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APPENDIX

A: Proofs for Section 6

In this section, for notational convenience, we will use:

$$\theta \triangleq (x_0+1)^{-\frac{1}{3}}.$$

Since $x_0 \geq 1$, we have: $\theta \in (0,1)$.

Proof of Lemma 1.

First, we have $\eta_0 \geq 0 \geq \eta_1$. So condition (2) holds.

Second, we have $\eta_0 t_1 + \eta_1 (T-t_1) = 0$. So condition (3) holds. ■

Proof of Lemma 2.

Conditional on $\mathcal{G}$, we have

$$\begin{aligned}
\hat{\rho}_1 &= \max\left\{\left(\rho^d - \eta_1 - \frac{\hat{\Delta}_0}{\lambda(T-t_1)}\right)^+, \hat{\rho}_0\right\} \\
&= \max\left\{\rho^d - \eta_1 - \frac{\hat{\Delta}_0}{\lambda(T-t_1)}, \hat{\rho}_0\right\} \\
&= \max\left\{\rho^d - \eta_1 - \frac{\hat{\Delta}_0}{\lambda(T-t_1)}, \rho^d - \eta_0\right\} \\
&= \rho^d - \eta_1 - \frac{\hat{\Delta}_0}{\lambda(T-t_1)}.
\end{aligned}$$

The first equality follows from the definition of $\hat{\rho}_1$. The second equality follows from the definition of $\mathcal{G}_1$. The third equality follows from the definition of $\hat{\rho}_0$. The fourth equality follows from the definition of $\mathcal{G}_0$. ■

Proof of Lemma 3.

We begin with proving Part 1 (i.e., the bound for A_1).

First, we establish a lower bound of $\mathsf{E}[\hat{\rho}_1 - \rho^d]$. We have

$$\begin{aligned}
\mathsf{E}\left[\hat{\rho}_1 - \rho^d\right] &= \mathsf{E}\left[\left(\hat{\rho}_1 - \rho^d\right)\mathbf{1}\{\mathcal{G}\}\right] + \mathsf{E}\left[\left(\hat{\rho}_1 - \rho^d\right)\mathbf{1}\{\mathcal{G}^c\}\right] \\
&= \mathsf{E}\left[\left(-\eta_1 - \frac{\hat{\Delta}_0}{\lambda(T-t_1)}\right)\mathbf{1}\{\mathcal{G}\}\right] + \mathsf{E}\left[\left(\hat{\rho}_1 - \rho^d\right)\mathbf{1}\{\mathcal{G}^c\}\right] \\
&= \mathsf{E}\left[\left(-\eta_1 - \frac{\hat{\Delta}_0}{\lambda(T-t_1)}\right)\mathbf{1}\{\mathcal{G}\}\right] + \mathsf{E}\left[\left(-\eta_1 - \frac{\hat{\Delta}_0}{\lambda(T-t_1)}\right)\mathbf{1}\{\mathcal{G}^c\}\right]
\end{aligned}$$

$$
\begin{aligned}
&+\mathsf{E}\left[\left(\hat{\rho}_1-\rho^d+\eta_1+\frac{\hat{\Delta}_0}{\lambda(T-t_1)}\right)\mathbf{1}\left\{\mathcal{G}^c\right\}\right]\\
&=\mathsf{E}\left[-\eta_1-\frac{\hat{\Delta}_0}{\lambda(T-t_1)}\right]+\mathsf{E}\left[\left(\hat{\rho}_1-\rho^d+\eta_1+\frac{\hat{\Delta}_0}{\lambda(T-t_1)}\right)\mathbf{1}\left\{\mathcal{G}^c\right\}\right]\\
&=-\eta_1+\mathsf{E}\left[\left(\hat{\rho}_1-\rho^d+\eta_1+\frac{\hat{\Delta}_0}{\lambda(T-t_1)}\right)\mathbf{1}\left\{\mathcal{G}^c\right\}\right]\\
&\geq-\eta_1-\mathsf{E}\left[\left(\rho^d-\eta_1-\frac{\hat{\Delta}_0}{\lambda(T-t_1)}\right)\mathbf{1}\left\{\mathcal{G}^c\right\}\right]\\
&\geq-\eta_1-\mathsf{E}\left[\left(\rho^d-\eta_1+\frac{t_1\hat{\rho}_0}{T-t_1}\right)\mathbf{1}\left\{\mathcal{G}^c\right\}\right]\\
&\geq-\eta_1-\mathsf{E}\left[\left(\rho^d-\eta_1+\frac{t_1}{T-t_1}\right)\mathbf{1}\left\{\mathcal{G}^c\right\}\right]\\
&=-\eta_1-\left(\rho^d-\eta_1+\frac{t_1}{T-t_1}\right)\mathbb{P}\left(\mathcal{G}^c\right)\\
&=-\eta_1-\left(\rho^d-\eta_1+\frac{1}{\theta}-1\right)\mathbb{P}\left(\mathcal{G}^c\right),
\end{aligned}
$$

where the fifth equality follows since $\mathsf{E}[\hat{\Delta}_0]=0$, the first inequality follows since $\hat{\rho}_1\geq 0$, the second inequality follows from the definition of $\hat{\Delta}_0$ and the property that $\hat{S}_0\geq 0$, the third inequality follows since $\hat{\rho}_0\leq 1$, and the last equality follows from the definition of t_1.

Second, we establish a lower bound of $\hat{\rho}_0-\rho^d$. We have

$$
\hat{\rho}_0-\rho^d=-\eta_0\;\geq\;-\eta_0-\left(\rho^d-\eta_0\right)\mathbb{P}\left(\mathcal{G}^c\right),
$$

where the equality and the inequality follow from the assumption that $\rho^d>\eta_0$.

