^*(t)$ indeed gives the optimal control. Before we proceed, we notice first that by the definition

$$-v^*(t)\beta+\lambda(t)\int_0^\infty e^{-\beta K(t,v^*(t),y)}f(y)dy\geq\lambda(t)\int_0^\infty e^{-\beta K(t,0,y)}f(y)dy=\lambda(t)\mathcal{L}(\beta),$$

and together with $\beta K(t,v^*(t),y)\geq 0$, we have

$$v^*(t)\leq\frac{\lambda(t)}{\beta}(1-\mathcal{L}(\beta)).$$

Since $\lambda(t)$ is continuous in t, $v^*(t)$ is bounded on $[0,T]$, and finally since $v^*(t)$ is deterministic, we have verified that $v^*(\cdot)\in\mathcal{V}$. Moreover, since K is continuous differentiable and $\lambda(t)$ is continuous, $v^*(t)$ is continuous, and hence $A(t)$ is continuously differentiable and $w(t,x)$ belongs to $C^{1,2}$. Therefore, $w(t,x)$ is indeed a solution of the HJB equation.

Next, we need to verify that $V^{\mathcal{V}}(t,x)=w(t,x)$. We first show that $w(t,x)\geq V^{\mathcal{V}}(t,x)$. Note that for any $v(t)\in\mathcal{V}$, we have

$$\begin{aligned}
\mathbb{E}[e^{\beta X_T}] &= \mathbb{E}[w(T,X_T)] \\
&= w(t,x)+\mathbb{E}\bigg[\int_t^T\bigg[\frac{\partial w}{\partial s}(s,X_s)-\left(r(s)-\frac{1}{2}\sigma(s)^2-v(s)\right)\frac{\partial w}{\partial x}(s,X_s)+\frac{1}{2}\sigma(s)^2\frac{\partial^2 w}{\partial x^2}(s,X_s) \\
&\qquad -\lambda(s)\int_0^\infty[w(s,X_s)-w(s,X_s-K(s,v(s),y))]f(y)dy\bigg]ds\bigg] \\
&\leq w(t,x).
\end{aligned} \tag{3}$$

The first equality in (3) holds due to the fact that $A(T)=1$. The second equality in (3) holds due to the Dynkin's formula (see e.g. Theorem 1.24 in Øksendal and Sulem (2005)) given that the following technical condition holds

$$\begin{aligned}
&\mathbb{E}[|w(T,X_T)|]+\mathbb{E}\bigg[\int_t^T\bigg|\frac{\partial w}{\partial s}(s,X_s)-\left(r(s)-\frac{1}{2}\sigma(s)^2-v(s)\right)\frac{\partial w}{\partial x}(s,X_s)+\frac{1}{2}\sigma(s)^2\frac{\partial^2 w}{\partial x^2}(s,X_s) \\
&-\lambda(s)\int_0^\infty[w(s,X_s)-w(s,X_s-K(s,v(s),y))]f(y)dy\bigg|ds\bigg] \\
&+\mathbb{E}\left[\left(\sigma(s)\frac{\partial w}{\partial x}(s,X_s)\right)^2ds\right] \\
&+\mathbb{E}\left[\int_t^T\lambda(s)\int_0^\infty[w(s,X_s)-w(s,X_s-K(s,v(s),y))]^2f(y)dyds\right]<\infty.
\end{aligned}$$

By the definition $w(t,x)=A(t)e^{\beta x}$, and since $K(s,v(s),y)\leq y$, $A(T)=1$, thus it suffices to show that

$$\begin{aligned}
&\mathbb{E}[e^{\beta X_T}]+\left(\max_{t\leq s\leq T}|A'(s)|+\max_{t\leq s\leq T}\frac{1}{2}\sigma(s)^2A(s)\beta^2\right)\int_t^T\mathbb{E}[e^{\beta X_s}]ds \\
&+\left(\max_{t\leq s\leq T}\left(r(s)+\frac{1}{2}\sigma(s)^2+\text{ess sup }u(s)\right)A(s)\beta+\max_{t\leq s\leq T}\lambda(s)(1-\mathcal{L}(\beta))\right)\int_t^T\mathbb{E}[e^{\beta X_s}]ds \\
&+\max_{t\leq s\leq T}\sigma(s)^2\beta^2\int_0^T\mathbb{E}[e^{2\beta X_s}]ds+\max_{t\leq s\leq T}\lambda(s)\int_0^\infty[1-e^{-\beta y}]^2f(y)dy\int_0^T\mathbb{E}[e^{2\beta X_s}]ds<\infty.
\end{aligned} \tag{4}$$

For any $v\in\mathcal{V}$ and $\beta\geq 0$,

$$\begin{aligned}
\mathbb{E}[e^{\beta X_t}] &\leq \mathbb{E}[e^{\beta(\int_0^t r(s)-\frac{1}{2}\sigma(s)^2)ds+\beta\int_0^t\sigma(s)dB_s}] \\
&= e^{\beta(\int_0^t r(s)-\frac{1}{2}\sigma(s)^2)ds+\frac{1}{2}\beta^2\int_0^t\sigma(s)^2ds}\leq e^{t\max_{0\leq t\leq T}|\beta r(t)-\beta\frac{1}{2}\sigma(t)^2+\frac{1}{2}\beta^2\sigma(t)^2|}.
\end{aligned}$$

Therefore, inequality (4) holds.

Finally, inequality (3) holds because

$$\mathbb{E}\bigg[\int_t^T\bigg[\frac{\partial w}{\partial s}(s,X_s)-\left(r(s)-\frac{1}{2}\sigma(s)^2-v(s)\right)\frac{\partial w}{\partial x}(s,X_s)+\frac{1}{2}\sigma(s)^2\frac{\partial^2 w}{\partial x^2}(s,X_s)$$

$$- \lambda(s) \int_0^\infty [w(s,X_s) - w(s, X_s - K(s, v(s), y))] f(y) dy \bigg] ds \bigg]$$

$$\leq \mathbb{E}\bigg[\int_t^T \sup_{v \geq 0} \bigg[\frac{\partial w}{\partial s}(s, X_s) - \left(r(s) - \frac{1}{2}\sigma(s)^2 - v \right) \frac{\partial w}{\partial x}(s, X_s) + \frac{1}{2}\sigma(s)^2 \frac{\partial^2 w}{\partial x^2}(s, X_s)$$

$$- \lambda(s) \int_0^\infty [w(s,X_s) - w(s, X_s - K(s, v, y))] f(y) dy \bigg] ds \bigg] = 0,$$

following from the HJB equation. Taking the supremum over $v(t) \in \mathcal{V}$ in (3), we obtain $V^{\mathcal{V}}(t,x) \leq w(t,x)$. Denoting $X_t^* = \log V_t^*$, with V_T^* being the asset value process with v^*, we have $V^{\mathcal{V}}(t,x) \geq \mathbb{E}[e^{\beta X_T^*}]$. Then from Dynkin's formula (we can check a similar technical condition as in (3)) it follows that

$$\mathbb{E}[e^{\beta X_T^*}] = e^{\beta x} + \int_t^T \bigg(\left(r(s) - \frac{1}{2}\sigma(s)^2 - v^*(s) \right) \beta + \frac{1}{2}\sigma(s)^2 \beta^2 - \lambda(s) \left(1 - \int_0^\infty e^{-\beta K(s, v^*(s), y)} f(y) dy \right) \bigg) \mathbb{E} e^{\beta X_s^*} ds,$$

which implies that $\mathbb{E}[e^{\beta X_T^*}] = A(t) e^{\beta x} = w(t,x)$.

