Additional Material

EC.1. Proofs

EC.1.1. Proof of Theorem 1

We show that every feasible solution of Problem (5b) is a feasible solution to Problem (4b). Let $\mathbf{\Phi}_i$ for all $i \in \mathcal{M}$ be any feasible policy in Problem (5b). Starting with $\chi_{i,1} = x_{i,1}$, the state of agent i at time t are given as

$$\begin{aligned} x_{i,t} &= A_{i,1}^t x_{i,1} + \sum_{\tau=1}^{t-1} \left(A_{i,\tau+1}^t B_{i,\tau} [\chi_{j,\tau}(\boldsymbol{\xi}_{\overline{\mathcal{N}}_j}^{\tau-1})]_{j\in\mathcal{N}_i} + A_{i,\tau+1}^t D_{i,\tau} \Phi_{i,\tau}(\boldsymbol{\xi}_{\overline{\mathcal{N}}_i}^{\tau-1}) + A_{i,\tau+1}^t E_{i,\tau} \xi_{i,\tau} \right) \\ &=: \chi_{i,t}(\boldsymbol{\xi}_{\overline{\mathcal{N}}_i}^{t-1}) \end{aligned} \tag{A.1}$$

where $A_{i,\tau}^t = A_{i,\tau} A_{i,\tau+1} \dots A_{i,t-1}$ for $\tau < t$ and $A_{i,t}^t = I$. The last implication follows from the fact that $\overline{\mathcal{N}}_i \supseteq \overline{\mathcal{N}}_j$ for all $j \in \mathcal{N}_i$ since the network admits a partially nested structure. Given (A.1), it is easy to verify that for each agent i its dynamics and constraints in Problem (5b) only depend on $\boldsymbol{\xi}_{\overline{\mathcal{N}}_i}$. Hence, any feasible solution to Problem (5b) is also feasible to the following optimization problem:

$$\begin{aligned} \text{minimize} \;& \sum_{i=1}^{M} \max_{\boldsymbol{\xi}_{\mathcal{M}}\in\Xi} J_i(\boldsymbol{x}_i, \boldsymbol{u}_i) \\ \text{subject to} \;& \left.\begin{aligned} &\boldsymbol{u}_i = \mathbf{\Phi}_i(\boldsymbol{\xi}_{\overline{\mathcal{N}}_i}) := [\Phi_{i,t}(\boldsymbol{\xi}_{\overline{\mathcal{N}}_i}^{t-1})]_{t\in\mathcal{T}} \\ &\boldsymbol{x}_i = f_i\big(\boldsymbol{x}_{\mathcal{N}_i}, \boldsymbol{u}_i, \boldsymbol{\xi}_i\big) \\ &(\boldsymbol{x}_i, \boldsymbol{u}_i) \in \mathcal{O}_i \end{aligned}\right\} \forall \boldsymbol{\xi}_{\mathcal{M}} \in \Xi_{\mathcal{M}},\ \forall i \in \mathcal{M} \end{aligned} \tag{A.2}$$

Additionally, they achieve the same objective value since they share the same objective function. This shows equivalence of Problem (5b) and Problem (A.2). The relation between Problem (4b) and Problem (5b) stated in the theorem now follows immediately since the two problems share the same constraints and objective function, and the policies in Problem (5b) are restricted compared to Problem (4b) as they are functions of $\boldsymbol{\xi}_{\overline{\mathcal{N}}_i}$ while the policies in Problem (4b) are functions of $\boldsymbol{\xi}_{\mathcal{M}}$.

□

EC.1.2. Proof of Theorem 2

The statement is proved by induction using similar theoretical tools to (Hadjiyiannis et al. 2011, Prop. 2.1). The statement holds for $t = 1$ since the initial state, $x_{i,1}$, is known for every $i \in \mathcal{M}$; therefore, functions $\psi_{i,1}(x_{i,1}, \boldsymbol{\zeta}_{\mathcal{N}_i,1})$ and $\Psi_{i,1}(\boldsymbol{\zeta}_{\mathcal{N}_i,1})$ can always be constructed such

$$\psi_{i,1}(x_{i,1}, \boldsymbol{\zeta}_{\mathcal{N}_i,1}) = \Psi_{i,1}(\boldsymbol{\zeta}_{\mathcal{N}_i,1}) \tag{A.3}$$

Assume now that the statement holds for all $1 < \tau \leq t-1$, i.e., there exist policies $\psi_i(\cdot)$ and $\Psi_i(\cdot)$ such that $\psi_{i,\tau}(\boldsymbol{x}_i^\tau, \boldsymbol{\zeta}_{\mathcal{N}_i}^\tau) = \Psi_{i,\tau}(\boldsymbol{\xi}_i^{\tau-1}, \boldsymbol{\zeta}_{\mathcal{N}_i}^\tau)$. In the sequel, we show that the statement also holds for $\tau = t$. From (1), we have that

$$\begin{aligned} x_{i,t} &= A_{i,1}^t x_{i,1} + \sum_{\tau=1}^{t-1} \Big(A_{i,\tau+1}^t B_{i,\tau} \boldsymbol{\zeta}_{\mathcal{N}_i,\tau} + A_{i,\tau+1}^t D_{i,\tau} \Psi_{i,\tau}(\boldsymbol{\xi}_i^{\tau-1}, \boldsymbol{\zeta}_{\mathcal{N}_i}^\tau) + A_{i,\tau+1}^t E_{i,\tau} \xi_{i,\tau} \Big) \\ &=: \chi_{i,t}(\boldsymbol{\xi}_i^{t-1}, \boldsymbol{\zeta}_{\mathcal{N}_i}^{t-1}), \end{aligned} \tag{A.4}$$

where $A_{i,\tau}^t = A_{i,\tau} A_{i,\tau+1} \ldots A_{i,t-1}$ for $\tau < t$ and $A_{i,t}^t = I$. Moreover, it holds that

$$\begin{aligned} \xi_{i,t-1} &= E_{i,t-1}^{+} \big(x_{i,t} - A_{i,t-1} x_{i,t-1} + B_{j,t-1} \boldsymbol{\zeta}_{\mathcal{N}_i,t-1} + D_{i,t-1} \psi_{i,t-1}(\boldsymbol{x}_i^{t-1}, \boldsymbol{\zeta}_{\mathcal{N}_i}^{t-1}) \big) \\ &=: \rho_{i,t}(\boldsymbol{x}_i^t, \boldsymbol{\zeta}_{\mathcal{N}_i}^{t-1}), \end{aligned} \tag{A.5}$$

where $E_{i,t}^+ := (E_{i,t}^\top E_{i,t})^{-1} E_{i,t}^\top$ is the left inverse of $E_{i,t}$ since it is full rank.

The relation (A.4) implies that given a feasible policy $\psi_{i,t}(\cdot)$ for Problem (6a), we can construct a feasible policy for Problem (6b) as

$$\psi_{i,t}(\boldsymbol{x}_i^t, \boldsymbol{\zeta}_{\mathcal{N}_i}^t) = \psi_{i,t}(\chi_i^t(\boldsymbol{\xi}_i^{t-1}, \boldsymbol{\zeta}_{\mathcal{N}_i}^t), \boldsymbol{\zeta}_{\mathcal{N}_i}^t) := \Psi_{i,t}(\boldsymbol{\xi}_i^{t-1}, \boldsymbol{\zeta}_{\mathcal{N}_i}^t). \tag{A.6}$$

The claim follows from the fact that the composition of continuous differentiable functions is a continuous differentiable function. Hence, the policy $\psi_{i,t}(\cdot)$ will also be feasible in Problem (6b) since the two problems have the same pointwise constraints. Additionally, they achieve the same objective value since they share the same objective function.

Similarly, the relation (A.5) implies that given a feasible policy $\Psi_i(\cdot)$ for Problem (6b), we can construct a feasible policy for Problem (6a) as

$$\Psi_{i,t}(\boldsymbol{\xi}_i^{t-1}, \boldsymbol{\zeta}_{\mathcal{N}_i}^t) = \Psi_{i,t}(\rho_i^t(\boldsymbol{x}_i^t, \boldsymbol{\zeta}_{\mathcal{N}_i}^{t-1}), \boldsymbol{\zeta}_{\mathcal{N}_i}^t) := \psi_{i,t}(\boldsymbol{x}_i^t, \boldsymbol{\zeta}_{\mathcal{N}_i}^t). \tag{A.7}$$

The claim follows from the fact that the composition of continuous differentiable functions is a continuous differentiable function. Hence, the policy $\Psi_{i,t}(\cdot)$ is also feasible in Problem (6a) since the two problems have the same pointwise constraints. Additionally, they achieve the same objective value since they share the same objective function. $\square$

EC.1.3. Proof of Theorem 3

We show that every feasible solution of Problem (6b) is feasible in Problem (5b). Let $(\mathbf{\Psi}_i, \mathcal{X}_i)$ for all $i \in \mathcal{M}$ be feasible in Problem (6b). Since the state of agent i evolve according to (1), we can conclude that at time t we have

$$\begin{aligned} x_{i,t} &= A_{i,1}^t x_{i,1} + \sum_{\tau=1}^{t-1} \Big(A_{i,\tau+1}^t B_{i,\tau} \boldsymbol{\zeta}_{\mathcal{N}_i,\tau} + A_{i,\tau+1}^t D_{i,\tau} \Psi_{i,\tau}(\boldsymbol{\xi}_i^{\tau-1}, \boldsymbol{\zeta}_{\mathcal{N}_i}^{\tau}) + A_{i,\tau+1}^t E_{i,\tau} \xi_{i,\tau} \Big) \\ &=: \chi_{i,t}(\boldsymbol{\xi}_i^{t-1}, \boldsymbol{\zeta}_{\mathcal{N}_i}^{t-1}), \end{aligned} \tag{A.8}$$

where where $A_{i,\tau}^t = A_{i,\tau} A_{i,\tau+1} \dots A_{i,t-1}$ for $\tau < t$ and $A_{i,t}^t = I$.

We note that $\boldsymbol{\chi}_i(\boldsymbol{\xi}_i, \boldsymbol{\zeta}_{\mathcal{N}_i}) = [\chi_{i,t}(\boldsymbol{\xi}_i^{t-1}, \boldsymbol{\zeta}_{\mathcal{N}_i}^{t-1})]_{t \in \mathcal{T}_+} \in \mathcal{X}_i$ for all $\boldsymbol{\xi}_i \in \Xi_i$ and $\boldsymbol{\zeta}_{\mathcal{N}_i} \in \mathcal{X}_{\mathcal{N}_i}$ due to the feasibility of Problem (6b). To show that $\mathbf{\Psi}_i$ is feasible in Problem (5b), we first construct the state of agent i which evolves according to $\boldsymbol{x}_i = f_i\big(\boldsymbol{x}_{\mathcal{N}_i}, \mathbf{\Psi}_i(\boldsymbol{\xi}_i, \boldsymbol{\zeta}_{\mathcal{N}_i}), \boldsymbol{\xi}_i\big)$. Starting with $\hat{\chi}_{i,1} = x_{i,1}$, we have that

$$\begin{aligned} x_{i,t} &= A_{i,1}^t x_{i,1} + \sum_{\tau=1}^{t-1} \Big(A_{i,\tau+1}^t B_{i,\tau} [\widehat{\chi}_{j,\tau}(\boldsymbol{\xi}_{\overline{\mathcal{N}}_j}^{\tau-1}]_{j \in \mathcal{N}_i} + A_{i,\tau+1}^t D_{i,\tau} \Psi_{i,\tau}(\boldsymbol{\xi}_i^{\tau-1}, \boldsymbol{\zeta}_{\mathcal{N}_i}^{\tau}) + A_{i,\tau+1}^t E_{i,\tau} \xi_{i,\tau} \Big) \\ &= A_{i,1}^t x_{i,1} + \sum_{\tau=1}^{t-1} \Big(A_{i,\tau+1}^t B_{i,\tau} [\widehat{\chi}_{j,\tau}(\boldsymbol{\xi}_{\overline{\mathcal{N}}_j}^{\tau-1}]_{j \in \mathcal{N}_i} + A_{i,\tau+1}^t D_{i,\tau} \Psi_{i,\tau}(\boldsymbol{\xi}_i^{\tau-1}, [\widehat{\boldsymbol{\chi}}_j^{\tau}(\boldsymbol{\xi}_{\overline{\mathcal{N}}_j}^{\tau-1}]_{j \in \mathcal{N}_i}) + A_{i,\tau+1}^t E_{i,\tau} \xi_{i,\tau} \Big) \\ &= \chi_{i,t}(\boldsymbol{\xi}_i^{t-1}, [\widehat{\chi}_j^{t-1}(\boldsymbol{\xi}_{\overline{\mathcal{N}}_j}^{t-2})]_{j \in \mathcal{N}_i}) \\ &=: \widehat{\chi}_{i,t}(\boldsymbol{\xi}_{\overline{\mathcal{N}}_i}^{t-1}). \end{aligned} \tag{A.9}$$

where the implication follows because $\widehat{\boldsymbol{\chi}}_i([\boldsymbol{\xi}_j^{t-1}]_{j \in \overline{\mathcal{N}}_i}) = [\widehat{\chi}_{i,t}(\boldsymbol{\xi}_{\overline{\mathcal{N}}_i}^{t-1})]_{t \in \mathcal{T}_+} \in \mathcal{X}_i$ for all $\boldsymbol{\xi}_{\overline{\mathcal{N}}_i} \in \Xi_{\overline{\mathcal{N}}_i}$ and $i \in \mathcal{M}$. For each $i \in \mathcal{M}$, we consider the decision $\hat{\mathbf{\Phi}}_i(\boldsymbol{\xi}_{\overline{\mathcal{N}}_i})$ defined through

$$\widehat{\Phi}_{i,t}(\boldsymbol{\xi}_{\overline{\mathcal{N}}_i}^{t-1}) := \Psi_{i,t}(\boldsymbol{\xi}_i^{t-1}, [\widehat{\chi}_j^{t}(\boldsymbol{\xi}_{\overline{\mathcal{N}}_j}^{t-1})]_{j \in \mathcal{N}_i}) \tag{A.10}$$