Therefore, we can bound:

$$
\begin{aligned}
A_1&\leq\lambda t_1\left[\eta_0+(\rho^d-\eta_0)\mathbb{P}\left(\mathcal{G}^c\right)\right]+\lambda(T-t_1)\left[\eta_1+\left(\rho^d-\eta_1+\frac{1}{\theta}-1\right)\mathbb{P}\left(\mathcal{G}^c\right)\right]\\
&=\lambda\left[t_1\rho^d+(T-t_1)\left(\rho^d+\frac{1}{\theta}-1\right)\right]\mathbb{P}\left(\mathcal{G}^c\right)\\
&=\lambda\left[T\rho^d+(T-t_1)\left(\frac{1}{\theta}-1\right)\right]\mathbb{P}\left(\mathcal{G}^c\right)\\
&=\lambda\left[T\rho^d+T\theta\left(\frac{1}{\theta}-1\right)\right]\mathbb{P}\left(\mathcal{G}^c\right)\\
&\leq\left(\rho^d+1\right)\lambda T\mathbb{P}\left(\mathcal{G}^c\right),
\end{aligned}
$$

where the first equality follows from condition (3), the second equality follows from the definition of t_1, and the second inequality follows since $\theta\in(0,1)$.

Next, we prove Part 2 (i.e., the bound for A_2).

First, we establish an upper bound of $\mathsf{E}[(\hat{\rho}_1-\rho^d)^2]$. We have:

$$\begin{aligned}
\mathsf{E}\left[\left(\hat{\rho}_1-\rho^d\right)^2\right] &= \mathsf{E}\left[\left(\hat{\rho}_1-\rho^d\right)^2\mathbf{1}\{\mathcal{G}\}\right]+\mathsf{E}\left[\left(\hat{\rho}_1-\rho^d\right)^2\mathbf{1}\{\mathcal{G}^c\}\right] \\
&= \mathsf{E}\left[\left(-\eta_1-\frac{\hat{\Delta}_0}{\lambda(T-t_1)}\right)^2\mathbf{1}\{\mathcal{G}\}\right]+\mathsf{E}\left[\left(\hat{\rho}_1-\rho^d\right)^2\mathbf{1}\{\mathcal{G}^c\}\right] \\
&= \mathsf{E}\left[\left(\eta_1+\frac{\hat{\Delta}_0}{\lambda(T-t_1)}\right)^2\mathbf{1}\{\mathcal{G}\}\right]+\mathsf{E}\left[\left(\eta_1+\frac{\hat{\Delta}_0}{\lambda(T-t_1)}\right)^2\mathbf{1}\{\mathcal{G}^c\}\right] \\
&\quad +\mathsf{E}\left[\left\{\left(\hat{\rho}_1-\rho^d\right)^2-\left(\eta_1+\frac{\hat{\Delta}_0}{\lambda(T-t_1)}\right)^2\right\}\mathbf{1}\{\mathcal{G}^c\}\right] \\
&= \underbrace{\mathsf{E}\left[\left(\eta_1+\frac{\hat{\Delta}_0}{\lambda(T-t_1)}\right)^2\right]}_{B_1}+\underbrace{\mathsf{E}\left[\left\{\left(\hat{\rho}_1-\rho^d\right)^2-\left(\eta_1+\frac{\hat{\Delta}_0}{\lambda(T-t_1)}\right)^2\right\}\mathbf{1}\{\mathcal{G}^c\}\right]}_{B_2}.
\end{aligned}$$

For B_1, we have

$$\begin{aligned}
B_1 &= \eta_1^2+\mathsf{E}\left[\left(\frac{\hat{\Delta}_0}{\lambda(T-t_1)}\right)^2\right]+2\eta_1\mathsf{E}\left[\frac{\hat{\Delta}_0}{\lambda(T-t_1)}\right] \\
&= \eta_1^2+\mathsf{E}\left[\left(\frac{\hat{\Delta}_0}{\lambda(T-t_1)}\right)^2\right] \\
&= \eta_1^2+\mathsf{E}\left[\frac{t_1\hat{\rho}_0}{\lambda\left(T-t_1\right)^2}\right] \\
&\le \eta_1^2+\frac{t_1}{\lambda\left(T-t_1\right)^2},
\end{aligned}$$

where the second equality follows since $\mathsf{E}[\hat{\Delta}_0]=0$.

Next, we bound B_2. We discuss the following two cases: $\hat{\rho}_1\le\rho^d$ and $\hat{\rho}_1>\rho^d$.

Case 1: $\hat{\rho}_1\le\rho^d$.

We have

$$\left(\hat{\rho}_1-\rho^d\right)^2-\left(\eta_1+\frac{\hat{\Delta}_0}{\lambda(T-t_1)}\right)^2\le\left(\hat{\rho}_1-\rho^d\right)^2\le\left(\rho^d\right)^2.$$

Case 2: $\hat{\rho}_1>\rho^d$.

We have:

$$\hat{\rho}_1 = \max\left\{\left(\rho^d - \eta_0\right)^+, \left(\rho^d - \eta_1 - \frac{\hat{\Delta}_0}{\lambda(T-t_1)}\right)^+\right\}$$
$$\leq \max\left\{\rho^d - \eta_0, \rho^d - \eta_1 - \frac{\hat{\Delta}_0}{\lambda(T-t_1)}\right\},$$

where the last equality follows since if $a = \max\{b^+, c^+\}$, where $(\cdot)^+$ is a projection to $[0,1]$, and $a > 0$, then at least one of b and c is strictly positive. Moreover, $a \leq \max\{b, c\}$.