Hence, $V^{\mathcal{V}}(t,x) = w(t,x)$, and the optimal strategy $v^*(t)$ is given by

$$v^*(t) = \arg\max_{v \geq 0} \left\{ -v\beta + \lambda(t) \int_0^\infty e^{-\beta K(t,v,y)} f(y) dy \right\}.$$

and the value function is given by

$$V^{\mathcal{V}}(t,x) = e^{\beta x} \exp\bigg\{ \int_t^T \bigg[\beta \left(r(s) - \frac{1}{2}\sigma(s)^2 - v^*(s) \right) + \frac{1}{2}\sigma(s)^2 \beta^2 - \lambda(s) \left(1 - \int_0^\infty e^{-\beta K(s, v^*(s), y)} f(y) dy \right) \bigg] ds \bigg\}.$$

This completes the proof. □

Proof of Proposition 4. First, from Theorem 3, we know that the optimal $v^*(t)$ is given by

$$v^*(t) = \arg\max_{v \geq 0} \left\{ -v\beta + \lambda(t) \int_0^\infty e^{-\beta K(t, v(t), y)} f(y) dy \right\}.$$

Therefore, at optimality, if $v^*(t) > 0$, we have

$$\frac{\beta}{\lambda(t)} = \int_0^\infty \frac{\partial}{\partial v} \left[e^{-\beta K(v(t), y, t)} \right] \bigg|_{v = v^*} f(y) dy.$$

Note that $e^{-\beta K(t, v(t), y)}$ is concave in $v(t)$ and thus $\partial e^{-\beta K(v(t), y, t)} / \partial v$ is decreasing in $v(t)$. Now, assuming $\beta_1 \geq \beta_2$, we claim that $v_1^* \leq v_2^*$. Assuming not, we would have $v_1^* > v_2^*$, and

$$\frac{\beta_1}{\lambda(t)} = \int_0^\infty \frac{\partial}{\partial v} \left[e^{-\beta_1 K(v(t), y, t)} \right] \bigg|_{v = v_1^*} f(y) dy$$

$$< \int_0^\infty \frac{\partial}{\partial v} \left[e^{-\beta_1 K(v(t), y, t)} \right] \bigg|_{v = v_2^*} f(y) dy$$

$$< \int_0^\infty \frac{\partial}{\partial v} \left[e^{-\beta_2 K(v(t), y, t)} \right] \bigg|_{v = v_2^*} f(y) dy = \frac{\beta_2}{\lambda(t)},$$

which is a contradiction. □

Proof of Proposition 5. First, from Theorem 3, we know that the optimal $v^*(t)$ is given by

$$v^*(t) = \arg\max_{v\geq 0}\left\{-v\beta + \lambda(t)\int_0^\infty e^{-\beta K(t,v(t),y)} f(y)dy\right\}.$$

Therefore, at optimality, if $v^*(t) > 0$, we have

$$\frac{\beta}{\lambda(t)} = \int_0^\infty \frac{\partial}{\partial v}\left[e^{-\beta K(v(t),y,t)}\right]\Big|_{v=v^*} f(y)dy.$$

Note that $e^{-\beta K(t,v(t),y)}$ is concave in $v(t)$ and thus $\partial e^{-\beta K(v(t),y,t)}/\partial v$ is decreasing in $v(t)$. Now, assuming $\lambda_1 \geq \lambda_2$, we claim that $v_1^* \geq v_2^*$. Assuming not, we would have $v_1^* < v_2^*$, and

$$\begin{aligned}\frac{\beta}{\lambda_1(t)} &= \int_0^\infty \frac{\partial}{\partial v}\left[e^{-\beta K(v(t),y,t)}\right]\Big|_{v=v_1^*} f(y)dy \\ &> \int_0^\infty \frac{\partial}{\partial v}\left[e^{-\beta K(v(t),y,t)}\right]\Big|_{v=v_2^*} f(y)dy = \frac{\beta}{\lambda_2(t)},\end{aligned}$$

which is a contradiction. □

Proof of Theorem 4. The proof is similar to the proof of Theorem 3, and is therefore omitted here. □

Proof of Theorem 5. Define the Hamilton-Jacobi-Bellman (HJB) equation:

$$\begin{aligned}\sup_{u,v\geq 0}\Bigg\{&\frac{\partial V}{\partial t} + \left(r(t) - \frac{1}{2}\sigma(t)^2 - u - v\right)\frac{\partial V}{\partial x} + \frac{1}{2}\sigma(t)^2\frac{\partial^2 V}{\partial x^2} \\ &- G(t,u,\lambda(t))\int_0^\infty [V(t,x) - V(t,x-K(t,v,y))]f(y)dy\Bigg\} = 0,\end{aligned}$$

with boundary condition $V(T,x) = e^{\beta x}$.

We will show that this HJB equation has a classical solution $w(t,x)$ which equals to the value function $V^{\mathcal{U},\mathcal{V}}(t,x)$.

We show that $w(t,x) \in C^{1,2}$ is a classical solution to the HJB equation, where $C^{1,2}$ denotes continuous differentiability in time and twice continuous differentiability in space, where $w(t,x)$ is of the form $w(t,x) = A(t)e^{\beta x}$. It is easy to see that $A(t)$ has to satisfy the ODE

$$\begin{aligned}A'(t) + \Bigg[&\beta\left(r(t) - \frac{1}{2}\sigma(t)^2 - u^*(t) - v^*(t)\right) + \frac{1}{2}\sigma(t)^2\beta^2 \\ &- G(t,u^*(t),\lambda(t))\left(1 - \int_0^\infty e^{-\beta K(t,v^*(t),y)} f(y)dy\right)\Bigg]A(t) = 0,\end{aligned}$$

with boundary condition $A(T)=1$, where $u^*(t),v^*(t)$ is defined as

$$(u^*(t),v^*(t))=\arg\max_{u,v\geq 0}\left\{-u\beta-v\beta-G(t,u,\lambda(t))\left(1-\int_0^\infty e^{-\beta K(t,v,y)}f(y)dy\right)\right\}.$$