Notice that (A.10) defines a valid policy construction since $\widehat{\boldsymbol{\chi}}_i([\boldsymbol{\xi}_j^{t-1}]_{j \in \overline{\mathcal{N}}_i}) \in \mathcal{X}_i$ for all $\boldsymbol{\xi}_{\overline{\mathcal{N}}_i} \in \Xi_{\overline{\mathcal{N}}_i}$. It remains to show that $\mathbf{\Psi}_i$ is feasible also for the constraints of Problem (5b). We do so using deduction, as follows:

$$\begin{aligned} &\big(\boldsymbol{\chi}_i(\boldsymbol{\xi}_i, \boldsymbol{\zeta}_{\mathcal{N}_i}), \mathbf{\Psi}_i(\boldsymbol{\xi}_i, \boldsymbol{\zeta}_{\mathcal{N}_i})\big) \in \mathcal{O}_i, && \forall \boldsymbol{\xi}_i \in \Xi_i, \forall \boldsymbol{\zeta}_{\mathcal{N}_i} \in \mathcal{X}_{\mathcal{N}_i}, \\ \implies &\big(\boldsymbol{\chi}_i(\boldsymbol{\xi}_i, [\widehat{\boldsymbol{\chi}}_j(\boldsymbol{\xi}_{\overline{\mathcal{N}}_j})]_{j \in \mathcal{N}_i}), \mathbf{\Psi}_i(\boldsymbol{\xi}_i, [\widehat{\boldsymbol{\chi}}_j(\boldsymbol{\xi}_{\overline{\mathcal{N}}_j})]_{j \in \mathcal{N}_i})\big) \in \mathcal{O}_i, && \forall \boldsymbol{\xi}_{\overline{\mathcal{N}}_i} \in \Xi_{\overline{\mathcal{N}}_i}, \\ \implies &\big(\widehat{\boldsymbol{\chi}}_i(\boldsymbol{\xi}_{\overline{\mathcal{N}}_i}), \widehat{\mathbf{\Phi}}_i(\boldsymbol{\xi}_{\overline{\mathcal{N}}_i})\big) \in \mathcal{O}_i, && \forall \boldsymbol{\xi}_{\overline{\mathcal{N}}_i} \in \Xi_{\overline{\mathcal{N}}_i}, \end{aligned} \tag{A.11}$$

The first implication follows from (A.9) and the fact that $\widehat{\boldsymbol{\chi}}_i(\boldsymbol{\xi}_{\overline{\mathcal{N}}_i}) \subseteq \mathcal{X}_i$ for all $\boldsymbol{\xi}_{\overline{\mathcal{N}}_i} \in \Xi_{\overline{\mathcal{N}}_i}$, while the second implication follows from (A.9) and (A.10). Finally, this feasible solution attains a value for the objective function of Problem (6b) which is equal or larger than the value attained for the objective function of Problem (5b), that is

$$\begin{aligned}
&\sum_{i=1}^{M} \max_{\boldsymbol{\xi}_i\in\Xi_i, \boldsymbol{\zeta}_{\mathcal{N}_i}\in\mathcal{X}_{\mathcal{N}_i}} J_i\big(\boldsymbol{\chi}_i(\boldsymbol{\xi}_i,\boldsymbol{\zeta}_{\mathcal{N}_i}),\boldsymbol{\Psi}_i(\boldsymbol{\xi}_i,\boldsymbol{\zeta}_{\mathcal{N}_i})\big)\\
\geq &\sum_{i=1}^{M} \max_{\boldsymbol{\xi}_{\overline{\mathcal{N}}_i}\in\Xi_{\overline{\mathcal{N}}_i}} J_i\big(\boldsymbol{\chi}_i(\boldsymbol{\xi}_i,[\widehat{\boldsymbol{\chi}}_j(\boldsymbol{\xi}_{\overline{\mathcal{N}}_j})]_{j\in\mathcal{N}_i}),\boldsymbol{\Psi}_i(\boldsymbol{\xi}_i,[\widehat{\boldsymbol{\chi}}_j(\boldsymbol{\xi}_{\overline{\mathcal{N}}_j})]_{j\in\mathcal{N}_i})\big)\\
= &\sum_{i=1}^{M} \max_{\boldsymbol{\xi}_{\overline{\mathcal{N}}_i}\in\Xi_{\overline{\mathcal{N}}_i}} J_i\big(\widehat{\boldsymbol{\chi}}_i(\boldsymbol{\xi}_{\overline{\mathcal{N}}_i}),\widehat{\boldsymbol{\Phi}}_i(\boldsymbol{\xi}_{\overline{\mathcal{N}}_i})\big),
\end{aligned}$$

where again the first implication follows from (A.9) and the fact that $\widehat{\boldsymbol{\chi}}_i(\boldsymbol{\xi}_{\overline{\mathcal{N}}_i})\subseteq\mathcal{X}_i$ for all $\boldsymbol{\xi}_{\overline{\mathcal{N}}_i}\in\Xi_{\overline{\mathcal{N}}_i}$, while the second implication follows from (A.9) and (A.10). $\square$

EC.1.4. Proof of Proposition 1

Theorem 3 already established that a feasible solution of Problem (6b) is feasible in Problem (5b). We will now show that a feasible solution in Problem (5a) is feasible in Problem (6a). The result will then follow due to the relation between Problems (5a) and (5b), see (Lin and Bitar 2016), and Problems (6a) to (6b), see Theorem 2.

In a directed acyclic bipartite network, the agents can be split in two groups, the first layer agents where $\overline{\mathcal{N}}_i=\{i\}$ and the second layer agents where $\overline{\mathcal{N}}_i=\{\mathcal{N}_i,i\}$. Sets $\mathcal{FL}$ and $\mathcal{SL}$ denote the first and second layer agents, respectively. Problem (6a) can then be explicitly written as

$$\begin{aligned}
\text{minimize}\;\; & \sum_{i\in\mathcal{FL}}\max_{\boldsymbol{\xi}_i\in\Xi_i} J_i(\boldsymbol{x}_i,\boldsymbol{u}_i)+\sum_{i\in\mathcal{SL}}\max_{\boldsymbol{\xi}_i\in\Xi_i,\boldsymbol{\zeta}_{\mathcal{N}_i}\in\mathcal{X}_{\mathcal{N}_i}} J_i(\boldsymbol{x}_i,\boldsymbol{u}_i)\\
\text{subject to}\;\; & \left.\begin{array}{l}\boldsymbol{u}_i=\boldsymbol{\psi}_i(\boldsymbol{x}_i):=[\psi_{i,t}(\boldsymbol{x}_i^t)]_{t\in\mathcal{T}}\\ \boldsymbol{x}_i=f_i\big(\boldsymbol{u}_i,\boldsymbol{\xi}_i\big)\\ (\boldsymbol{x}_i,\boldsymbol{u}_i)\in\mathcal{O}_i\\ \boldsymbol{x}_i\in\mathcal{X}_i,\ \mathcal{X}_i\in\mathcal{F}(\mathcal{S}_i)\end{array}\right\}\forall\boldsymbol{\xi}_i\in\Xi_i,\ \forall i\in\mathcal{FL},\\
& \left.\begin{array}{l}\boldsymbol{u}_i=\boldsymbol{\psi}_i(\boldsymbol{x}_i,\boldsymbol{\zeta}_{\mathcal{N}_i}):=[\psi_{i,t}(\boldsymbol{x}_i^t,\boldsymbol{\zeta}_{\mathcal{N}_i}^t)]_{t\in\mathcal{T}}\\ \boldsymbol{x}_i=f_i\big(\boldsymbol{\zeta}_{\mathcal{N}_i},\boldsymbol{u}_i,\boldsymbol{\xi}_i\big)\\ (\boldsymbol{x}_i,\boldsymbol{u}_i)\in\mathcal{O}_i\end{array}\right\}\begin{array}{l}\forall\boldsymbol{\zeta}_{\mathcal{N}_i}\in\mathcal{X}_{\mathcal{N}_i}\\ \forall\boldsymbol{\xi}_{\mathcal{M}}\in\Xi_{\mathcal{M}}\end{array}\ \forall i\in\mathcal{SL},
\end{aligned}\tag{A.12}$$

where the constraints distinguish between first and second layer agents. Notice that agents in the first layer are the ones communicating their state forecast sets $\mathcal{X}_i$ to their neighbors (constraint $\boldsymbol{x}_i\in\mathcal{X}_i$), while the agents on the second layer only receive these sets ($\boldsymbol{x}_i=f_i\big(\boldsymbol{\zeta}_{\mathcal{N}_i},\boldsymbol{u}_i,\boldsymbol{\xi}_i\big)$, $\forall\boldsymbol{\zeta}_{\mathcal{N}_i}\in\mathcal{X}_{\mathcal{N}_i}$).

Let ϕ_i for all $i \in \mathcal{M}$ be a feasible policy for Problem (5a). It is easy to see that policy is feasible in $(f_i(\phi_i(\boldsymbol{x}_i), \boldsymbol{\xi}_i), \boldsymbol{u}_i) \in \mathcal{O}_i \quad \forall \boldsymbol{\xi}_i \in \Xi_i,\ i \in \mathcal{FL}$ as Problems (5a) and (6a) share the same constraints. We set

$$\mathcal{X}_i := \{\boldsymbol{x}_i \in \mathbb{R}^{N_{x,i}} \,:\, \boldsymbol{x}_i = f_i\big(\phi_i(\boldsymbol{x}_i), \boldsymbol{\xi}_i\big)\ \forall \boldsymbol{\xi}_i \in \Xi_i\} \quad \forall i \in \mathcal{FL}, \tag{A.13}$$

which satisfies $\mathcal{X}_i \in \mathcal{F}(\mathcal{S}_i)$ where $\mathcal{S}_i = \mathbb{R}^{N_{x,i}}$ for all $i \in \mathcal{FL}$. By the feasibility of ϕ_i in Problem (5a), we have

$$\begin{aligned}
&\left.\begin{array}{l}\boldsymbol{x}_j = f_j\big(\phi_j(\boldsymbol{x}_j), \boldsymbol{\xi}_j\big),\ \forall j \in \mathcal{N}_i \\ (f_i(\boldsymbol{x}_{\mathcal{N}_i}, \phi_i(\boldsymbol{x}_i, \boldsymbol{x}_{\mathcal{N}_i}), \boldsymbol{\xi}_i), \phi_i(\boldsymbol{x}_i, \boldsymbol{x}_{\mathcal{N}_i})) \in \mathcal{O}_i\end{array}\right\} \forall \boldsymbol{\xi}_{\overline{\mathcal{N}}_i} \in \Xi_{\overline{\mathcal{N}}_i},\ \forall i \in \mathcal{SL}, \\
\iff\ & (f_i(\boldsymbol{x}_{\mathcal{N}_i}, \phi_i(\boldsymbol{x}_i, \boldsymbol{x}_{\mathcal{N}_i}), \boldsymbol{\xi}_i), \phi_i(\boldsymbol{x}_i, \boldsymbol{x}_{\mathcal{N}_i})) \in \mathcal{O}_i \Big\}\ \begin{array}{l}\forall [\boldsymbol{x}_j]_{j \in \mathcal{N}_i} = [f_j\big(\phi_j(\boldsymbol{x}_j), \boldsymbol{\xi}_j\big)]_{j \in \mathcal{N}_i} \\ \forall \boldsymbol{\xi}_{\overline{\mathcal{N}}_i} \in \Xi_{\overline{\mathcal{N}}_i}\end{array},\ \forall i \in \mathcal{SL}, \\
\iff\ & (f_i(\boldsymbol{\zeta}_{\mathcal{N}_i}, \phi_i(\boldsymbol{x}_i, \boldsymbol{\zeta}_{\mathcal{N}_i}), \boldsymbol{\xi}_i), \phi_i(\boldsymbol{x}_i, \boldsymbol{\zeta}_{\mathcal{N}_i})) \in \mathcal{O}_i \Big\}\ \begin{array}{l}\forall \boldsymbol{\zeta}_{\mathcal{N}_i} \in \mathcal{X}_{\mathcal{N}_i} \\ \forall \boldsymbol{\xi}_{\mathcal{M}} \in \Xi_{\mathcal{M}}\end{array},\ \forall i \in \mathcal{SL},
\end{aligned}$$

where the first equivalence hold since for fixed policy ϕ_i the dynamics of all agents in the first layer are uniquely determined by the uncertain parameters $\boldsymbol{\xi}_i \in \Xi_i$, $i \in \mathcal{FL}$, while the second equivalence holds by the definition of $\mathcal{X}_i$ in (A.13). Thus, ϕ_i is feasible in Problem (6a).