Equivalently,

$$\hat{\rho}_1 - \rho^d \leq \max\left\{-\eta_0, -\eta_1 - \frac{\hat{\Delta}_0}{\lambda(T-t_1)}\right\}.$$

Since $\hat{\rho}_1 - \rho^d > 0$ and $\eta_0 > 0$, we must have

$$0 < \hat{\rho}_1 - \rho^d \leq -\eta_1 - \frac{\hat{\Delta}_0}{\lambda(T-t_1)},$$

which implies

$$\left(\hat{\rho}_1 - \rho^d\right)^2 - \left(\eta_1 + \frac{\hat{\Delta}_0}{\lambda(T-t_1)}\right)^2 \leq 0.$$

Therefore, the two cases above jointly imply

$$\left(\hat{\rho}_1 - \rho^d\right)^2 - \left(\eta_1 + \frac{\hat{\Delta}_0}{\lambda(T-t_1)}\right)^2 \leq \left(\rho^d\right)^2,$$

which gives

$$B_2 \leq \left(\rho^d\right)^2 \mathsf{E}\left[\mathbf{1}\left\{\mathcal{G}^c\right\}\right] = \left(\rho^d\right)^2 \mathbb{P}\left(\mathcal{G}^c\right).$$

Using the bounds for B_1 and B_2, we can upper bound A_2 with

$$\lambda\left(t_1\eta_0^2 + (T-t_1)\,\eta_1^2\right) + \lambda\,(T-t_1)\,\frac{t_1}{\lambda\,(T-t_1)^2} + \lambda T\left(\rho^d\right)^2 \mathbb{P}\left(\mathcal{G}^c\right). \tag{EC.1}$$

Now, we bound the first and the second terms in (EC.1). For the first term in (EC.1), by the definition of η_0, η_1 and t_1, we have:

$$\lambda\left(t_1\eta_0^2 + (T-t_1)\,\eta_1^2\right) = \frac{\left(\ln\left(x_0+1\right)\right)^{2\alpha}}{t_1}\left(t_1 + \frac{t_1^2}{T-t_1}\right)$$

$$= \frac{(\ln(x_0+1))^{2\alpha} T}{T-t_1}$$
$$= \frac{(\ln(x_0+1))^{2\alpha}}{\theta}.$$

For the second term in (EC.1), we have:

$$\lambda(T-t_1)\frac{t_1}{\lambda(T-t_1)^2} = \frac{t_1}{T-t_1} = \frac{1-\theta}{\theta} \leq \frac{1}{\theta}.$$

Combining the bounds for the first and the second terms in (EC.1) yields

$$\begin{aligned} A_2 &\leq \frac{(\ln(x_0+1))^{2\alpha}}{\theta} + \frac{1}{\theta} + \lambda T\left(\rho^d\right)^2 \mathbb{P}(\mathcal{G}^c) \\ &= \frac{(\ln(x_0+1))^{2\alpha}+1}{\theta} + \lambda T\left(\rho^d\right)^2 \mathbb{P}(\mathcal{G}^c). \end{aligned}$$

■

Proof of Lemma 4.

Conditional on $\mathcal{G}$, we have:

$$\hat{\rho}_1 = \rho^d - \eta_1 - \frac{\hat{\Delta}_0}{\lambda(T-t_1)}, \quad \text{and} \quad \hat{\rho}_1 \in (0,1).$$

Therefore, we can write

$$\begin{aligned} x_0 - \hat{S}_0 &= \lambda T\rho^d - \left(\lambda t_1 \hat{\rho}_0 + \hat{\Delta}_0\right) \\ &= \lambda T\rho^d - \left[\lambda t_1\left(\rho^d - \eta_0\right) + \hat{\Delta}_0\right] \\ &= \lambda T\rho^d - \lambda t_1 \rho^d + \lambda t_1 \eta_0 - \hat{\Delta}_0 \\ &= \lambda T\rho^d - \lambda t_1 \rho^d - \lambda(T-t_1)\eta_1 - \hat{\Delta}_0 \\ &= \lambda(T-t_1)\rho^d - \lambda(T-t_1)\eta_1 - (T-t_1)\frac{\hat{\Delta}_0}{T-t_1} \\ &= \lambda(T-t_1)\left(\rho^d - \eta_1 - \frac{\hat{\Delta}_0}{\lambda(T-t_1)}\right) \\ &= \lambda(T-t_1)\hat{\rho}_1 \\ &= \lambda T(x_0+1)^{-1/3}\hat{\rho}_1 \\ &> 0, \end{aligned}$$

where the fourth equality follows since $\sum_{n=0}^{1}(t_{n+1}-t_n)\eta_n = 0$. ■

Proof of Lemma 5.

We have:

$$\mathsf{E}\left[\left(\sum_{n=0}^{1}\hat{S}_n - x_0\right)^+\right] = \underbrace{\mathsf{E}\left[\left(\sum_{n=0}^{1}\hat{S}_n - x_0\right)^+ \mathbf{1}\left\{\mathcal{G}\right\}\right]}_{C_1} + \underbrace{\mathsf{E}\left[\left(\sum_{n=0}^{1}\hat{S}_n - x_0\right)^+ \mathbf{1}\left\{\mathcal{G}^c\right\}\right]}_{C_2}.$$

First, we establish an upper bound for C_1. Note that

$$\begin{aligned}
C_1 &= \mathsf{E}\left[\left(\sum_{n=0}^{1}\hat{S}_n - x_0\right)^+ \middle| \mathcal{G}\right]\mathbb{P}\left(\mathcal{G}\right) \\
&= \mathsf{E}\left[\left(\hat{S}_1 - X_{t_1-}\right)^+ \middle| \mathcal{G}\right]\mathbb{P}\left(\mathcal{G}\right) \\
&= \mathsf{E}\left[\left(\hat{S}_1 - \lambda\left(T-t_1\right)\hat{\rho}_1\right)^+ \middle| \mathcal{G}\right]\mathbb{P}\left(\mathcal{G}\right) \\
&= \mathsf{E}\left[\left(\hat{S}_1 - \lambda\left(T-t_1\right)\hat{\rho}_1\right)^+\right] \\
&\le \frac{1}{2}\lambda^{1/2}\left(T-t_1\right)^{1/2} \\
&= \frac{1}{2}\left(\lambda T\right)^{1/2}\left(x_0+1\right)^{-1/6},
\end{aligned}$$

where the third equality follows from Lemma 4 and the inequality follows since $\hat{\rho}_1 \in [0,1]$ combined with the fact that if a random variable X has mean μ and standard deviation σ, then $\mathsf{E}[(X-\mu)^+] \le \frac{\sigma}{2}$ (see equation (18) in Gallego and Van Ryzin (1994)).