We can solve this ODE and get

$$A(t)=\exp\left\{\int_t^T\left[\beta\left(r(s)-\frac{1}{2}\sigma(s)^2-u^*(s)-v^*(s)\right)+\frac{1}{2}\sigma(s)^2\beta^2\right.\right.$$
$$\left.\left.-G(s,u^*(s),\lambda(s))\left(1-\int_0^\infty e^{-\beta K(s,v^*(s),y)}f(y)dy\right)\right]ds\right\}.$$

We will show later that $(u^*(t),v^*(t))$ indeed gives the optimal control. Before we proceed, we notice first that by the definition

$$\begin{aligned}&-u^*(t)\beta-v^*(t)\beta-G(t,u^*(t),\lambda(t))\left(1-\int_0^\infty e^{-\beta K(t,v^*(t),y)}f(y)dy\right)\\&\geq -G(t,0,\lambda(t))\left(1-\int_0^\infty e^{-\beta K(t,0,y)}f(y)dy\right)\\&\geq -\lambda(t)(1-\mathcal{L}(\beta)),\end{aligned}$$

which implies that

$$u^*(t)+v^*(t)\leq\frac{\lambda(t)}{\beta}(1-\mathcal{L}(\beta)).$$

Since $\lambda(t)$ is continuous in t, $u^*(t),v^*(t)$ is bounded on $[0,T]$, and finally since $u^*(t),v^*(t)$ is deterministic, we have verified that $u^*(\cdot)\in\mathcal{U}$, $v^*(\cdot)\in\mathcal{V}$. Moreover, since G and K are continuous differentiable and $\lambda(t)$ is continuous, $u^*(t),v^*(t)$ are continuous, and hence $A(t)$ is continuously differentiable and $w(t,x)$ belongs to $C^{1,2}$. Therefore, $w(t,x)$ is indeed a solution of the HJB equation.

Next, we need to verify that $V^{\mathcal{U},\mathcal{V}}(t,x)=w(t,x)$. We first show that $w(t,x)\geq V^{\mathcal{U},\mathcal{V}}(t,x)$. Note that for any $u(t)\in\mathcal{U}$, $v(t)\in\mathcal{V}$, we have

$$\begin{aligned}&\mathbb{E}[e^{\beta X_T}]=\mathbb{E}[w(T,X_T)]\\&=w(t,x)+\mathbb{E}\left[\int_t^T\left[\frac{\partial w}{\partial s}(s,X_s)-\left(r(s)-\frac{1}{2}\sigma(s)^2-u(s)-v(s)\right)\frac{\partial w}{\partial x}(s,X_s)+\frac{1}{2}\sigma(s)^2\frac{\partial^2 w}{\partial x^2}(s,X_s)\right.\right.\\&\qquad\left.\left.-G(s,u(s),\lambda(s))\int_0^\infty[w(s,X_s)-w(s,X_s-K(s,v(s),y))]f(y)dy\right]ds\right]\\&\leq w(t,x).\end{aligned}\tag{5}$$

The first equality in (5) holds due to the fact that $A(T)=1$. The second equality in (5) holds due to the Dynkin's formula (see e.g. Theorem 1.24 in Øksendal and Sulem (2005)) given that the following technical condition holds

$$\mathbb{E}[|w(T,X_T)|]+\mathbb{E}\left[\int_t^T\left|\frac{\partial w}{\partial s}(s,X_s)-\left(r(s)-\frac{1}{2}\sigma(s)^2-u(s)-v(s)\right)\frac{\partial w}{\partial x}(s,X_s)+\frac{1}{2}\sigma(s)^2\frac{\partial^2 w}{\partial x^2}(s,X_s)\right.\right.$$

$$
\begin{aligned}
&-G(s,u(s),\lambda(s))\int_0^\infty [w(s,X_s)-w(s,X_s-K(s,v(s),y))]f(y)dy\bigg| ds\bigg] \\
&+\mathbb{E}\left[\left(\sigma(s)\frac{\partial w}{\partial x}(s,X_s)\right)^2 ds\right] \\
&+\mathbb{E}\left[\int_t^T G(s,u(s),\lambda(s))\int_0^\infty [w(s,X_s)-w(s,X_s-K(s,v(s),y))]^2 f(y)dyds\right]<\infty.
\end{aligned}
$$

By the definition $w(t,x)=A(t)e^{\beta x}$, and since $K(s,v(s),y)\le y$ and $G(s,u(s),\lambda(s))\le\lambda(s)$, $A(T)=1$, thus it suffices to show that

$$
\begin{aligned}
&\mathbb{E}[e^{\beta X_T}]+\left(\max_{t\le s\le T}|A'(s)|+\max_{t\le s\le T}\frac{1}{2}\sigma(s)^2A(s)\beta^2\right)\int_t^T \mathbb{E}[e^{\beta X_s}]ds \\
&\quad+\left(\max_{t\le s\le T}\left(r(s)+\frac{1}{2}\sigma(s)^2+\text{ess sup } u(s)\right)A(s)\beta+\max_{t\le s\le T}\lambda(s)(1-\mathcal{L}(\beta))\right)\int_t^T \mathbb{E}[e^{\beta X_s}]ds \\
&\quad+\max_{t\le s\le T}\sigma(s)^2\beta^2\int_0^T \mathbb{E}[e^{2\beta X_s}]ds+\max_{t\le s\le T}\lambda(s)\int_0^\infty [1-e^{-\beta y}]^2 f(y)dy\int_0^T \mathbb{E}[e^{2\beta X_s}]ds<\infty. \qquad (6)
\end{aligned}
$$

For any $u\in\mathcal{U}$, $v\in\mathcal{V}$ and $\beta\ge 0$,

$$
\begin{aligned}
\mathbb{E}[e^{\beta X_t}] &\le \mathbb{E}[e^{\beta(\int_0^t r(s)-\frac{1}{2}\sigma(s)^2)ds+\beta\int_0^t\sigma(s)dB_s}] \\
&= e^{\beta(\int_0^t r(s)-\frac{1}{2}\sigma(s)^2)ds+\frac{1}{2}\beta^2\int_0^t\sigma(s)^2ds}\le e^{t\max_{0\le t\le T}|\beta r(t)-\beta\frac{1}{2}\sigma(t)^2+\frac{1}{2}\beta^2\sigma(t)^2|}.
\end{aligned}
$$

Therefore, the inequality (6) holds.