We next show that ϕ_i achieves the same optimal value in both problems.

$$\begin{aligned}
&\sum_{i=1}^{M} \max_{\boldsymbol{\xi}_i \in \Xi_{\overline{\mathcal{N}}_i}} J_i(\boldsymbol{x}_i, \boldsymbol{\phi}_i(\boldsymbol{x}_{\overline{\mathcal{N}}_i})) \\
&= \sum_{i \in \mathcal{FL}} \max_{\boldsymbol{\xi}_i \in \Xi_i} J_i(\boldsymbol{x}_i, \boldsymbol{\phi}_i(\boldsymbol{x}_i)) + \sum_{i \in \mathcal{SL}} \max_{\boldsymbol{\xi}_{\overline{\mathcal{N}}_i} \in \Xi_{\overline{\mathcal{N}}_i}} J_i(\boldsymbol{x}_i, \boldsymbol{\phi}_i(\boldsymbol{x}_i, \boldsymbol{x}_{\mathcal{N}_i})) \\
&= \sum_{i \in \mathcal{FL}} \max_{\boldsymbol{\xi}_i \in \Xi_i} J_i(\boldsymbol{x}_i, \boldsymbol{\phi}_i(\boldsymbol{x}_i)) + \sum_{i \in \mathcal{SL}} \max_{\boldsymbol{\xi}_{\overline{\mathcal{N}}_i} \in \Xi_{\overline{\mathcal{N}}_i},\, [\boldsymbol{x}_j]_{j \in \mathcal{N}_i} = [f_j\left(\phi_j(\boldsymbol{x}_j), \boldsymbol{\xi}_j\right)]_{j \in \mathcal{N}_i}} J_i(\boldsymbol{x}_i, \boldsymbol{\phi}_i(\boldsymbol{x}_i, \boldsymbol{x}_{\mathcal{N}_i})) \\
&= \sum_{i \in \mathcal{FL}} \max_{\boldsymbol{\xi}_i \in \Xi_i} J_i(\boldsymbol{x}_i, \boldsymbol{\phi}_i(\boldsymbol{x}_i)) + \sum_{i \in \mathcal{SL}} \max_{\boldsymbol{\xi}_i \in \Xi_i,\, \boldsymbol{\zeta}_{\mathcal{N}_i} \in \mathcal{X}_{\mathcal{N}_i}} J_i(\boldsymbol{x}_i, \boldsymbol{\phi}_i(\boldsymbol{x}_i, \boldsymbol{\zeta}_{\mathcal{N}_i}))
\end{aligned} \tag{A.14}$$

Starting from the objective of Problem (5a), the first equivalence rewrites the objective function it in terms of the first and second layer agents. Since for fixed ϕ_i the dynamics of the first layer agents are uniquely determined by the uncertain parameters $\boldsymbol{\xi}_i \in \Xi_i$, $i \in \mathcal{FL}$, the second equality re-expresses the dynamics of $\boldsymbol{x}_{\mathcal{N}_i}$ in terms of $\boldsymbol{\xi}_i \in \Xi_i$, $i \in \mathcal{FL}$. Finally, in the third equality we substitute the definition of $\mathcal{X}_i$ in (A.13). Notice that the uncertain parameters governing $\mathcal{X}_i$ are those affecting first layer agents, thus making $\mathcal{X}_{\mathcal{FL}}$ and $\Xi_{\mathcal{SL}}$ rectangular. The last expression in (A.14) coincides with the objective of Problem (6a).

EC.1.5. Proof of Proposition 2

The recession cone of the set $\mathcal{S}_i$ is defined as $\mathrm{recc}(\mathcal{S}_i) = \{\boldsymbol{\nu}_i \in \mathbb{R}^{n_x^i} : \boldsymbol{s}_i + \lambda\boldsymbol{\nu}_i \in \mathcal{S}_i, \forall \boldsymbol{s}_i \in \mathcal{S}_i,\, \lambda \geq 0\}$. The fact that $\mathcal{S}_i$ is bounded implies that the recession cone of $\mathcal{S}_i$ is empty, i.e., $\mathrm{recc}(\mathcal{S}_i) = \{0\}$. We now show that,

$$\mathcal{X}_{\mathcal{FS}} = \left\{ (\boldsymbol{x}_i, y_i, z_i) \,:\, \exists \boldsymbol{s}_i \in \mathbb{R}^{N_{x,i}} \text{ s.t. } \boldsymbol{x}_i = \sum_{k=1}^{K_i} y_{i,k} P_{i,k} \boldsymbol{s}_i + z_i,\; G_{i,k} P_{i,k} \boldsymbol{s}_i \preceq_{\mathcal{K}_{i,k}} g_{i,k},\; k = 1, \ldots, K_i \right\}$$

is equivalent to

$$\widehat{\mathcal{X}}_{\mathcal{FS}} = \left\{ (\boldsymbol{x}_i, y_i, z_i) \,:\, \exists \boldsymbol{\nu}_{i,k} \in \mathbb{R}^{N_{x,i}} \text{ s.t. } \boldsymbol{x}_i = \sum_{k=1}^{K_i} P_{i,k} \boldsymbol{\nu}_{i,k} + z_i,\; G_{i,k} P_{i,k} \boldsymbol{\nu}_{i,k} \preceq_{\mathcal{K}_{i,k}} y_{i,k} g_{i,k},\; k = 1, \ldots, K_i \right\}$$

It is easy to verify that this in the case where $y_{i,k}$ are positive scalar by using the substitution $\boldsymbol{\nu}_{i,k} = y_{i,k}\boldsymbol{s}_i$. In the case that any $y_{i,k} = 0$ then it remains to show that the only feasible solution is $\boldsymbol{\nu}_{i,k} = 0$ so that the equality $\boldsymbol{\nu}_{i,k} = y_{i,k}\boldsymbol{s}_i$ holds. Assume that this is not the case, i.e., there exist $\boldsymbol{\nu}_{i,k} \neq 0$. Then, $\boldsymbol{\nu}_{i,k} \in \mathrm{recc}(S_i)$ which means that the $\mathcal{S}_i$ recedes in the direction of $\boldsymbol{\nu}_{i,k}$. However, this is a contradicts the boundedness of $\mathcal{S}_i$. The substitution $\boldsymbol{\nu}_{i,k} = y_{i,k}\boldsymbol{s}_i$ was first proposed by George Dantzig in (Dantzig 2016), and a similar proof also appear in (Gorissen et al. 2014).

EC.1.6. Preliminaries for the proof of Theorem 4

The following lemma establishes constraint satisfaction between Problem (8a) and Problem (8b).

Lemma 1 *Given vectors y_i and z_i such that $\widehat{\mathcal{X}}_i(y_i, z_i)$, then for any two functions $f_{i,t}$ and $g_{i,t}$, it holds:*

$$\begin{aligned} &f_{i,t}(\boldsymbol{\zeta}^t_{\mathcal{N}_i}, \boldsymbol{\xi}^t_i) \leq 0, \qquad \forall \boldsymbol{\zeta}_{\mathcal{N}_i} \in \widehat{\mathcal{X}}_{\mathcal{N}_i}(y_{\mathcal{N}_i}, z_{\mathcal{N}_i}),\, \forall \boldsymbol{\xi}_i \in \Xi_i, \\ \Rightarrow\; &f_{i,t}\big([R^t_j(\boldsymbol{s}^t_j)]_{j\in\mathcal{N}_i}), \boldsymbol{\xi}^t_i\big) \leq 0,\, \forall \boldsymbol{s}_{\mathcal{N}_i} \in \mathcal{S}_{\mathcal{N}_i},\, \forall \boldsymbol{\xi}_i \in \Xi_i, \end{aligned} \tag{A.15}$$

and

$$\begin{aligned} &g_{i,t}(\boldsymbol{s}^t_{\mathcal{N}_i}, \boldsymbol{\xi}^t) \leq 0, \qquad \forall \boldsymbol{s}_{\mathcal{N}_i} \in \mathcal{S}_{\mathcal{N}_i},\, \forall \boldsymbol{\xi}_i \in \Xi_i, \\ \Rightarrow\; &g_{i,t}\big([L^t_j(\boldsymbol{\zeta}^t_j)]_{j\in\mathcal{N}_i}, \boldsymbol{\xi}^t_i\big) \leq 0,\, \forall \boldsymbol{\zeta}_{\mathcal{N}_i} \in \widehat{\mathcal{X}}_{\mathcal{N}_i}(y_{\mathcal{N}_i}, z_{\mathcal{N}_i}),\, \forall \boldsymbol{\xi}_i \in \Xi_i. \end{aligned} \tag{A.16}$$

Proof of Lemma 1 We prove (A.15) by contradiction. Assume that $f_{i,t}(\boldsymbol{\zeta}^t_{\mathcal{N}_i}, \boldsymbol{\xi}^t_i) \leq 0, \forall \boldsymbol{\zeta}_{\mathcal{N}_i} \in \widehat{\mathcal{X}}_{\mathcal{N}_i}(y_{\mathcal{N}_i}, z_{\mathcal{N}_i})$, $\forall \boldsymbol{\xi}_i \in \Xi_i$, and there exist $\boldsymbol{s}_{\mathcal{N}_i} \in \mathcal{S}_{\mathcal{N}_i}$ such that $f_{i,t}\big([R^t_j(\boldsymbol{s}^t_j)]_{j\in\mathcal{N}_i}), \boldsymbol{\xi}^t_i\big) > 0$. Considering that $[R^t_j(\boldsymbol{s}^t_j)]_{j\in\mathcal{N}_i} \in \widehat{\mathcal{X}}(y_{\mathcal{N}_i}, z_{\mathcal{N}_i})$ for all $\boldsymbol{s}_{\mathcal{N}_i} \in \mathcal{S}_{\mathcal{N}_i}$ by construction, this leads to a contradiction. The proof of (A.16) follows similar arguments. □

EC.1.7. Proof of Theorem 4

We show that every feasible solution of Problem (8b) is feasible in Problem (8a). Let $(\widehat{\mathbf{\Gamma}}_i, \widehat{\mathcal{X}}_i)$ for all $i \in \mathcal{M}$ be a feasible solution in Problem (8b). Since the state of agent i evolve according to (1), we can conclude that at time t we have

$$\begin{aligned}
x_{i,t} &= A^t_{i,1} x_{i,1} + \sum_{\tau=1}^{t-1} \Big(A^t_{i,\tau+1} B_{i,\tau} \boldsymbol{\zeta}_{\mathcal{N}_i,\tau} + A^t_{i,\tau+1} D_{i,\tau} \widehat{\Gamma}_{i,\tau}(\boldsymbol{\xi}_i^{\tau-1}, \boldsymbol{s}^{\tau}_{\mathcal{N}_i}) + A^t_{i,\tau+1} E_{i,\tau} \xi_{i,\tau} \Big) \\
&= A^t_{i,1} x_{i,1} + \sum_{\tau=1}^{t-1} \Big(A^t_{i,\tau+1} B_{i,\tau} (Y_{\mathcal{N}_i,\tau} \boldsymbol{s}_{\mathcal{N}_i,\tau} + z_{\mathcal{N}_i,\tau}) + A^t_{i,\tau+1} D_{i,\tau} \widehat{\Gamma}_{i,\tau}(\boldsymbol{\xi}_i^{\tau-1}, \boldsymbol{s}^{\tau}_{\mathcal{N}_i}) + A^t_{i,\tau+1} E_{i,\tau} \xi_{i,\tau} \Big) \\
&=: \widehat{\chi}_{i,t}(\boldsymbol{\xi}_i^{t-1}, \boldsymbol{s}^{t-1}_{\mathcal{N}_i}),
\end{aligned}$$

where $A^t_{i,\tau} = A_{i,\tau} A_{i,\tau+1} \dots A_{i,t-1}$ for $\tau < t$ and $A^t_{i,t} = I$. To show that $\widehat{\mathbf{\Gamma}}_i$ is feasible in Problem (8a), we first construct the state of agent i which evolves according to $\boldsymbol{x}_i = f_i\big(\boldsymbol{\zeta}_{\mathcal{N}_i}, \widehat{\mathbf{\Gamma}}_i(\boldsymbol{\xi}_i, \boldsymbol{s}_{\mathcal{N}_i}), \boldsymbol{\xi}_i\big)$. Starting with $\widetilde{\chi}_{i,1} = x_{i,1}$, we have that

$$\begin{aligned}
x_{i,t} &= A^t_{i,1} x_{i,1} + \sum_{\tau=1}^{t-1} \Big(A^t_{i,\tau+1} B_{i,\tau} \boldsymbol{\zeta}_{\mathcal{N}_i,\tau} + A^t_{i,\tau+1} D_{i,\tau} \widehat{\Gamma}_{i,\tau}(\boldsymbol{\xi}_i^{\tau-1}, \boldsymbol{s}^{\tau}_{\mathcal{N}_i}) + A^t_{i,\tau+1} E_{i,\tau} \xi_{i,\tau} \Big) \\
&= A^t_{i,t-1} x_{i,1} + \sum_{\tau=1}^{t-1} \Big(A^t_{i,\tau+1} B_{i,\tau} \boldsymbol{\zeta}_{\mathcal{N}_i} + A^t_{i,\tau+1} D_{i,\tau} \widehat{\Gamma}_{i,\tau}(\boldsymbol{\xi}_i^{\tau-1}, [L^{\tau}_j(\boldsymbol{\zeta}^{\tau}_j)]_{j\in\mathcal{N}_i}) + A^t_{i,\tau+1} E_{i,\tau} \xi_{i,\tau} \Big) \\
&= \widehat{\chi}_{i,t} \left(\boldsymbol{\xi}_i^{t-1}, [L^{t-1}_j(\boldsymbol{\zeta}^{t-1}_j)]_{j\in\mathcal{N}_i} \right) \\
&=: \widetilde{\chi}_{i,t} \left(\boldsymbol{\xi}_i^{t-1}, \boldsymbol{\zeta}^{t-1}_{\mathcal{N}_i} \right),
\end{aligned} \tag{A.17}$$

where the implications follow due to the mapping (9b). For each $i \in \mathcal{M}$, we consider the decision $\widetilde{\mathbf{\Psi}}_i(\boldsymbol{\xi}_i, \boldsymbol{\zeta}_{\mathcal{N}_i})$ defined through

$$\widetilde{\Psi}_{i,t} \left(\boldsymbol{\xi}_i^{t-1}, \boldsymbol{\zeta}^{t}_{\mathcal{N}_i} \right) = \widehat{\Gamma}_{i,t} \left(\boldsymbol{\xi}_i^{t-1}, [L^{t}_j(\boldsymbol{\zeta}^{t}_j)]_{j\in\mathcal{N}_i} \right). \tag{A.18}$$