Next, we establish an upper bound for C_2. Let $\tilde{Z}$ denote the total number of customers arriving to the system over $[0,T]$. Then, $\tilde{Z}$ is a Poisson random variable with parameter λT. We have:

$$\begin{aligned}
C_2 &\le \mathsf{E}\left[\left(\tilde{Z} - x_0\right)^+ \mathbf{1}\left\{\mathcal{G}^c\right\}\right] \\
&\le \mathsf{E}\left[\tilde{Z}\mathbf{1}\left\{\mathcal{G}^c\right\}\right] \\
&\le \mathsf{E}[\tilde{Z}^2]^{\frac{1}{2}}\,\mathsf{E}[\mathbf{1}\left\{\mathcal{G}^c\right\}]^{\frac{1}{2}} \\
&= \left(\lambda^2T^2 + \lambda T\right)^{\frac{1}{2}}\mathbb{P}\left(\mathcal{G}^c\right)^{\frac{1}{2}} \\
&\le \left(\lambda T + 1\right)\mathbb{P}\left(\mathcal{G}^c\right)^{\frac{1}{2}},
\end{aligned}$$

where the first inequality follows since $\sum_{n=0}^{1}\hat{S}_n \le \tilde{Z}$ and the third inequality follows from Cauchy-Schwarz inequality.

Putting the bounds for C_1 and C_2 together completes the proof of Lemma EC.3. ■

Proof of Lemma 6.

In proving Lemma 6, we will use some results proved in Appendix C.

First, we establish an upper bound for $\mathbb{P}(\mathcal{G}_0^c)$. We have

$$\begin{aligned}
\mathbb{P}(\mathcal{G}_0^c) &= \mathbb{P}\left(\frac{\hat{\Delta}_0}{\lambda(T-t_1)} \geq \eta_0 - \eta_1\right) \\
&\leq \exp\left(-0.1\lambda(T-t_1)(\eta_0-\eta_1)\min\left\{\frac{\lambda(T-t_1)(\eta_0-\eta_1)}{\lambda(t_1-t_0)\hat{\rho}_0}, 1\right\}\right) \\
&\leq \exp\left(-0.1\lambda(T-t_1)(\eta_0-\eta_1)\min\left\{\frac{T-t_1}{t_1}(\eta_0-\eta_1), 1\right\}\right) \\
&= \exp\left(-0.1\lambda T\theta(\eta_0-\eta_1)\min\left\{\frac{\theta}{1-\theta}(\eta_0-\eta_1), 1\right\}\right) \\
&= \exp\left(-0.1(\lambda T)^{\frac{1}{2}}(\ln(x_0+1))^{\alpha}\frac{1}{(1-\theta)^{1/2}}\min\left\{\frac{(\ln(x_0+1))^{\alpha}}{(\lambda T)^{1/2}}\frac{1}{(1-\theta)^{3/2}}, 1\right\}\right) \\
&\leq \exp\left(-0.1(\lambda T)^{\frac{1}{2}}(\ln(x_0+1))^{\alpha}\min\left\{\frac{(\ln(x_0+1))^{\alpha}}{(\lambda T)^{1/2}}, 1\right\}\right) \\
&= \exp\left(-0.1(\ln(x_0+1))^{2\alpha}\min\left\{1, \frac{(\lambda T)^{1/2}}{(\ln(x_0+1))^{\alpha}}\right\}\right),
\end{aligned}$$

where the first inequality follows from Lemma EC.5, the second inequality follows since $\hat{\rho}_0 \leq 1$, the third equality follows from the property that

$$\eta_0 - \eta_1 = \frac{(\ln(x_0+1))^{\alpha}}{\lambda^{1/2}}\frac{1}{T^{1/2}\theta(1-\theta)^{1/2}},$$

and the third inequality follows since $\theta \in (0,1)$.

Second, we establish an upper bound for $\mathbb{P}(\mathcal{G}_1^c)$. We can write $\mathcal{G}_1^c = E_1 \cup E_2$ where

$$\begin{aligned}
E_1 &\triangleq \left\{\rho^d - \eta_1 - \frac{\hat{\Delta}_0}{\lambda(T-t_1)} \leq 0\right\}, \\
E_2 &\triangleq \left\{\rho^d - \eta_1 - \frac{\hat{\Delta}_0}{\lambda(T-t_1)} \geq 1\right\}.
\end{aligned}$$

The two events E_1 and E_2 are independent. So, $\mathbb{P}(\mathcal{G}_1^c) = \mathbb{P}(E_1) + \mathbb{P}(E_2)$. We start with deriving an upper bound for $\mathbb{P}(E_1)$.