Finally, the inequality in (5) holds because

$$
\begin{aligned}
&\mathbb{E}\bigg[\int_t^T\bigg[\frac{\partial w}{\partial s}(s,X_s)-\left(r(s)-\frac{1}{2}\sigma(s)^2-u(s)-v(s)\right)\frac{\partial w}{\partial x}(s,X_s)+\frac{1}{2}\sigma(s)^2\frac{\partial^2 w}{\partial x^2}(s,X_s) \\
&\quad-G(s,u(s),\lambda(s))\int_0^\infty (w(s,X_s)-w(s,X_s-K(s,v(s),y)))f(y)dy\bigg]ds\bigg] \\
&\le\mathbb{E}\bigg[\int_t^T\sup_{u(s),v(s)\ge 0}\bigg[\frac{\partial w}{\partial s}(s,X_s)-\left(r(s)-\frac{1}{2}\sigma(s)^2-u(s)-v(s)\right)\frac{\partial w}{\partial x}(s,X_s)+\frac{1}{2}\sigma(s)^2\frac{\partial^2 w}{\partial x^2}(s,X_s) \\
&\quad-G(s,u(s),\lambda(s))\int_0^\infty [w(s,X_s)-w(s,X_s-K(s,v(s),y))]f(y)dy\bigg]ds\bigg]=0,
\end{aligned}
$$

following from the HJB equation. Taking the supremum over $u(t)\in\mathcal{U}$, $v(t)\in\mathcal{V}$ in (5), we obtain $V^{\mathcal{U},\mathcal{V}}(t,x)\le w(t,x)$. Denoting $X_t^*=\log V_t^*$, with V_T^* being the asset value process with u^*,v^*, we have $V^{\mathcal{U},\mathcal{V}}(t,x)\ge\mathbb{E}[e^{\beta X_T^*}]$. Then by Dynkin's formula (we can check a similar technical condition as in (5)) we have

$$
\begin{aligned}
\mathbb{E}[e^{\beta X_T^*}]=e^{\beta x}+\int_t^T\bigg(&\left(r(s)-\frac{1}{2}\sigma(s)^2-u^*(s)-v^*(s)\right)\beta+\frac{1}{2}\sigma(s)^2\beta^2 \\
&-G(s,u^*,\lambda(s))\left(1-\int_0^\infty e^{-\beta K(s,v^*(s),y)}f(y)dy\right)\bigg)\mathbb{E}e^{\beta X_s^*}ds,
\end{aligned}
$$

which implies that $\mathbb{E}[e^{\beta X_T^*}] = A(t)e^{\beta x} = w(t,x)$.

Hence, we conclude that $V^{\mathcal{U},\mathcal{V}}(t,x) = w(t,x)$, and the optimal strategy $(u^*(t), v^*(t))$ is given by

$$(u^*(t), v^*(t)) = \arg\max_{u,v\geq 0} \left\{ -(u+v)\beta - G(t,u,\lambda(t)) \left(1 - \int_0^\infty e^{-\beta K(t,v,y)} f(y)dy \right) \right\},$$

and the value function is given by

$$V^{\mathcal{U},\mathcal{V}}(t,x) = e^{\beta x} \exp\left\{ \int_t^T \left[\beta \left(r(s) - \frac{1}{2}\sigma(s)^2 - u^*(s) - v^*(s) \right) + \frac{1}{2}\sigma(s)^2\beta^2 \right.\right.$$
$$\left.\left. - G(s,u^*(s),\lambda(s)) \left(1 - \int_0^\infty e^{-\beta K(s,v^*(s),y)} f(y)dy \right) \right] ds \right\}.$$

This completes the proof. □

Proof of Theorem 6. The proof is similar to the proof of Theorem 5, and is thus omitted here. □

Proof of Proposition 6. When $\beta \leq \lambda\delta_1(1-\delta_1/\delta_2)$ or $\beta > \delta_2/\delta_1 - 1 > 0$ does not hold, at least one of u^* and v^* is zero.

When only the preventive control is available, then $v^* = 0$ and by applying Theorem 1, we have

$$u^* = \frac{1}{\delta_1} \log\left(\mathcal{N}(\delta_1, \beta, \lambda, \mathbb{E}[Y]) \right),$$

when $\mathcal{N}(\delta_1, \beta, \lambda, \mathbb{E}[Y]) \geq 1$ and $u^* \equiv 0$ otherwise.

When only the corrective control is available, then $u^* = 0$ and we can apply Theorem 5. We can compute that

$$\int_0^\infty e^{-\beta K(v,y)} f(y)dy = \frac{\frac{1}{\mathbb{E}[Y]}}{\frac{1}{\mathbb{E}[Y]} + \beta e^{-\delta_2 v}},$$

and

$$\frac{\partial^2}{\partial v^2} \frac{\frac{1}{\mathbb{E}[Y]}}{\frac{1}{\mathbb{E}[Y]} + \beta e^{-\delta_2 v}} = -\mathbb{E}[Y] \frac{\beta\delta_2^2 e^{-\delta_2 v}}{(\frac{1}{\mathbb{E}[Y]} + \beta e^{-\delta_2 v})^3} \left(\frac{1}{\mathbb{E}[Y]} - \beta e^{-\delta_2 v} \right) > 0 (< 0),$$

for every $v < (>) (1/\delta_2) \log(\mathbb{E}[Y]\beta)$. Therefore, by applying Theorem 5, we can see that

$$v^\dagger = \frac{1}{\delta_2} \log\left(\mathcal{M}(\delta_2, \beta, \lambda, \mathbb{E}[Y]) \right),$$

when $\mathcal{M}(\delta_2, \beta, \lambda, \mathbb{E}[Y]) \geq 1$, provides a local maximum and the global maximum is obtained by comparing $v^* = v^\dagger$ with $v^* = 0$.

Hence, it follows that:

If $\mathcal{N}(\delta_1, \beta, \lambda, \mathbb{E}[Y]) < 1$ and $\mathcal{M}(\delta_2, \beta, \lambda, \mathbb{E}[Y]) < 1$, then $u^* = v^* = 0$.

If $\mathcal{N}(\delta_1, \beta, \lambda, \mathbb{E}[Y]) \geq 1$ and $\mathcal{M}(\delta_2, \beta, \lambda, \mathbb{E}[Y]) < 1$, then $v^* = 0$ and

$$u^* = \frac{1}{\delta_1} \log(\mathcal{N}(\delta_1, \beta, \lambda, \mathbb{E}[Y])).$$

If $\mathcal{N}(\delta_1,\beta,\lambda,\mathbb{E}[Y])<1$ and $\mathcal{M}(\delta_2,\beta,\lambda,\mathbb{E}[Y])\geq 1$ then $u^*=0$ and

$$v^* = \frac{1}{\delta_2}\log\left(\mathcal{M}(\delta_2,\beta,\lambda,\mathbb{E}[Y])\right),$$

if $C(0,v^\dagger)<C(0,0)$ and otherwise $v^*=0$.