Notice that (A.18) defines a valid policy construction due to the mapping (9b). It remains to show that $\widehat{\mathbf{\Gamma}}_i$ is feasible also for the constraints of Problem (8a). We do so using deduction, as follows:

$$\begin{aligned}
&\big(\widehat{\boldsymbol{\chi}}_i(\boldsymbol{\xi}_i, \boldsymbol{s}_{\mathcal{N}_i}), \widehat{\mathbf{\Gamma}}_i(\boldsymbol{\xi}_i, \boldsymbol{s}_{\mathcal{N}_i})\big) \in \mathcal{O}_i, && \forall \boldsymbol{\xi}_i \in \Xi_i, \forall \boldsymbol{s}_{\mathcal{N}_i} \in \mathcal{S}_{\mathcal{N}_i}, \\
\implies &\big(\widehat{\boldsymbol{\chi}}_i(\boldsymbol{\xi}_i, [L_j(\boldsymbol{\zeta}_j)]_{j\in\mathcal{N}_i}), \widehat{\mathbf{\Gamma}}_i(\boldsymbol{\xi}_i, [L_j(\boldsymbol{\zeta}_j)]_{j\in\mathcal{N}_i})\big) \in \mathcal{O}_i, && \forall \boldsymbol{\xi}_i \in \Xi_i, \forall \boldsymbol{\zeta}_{\mathcal{N}_i} \in \widehat{\mathcal{X}}_{\mathcal{N}_i}, \\
\implies &\big(\widetilde{\boldsymbol{\chi}}_i(\boldsymbol{\xi}_i, \boldsymbol{\zeta}_{\mathcal{N}_i}), \widetilde{\mathbf{\Psi}}_i(\boldsymbol{\xi}_i, \boldsymbol{\zeta}_{\mathcal{N}_i})\big) \in \mathcal{O}_i, && \forall \boldsymbol{\xi}_i \in \Xi_i, \forall \boldsymbol{\zeta}_{\mathcal{N}_i} \in \widehat{\mathcal{X}}_{\mathcal{N}_i},
\end{aligned} \tag{A.19}$$

where the implications directly follow from (A.17) and (A.18), and Lemma 1. Same reasoning applies to all constraints in the problem formulation. This feasible solution attains the same value, ℓ, for the objective functions of Problem (8b) and Problem (8a), that is:

$$\begin{aligned}
\ell &= \sum_{i=1}^{M} \max_{\boldsymbol{\xi}_i \in \Xi, \boldsymbol{s}_{\mathcal{N}_i} \in \mathcal{S}_{\mathcal{N}_i}} J_i\big(\widehat{\boldsymbol{\chi}}_i(\boldsymbol{\xi}_i, \boldsymbol{s}_{\mathcal{N}_i}), \widehat{\boldsymbol{\Gamma}}_i(\boldsymbol{\xi}_i, \boldsymbol{s}_{\mathcal{N}_i})\big) \\
&= \left\{ \begin{array}{l} J_i\big(\widehat{\boldsymbol{\chi}}_i(\boldsymbol{\xi}_i, \boldsymbol{s}_{\mathcal{N}_i}), \widehat{\boldsymbol{\Gamma}}_i(\boldsymbol{\xi}_i, \boldsymbol{s}_{\mathcal{N}_i})\big) \le \ell_i, \;\; \forall \boldsymbol{\xi}_i \in \Xi_i, \forall \boldsymbol{s}_{\mathcal{N}_i} \in \mathcal{S}_{\mathcal{N}_i} \\ \sum_{i=1}^{M} \ell_i = \ell, \end{array} \right\} \\
&= \left\{ \begin{array}{l} J_i\big(\widehat{\boldsymbol{\chi}}_i(\boldsymbol{\xi}_i, [L_j(\boldsymbol{\zeta}_j)]_{j\in\mathcal{N}_i}), \widehat{\boldsymbol{\Gamma}}_i(\boldsymbol{\xi}_i, [L_j(\boldsymbol{\zeta}_j)]_{j\in\mathcal{N}_i})\big) \le \ell_i, \;\; \forall \boldsymbol{\xi}_i \in \Xi_i, \forall \boldsymbol{\zeta}_{\mathcal{N}_i} \in \widehat{\mathcal{X}}_{\mathcal{N}_i} \\ \sum_{i=1}^{M} \ell_i = \ell, \end{array} \right\} \\
&= \sum_{i=1}^{M} \max_{\boldsymbol{\xi}_i \in \Xi_i, \boldsymbol{\zeta}_{\mathcal{N}_i} \in \widehat{\mathcal{X}}_{\mathcal{N}_i}} J_i\big(\widetilde{\boldsymbol{\chi}}_i(\boldsymbol{\xi}_i, \boldsymbol{\zeta}_{\mathcal{N}_i}), \widetilde{\boldsymbol{\Psi}}_i(\boldsymbol{\xi}_i, \boldsymbol{\zeta}_{\mathcal{N}_i})\big) = \ell.
\end{aligned} \tag{A.20}$$

The implications directly follow from (A.17) and (A.18), and Lemma 1.

Similarly, we now show that every feasible solution of Problem (8a) is feasible in Problem (8b). Let $(\widetilde{\boldsymbol{\Psi}}_i, \widehat{\mathcal{X}}_i)$ for all $i \in \mathcal{M}$ be feasible in Problem (8a). Since the state of agent i evolve according to (1), we can conclude that at time t we have

$$\begin{aligned}
x_{i,t} &= A_{i,1}^t x_{i,1} + \sum_{\tau=1}^{t-1} \Big(A_{i,\tau+1}^t B_{i,\tau} \boldsymbol{\zeta}_{\mathcal{N}_i,\tau} + A_{i,\tau+1}^t D_{i,\tau} \widetilde{\Psi}_{i,\tau}(\boldsymbol{\xi}_i^{\tau-1}, \boldsymbol{\zeta}_{\mathcal{N}_i}^{\tau}) + A_{i,\tau+1}^t E_{i,\tau} \xi_{i,\tau} \Big) \\
&=: \widetilde{\chi}_{i,t}(\boldsymbol{\xi}_i^{t-1}, \boldsymbol{\zeta}_{\mathcal{N}_i}^{t-1})
\end{aligned} \tag{A.21}$$

To show that $\widetilde{\boldsymbol{\Psi}}_i$ is feasible in Problem (8b), we first construct the state of agent i which evolves according to $\boldsymbol{x}_i = f_i\big(Y_{\mathcal{N}_i} \boldsymbol{s}_{\mathcal{N}_i} + z_{\mathcal{N}_i}, \widetilde{\boldsymbol{\Psi}}_i(\boldsymbol{\xi}_i, \boldsymbol{\zeta}_{\mathcal{N}_i}), \boldsymbol{\xi}_i\big)$. Starting with $\widehat{\chi}_{i,1} = x_{i,1}$, we have that

$$\begin{aligned}
x_{i,t} &= A_{i,1}^t x_{i,1} + \sum_{\tau=1}^{t-1} \Big(A_{i,\tau+1}^t B_{i,\tau} (Y_{\mathcal{N}_i,\tau} \boldsymbol{s}_{\mathcal{N}_i,\tau} + z_{\mathcal{N}_i,\tau}) + A_{i,\tau+1}^t D_{i,\tau} \widetilde{\Psi}_{i,t}(\boldsymbol{\xi}_i^{\tau-1}, \boldsymbol{\zeta}_{\mathcal{N}_i}^{\tau}) + A_{i,\tau+1}^t E_{i,\tau} \xi_{i,\tau} \Big) \\
&= A_{i,1}^t x_{i,1} + \sum_{\tau=1}^{t-1} \Big(A_{i,\tau+1}^t B_{i,\tau} [R_{j,\tau}(\boldsymbol{s}_{j,\tau})]_{j\in\mathcal{N}_i} + A_{i,\tau+1}^t D_{i,\tau} \widetilde{\Psi}_{i,t}(\boldsymbol{\xi}_i^{\tau-1}, [R_j^{\tau}(\boldsymbol{s}_j^{\tau})]_{j\in\mathcal{N}_i}) + A_{i,\tau+1}^t E_{i,\tau} \xi_{i,\tau} \Big) \\
&= \widetilde{\chi}_{i,t}\left(\boldsymbol{\xi}_i^{t-1}, [R_j^{t-1}(\boldsymbol{s}_j^{t-1})]_{j\in\mathcal{N}_i}\right) \\
&=: \widehat{\chi}_{i,t}\left(\boldsymbol{\xi}_i^{t-1}, \boldsymbol{s}_{\mathcal{N}_i}^{t-1}\right).
\end{aligned} \tag{A.22}$$

where the implications follow due to the mapping (9a). For each $i \in \mathcal{M}$, we consider the decision $\widehat{\boldsymbol{\Gamma}}_i(\boldsymbol{\xi}_i, \boldsymbol{s}_{\mathcal{N}_i})$ defined through

$$\widehat{\Gamma}_{i,t}\left(\boldsymbol{\xi}_i^{t-1}, \boldsymbol{s}_{\mathcal{N}_i}^{t}\right) = \widetilde{\Psi}_{i,t}\left(\boldsymbol{\xi}_i^{t-1}, [R_j^t(\boldsymbol{s}_j^t)]_{j\in\mathcal{N}_i}\right). \tag{A.23}$$

Notice that (A.23) defines a valid policy construction due to the mapping (9a). It remains to show that $\widehat{\mathbf{\Gamma}}_i$ is feasible also for the constraints of Problem (8b). We do so using deduction, as follows:

$$\begin{aligned} &\big(\widetilde{\chi}_i(\boldsymbol{\xi}_i,\boldsymbol{\zeta}_{\mathcal{N}_i}),\widetilde{\mathbf{\Psi}}_i(\boldsymbol{\xi}_i,\boldsymbol{\zeta}_{\mathcal{N}_i})\big)\in\mathcal{O}_i, && \forall\boldsymbol{\xi}_i\in\Xi_i,\forall\boldsymbol{\zeta}_{\mathcal{N}_i}\in\widehat{\mathcal{X}}_{\mathcal{N}_i},\\ \Longrightarrow\ &\big(\widetilde{\chi}_i(\boldsymbol{\xi}_i,[R_j(\boldsymbol{s}_j)]_{j\in\mathcal{N}_i}),\widetilde{\mathbf{\Psi}}_i(\boldsymbol{\xi}_i,[R_j(\boldsymbol{s}_j)]_{j\in\mathcal{N}_i})\big)\in\mathcal{O}_i, && \forall\boldsymbol{\xi}_i\in\Xi_i,\forall\boldsymbol{s}_{\mathcal{N}_i}\in\mathcal{S}_{\mathcal{N}_i},\\ \Longrightarrow\ &\big(\widehat{\chi}_i(\boldsymbol{\xi}_i,\boldsymbol{s}_{\mathcal{N}_i}),\widehat{\mathbf{\Gamma}}_i(\boldsymbol{\xi}_i,\boldsymbol{s}_{\mathcal{N}_i})\big)\in\mathcal{O}_i, && \forall\boldsymbol{\xi}_i\in\Xi_i,\forall\boldsymbol{s}_{\mathcal{N}_i}\in\mathcal{S}_{\mathcal{N}_i}, \end{aligned}\tag{A.24}$$

where the implications directly follow from (A.22) and (A.23), and Lemma 1. Same reasoning applies to all constraints in the problem formulation. This feasible solution attains the same value, ℓ, for the objective functions of Problem (8a) and Problem (8b), that is:

$$\begin{aligned} \ell &= \sum_{i=1}^{M}\max_{\boldsymbol{\xi}_i\in\Xi_i,\boldsymbol{\zeta}_{\mathcal{N}_i}\in\widehat{\mathcal{X}}_{\mathcal{N}_i}} J_i\big(\widetilde{\chi}_i(\boldsymbol{\xi}_i,\boldsymbol{\zeta}_{\mathcal{N}_i}),\widetilde{\mathbf{\Psi}}_i(\boldsymbol{\xi}_i,\boldsymbol{\zeta}_{\mathcal{N}_i})\big)\\ &= \left\{\begin{array}{l} J_i\big(\widetilde{\chi}_i(\boldsymbol{\xi}_i,\boldsymbol{\zeta}_{\mathcal{N}_i}),\widetilde{\mathbf{\Psi}}_i(\boldsymbol{\xi}_i,\boldsymbol{\zeta}_{\mathcal{N}_i})\big)\le\ell_i,\ \ \forall\boldsymbol{\xi}_i\in\Xi_i,\forall\boldsymbol{\zeta}_{\mathcal{N}_i}\in\widehat{\mathcal{X}}_{\mathcal{N}_i},\\ \sum_{i=1}^{M}\ell_i=\ell, \end{array}\right\}\\ &= \left\{\begin{array}{l} J_i\big(\widetilde{\chi}_i(\boldsymbol{\xi}_i,[R_j(\boldsymbol{s}_j)]_{j\in\mathcal{N}_i}),\widetilde{\mathbf{\Psi}}_i(\boldsymbol{\xi}_i,[R_j(\boldsymbol{s}_j)]_{j\in\mathcal{N}_i})\big)\le\ell_i,\ \ \forall\boldsymbol{\xi}_i\in\Xi_i,\forall\boldsymbol{s}_{\mathcal{N}_i}\in\mathcal{S}_{\mathcal{N}_i},\\ \sum_{i=1}^{M}\ell_i=\ell, \end{array}\right\}\\ &= \sum_{i=1}^{M}\max_{\boldsymbol{\xi}_i\in\Xi_i,\boldsymbol{s}_{\mathcal{N}_i}\in\mathcal{S}_{\mathcal{N}_i}} J_i\big(\widehat{\chi}_i(\boldsymbol{\xi}_i,\boldsymbol{s}_{\mathcal{N}_i}),\widehat{\mathbf{\Gamma}}_i(\boldsymbol{\xi}_i,\boldsymbol{s}_{\mathcal{N}_i})\big)=\ell. \end{aligned}\tag{A.25}$$

The implications directly follow from (A.22) and (A.23), and Lemma 1. □

EC.1.8. Proof of Corollary 2

To demonstrate the result, it is sufficient to show that approximation (7) has sufficient degrees of freedom to represent the optimal state forecast set $\mathcal{X}_i$.

We first need to determine the complexity of $\mathcal{X}_i$ in Problem (6b). For fixed $\mathcal{X}_i$ and an appropriate linearization of the piecewise objective function $J_i(\boldsymbol{x}_i,\boldsymbol{u}_i)$ using epigraph variables, Problem (6b) falls into the class of linear multistage robust optimization problem with right-hand-side uncertainty. This implies that we can replace the for all $\boldsymbol{\xi}_i\in\Xi_i$ and $\boldsymbol{\zeta}_{\mathcal{N}_i}\in\mathcal{X}_{\mathcal{N}_i}$, with for all *extreme points* $\boldsymbol{\xi}_i\in\text{ext}(\Xi_i)$ and $\boldsymbol{\zeta}_{\mathcal{N}_i}\in\text{ext}(\mathcal{X}_{\mathcal{N}_i})$, without affecting the optimal value of the problem, see Georghiou et al. (2019). Since the choice of $\mathcal{X}_i$ at the beginning of the argument was arbitrary, we can conclude that we can always make this replacement without loss of generality. Moreover, due to the arborescence network structure and the linearity of the dynamics, the state $\boldsymbol{x}_{\text{root}}$ of the root node can take at most $|\text{ext}(\Xi_{\text{root}})|$ unique values. Hence, due to the convexity of the objective function, the smallest state forecast set $\mathcal{X}_{\text{root}}$ can be described with a convex combination of at most $|\text{ext}(\Xi_{\text{root}})|$ points. Using a recursive construction, the states $\boldsymbol{x}_i$ all agents in the tree can take at most $\prod_{j\in\overline{\mathcal{N}}_i}|\text{ext}(\Xi_j)|$ unique values, hence the state forecast set $\mathcal{X}_i$ can be described with a convex combination of at most $\prod_{j\in\overline{\mathcal{N}}_i}|\text{ext}(\Xi_j)|$ points.

The above arguments shows that the maximum degrees of freedom needed to describe $\mathcal{X}_i$ is $\prod_{j\in\overline{\mathcal{N}}_i}|\text{ext}(\Xi_j)|$. Hence, by setting $\mathcal{S}_i$ to be a simplex of dimension $\prod_{j\in\overline{\mathcal{N}}_i}|\text{ext}(\Xi_i)|$, then for each $i\in\mathcal{M}$ the affine mapping in (7) can project $\mathcal{S}_i$ to a set of at most $\prod_{j\in\overline{\mathcal{N}}_i}|\text{ext}(\Xi_i)|$. Hence the result follows. □

EC.2. Supply chain with quantity flexibility contracts

In this section, we evaluate the performance of the proposed method in a contract design mechanism for supply chains with decentralized operations. The proposed contract design is based on the structure of quantity flexibility (QF) contracts described in (Tsay and Lovejoy 1999). Decentralized supply chains are the norm in modern businesses since agents around the world cooperate to deliver multiple products. Due to this decentralized structure, each manufacturer (supplier) knows only what its immediate retailer (manufacturer) has requested, and is only concerned with its own performance cost. This, however, leads to "mutual deception" situations in which, for instance, some buyers inflate demand only to later disavow any undesired product (Lee et al. 1997), which increases uncertainty and operational costs in decentralized supply change networks (Magee and Boodman 1967, Lovejoy 1998).

To address this problem, QF contracts are used in the industry to coordinate the flow of materials and information in distributed supply chains over a fixed period of time. In this setting, for a given product $p\in\mathcal{P}$ the QF contract between the pair manufacturer-retailer is parametrized by lower and upper bounds $\underline{\boldsymbol{b}}^p=[\underline{b}_1^p,\ldots,\underline{b}_T^p]$ and $\overline{\boldsymbol{b}}^p=[\overline{b}_1^p,\ldots,\overline{b}_T^p]$, respectively. Every period $t\in\mathcal{T}$, the retailer has the right to request delivery of any quantity of product $p\in\mathcal{P}$ within the agreed bounds $[\underline{b}_t^p,\overline{b}_t^p]$, and the manufacturer has the obligation to deliver it. QF contracts exist between suppliers and manufacturers as well. In this way, the contract provides some flexibility for the retailer, helping to mitigate the uncertainties of future demand, as well as provide strong indications to the manufacturer about how to schedule the production line. In practice, as time passes and the actual demand faced by the retailer is revealed, the two parties are allowed to revise their contracts within pre-agreed percentage changes of the lower and upper bounds. Figure EC.1 shows

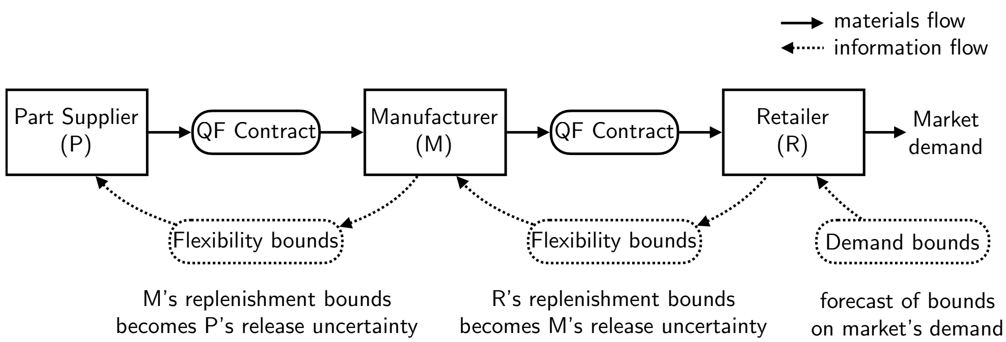


Figure EC.1 Supply chain design with quantitative flexibility contracts (Tsay and Lovejoy 1999)

graphically how a serial supply chain with a supplier, a manufacturer and a retailer operates using QF contracts to move a single product.

The design of QF contracts fits perfectly within the proposed framework where the forecast sets are in fact the upper and lower bounds that define the QF contracts. The seminal work of (Lee et al. 1997) first proposes the use of optimization techniques for designing QF contracts in a so-called open-loop feedback control system, in which uncertain exogenous values are assumed to be known. Our proposed framework, however, truthfully models the uncertainty and incorporates it within the optimization framework. In the following, we discuss the design of such QF contracts.

EC.2.1. Problem Formulation

First consider a supply chain with $M = 3$ agents as depicted in Figure EC.1. In this simple example, the supplier is $i = 1$, the manufacturer is $i = 2$ and the retailer is $i = 3$, with $\mathcal{N}_1 = \{2\}$, $\mathcal{N}_2 = \{3\}$ and $\mathcal{N}_3 = \emptyset$. We next partition the index set M into three disjoint sets $\mathcal{M}_s = \{1\}$, $\mathcal{M}_m = \{2\}$, and $M_r = \{3\}$ to represent the indices associated with the supplier, the manufacturer, and the supplier, respectively. The inventory dynamics for product $p \in \mathcal{P} = \{1, \ldots, P\}$ and for every $t \in \mathcal{T}$ and $i \in \mathcal{M}$ are expressed through

$$\begin{cases} I^p_{1,t+1} = I^p_{1,t} + R^p_{1,t} - U^p_{2,t} & \text{(supplier)}, \\ I^p_{2,t+1} = I^p_{2,t} + R^p_{2,t} - U^p_{3,t} & \text{(manufacturer)}, \\ I^p_{3,t+1} = I^p_{3,t} + R^p_{3,t} - D^p_{t} & \text{(retailer)}, \end{cases} \tag{B.26}$$

where $I^p_{i,t}$ denotes the inventory stock of product $p \in \mathcal{P}$ held by agent $i \in \mathcal{M}$ at time t. Furthermore, $R^p_{i,t}$ is the replenishment decision defined as

$$R^p_{i,t} = \sum_{p=1}^{P} B^p_i U^p_{i,t} + \xi^p_{i,\text{production},t},$$

where B^p_i is the blending coefficients, $U^p_{i,t} \in [\underline{b}^p_{i,t}, \overline{b}^p_{i,t}]$ denotes the quantity of product p that agent i will request from its neighbor at time t with $\underline{b}^p_{i,t}$ and $\overline{b}^p_{i,t}$ being the lower bound and upper bound of QF contracts, and $\xi^p_{i,\text{production},t}$ is an uncertain vector capturing materials loss. Finally, D^p_t denotes the product demand for item p at time t originating from the market, and it is assumed to be periodic and governed by a factor model with K factors of the form

$$D^p_t = \begin{cases} 2 + \sin\left(2\pi \dfrac{t}{T-1}\right) + \dfrac{1}{K}\sum\limits_{k=1}^{K} F^p_k \xi_{k,t,\text{demand}} & \text{for } p \text{ even} \\ 2 + \cos\left(2\pi \dfrac{t}{T-1}\right) + \dfrac{1}{K}\sum\limits_{k=1}^{K} F^p_k \xi_{k,t,\text{demand}} & \text{for } p \text{ odd,} \end{cases} \tag{B.27}$$

where F^k_p captures correlations amongst the products and $\xi_{k,\text{demand},t}$ captures uncertainty in the factor k, similar to (Georghiou et al. 2019).