Consider $\mu \in (0, \lambda)$ and suppose that $\eta_1 < \rho^d$ and $\theta < \frac{2}{\rho^d - \eta_1}$[5]. We have:

$$\begin{aligned}
\mathbb{P}(E_1) &= \mathbb{P}\left(\frac{\hat{\Delta}_0}{\lambda(T-t_1)} \geq \rho^d - \eta_1\right) \\
&= \mathbb{P}\left(\frac{\hat{\Delta}_0}{\lambda} \geq (T-t_1)(\rho^d - \eta_1)\right) \\
&= \mathbb{P}\left(\exp\left(\frac{\mu}{\lambda}\hat{\Delta}_0\right) \geq \exp\left(\mu(T-t_1)\left(\rho^d - \eta_1\right)\right)\right) \\
&\leq \mathsf{E}\left[\exp\left(\frac{\mu}{\lambda}\hat{\Delta}_0 - \mu(T-t_1)\left(\rho^d - \eta_1\right)\right)\right] \\
&\leq \exp\left(\frac{\mu^2}{\lambda}t_1 - \mu(T-t_1)(\rho^d - \eta_1)\right) \\
&= \exp\left(\frac{\mu^2 T}{\lambda}(1-\theta) - \mu T\theta(\rho^d - \eta_1)\right) \\
&\leq \exp\left(\mu^2\frac{T}{\lambda} - \mu T\theta(\rho^d - \eta_1)\right).
\end{aligned}$$

The first inequality follows from Markov inequality. The second inequality holds due to the following reasons: First, if X is a Poisson random variable with parameter λ, then the moment generating function is given by $\mathsf{E}[e^{\tilde{\mu}X}] = \exp(\lambda(e^{\tilde{\mu}} - 1))$; second, we have the property that $e^{\tilde{\mu}} \leq 1 + \tilde{\mu} + \tilde{\mu}^2$ for $\tilde{\mu} \in [0,1]$; third, these two results jointly imply $\mathsf{E}\left[e^{\tilde{\mu}(X-\lambda)}\right] \leq e^{\lambda\tilde{\mu}^2}$ for $\tilde{\mu} \in [0,1]$; and, fourth, the condition $\mu \in (0, \lambda)$ guarantees that $\frac{\mu}{\lambda} \in (0,1)$.

The last bound for $\mathbb{P}(E_1)$ is minimized at

$$\mu = \frac{\theta\lambda(\rho^d - \eta_1)}{2},$$

which is in $(0, \lambda)$ since $\theta < \frac{2}{\rho^d - \eta_1}$, and yields the following bound:

$$\mathbb{P}(E_1) \leq \exp\left(-\frac{\lambda T\theta^2(\rho^d - \eta_1)^2}{4}\right).$$

Now, we derive an upper bound for $\mathbb{P}(E_2)$. We can use a similar argument as above. Consider again $\mu \in (0, \lambda)$ and suppose that $\eta_1 < \rho^d$ and $\theta < \frac{2}{1 - \rho^d + \eta_1}$. Then,

$$\begin{aligned}
\mathbb{P}(E_2) &= \mathbb{P}\left(\frac{-\hat{\Delta}_0}{\lambda(T-t_1)} \geq 1 - (\rho^d - \eta_1)\right) \\
&\leq \mathsf{E}\left[\exp\left(-\frac{\mu}{\lambda}\hat{\Delta}_0 - \mu(T-t_1)\left(1 - \rho^d + \eta_1\right)\right)\right]
\end{aligned}$$

[5] This condition is satisfied in the asymptotic regime that we will analyze in the later part of this proof. As we will show momentarily, $\lim_{k\to\infty} \theta^{(k)} = 0$ and $\lim_{k\to\infty} \eta_1^{(k)} = 0$. Therefore, this condition is guaranteed to hold for sufficiently large k.

$$\begin{aligned}
&\le \exp\left(\frac{\mu^2}{\lambda}t_1 - \mu(T-t_1)(1-\rho^d+\eta_1)\right)\\
&\le \exp\left(\mu^2\frac{T}{\lambda} - \mu T\theta(1-\rho^d+\eta_1)\right).
\end{aligned}$$

The second inequality holds due to the following reasons: First, if X is a Poisson random variable with parameter λ, then $\mathsf{E}[e^{-\tilde{\mu}X}] = \exp(\lambda(e^{-\tilde{\mu}}-1))$; second, $e^{-\tilde{\mu}} \le 1-\tilde{\mu}+\tilde{\mu}^2$ for $\tilde{\mu}\in[0,1]$; third, the first and the second points jointly imply $\mathsf{E}\left[e^{-\tilde{\mu}(X-\lambda)}\right] \le e^{\lambda\tilde{\mu}^2}$ for $\tilde{\mu}\in[0,1]$; and, fourth, the condition $\mu\in(0,\lambda)$ guarantees that $\frac{\mu}{\lambda}\in(0,1)$.

The last bound for $\mathbb{P}(E_2)$ is minimized at

$$\mu = \frac{\theta\lambda(1-\rho^d+\eta_1)}{2},$$

which is in $(0,\lambda)$ since $\theta < \frac{2}{1-\rho^d+\eta_1}$, and yields the following bound:

$$\mathbb{P}(E_2) \le \exp\left(-\frac{\lambda T\theta^2(1-\rho^d+\eta_1)^2}{4}\right).$$

We conclude that as long as $\theta < \min\left\{\frac{2}{\rho^d-\eta_1}, \frac{2}{1-\rho^d+\eta_1}\right\}$, we have:

$$\begin{aligned}
\mathbb{P}(\mathcal{G}^c) &\le \sum_{n=0}^{1}\mathbb{P}(\mathcal{G}_n^c)\\
&\le \exp\left(-0.1(\ln(x_0+1))^{2\alpha}\min\left\{1, \frac{(\lambda T)^{1/2}}{(\ln(x_0+1))^{\alpha}}\right\}\right)\\
&\quad + \exp\left(-\frac{\lambda T\theta^2(\rho^d-\eta_1)^2}{4}\right) + \exp\left(-\frac{\lambda T\theta^2(1-\rho^d+\eta_1)^2}{4}\right).
\end{aligned}$$

Now, we claim that

$$\lim_{k\to\infty}\theta^{(k)} = 0, \quad \lim_{k\to\infty}\frac{(\lambda T^{(k)})^{1/2}\theta^{(k)}}{(\ln(x_0^{(k)}+1))^{\alpha}} = \infty, \quad \lim_{k\to\infty}\eta_1^{(k)} = 0.$$