If $\mathcal{N}(\delta_1,\beta,\lambda,\mathbb{E}[Y])\geq 1$ and $\mathcal{M}(\delta_2,\beta,\lambda,\mathbb{E}[Y])\geq 1$, then we recall that

$$u^\dagger = \frac{1}{\delta_1}\log(\mathcal{N}(\delta_1,\beta,\lambda,\mathbb{E}[Y])),$$
$$v^\dagger = \frac{1}{\delta_2}\log\left(\mathcal{M}(\delta_2,\beta,\lambda,\mathbb{E}[Y])\right).$$

We define $V(u,v)$ as the value function with emphasis on the dependence on the constant controls u and v:

$$\begin{aligned}V(u,v) &= e^{\beta x}\exp\left\{\int_t^T\left[\beta\left(r(s)-\frac{1}{2}\sigma(s)^2-u-v\right)+\frac{1}{2}\sigma(s)^2\beta^2\right.\right.\\ &\qquad\left.\left.-\lambda e^{-\delta_1 u}\left(1-\int_0^\infty e^{-\beta y e^{-\delta_2 v}}f(y)dy\right)\right]ds\right\}\\ &= e^{\beta x}\exp\left\{\int_t^T\left[\beta\left(r(s)-\frac{1}{2}\sigma(s)^2\right)+\frac{1}{2}\sigma(s)^2\beta^2\right]ds-(T-t)C(u,v)\right\},\end{aligned}$$

where

$$C(u,v) := u+v+\lambda e^{-\delta_1 u}\frac{\beta e^{-\delta_2 v}\mathbb{E}[Y]}{1+\beta e^{-\delta_2 v}\mathbb{E}[Y]}.$$

Note that $V(u_1,v_1)\geq V(u_2,v_2)$ if and only if $C(u_1,v_1)\leq C(u_2,v_2)$. Thus, the optimal (u^*,v^*) satisfies:

$$\begin{aligned}(u^*,v^*) &= (u^\dagger,0) \qquad \text{if} \quad C(u^\dagger,0)\leq C(0,v^\dagger),\\ (u^*,v^*) &= (0,v^\dagger) \qquad \text{if} \quad \min\{C(0,0),C(u^\dagger,0)\}>C(0,v^\dagger).\end{aligned}$$

When $\beta\leq\lambda\delta_1(1-\delta_1/\delta_2)$ and $\beta>\delta_2/\delta_1-1>0$,

$$(u^{**},v^{**}) = \left(\frac{1}{\delta_1}\log\left(\frac{\lambda\delta_1}{\beta}\left(1-\frac{\delta_1}{\delta_2}\right)\right),\frac{1}{\delta_2\mathbb{E}[Y]}\log\left(\frac{\beta}{\frac{\delta_2}{\delta_1}-1}\right)\right)$$

gives a critical point. By comparing with the preventive control only, and corrective control only, we get the desired result. □

Proof of Proposition 7. (i) and (ii) are easy to see. To see (iii), we can compute that

$$\frac{\partial u^*}{\partial\delta_1} = \frac{1-\log(\mathcal{N}(\delta_1,\beta,\lambda,\mathbb{E}[Y]))}{\delta_1^2}, \qquad \frac{\partial^2 u^*}{\partial\delta_1^2} = \frac{\delta_1-2\delta_1(1-\log(\mathcal{N}(\delta_1,\beta,\lambda,\mathbb{E}[Y])))}{\delta_1^4}.$$

Thus u^* is increasing in δ_1 if $\mathcal{N}(\delta_1,\beta,\lambda,\mathbb{E}[Y])<e$ and it is decreasing otherwise. u^* is convex in δ_1 if $\mathcal{N}(\delta_1,\beta,\lambda,\mathbb{E}[Y])>\sqrt{e}$ and it is concave otherwise.

To see (iv), we can compute that

$$\frac{\partial u^*}{\partial \mathbb{E}[Y]} = \frac{1}{\delta_1}\frac{1}{(\mathbb{E}[Y])^2\beta + \mathbb{E}[Y]} > 0, \qquad \frac{\partial^2 u^*}{\partial (\mathbb{E}[Y])^2} = \frac{-1}{\delta_1}\frac{2\mathbb{E}[Y]\beta + 1}{((\mathbb{E}[Y])^2\beta + \mathbb{E}[Y])^2} < 0.$$

Thus u^* is increasing and concave in $\mathbb{E}[Y]$. □

Proof of Proposition 8. (iii) and (v) trivially hold. Let us prove (i), (ii) and (iv).

(i) We can compute that

$$\frac{\partial v^*}{\partial \lambda} = \frac{1}{\delta_2}\frac{\delta_2 + \frac{\delta_2(\lambda\delta_2 - 2\beta)}{\sqrt{(\lambda\delta_2-2\beta)^2-4\beta^2}}}{\lambda\delta_2 - 2\beta + \sqrt{(\lambda\delta_2 - 2\beta)^2 - 4\beta^2}} = \frac{1}{\sqrt{(\lambda\delta_2 - 2\beta)^2 - 4\beta^2}} > 0,$$

and

$$\frac{\partial^2 v^*}{\partial \lambda^2} = -\frac{\delta_2(\lambda\delta_2 - 2\beta)}{((\lambda\delta_2 - 2\beta)^2 - 4\beta^2)^{3/2}} < 0.$$

Thus v^* is increasing and concave in λ.

(ii) We can compute that

$$\frac{\partial v^*}{\partial \beta} = \frac{1}{\delta_2}\frac{-2 - \frac{2\lambda\delta_2}{\sqrt{(\lambda\delta_2-2\beta)^2-4\beta^2}}}{\lambda\delta_2 - 2\beta + \sqrt{(\lambda\delta_2 - 2\beta)^2 - 4\beta^2}} = \frac{-2}{\delta_2}\frac{1 + \frac{\lambda\delta_2}{\sqrt{(\lambda\delta_2-2\beta)^2-4\beta^2}}}{\lambda\delta_2 - 2\beta + \sqrt{(\lambda\delta_2 - 2\beta)^2 - 4\beta^2}} < 0,$$

and

$$\frac{\partial^2 v^*}{\partial \beta^2} = \frac{-4}{\delta_2}\frac{\lambda^2\delta_2^2\frac{\lambda\delta_2-2\beta}{\sqrt{(\lambda\delta_2-2\beta)^2-4\beta^2}} + \lambda^2\delta_2^2 + (1 + \frac{\lambda\delta_2}{\sqrt{(\lambda\delta_2-2\beta)^2-4\beta^2}})^2}{(\lambda\delta_2 - 2\beta + \sqrt{(\lambda\delta_2 - 2\beta)^2 - 4\beta^2})^2((\lambda\delta_2 - 2\beta)^2 - 4\beta^2)} < 0.$$

Thus v^* is decreasing and concave in β.