Since the market demand and material loss are uncertain, we seek to design QF contracts that minimize the worst-case sum of backlog and inventory holding costs across all agents. Moreover, we assume that agents do not wish to disclose their actual demands from their neighbors due to privacy concerns. However, they are willing to share lower and upper limits of what they need from each other. In this case, the inventory dynamic (B.26) translates to the following compact representation

$$I^p_{i,t+1} = I^p_{i,t} + R^p_{i,t} - D^p_{i,t},$$

where $D^p_{i,t} = \zeta^p_{\mathcal{N}_i,t}$ for every $i \in \mathcal{M}_s \bigcup \mathcal{M}_m$ with $\zeta^p_{i,t} \in \mathcal{U}^p_{i,t} = [\underline{b}^p_{i,t}, \overline{b}^p_{i,t}]$ being the introduced auxiliary uncertainty to represent the unknown demand from the neighbor and $D^p_{i,t} = D^p_t$ for $i \in M_r$. The objective of agent $i \in \mathcal{M}$ is to determine an ordering policy $\boldsymbol{U}_{i,t}$ for every $t \in \mathcal{T}$ based on his inventory level up to time t, namely $[I_{i,1}, \dots, I_{i,t}]$, and the uncertain demand from his neighbor up to time t, namely $[\zeta_{i+1,1}, \dots, \zeta_{i+1,t}]$, if additionally $i \in \mathcal{M}_s \bigcup \mathcal{M}_m$. The overall optimization problem for designing the QF contracts is an instance of Problem (6a) and is formulated as

$$
\begin{array}{lll}
\text{minimize} & \displaystyle\sum_{i=1}^{M} \max_{\boldsymbol{\xi}_i\in\Xi_i,\boldsymbol{\zeta}_{\mathcal{N}_i}\in\mathcal{U}_{\mathcal{N}_i}} \sum_{t=1}^{T+1}\sum_{p=1}^{P} c_H\left[I_{i,t}^p\right]_+ + c_B\left[-I_{i,t}^p\right]_+ & \\
\text{subject to} & \boldsymbol{U}_i=\boldsymbol{\psi}_i(\boldsymbol{I}_i,\boldsymbol{\zeta}_{\mathcal{N}_i}) & \forall\boldsymbol{\xi}_i\in\Xi_i,\forall\boldsymbol{\zeta}_{\mathcal{N}_i}\in\mathcal{U}_{\mathcal{N}_i},\forall i\in\mathcal{M}\\
& \underline{\boldsymbol{b}}_i=[\underline{b}_i^1,\ldots,\underline{b}_i^P]\in\mathbb{R}^{T\times P} & \forall i\in\mathcal{M}_m\bigcup\mathcal{M}_r\\
& \overline{\boldsymbol{b}}_i=[\overline{b}_i^1,\ldots,\overline{b}_i^P]\in\mathbb{R}^{T\times P} & \forall i\in\mathcal{M}_m\bigcup\mathcal{M}_r\\
& \boldsymbol{U}_i\in\mathcal{U}_i=[\underline{b}_i^1,\,\overline{b}_i^1]\times\cdots\times[\underline{b}_i^P,\,\overline{b}_i^P] & \forall\boldsymbol{\xi}_i\in\Xi_i,\forall\boldsymbol{\zeta}_{\mathcal{N}_i}\in\mathcal{U}_{\mathcal{N}_i},\forall i\in\mathcal{M}_m\bigcup\mathcal{M}_r\\
& R_{i,t}^p=\displaystyle\sum_{p=1}^{P}B_i^pU_{i,t}^p+\xi_{i,\text{production},t}^p & \forall t\in\mathcal{T},\forall p\in\mathcal{P},\forall\boldsymbol{\xi}_i\in\Xi_i,\forall\boldsymbol{\zeta}_{\mathcal{N}_i}\in\mathcal{U}_{\mathcal{N}_i},\forall i\in\mathcal{M}\\
& I_{i,t+1}^p=I_{i,t}^p+R_{i,t}^p-D_{i,t}^p & \forall t\in\mathcal{T},\forall p\in\mathcal{P},\forall\boldsymbol{\xi}_i\in\Xi_i,\forall\boldsymbol{\zeta}_{\mathcal{N}_i}\in\mathcal{U}_{\mathcal{N}_i},\forall i\in\mathcal{M}
\end{array}
\tag{B.28}
$$

where $[\cdot]_+=\max\{0,\cdot\}$ and the coefficients c_B, c_H denote the backlogging and inventory holding costs, respectively. By construction of Problem (B.28), for each agent i and product p we have $\mathcal{U}_i^p=\mathcal{U}_{i,1}^p\times\cdots\times\mathcal{U}_{i,T}^p$ and $\mathcal{U}_i=\mathcal{U}_i^1\times\cdots\times\mathcal{U}_i^P$. Thus $\mathcal{U}_i$ is a hyper-rectangles which is controlled coordinate wise. Hence, it can be exactly represented by the primitive sets $\mathcal{S}_{i,t}^p=[-1,\,1]$ and

$$
\mathcal{U}_{i,t}^p(y_{i,t}^p,z_{i,t}^p)=\left\{U_{i,t}^p\in\mathbb{R}\,:\,\exists s_{i,t}^p\in\mathcal{S}_{i,t}^p\text{ s.t. }U_{i,t}^p=y_{i,t}^ps_{i,t}^p+z_{i,t}^p\right\},
$$

as discussed in Remark 1. Applying Theorem 4, Problem (B.28) can thus be reformulated as an instance of Problem (8b) where sets $\mathcal{S}_i=\prod_{p=1}^{P}[-1,\,1]^T$.

$$
\begin{array}{lll}
\text{minimize} & \displaystyle\sum_{i=1}^{M} \max_{\boldsymbol{\xi}_i\in\Xi_i,\boldsymbol{s}_{\mathcal{N}_i}\in\mathcal{S}_{\mathcal{N}_i}} \sum_{t=1}^{T+1}\sum_{p=1}^{P} c_H\left[I_{i,t}^p\right]_+ + c_B\left[-I_{i,t}^p\right]_+ & \\
\text{subject to} & \boldsymbol{U}_i=\boldsymbol{\Gamma}_i(\boldsymbol{\xi}_i,\boldsymbol{s}_{\mathcal{N}_i}) & \forall\boldsymbol{\xi}_i\in\Xi_i,\forall\boldsymbol{s}_{\mathcal{N}_i}\in\mathcal{S}_{\mathcal{N}_i},\forall i\in\mathcal{M}\\
& \underline{\boldsymbol{b}}_i,\,\overline{\boldsymbol{b}}_i,\,\boldsymbol{z}_i\in\mathbb{R}^{T\times P},\,\boldsymbol{y}_i\in\mathbb{R}_+^{T\times P} & \forall i\in\mathcal{M}_m\bigcup\mathcal{M}_r\\
& \underline{\boldsymbol{b}}_i=\boldsymbol{z}_i-\boldsymbol{y}_i,\,\overline{\boldsymbol{b}}_i=\boldsymbol{z}_i+\boldsymbol{y}_i & \forall i\in\mathcal{M}_m\bigcup\mathcal{M}_r\\
& \boldsymbol{U}_i\in\mathcal{U}_i=[\underline{b}_i^1,\,\overline{b}_i^1]\times\cdots\times[\underline{b}_i^P,\,\overline{b}_i^P] & \forall\boldsymbol{\xi}_i\in\Xi_i,\forall\boldsymbol{s}_{\mathcal{N}_i}\in\mathcal{S}_{\mathcal{N}_i},\forall i\in\mathcal{M}_m\bigcup\mathcal{M}_r\\
& \zeta_{i,t}^p=y_{i,t}^ps_{i,t}^p+z_{i,t}^p & \forall t\in\mathcal{T},\forall p\in\mathcal{P},\forall\boldsymbol{s}_i\in\mathcal{S}_i,\forall i\in\mathcal{M}_m\bigcup\mathcal{M}_r\\
& R_{i,t}^p=\displaystyle\sum_{p=1}^{P}B_i^pU_{i,t}^p+\xi_{i,\text{production},t}^p & \forall t\in\mathcal{T},\forall p\in\mathcal{P},\forall\boldsymbol{\xi}_i\in\Xi_i,\forall\boldsymbol{s}_{\mathcal{N}_i}\in\mathcal{S}_{\mathcal{N}_i},\forall i\in\mathcal{M}\\
& I_{i,t+1}^p=I_{i,t}^p+R_{i,t}^p-D_{i,t}^p & \forall t\in\mathcal{T},\forall p\in\mathcal{P},\forall\boldsymbol{\xi}_i\in\Xi_i,\forall\boldsymbol{s}_{\mathcal{N}_i}\in\mathcal{S}_{\mathcal{N}_i},\forall i\in\mathcal{M}
\end{array}
\tag{B.29}
$$

The corresponding centralized supply chain problem that *does not involve the quantity flexibility contracts* can be written as an instance of Problem (4a) in which all agents have access to the inventory levels of all other agents. This results in the following optimization problem

$$\begin{aligned}
\text{minimize } & \sum_{i=1}^{M} \max_{\boldsymbol{\xi}_{\mathcal{M}} \in \Xi_{\mathcal{M}}} \sum_{t=1}^{T+1} \sum_{p=1}^{P} c_H \left[I_{i,t}^p \right]_+ + c_B \left[- I_{i,t}^p \right]_+ \\
\text{subject to } & \boldsymbol{U}_i = \boldsymbol{\pi}_i(\boldsymbol{I}_{\mathcal{M}}) && \forall \boldsymbol{\xi}_{\mathcal{M}} \in \Xi_{\mathcal{M}}, \forall i \in \mathcal{M} \\
& R_{i,t}^p = \sum_{p=1}^{P} B_i^p U_{i,t}^p + \xi_{i,\text{production},t}^p && \forall t \in \mathcal{T}, \forall p \in \mathcal{P}, \forall \boldsymbol{\xi}_{\mathcal{M}} \in \Xi_{\mathcal{M}}, \forall i \in \mathcal{M} \\
& I_{i,t+1}^p = I_{i,t}^p + R_{i,t}^p - D_{i,t}^p && \forall t \in \mathcal{T}, \forall p \in \mathcal{P}, \forall \boldsymbol{\xi}_{\mathcal{M}} \in \Xi_{\mathcal{M}}, \forall i \in \mathcal{M},
\end{aligned} \tag{B.30}$$

where, with slight abuse of nation, $D_{i,t}^p = U_{\mathcal{N}_i,t}^p$ for every $i \in \mathcal{M}_s \bigcup \mathcal{M}_m$ and $D_{i,t}^p = D_t^p$ for $i \in \mathcal{M}_r$. Problem (B.30) is in turn can be reformulated as an instance of Problem (4b).

In the following numerical experiments, we assume that the initial inventory levels are zero, the uncertain demand (B.27) are produced by $K = 4$ factors. We generate random instance of Problem (B.29) by uniformly generating the coefficients F_p^k from the interval $[-1, 1]$, the backlogging and holding coefficients, c_B and c_H respectively, are randomly generated from $[0, 1]$, and the blending coefficients B_i^p are uniformly generated from $[0.5, 1]^P$ for all $i \in \mathcal{M}$ and $p \in \mathcal{P}$. We also assume that the production and demand uncertainties and the optimization belong to $\xi_{i,\text{production},t}^p \in [-0.1, 0]$ and $\xi_{k,\text{demand},t} \in [-\theta, \theta]$ with θ denoting the level of uncertainty, respectively. In this way, the uncertainty is characterized by the set $\Xi_i = \prod_{t=1}^{T} [-0.1, 0]^P$ for agents $i \in \mathcal{M}_s \bigcup \mathcal{M}_m$ and by the set $\Xi_i = \prod_{t=1}^{T} [-0.1, 0]^P \times [-\theta, \theta]^K$ for agents $i \in \mathcal{M}_r$. Finally, we approximate Problems (B.29) and (B.30) using affine policies, while the maximum operator $[\cdot]_+$ is linearized using epigraphical variables similar to (Bertsimas and Georghiou 2015, Section 5.1).

EC.2.2. Decentralized Optimization via ADMM

We demonstrate how an ADMM algorithm can be applied to solve the Problem (B.29), which promotes decentralized computation and provides significant privacy to all agents. For illustration, we consider the system depicted in Figure EC.1 with $M = 3$ agents, $T = 20$ horizon length, $P = 1$ product, and the market demand parameter $\theta = 1$. Notice that the decision variables $\underline{\boldsymbol{b}}_i, \overline{\boldsymbol{b}}_i$ in the optimization problem (B.29) are auxiliary. Removing these auxiliary variables yields the following reformulation

$$\min_{\boldsymbol{y}_2, \boldsymbol{y}_3 \in \mathbb{R}_+^T, \boldsymbol{z}_2, \boldsymbol{z}_3 \in \mathbb{R}^T} J_1(\boldsymbol{y}_2, \boldsymbol{z}_2) + J_2(\boldsymbol{y}_2, \boldsymbol{z}_2, \boldsymbol{y}_3, \boldsymbol{z}_3) + J_3(\boldsymbol{y}_3, \boldsymbol{z}_3),$$

where the implicit functions J_i are defined as

$$J_1(\boldsymbol{y}_2,\boldsymbol{z}_2)=\begin{cases}\text{minimize} \displaystyle\max_{\boldsymbol{\xi}_1\in\Xi_1,\boldsymbol{s}_2\in\mathcal{S}_2}\sum_{t=1}^{T+1}\sum_{p=1}^{P} c_H\left[I_{1,t}^p\right]_+ + c_B\left[-I_{1,t}^p\right]_+ & \\ \text{subject to } \boldsymbol{U}_1=\boldsymbol{\Gamma}_1(\boldsymbol{\xi}_1,\boldsymbol{s}_2) & \forall\boldsymbol{\xi}_1\in\Xi_1,\forall\boldsymbol{s}_2\in\mathcal{S}_2\\ \qquad\zeta_{2,t}^p=y_{2,t}^p s_{2,t}^p+z_{2,t}^p & \forall t\in\mathcal{T},\forall p\in\mathcal{P},\forall\boldsymbol{s}_2\in\mathcal{S}_2\\ \qquad R_{1,t}^p=\displaystyle\sum_{p=1}^{P}B_1^pU_{1,t}^p+\xi_{1,\text{production},t}^p & \forall t\in\mathcal{T},\forall p\in\mathcal{P},\forall\boldsymbol{\xi}_1\in\Xi_1,\forall\boldsymbol{s}_2\in\mathcal{S}_2\\ \qquad I_{1,t+1}^p=I_{1,t}^p+R_{1,t}^p-\zeta_{2,t}^p & \forall t\in\mathcal{T},\forall p\in\mathcal{P},\forall\boldsymbol{\xi}_1\in\Xi_1,\forall\boldsymbol{s}_2\in\mathcal{S}_2,\end{cases}$$