The first one is immediate by the definition of θ. The second one is also immediate since the numerator is on the order of $k^{1/6}$ whereas the denominator is only poly-logarithmic in k. (Both the first and the second limits also imply $\lim_{k\to\infty}\frac{(\lambda T^{(k)})^{1/2}}{(\ln(x_0^{(k)}+1))^{\alpha}} = \infty$.) As for the third one, note that

$$\begin{aligned}
\eta_1 &= -\frac{(\ln(x_0+1))^{\alpha}}{\lambda^{1/2}}\frac{t_1^{1/2}}{T-t_1}\\
&= -\frac{(\ln(x_0+1))^{\alpha}}{\lambda^{1/2}}\frac{(1-\theta)^{1/2}}{T^{1/2}\theta}
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{(\ln(x_0+1))^{\alpha}(1-\theta)^{1/2}}{(\lambda T)^{1/2}\theta} \\
&\geq -\frac{(\ln(x_0+1))^{\alpha}}{(\lambda T)^{1/2}\theta} \\
&= -\frac{(\ln(x_0+1))^{\alpha}(x_0+1)^{1/3}}{(\lambda T)^{1/2}}.
\end{aligned}
$$

The term after the last equality goes to 0 as $k \to \infty$. So, $\eta_1^{(k)} \geq 0$ as $k \to \infty$. But since $\eta_1^{(k)} \leq 0$ by definition, we must have $\eta_1^{(k)} \to 0$ as $k \to \infty$.

Let $\hat{\beta} \triangleq \min\{\rho^d, 1-\rho^d\}/2$. (Note that $\hat{\beta} \in (0,1)$.) Based on our limit results above, we conclude there exists a constant $K > 0$ such that, for all $k > K$, we have:

$$
\begin{aligned}
\lambda T^{(k)} &\geq (\ln(x_0^{(k)}+1))^{2\alpha}, \\
\lambda T^{(k)}(\theta^{(k)})^2 &\geq \frac{0.4}{\hat{\beta}^2} \cdot (\ln(x_0^{(k)}+1))^{2\alpha}, \\
\rho^d - \eta_1^{(k)} &\geq \hat{\beta}, \\
1-\rho^d+\eta_1^{(k)} &\geq \hat{\beta}
\end{aligned}
$$

We conclude that, for all $k > K$, we can bound:

$$
\mathbb{P}(\mathcal{G}^c) \leq \sum_{n=0}^{1} \mathbb{P}(\mathcal{G}_n^c) \leq 3\exp\left(-0.1\left(\ln\left(x_0^{(k)}+1\right)\right)^{2\alpha}\right). \quad \blacksquare
$$

B: Proofs for Section 7

In this section, we will use:

$$
\theta \triangleq (x_0+1)^{-\frac{1}{N+2}}.
$$

We have: $\theta \in (0,1)$.

Proof of Lemma 7.

First, we prove that condition (2) is satisfied.

For any $n = 0, \cdots, N-2$, by definition, we have:

$$
\begin{aligned}
\eta_n - \eta_{n+1} &= \frac{(\ln(x_0+1))^{\alpha}}{(\lambda T)^{1/2}}\left[\frac{T^{1/2}}{(t_{n+1}-t_n)^{1/2}} - \frac{T^{1/2}}{(t_{n+2}-t_{n+1})^{1/2}} + \frac{T^{1/2}(t_{n+1}-t_n)^{1/2}}{T-t_{n+1}}\right] \\
&= \frac{(\ln(x_0+1))^{\alpha}}{(\lambda T)^{1/2}}\left[\frac{1}{\theta^{n/2}(1-\theta)^{1/2}} - \frac{1}{\theta^{(n+1)/2}(1-\theta)^{1/2}} + \frac{(1-\theta)^{1/2}}{\theta^{(n+2)/2}}\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{(\ln(x_0+1))^\alpha}{(\lambda T)^{1/2}} \frac{1-\theta^{1/2}}{\theta^{(n+2)/2}(1-\theta)^{1/2}} \\
&> 0.
\end{aligned}
$$

For $n = N-1$, we have:

$$
\begin{aligned}
\eta_{N-1} - \eta_N &= \frac{(\ln(x_0+1))^\alpha}{(\lambda T)^{1/2}} \left[\frac{T^{1/2}}{(t_N - t_{N-1})^{1/2}} + \frac{T^{1/2}(t_N - t_{N-1})^{1/2}}{T - t_N} \right] \\
&= \frac{(\ln(x_0+1))^\alpha}{\lambda^{1/2}} \frac{1}{\theta^{(N+1)/2}(1-\theta)^{1/2}} \\
&> 0.
\end{aligned}
$$

Second, we prove that condition (3) is satisfied. Note that

$$
\begin{aligned}
\sum_{n=0}^{N} (t_{n+1} - t_n)\,\eta_n &= \frac{(\ln(x_0+1))^\alpha}{\lambda^{1/2}} \sum_{n=0}^{N-1} (t_{n+1} - t_n)^{\frac{1}{2}} - \frac{(\ln(x_0+1))^\alpha}{\lambda^{1/2}} \sum_{n=0}^{N} \sum_{m=0}^{n-1} (t_{n+1} - t_n) \frac{(t_{m+1} - t_m)^{1/2}}{T - t_{m+1}} \\
&= \frac{(\ln(x_0+1))^\alpha}{\lambda^{1/2}} \sum_{n=0}^{N-1} (t_{n+1} - t_n)^{\frac{1}{2}} - \frac{(\ln(x_0+1))^\alpha}{\lambda^{1/2}} \sum_{m=0}^{N-1} \left[\sum_{n=m+1}^{N} (t_{n+1} - t_n) \right] \frac{(t_{m+1} - t_m)^{1/2}}{T - t_{m+1}} \\
&= \frac{(\ln(x_0+1))^\alpha}{\lambda^{1/2}} \sum_{n=0}^{N-1} (t_{n+1} - t_n)^{\frac{1}{2}} - \frac{(\ln(x_0+1))^\alpha}{\lambda^{1/2}} \sum_{m=0}^{N-1} (t_{m+1} - t_m)^{1/2} \\
&= 0.
\end{aligned}
$$

This completes the proof of Lemma 7. ■

B.1: Outline of the Proof of Theorem 4

In this subsection, we provide an outline of the proof of Theorem 4. The remaining details of the proof can be found in Appendix B.2.