(iv) We can write v^* as

$$v^* = \frac{1}{\delta_2}\log\left(\frac{\mathbb{E}[Y]}{2}\right) + \frac{1}{\delta_2}\log\left(\lambda\delta_2 - 2\beta + \sqrt{(\lambda\delta_2 - 2\beta)^2 - 4\beta^2}\right).$$

Thus, it follows that

$$\begin{aligned}\frac{\partial v^*}{\partial \delta_2} = -\frac{1}{\delta_2^2}\log\left(\frac{\mathbb{E}[Y]}{2}\right) - \frac{1}{\delta_2^2}\log\left(\lambda\delta_2 - 2\beta + \sqrt{(\lambda\delta_2 - 2\beta)^2 - 4\beta^2}\right) \\ + \frac{1}{\delta_2}\frac{\lambda}{\sqrt{(\lambda\delta_2 - 2\beta)^2 - 4\beta^2}} = -\frac{1}{\delta_2}v^* + \frac{1}{\delta_2}\frac{\lambda}{\sqrt{(\lambda\delta_2 - 2\beta)^2 - 4\beta^2}}.\end{aligned}$$

Thus v^* is increasing in δ_2 if

$$\log\left(\frac{\mathbb{E}[Y]}{2}\right) + \log\left(\lambda\delta_2 - 2\beta + \sqrt{(\lambda\delta_2 - 2\beta)^2 - 4\beta^2}\right) < \frac{\lambda\delta_2}{\sqrt{(\lambda\delta_2 - 2\beta)^2 - 4\beta^2}},$$

and it is decreasing otherwise. We can further compute that

$$\begin{aligned}\frac{\partial^2 v^*}{\partial \delta_2^2} &= \frac{1}{\delta_2^2}v^* - \frac{1}{\delta_2}\left[-\frac{1}{\delta_2}v^* + \frac{1}{\delta_2}\frac{\lambda}{\sqrt{(\lambda\delta_2-2\beta)^2-4\beta^2}}\right] \\ &\quad - \frac{\lambda^2}{\delta_2}\frac{2\lambda\delta_2-6\beta}{((\lambda\delta_2-2\beta)^2-4\beta^2)^{3/2}} \\ &= \frac{2v^*}{\delta_2^2} - \frac{3\lambda^3\delta_2^2-10\lambda^2\beta\delta_2}{\delta_2^2((\lambda\delta_2-2\beta)^2-4\beta^2)^{3/2}}.\end{aligned}$$

Thus v^* is convex in δ_2 if

$$\frac{2}{\delta_2}\log\left(\frac{\mathbb{E}[Y]}{2}\right) + \frac{1}{\delta_2}\log(\lambda\delta_2-2\beta+\sqrt{(\lambda\delta_2-2\beta)^2-4\beta^2})((\lambda\delta_2-2\beta)^2-4\beta^2)^{3/2} > 3\lambda^3\delta_2^2-10\lambda^2\beta\delta_2,$$

and it is concave in δ_2 otherwise. □

Proof of Proposition 9. (i), (ii), (iii) here are easy to see. To see (iv), we can compute that

$$\begin{aligned}\frac{\partial u^*}{\partial \delta_1} &= \frac{\frac{\delta_2-2\delta_1}{\delta_2-\delta_1} - \log(\frac{\lambda\delta_1}{\beta}(1-\frac{\delta_1}{\delta_2}))}{\delta_1^2}, \\ \frac{\partial^2 u^*}{\partial \delta_1^2} &= \frac{(\frac{-\delta_2}{(\delta_2-\delta_1)^2} - \frac{1}{\delta_1} + \frac{1}{\delta_2-\delta_1})\delta_1^2 - (\frac{\delta_2-2\delta_1}{\delta_2-\delta_1} - \log(\frac{\lambda\delta_1}{\beta}(1-\frac{\delta_1}{\delta_2})))2\delta_1}{\delta_1^4}.\end{aligned}$$

Thus, u^* is increasing in δ_1 if

$$\frac{\delta_2-2\delta_1}{\delta_2-\delta_1} > \log\left(\frac{\lambda\delta_1}{\beta}\left(1-\frac{\delta_1}{\delta_2}\right)\right),$$

and it is decreasing otherwise; u^* is convex in δ_1 if

$$\frac{-\delta_2}{(\delta_2-\delta_1)^2} - \frac{1}{\delta_1} + \frac{1}{\delta_2-\delta_1} > \left(\frac{\delta_2-2\delta_1}{\delta_2-\delta_1} - \log\left(\frac{\lambda\delta_1}{\beta}\left(1-\frac{\delta_1}{\delta_2}\right)\right)\right)\frac{2}{\delta_1},$$

and it is concave otherwise.

We can also compute that

$$\frac{\partial v^*}{\partial \delta_1} = \frac{1}{\delta_2}\frac{\delta_2}{\delta_2\delta_1-\delta_1^2} > 0, \qquad \frac{\partial^2 v^*}{\partial \delta_1^2} = \frac{1}{\delta_2}\frac{(2\delta_1-\delta_2)\delta_2}{(\delta_2\delta_1-\delta_1^2)^2},$$

Thus v^* is increasing in δ_1 and it is convex in δ_1 if $\delta_1 > \frac{1}{2}\delta_2$ and concave in δ_1 otherwise.

To see (v), we can compute that

$$\frac{\partial u^*}{\partial \delta_2} = \frac{\delta_1}{\delta_2^2-\delta_1\delta_2} > 0, \qquad \frac{\partial^2 u^*}{\partial \delta_2^2} = \frac{\delta_1(\delta_1-2\delta_2)}{(\delta_2^2-\delta_1\delta_2)^2} < 0.$$

Thus u^* is increasing and concave in δ_2. Moreover, we can compute that

$$\frac{\partial v^*}{\partial \delta_2} = \frac{-\frac{\delta_2}{\delta_2-\delta_1} - \log(\frac{\beta\mathbb{E}[Y]\delta_1}{\delta_2-\delta_1})}{\delta_2^2} < 0, \qquad \frac{\partial^2 v^*}{\partial \delta_2^2} = \frac{(\frac{1}{(\delta_2-\delta_1)^2} + \frac{1}{\delta_2-\delta_1})\delta_2^2 + (\frac{\delta_2}{\delta_2-1} + \log(\frac{\beta\mathbb{E}[Y]\delta_1}{\delta_2-\delta_1}))2\delta_2}{\delta_2^4} > 0.$$

Thus v^* is decreasing and convex in δ_2. □

Proof of Proposition 10. Note that if λ changes, only u^* changes, and there is no effect on v^*. If $\mathbb{E}[Y]$ changes, only v^* changes, and there is no effect on u^*. □

Appendix B: Numerical Experiments

In our paper, we conduct numerical experiments to characterize the optimal preventive and corrective controls in the different investment regions as discussed in Section 6.2, and analyze how different parameters affect the u^* and v^* respectively as discussed in Sections 6.2 and 6.3. The key parameters that determine the optimal investments here are λ, $\mathbb{E}[Y]$, β, δ_1, and δ_2, as discussed before. Now in our first experiment, we vary the variables within the ranges shown in Table 1. We show our numerical results in Figure 2.