$$J_2(\boldsymbol{y}_2,\boldsymbol{z}_2,\boldsymbol{y}_3,\boldsymbol{z}_3)=\begin{cases}\text{minimize} \displaystyle\max_{\boldsymbol{\xi}_2\in\Xi_2,\boldsymbol{s}_3\in\mathcal{S}_3}\sum_{t=1}^{T+1}\sum_{p=1}^{P} c_H\left[I_{2,t}^p\right]_+ + c_B\left[-I_{2,t}^p\right]_+ & \\ \text{subject to } \boldsymbol{U}_2=\boldsymbol{\Gamma}_2(\boldsymbol{\xi}_2,\boldsymbol{s}_3) & \forall\boldsymbol{\xi}_2\in\Xi_2,\forall\boldsymbol{s}_3\in\mathcal{S}_3\\ \qquad\zeta_{3,t}^p=y_{3,t}^p s_{3,t}^p+z_{3,t}^p & \forall t\in\mathcal{T},\forall p\in\mathcal{P},\forall\boldsymbol{s}_3\in\mathcal{S}_3\\ \qquad U_{2,t}^p\in\left[z_{2,t}^p-y_{2,t}^p,\ z_{2,t}^p+y_{2,t}^p\right] & \forall t\in\mathcal{T},\forall p\in\mathcal{P},\forall\boldsymbol{\xi}_2\in\Xi_2,\forall\boldsymbol{s}_3\in\mathcal{S}_3\\ \qquad R_{2,t}^p=\displaystyle\sum_{p=1}^{P}B_2^pU_{2,t}^p+\xi_{2,\text{production},t}^p & \forall t\in\mathcal{T},\forall p\in\mathcal{P},\forall\boldsymbol{\xi}_2\in\Xi_2,\forall\boldsymbol{s}_3\in\mathcal{S}_3\\ \qquad I_{2,t+1}^p=I_{2,t}^p+R_{2,t}^p-\zeta_{3,t}^p & \forall t\in\mathcal{T},\forall p\in\mathcal{P},\forall\boldsymbol{\xi}_2\in\Xi_2,\forall\boldsymbol{s}_3\in\mathcal{S}_3,\end{cases}$$

$$J_3(\boldsymbol{y}_2,\boldsymbol{z}_2)=\begin{cases}\text{minimize} \displaystyle\max_{\boldsymbol{\xi}_3\in\Xi_i}\sum_{t=1}^{T+1}\sum_{p=1}^{P} c_H\left[I_{3,t}^p\right]_+ + c_B\left[-I_{3,t}^p\right]_+ & \\ \text{subject to } \boldsymbol{U}_3=\boldsymbol{\Gamma}_3(\boldsymbol{\xi}_3) & \forall\boldsymbol{\xi}_3\in\Xi_3\\ \qquad U_{3,t}^p\in\left[z_{3,t}^p-y_{3,t}^p,\ z_{3,t}^p+y_{3,t}^p\right] & \forall t\in\mathcal{T},\forall p\in\mathcal{P},\forall\boldsymbol{\xi}_3\in\Xi_3\\ \qquad R_{3,t}^p=\displaystyle\sum_{p=1}^{P}B_3^pU_{3,t}^p+\xi_{3,\text{production},t}^p & \forall t\in\mathcal{T},\forall p\in\mathcal{P},\forall\boldsymbol{\xi}_3\in\Xi_3\\ \qquad I_{3,t+1}^p=I_{3,t}^p+R_{3,t}^p-D_{3,t}^p & \forall t\in\mathcal{T},\forall p\in\mathcal{P},\forall\boldsymbol{\xi}_3\in\Xi_3.\end{cases}$$

Define next the global decision variable $\boldsymbol{\alpha}=[\boldsymbol{y}_2^\top,\boldsymbol{z}_2^\top,\boldsymbol{y}_3^\top,\boldsymbol{y}_4^\top]^\top$. With slight abuse of notation, we can now reformulate the above optimization problem as the following optimization problem

$$\min_{\boldsymbol{\beta}_1,\boldsymbol{\beta}_2,\boldsymbol{\beta}_3,\boldsymbol{\alpha}}\left\{\sum_{i\in\mathcal{M}}J_i(\boldsymbol{\beta}_i):\boldsymbol{\beta}_i=\widetilde{\boldsymbol{\alpha}}_i\ \forall i\in\mathcal{M}\right\}, \tag{B.31}$$

where given a global decision variable $\boldsymbol{\alpha}=[\boldsymbol{y}_2^\top,\boldsymbol{z}_2^\top,\boldsymbol{y}_3^\top,\boldsymbol{y}_4^\top]^\top$, we split $\boldsymbol{\alpha}$ to the (overlapping) chunks $\widetilde{\boldsymbol{\alpha}}_1=[\boldsymbol{y}_2^\top,\boldsymbol{z}_2^\top]^\top$, $\widetilde{\boldsymbol{\alpha}}_2=[\boldsymbol{y}_2^\top,\boldsymbol{z}_2^\top,\boldsymbol{y}_3^\top,\boldsymbol{y}_4^\top]^\top$, and $\widetilde{\boldsymbol{\alpha}}_3=[\boldsymbol{y}_3^\top,\boldsymbol{z}_3^\top]^\top$. Figure EC.2 (a) depicts the underlying graph structure of Problem (B.31), whose augmented Lagrangian is of the form

$$\mathcal{L}(\boldsymbol{\beta}_1,\boldsymbol{\beta}_2,\boldsymbol{\beta}_3,\boldsymbol{\alpha},\boldsymbol{\gamma}_1,\boldsymbol{\gamma}_2,\boldsymbol{\gamma}_3)=\sum_{i\in\mathcal{M}} J_i(\boldsymbol{\beta}_i)+\boldsymbol{\gamma}_i^\top(\boldsymbol{\beta}_i-\widetilde{\boldsymbol{\alpha}}_i)+\frac{\rho}{2}\|\boldsymbol{\beta}_i-\widetilde{\boldsymbol{\alpha}}_i\|^2,$$

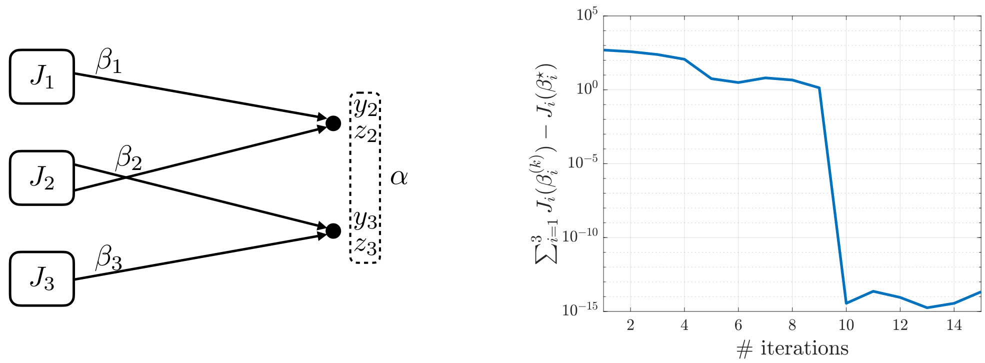
with $\boldsymbol{\gamma}_i$ being the dual variable associated with the constraint $\boldsymbol{\beta}_i=\widetilde{\boldsymbol{\alpha}}_i$, and ρ being a positive constant. Then, the ADMM update at iteration k follows the form

$$\begin{aligned}
\boldsymbol{\beta}_i^{(k+1)} &\leftarrow \arg\min_{\boldsymbol{\beta}_i}\left\{J_i(\boldsymbol{\beta}_i)+\left(\gamma_i^{(k)}\right)^\top\boldsymbol{\beta}_i+\frac{\rho}{2}\left\|\boldsymbol{\beta}_i-\widetilde{\boldsymbol{\alpha}}_i^{(k)}\right\|^2\right\}\\
\boldsymbol{\alpha}^{(k+1)} &\leftarrow \arg\min_{\boldsymbol{\alpha}}\left\{\sum_{i\in\mathcal{M}}-\left(\boldsymbol{\gamma}_i^{(k)}\right)^\top\widetilde{\boldsymbol{\alpha}}_i+\frac{\rho}{2}\left\|\boldsymbol{\beta}_i^{(k+1)}-\widetilde{\boldsymbol{\alpha}}_i\right\|^2\right\}\\
\boldsymbol{\gamma}_i^{(k+1)} &\leftarrow \boldsymbol{\gamma}_i^{(k)}+\rho\left(\boldsymbol{\beta}_i^{(k+1)}-\widetilde{\boldsymbol{\alpha}}_i^{(k+1)}\right).
\end{aligned}$$

Notice that the updates of decision variables $\boldsymbol{\beta}_i$ and $\boldsymbol{\gamma}_i$ can be carried out locally for every agent $i\in\mathcal{M}$, which implies that the structure of J_i is only known to agent i. This salient feature promotes privacy amongst the agents. In addition, the update of the decision variable $\boldsymbol{\alpha}$ involves solving an unconstrained quadratic minimization problem that can be solved analytically; see (Boyd et al. 2011, § 7.2). The analytic expression constitutes local averaging rather than global averaging, and therefore, it can be accomplished by local information exchange. This indicates that all ADMM updates can be performed locally up to the information exchange $\widetilde{\boldsymbol{\alpha}}_i$ between neighbors. Figure EC.2 (b) reports the convergence behavior of the average performance of the ADMM algorithm for solving 10 random instances of the Problem (B.29) where the functions J_i are approximated via affine decision rules. In the experiments we set set $\rho=0.1$ and decision variables $\widetilde{\boldsymbol{\alpha}}_i^{(0)}$, $\boldsymbol{\gamma}_i^{(0)},\boldsymbol{\beta}_i^{(0)}$ are initialized at zero for all $i\in\mathcal{M}$. We observe that the algorithm converges to an optimal solution and achieves the machine precision in 10 iterations.

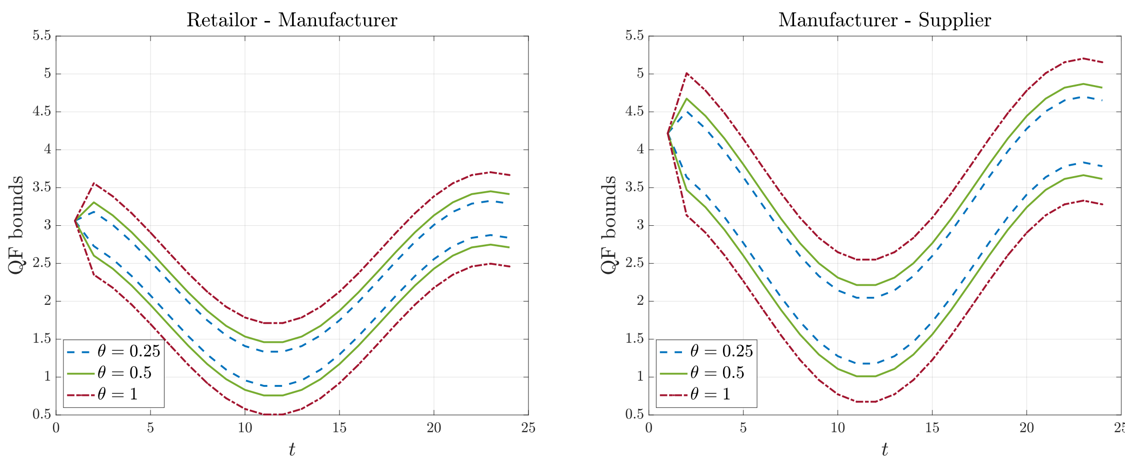
EC.2.3. Numerical Results

In the first experiment, we investigate how the degree of uncertainty in the market demand affects the QF bounds. We consider the system depicted in Figure EC.1 with $M=3$ agents and $P=1$ product. We solve the optimization problem (B.29) with $\theta=\{0.25,0.5,1\}$ for a horizon length of $T=24$. The QF contracts between retailer/manufacturer and manufacturer/supplier pairs are depicted in Figure EC.3 when we set $F_p^k=(-1)^k/2, c_B=c_H=1$ and $B_1^p=B_2^p=B_3^p=1$. We observe that the size of the QF contrasts increases as the uncertainty in the market demand increases.



(a) (b)

Figure EC.2 (a) Illustration of the graph structure with 3 agents. Local objective functions are on the left; global variable components are on the right. The bipartite graph can be viewed as a consistency constraint that links local variables and global variables. (b) Convergence behavior of the ADMM algorithm.



(a) (b)

Figure EC.3 Effect of uncertainty on (a) retailer-manufacturer and (b) manufacturer-supplier QF contracts over a $T = 24$ horizon length.

However, due to the adaptive nature of the recourse decisions, the size of the QF bounds does not substantially increase over time. Moreover, we observe that the QF bounds between manufacturers and suppliers are wider than those between manufacturers and retailers. This observation is consistent with the bullwhip effect in (Lee et al. 1997), a theory that describes how small fluctuations in demand at the retail level can cause progressively larger fluctuations at the manufacturer and supplier levels.