We proceed in a similar way as in the outline of the proof of Theorem 3 in Section 6. Without loss of generality, we assume that $\rho^d > \eta_0$ and $\lambda T > 1$. We also define $\hat{\rho}_n$ as follows:

$$
\hat{\rho}_n \triangleq \begin{cases} (\rho^d - \eta_0)^+ & \text{if } n = 0, \\ \max\left\{ \left(\rho^d - \eta_n - \sum_{m=0}^{n-1} \frac{\hat{\Delta}_m}{\lambda(T - t_{m+1})} \right)^+, \hat{\rho}_{n-1} \right\} & \text{if } n = 1, \cdots, N \end{cases}
$$

where $\hat{S}_n$ is a Poisson random variable with mean $\lambda(t_{n+1} - t_n)\hat{\rho}_n$ and $\hat{\Delta}_m \triangleq \hat{S}_m - \lambda(t_{m+1} - t_m)\hat{\rho}_m$.

We begin by noting the following lower bound:

$$J^{\pi^{\text{Re}}_{\searrow}}(x_0,T) \geq \sum_{n=0}^{N} \mathsf{E}\left[\bar{F}^{-1}(\hat{\rho}_n)\hat{S}_n\right] - V\mathsf{E}\left[\left(\sum_{n=0}^{N}\hat{S}_n - x_0\right)^+\right]. \tag{EC.2}$$

The first term on the R.H.S. can be bounded as follows:

$$\begin{aligned}
\sum_{n=0}^{N} \mathsf{E}\left[\bar{F}^{-1}(\hat{\rho}_n)\hat{S}_n\right] &= \sum_{n=0}^{N} \mathsf{E}\left[\bar{F}^{-1}(\hat{\rho}_n)\mathsf{E}\left[\hat{S}_n \middle| \mathcal{F}_{t_n-}\right]\right] \\
&= \sum_{n=0}^{N} \mathsf{E}\left[\bar{F}^{-1}(\hat{\rho}_n)\lambda(t_{n+1}-t_n)\hat{\rho}_n\right] \\
&= \sum_{n=0}^{N} \lambda(t_{n+1}-t_n)\mathsf{E}\left[R(\hat{\rho}_n)\right] \\
&\geq \bar{J}(x_0,T) - R'\left(\rho^d\right)\underbrace{(-\lambda)\sum_{n=0}^{N}(t_{n+1}-t_n)\mathsf{E}\left[\left(\hat{\rho}_n-\rho^d\right)\right]}_{A_1} \\
&\quad -\frac{C}{2}\underbrace{\lambda\sum_{n=0}^{N}(t_{n+1}-t_n)\mathsf{E}\left[\left(\hat{\rho}_n-\rho^d\right)^2\right]}_{A_2}.
\end{aligned}$$

As our analysis in Section 6, we define the "good event" $\mathcal{G} \triangleq \cap_{n=1}^{N}(\mathcal{G}_{n0}\cap\mathcal{G}_{n1})$, where

$$\begin{aligned}
\mathcal{G}_{n0} &\triangleq \left\{\eta_{n-1}-\eta_n > \frac{\hat{\Delta}_{n-1}}{\lambda(T-t_n)}\right\}, \\
\mathcal{G}_{n1} &\triangleq \left\{\rho^d - \eta_n - \sum_{m=0}^{n-1}\frac{\hat{\Delta}_m}{\lambda(T-t_{m+1})} \in (0,1)\right\}.
\end{aligned}$$

It is not difficult to see that, conditional on $\mathcal{G}$, we have

$$\hat{\rho}_n = \rho^d - \eta_n - \sum_{m=0}^{n-1}\frac{\hat{\Delta}_m}{\lambda(T-t_{m+1})} > 0 \tag{EC.3}$$

for all $n = 0,\cdots,N$. The proof is very similar to the proof of Lemma 2; we omit the details.

The following lemma gives us a bound for each A_1 and A_2.

LEMMA EC.1. *In the setting of Theorem 4, we can bound:*

$$\begin{aligned}
A_1 &\leq (\rho^d+1)\lambda T\mathbb{P}(\mathcal{G}^c), \\
A_2 &\leq 5\left(\ln(x_0+1)\right)^{2\alpha}\frac{N+1}{\theta} + \frac{N}{\theta}
\end{aligned}$$

$$+ \lambda T \left(\max \left\{ \rho^d, N \left(\ln\left(x_0+1\right)\right)^{\alpha} \frac{\left(x_0+1\right)^{1/2}}{\left(\lambda T\right)^{1/2}} + N\left(x_0+1\right)^{\frac{1}{N+2}} \right\} \right)^2 \mathbb{P}\left(\mathcal{G}^c\right).$$

Event $\mathcal{G}$ also enjoys the following inventory balancing property.

LEMMA EC.2. *(Inventory balancing) Conditional on $\mathcal{G}$, we have:*

$$X_{t_N-} = x_0 - \sum_{n=0}^{N-1} \hat{S}_n = \lambda T \left(x_0+1\right)^{\frac{-N}{N+2}} \hat{\rho}_N > 0.$$

The above lemma tells us that, conditional on event $\mathcal{G}$, the seller never runs out of inventory before his last price update time t_N. So, our policy depletes inventory at an appropriate rate that guarantees inventory is available for most of the time throughout the horizon.

Our next lemma gives us a bound for the second term on the R.H.S. of (EC.2).