Table 1 Experiment 1

Figure	Variable under study	Fixed variables
2(a)	$\lambda \in [1, 10]$	$\mathbb{E}[Y] = 50, \beta = 0.5, \delta_1 = 0.5, \delta_2 = 0.5$
2(b)	$\beta \in [0.0001, 0.9999]$	$\lambda = 5, \mathbb{E}[Y] = 50, \delta_1 = 0.5, \delta_2 = 0.5$
2(c)	$\mathbb{E}[Y] \in [1, 100]$	$\lambda = 5, \beta = 0.5, \delta_1 = 0.5, \delta_2 = 0.5$
2(d)	$\delta_2 \in [0, 1]$	$\lambda = 5, \mathbb{E}[Y] = 50, \beta = 0.5, \delta_1 = 0.5$
2(e)	$\delta_1 \in [0, 1]$	$\lambda = 5, \mathbb{E}[Y] = 50, \beta = 0.5, \delta_2 = 0.5$

In Figure 2, we find that the investment strategies in Figures 2(a) and (b) fall within the Preventive Control region, i.e., $v^* = 0$, and the investment strategy in Figure 2(e) first falls within the No Investment Region ($u^*, v^* = 0$), but then moves into the Preventive Region. Moreover, the structural properties of the optimal preventive control u^* in Figure 2 are consistent with Proposition 7. First, in Figure 2(a), u^* increases concavely in λ. Second, in Figure 2(b), u^* decreases in β. Third, in Figure 2(e), u^* is at first 0 and then moves into the Preventive Control Region: at first increasing in δ_1 and then decreasing in δ_1.

In Figures 2 (c) and (d), we plot u^* and v^* separately in order to be able to see v^* more clearly as it is not always 0 now. In Figure 2 (c), the optimal investment strategy first lies within the Corrective Control Region when $\mathbb{E}[Y]$ is small and then moves into the Preventive Control Region when $\mathbb{E}[Y]$ gets larger. In the Preventive Control Region, u^* increases concavely in $\mathbb{E}[Y]$, which is consistent with Proposition 7. In Figure 2 (d), when δ_2 is small, the optimal investment strategy falls within the Preventive Control Region, as investing in corrective control is not efficient in its loss reduction, and we can see that u^* is independent of δ_2. When δ_2 increases, then the optimal investment strategy moves into the Joint Investment Region, and we can see here the consistency with Proposition 9: u^* is increasing concavely in δ_2 and v^* is decreasing convexly in δ_2. Moreover, in the Joint Investment Region of Figure 2 (d), u^* and v^* are complementing each other. Finally, when δ_2 increases even more, then the optimal investment strategy moves into the Preventive Control Region again, and we see in Figure 2(d) that u^* assumes the same value as when δ_2 is very small, since now u^* is again independent of δ_2.

A similar experiment as Experiment 1 described above, but with δ_2 much smaller (i.e., 0.05 instead of 0.5), has yielded results that are similar to the results obtained above.

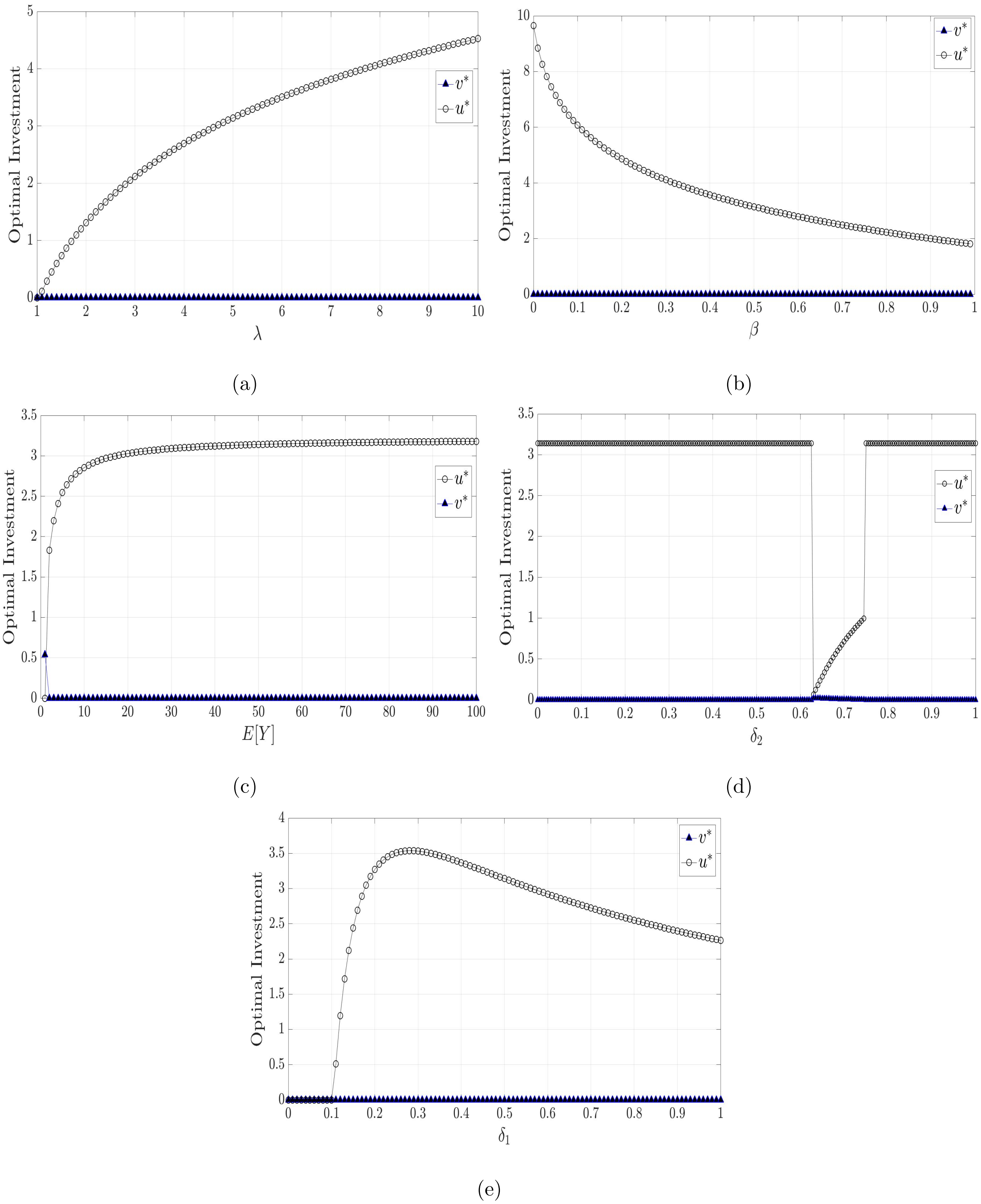

Figure 1 Optimal u^* and v^* for Experiment 1.