In the second experiment, we investigate the effects of horizon length and number of agents in the network with a single supplier, N intermediate manufacturers, and a single retailer. We compare our proposed local information exchange policy design to the centralized one over 10 randomly generated instances. Throughout the experiment, we fix the number of products at $P=2$ and the market demand parameter constant at $\theta=1$. First, we compare the optimal values of the local problem (B.29) and the centralized problem (B.30). Denoting by obj_L the objective value of (B.29) and by obj_C the objective of the centralized (B.30), we define the (percentage) suboptimality as $100\times(\text{obj}_L-\text{obj}_C)/\text{obj}_C$. We evaluate the effect of the horizon length and the number of manufacturers on the quality of the solution in terms of the suboptimality metric. In addition, we examine the impact of demand-side delays on the solution of the local and centralized policy designs, which occur when agents report their demands to their preceding agents at the end of the time interval rather than submitting their requests at the beginning. Specifically, we assume that the inventory dynamic is of the form

$$I^p_{i,t+1} = I^p_{i,t+1} + R^p_{i,t} - D^p_{i,t-1}$$

for every $i\in\mathcal{M}$, where the demand term $D^p_{i,t-1}$ is modified to incorporate the demand-side delay. The introduction of the delay yields policies of the form $U_{i,t}=\boldsymbol{\Psi}(\boldsymbol{\xi}_i^{t-1},\boldsymbol{s}_{\mathcal{N}_i}^{t-1})$ and $U_{i,t}=\boldsymbol{\Pi}(\boldsymbol{\xi}_i^{t-1},\boldsymbol{\xi}_{\mathcal{M}\setminus\{i\}}^{t-1})$ for every $t\in\mathcal{T}$ and $i\in\mathcal{M}$ in the decentralized and centralized settings, respectively, and in turn approximated by affine decision rules. We consider two different cases. In the first case, we consider $N=1$ intermediate manufacturer and increase the time horizon T up to the horizon length 10. In the second case, we fix the time horizon to $T=5$, while changing the number of manufacturers N from 1 to 10. Figure EC.4 summarizes the results. In the absence of a time delay in the system, we observe that the suboptimality is zero. Interestingly, the introduction of QF contracts between agents not only preserves their privacy but also does not impact performance. In contrast, time delay has a significant impact on the quality of the solutions. Specifically, we observe that an increase in horizon length can have an adverse effect on the quality of the local information problem. This is to be expected, as the uncertainty faced by each agent increases with increasing the horizon length in the local information problem. However, as the horizon length increases, the suboptimality becomes saturated, and the decentralized policy absorbs the effect of delay, as agents begin mitigating uncertainty through the use of local information exchanges. In

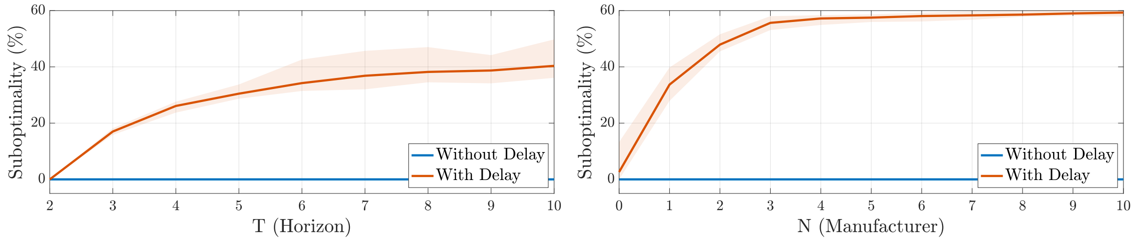


Figure EC.4 Suboptimality of local vs centralized problem as a function of the horizon length (left) and number of agents (right). Solid lines (shaded regions) represent averages (ranges) across 10 independent simulations.

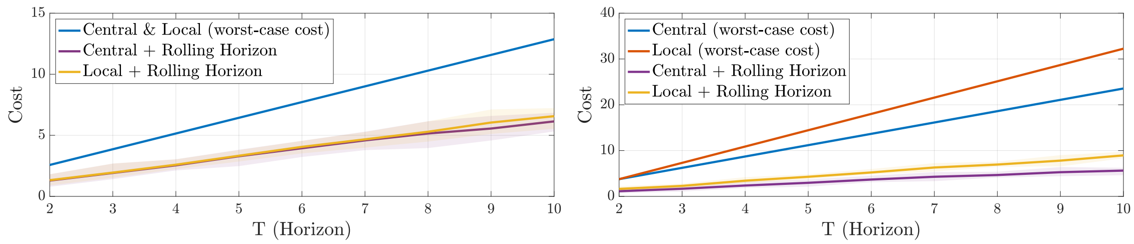


Figure EC.5 Effect of the rolling horizon in a system with no delay (left) and with delay (right). The graphs report the worst-case cost computed by solving Problems (B.29) and (B.30), and the cost from the rolling horizon scheme. Solid lines (shaded regions) represent averages (ranges) across 10 independent simulations.

addition, as the number of agents increases beyond 4, there is essentially no difference in terms of suboptimality as the function of the number of manufacturers.

Next, we use a rolling horizon scheme to compare the average performance of the centralized and local information problems. In this experiment, we fix all parameters as in the first experiment. We then generate a random realization of the uncertainty $\boldsymbol{\xi} \in \Xi$. Next, we solve the centralized and local information for the horizon length $T = 10$ and update the inventory levels according to the realization $\boldsymbol{\xi}_1$ and the first stage decisions $U^p_{i,1}$. We repeat the process until we reach the end of the horizon. Specifically, at any time $t = 2, \ldots, T$, we resolve each problem for the shorter horizon length $T - t$ and the initial inventory stocks $I^p_{i,t-1}$ for every $i \in \mathcal{M}$. We then update the inventory levels according to realization $\boldsymbol{\xi}_t$ and corresponding first stage decisions $U^p_{i,t}$. Figure EC.5 summarizes our results for 10 randomly generated realization of uncertainty. We observe that the average performances of the local and centralized information problems are significantly improved compared to their worst-case performance obtained by solving (B.29) and (B.30), respectively. The improvement is more significant when there is a delay in the system.

Finally, we compare the optimization time required to solve the local and centralized problems. Figure EC.6 reports the execution times required by Gurobi to solve 10 randomly generated in-

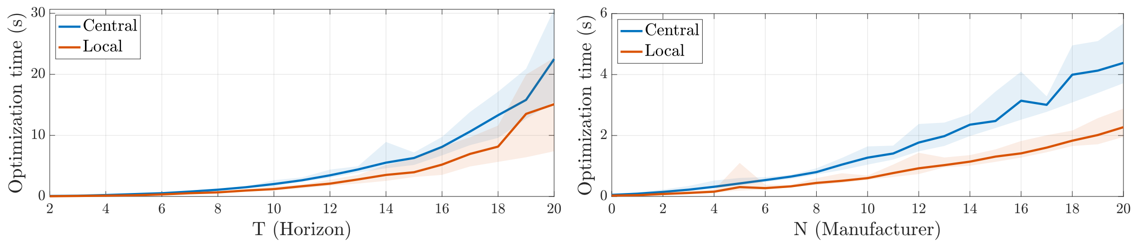


Figure EC.6 Comparison of the runtime of the centralized and local information problems. Solid lines (shaded regions) represent averages (ranges) across 10 randomly generate instances.

stances of the centralized and local information exchange designs. Figure EC.6 (left) the number of manufacturers is fixed to $N = 5$, whereas in Figure EC.6 (right) the horizon length is fixed to $T = 5$. In both cases, the time required to solve the local information problem (B.29) is nearly half the time required to solve the centralized optimization problem (B.30). This can be attributed to the nearly decoupled structure of the problem, in which only the QF bounds link dynamics and constraints.

EC.3. Summary of major notation

Index sets: We use $\mathcal{M}$ to denote the set of all agents (Section 2), $\mathcal{N}_i$ to denote the set of neighbors of agent i (Section 2), and $\overline{\mathcal{N}}_i$ the set that includes agent i and all its precedent agents (Section 2.3).
Vectors concatenation: For given vectors $v_i \in \mathbb{R}^{k_i}$ with $k_i \in \mathbb{N}$, $i \in \mathcal{M}$, we define $\boldsymbol{v}_{\mathcal{M}} = [v_i]_{i\in\mathcal{M}} = [v_1^\top \dots v_M^\top]^\top \in \mathbb{R}^k$ with $k = \sum_{i=1}^M k_i$ as their vector concatenation. Given time dependent vectors $\nu_{i,t} \in \mathbb{R}^{\ell_i}$ with $i \in \mathcal{M}$, $t \in \mathcal{T}$ and $\ell_i \in \mathbb{N}$, we define $\boldsymbol{\nu}_{\mathcal{M},t} = [\nu_{i,t}]_{i\in\mathcal{M}}$ as the concatenated vector at time t, $\boldsymbol{\nu}_i^t = [\nu_{i,1}^\top \dots \nu_{i,t}^\top]^\top$ as the history of the i-th vector up to time t, and $\boldsymbol{\nu}_{\mathcal{M}}^t = [\boldsymbol{\nu}_i^t]_{i\in\mathcal{M}}$ as the history of the concatenated vector up to time t.
Concatenated Vectors: The linear dynamics of the agent i is written in the compact form $\boldsymbol{x}_i = f_i(\boldsymbol{x}_{\mathcal{N}_i}, \boldsymbol{u}_i, \boldsymbol{\xi}_i)$, where $\boldsymbol{x}_i := [x_{i,t}]_{t\in\{0\}\cup\mathcal{T}}$, $\boldsymbol{u}_i := [u_{i,t}]_{t\in\mathcal{T}}$, $\boldsymbol{\xi}_i := [\xi_{i,t}]_{t\in\mathcal{T}}$ and $\boldsymbol{x}_{\mathcal{N}_i} := [\boldsymbol{x}_{\mathcal{N}_i,t}]_{t\in\mathcal{T}}$. Here, $\boldsymbol{x}_i$, $\boldsymbol{u}_i$, $\boldsymbol{\xi}_i$ and $\boldsymbol{x}_{\mathcal{N}_i}$ denote the state, input, exogenous uncertainty, and the state of neighbors affecting agents i, respectively (Section 2.1). Vector $\boldsymbol{\zeta}_j$ denote belief states of the neighbor agent $j \in \mathcal{N}_i$, i.e., the dynamic of agent i are affected by its belief $\boldsymbol{\zeta}_j \in \mathcal{X}_j$ of what values the states of agent j will take (Section 3). Vector $\boldsymbol{s}_j \in \mathcal{S}_j$ is used in the construction of the approximation (7) (Section 4).
Optimization variables: The paper analyzes four pairs of problems. A major distinction between problems is the information available to the policies and the present of sets as decision variables. Below we summarize this information.

- **Centralized information exchange (Section 2.2):** Problem (4a) has optimization variables the state feedback policies denoted with lower case $\boldsymbol{\pi}_i(\boldsymbol{x}_{\mathcal{M}}) := [\pi_{i,t}(\boldsymbol{x}^t_{\mathcal{M}})]_{t\in\mathcal{T}}$, and Problem (4b) has the uncertainty feedback policies denoted by upper case $\boldsymbol{\Pi}_i(\boldsymbol{\xi}_{\mathcal{M}}) := [\Pi_{i,t}(\boldsymbol{\xi}^{t-1}_{\mathcal{M}})]_{t\in\mathcal{T}}$.
- **Partially nested information exchange (Section 2.3):** Problem (5a) has optimization variables the state feedback policies denoted with lower case $\boldsymbol{\phi}_i(\boldsymbol{x}_{\overline{\mathcal{N}}_i}) := [\phi_{i,t}(\boldsymbol{x}^t_{\overline{\mathcal{N}}_i})]_{t\in\mathcal{T}}$, and Problem (5b) has the uncertainty feedback policies denoted with upper case $\boldsymbol{\Phi}_i(\boldsymbol{\xi}_{\overline{\mathcal{N}}_i}) := [\Phi_{i,t}(\boldsymbol{\xi}^{t-1}_{\overline{\mathcal{N}}_i})]_{t\in\mathcal{T}}$.
- **Local information exchange (Section 3):** Problem (6a) has optimization variables the state feedback policies denoted with lower case $\boldsymbol{\psi}_i(\boldsymbol{x}_i, \boldsymbol{\zeta}_{\mathcal{N}_i}) := [\psi_{i,t}(\boldsymbol{x}^t_i, \boldsymbol{\zeta}^t_{\mathcal{N}_i})]_{t\in\mathcal{T}}$ and the state forecast sets $\mathcal{X}_i$. Besides, Problem (6b) has optimization variables the uncertainty feedback policies denoted with upper case $\boldsymbol{\Psi}_i(\boldsymbol{\xi}_i, \boldsymbol{\zeta}_{\mathcal{N}_i}) := [\Psi_{i,t}(\boldsymbol{\xi}^{t-1}_i, \boldsymbol{\zeta}^t_{\mathcal{N}_i})]_{t\in\mathcal{T}}$ and the state forecast sets $\mathcal{X}_i$.
- **Approximation of Problem** (6b) **(Section 4):** Problem (8a) has optimization variables the uncertainty feedback policies $\boldsymbol{\Psi}_i(\boldsymbol{\xi}_i, \boldsymbol{\zeta}_{\mathcal{N}_i})$ and the state forecast sets $\mathcal{X}_i$ which are parameterized by approximation (7) through matrices Y_i and vector z_i and a given set $\mathcal{S}_i$. Problem (8b) has optimization variables the policy $\boldsymbol{\Gamma}_i(\boldsymbol{\xi}_i, \boldsymbol{s}_{\mathcal{N}_i}) := [\Gamma_{i,t}(\boldsymbol{\xi}^{t-1}_i, \boldsymbol{s}^t_{\mathcal{N}_i})]_{t\in\mathcal{T}}$ and the state forecast sets $\mathcal{X}_i$ which are also parameterized by approximation (7) through matrices Y_i and vector z_i and a given set $\mathcal{S}_i$.

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