From Experiment 1, we can see that when δ_1 and δ_2 are similar in scale, one would prefer in most cases to invest in preventive control. Therefore, in order to do a more in-depth study of corrective control, we consider in Experiment 2 a much smaller frequency reduction efficiency parameter δ_1, i.e., $\delta_1 = 0.05$. We show the value or range of our key parameters in Table 2, and our numerical results in Figure 3.

It is clear that the optimal investment strategies in Figures 3(a), (c), and (d) lie mainly in two regions: the Corrective Control Region and the No Investment Region, since now the investment in frequency reduction would be very inefficient given a very small δ_1. The structural properties of v^* in the Corrective Control Region are consistent with Proposition 8. First, in Figure 3(a), v^* first lies in the No Investment Region, and then moves into the Corrective Control Region while increasing concavely in λ. Second, in Figure 3(b), v^* first decreases in β in the Corrective Control Region; and then for a certain range of β, even when δ_1 is very small, the optimal investment strategy still falls into the Preventive Control Region and $v^* = 0$. However, when β gets larger and larger, in Figure 3(b), v^* moves back into the Corrective Control Region while decreasing concavely in β, and then with very large β the optimal investment strategy ends up in the No Investment Region. Third, in Figure 3(c), v^* increases concavely in $\mathbb{E}[Y]$ in the Corrective Control Region. Finally, in Figure 3(d), v^* first falls into the No Investment Region, as δ_2 is very small, and when δ_2 increases above a certain threshold as characterized in Proposition 6 (III) then v^* falls into the Corrective Control Region. Consistent with Proposition 8 (iii), in the Corrective Control Region of Figure 3(d), v^* first increases in δ_2 and then decreases.

Table 2 Experiment 2

Figure	Variable under study	Fixed variables
4(a)	$\lambda \in [1, 10]$	$\mathbb{E}[Y] = 50, \beta = 0.5, \delta_1 = 0.05, \delta_2 = 0.5$
4(b)	$\beta \in [0.0001, 0.9999]$	$\lambda = 5, \mathbb{E}[Y] = 50, \delta_1 = 0.05, \delta_2 = 0.5$
4(c)	$\mathbb{E}[Y] \in [1, 100]$	$\lambda = 5, \beta = 0.5, \delta_1 = 0.05, \delta_2 = 0.5$
4(d)	$\delta_2 \in [0, 1]$	$\lambda = 5, \mathbb{E}[Y] = 50, \beta = 0.5, \delta_1 = 0.05$

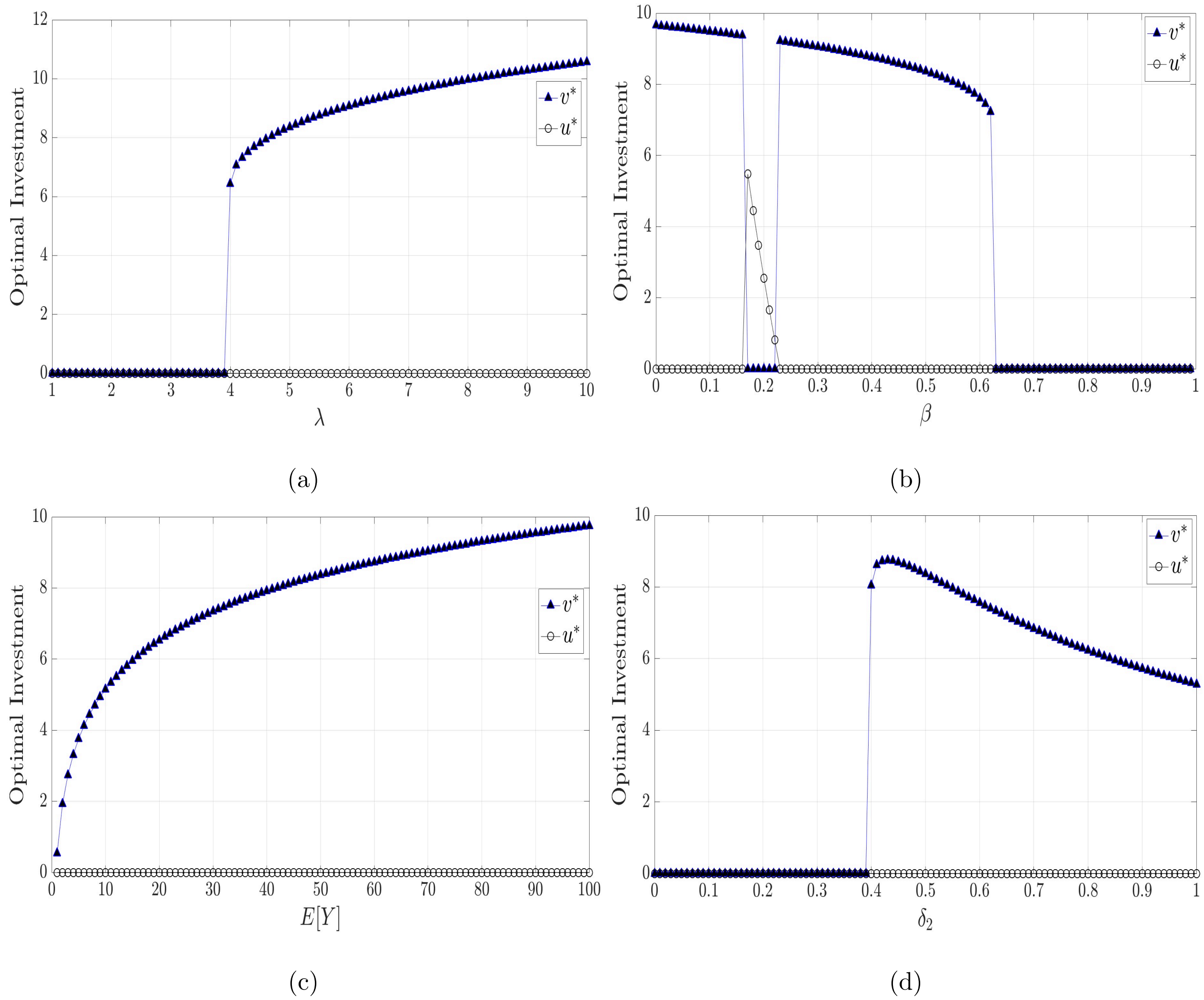

Figure 2 Optimal u^* and v^* for experiment 2.