LEMMA EC.3. *In the setting of Theorem 4, we have:*

$$\mathsf{E}\left[\left(\sum_{n=0}^{N} \hat{S}_n - x_0\right)^+\right] \le \frac{1}{2}\left(\lambda T\right)^{\frac{1}{2}}\left(x_0+1\right)^{-\frac{N}{2(N+2)}} + \left(\lambda T+1\right)\mathbb{P}\left(\mathcal{G}^c\right)^{\frac{1}{2}}.$$

Finally, our last lemma provides a bound for the probability of bad event $\mathcal{G}^c$.

LEMMA EC.4. *Consider the asymptotic setting in Theorem 4 and assume that N satisfies the condition stated in the theorem. There exists a constant $K > 0$ such that, for all $k > K$, we have:*

$$\mathbb{P}\left(\mathcal{G}^c\right) \le 3N \exp\left(-0.05\left(\ln(x_0^{(k)}+1)\right)^{2\alpha}\right).$$

Putting the bounds in Lemmas EC.1 to EC.4 immediately yields the bound in Theorem 4.

B.2: Proofs of Intermediate Results in Appendix B.1

Proof of Lemma EC.1.

We begin with proving Part 1 (i.e., the bound for A_1).

First, we establish a lower bound of $\mathsf{E}\left[\hat{\rho}_n - \rho^d\right]$. We have:

$$\mathsf{E}\left[\hat{\rho}_n - \rho^d\right] = \mathsf{E}\left[\left(\hat{\rho}_n - \rho^d\right)\mathbf{1}\{\mathcal{G}\}\right] + \mathsf{E}\left[\left(\hat{\rho}_n - \rho^d\right)\mathbf{1}\{\mathcal{G}^c\}\right]$$

$$
\begin{aligned}
&= \mathsf{E}\left[\left(-\eta_n - \sum_{m=0}^{n-1} \frac{\hat{\Delta}_m}{\lambda(T-t_{m+1})}\right)\mathbf{1}\{\mathcal{G}\}\right] + \mathsf{E}\left[\left(\hat{\rho}_n - \rho^d\right)\mathbf{1}\{\mathcal{G}^c\}\right] \\
&= \mathsf{E}\left[\left(-\eta_n - \sum_{m=0}^{n-1} \frac{\hat{\Delta}_m}{\lambda(T-t_{m+1})}\right)\mathbf{1}\{\mathcal{G}\}\right] + \mathsf{E}\left[\left(-\eta_n - \sum_{m=0}^{n-1} \frac{\hat{\Delta}_m}{\lambda(T-t_{m+1})}\right)\mathbf{1}\{\mathcal{G}^c\}\right] \\
&\quad + \mathsf{E}\left[\left(\hat{\rho}_n - \rho^d + \eta_n + \sum_{m=0}^{n-1} \frac{\hat{\Delta}_m}{\lambda(T-t_{m+1})}\right)\mathbf{1}\{\mathcal{G}^c\}\right] \\
&= \mathsf{E}\left[-\eta_n - \sum_{m=0}^{n-1} \frac{\hat{\Delta}_m}{\lambda(T-t_{m+1})}\right] + \mathsf{E}\left[\left(\hat{\rho}_n - \rho^d + \eta_n + \sum_{m=0}^{n-1} \frac{\hat{\Delta}_m}{\lambda(T-t_{m+1})}\right)\mathbf{1}\{\mathcal{G}^c\}\right] \\
&= -\eta_n + \mathsf{E}\left[\left(\hat{\rho}_n - \rho^d + \eta_n + \sum_{m=0}^{n-1} \frac{\hat{\Delta}_m}{\lambda(T-t_{m+1})}\right)\mathbf{1}\{\mathcal{G}^c\}\right] \\
&\geq -\eta_n - \mathsf{E}\left[\left(\rho^d - \eta_n - \sum_{m=0}^{n-1} \frac{\hat{\Delta}_m}{\lambda(T-t_{m+1})}\right)\mathbf{1}\{\mathcal{G}^c\}\right] \\
&\geq -\eta_n - \mathsf{E}\left[\left(\rho^d - \eta_n + \sum_{m=0}^{n-1} \frac{(t_{m+1}-t_m)\,\hat{\rho}_m}{T-t_{m+1}}\right)\mathbf{1}\{\mathcal{G}^c\}\right] \\
&\geq -\eta_n - \mathsf{E}\left[\left(\rho^d - \eta_n + \sum_{m=0}^{n-1} \frac{t_{m+1}-t_m}{T-t_{m+1}}\right)\mathbf{1}\{\mathcal{G}^c\}\right] \\
&= -\eta_n - \left(\rho^d - \eta_n + \sum_{m=0}^{n-1} \frac{t_{m+1}-t_m}{T-t_{m+1}}\right)\mathbb{P}(\mathcal{G}^c) \\
&= -\eta_n - \left(\rho^d - \eta_n + n\left(\frac{1}{\theta}-1\right)\right)\mathbb{P}(\mathcal{G}^c),
\end{aligned}
$$

where the first inequality follows from the property that $\hat{\rho}_n \geq 0$, the second inequality follows from the definition of $\hat{\Delta}_m$ and the property that $\hat{S}_m \geq 0$, the third inequality follows from the property that $\hat{\rho}_m \leq 1$, and the last equality follows from the definition of t_n. Therefore, we can bound:

$$
\begin{aligned}
-\lambda\sum_{n=0}^{N}(t_{n+1}-t_n)\,\mathsf{E}\left[\left(\hat{\rho}_n - \rho^d\right)\right] &\leq \lambda\sum_{n=0}^{N}(t_{n+1}-t_n)\,\eta_n + \lambda\sum_{n=0}^{N}(t_{n+1}-t_n)\left(\rho^d - \eta_n + n\left(\frac{1}{\theta}-1\right)\right)\mathbb{P}(\mathcal{G}^c) \\
&= \lambda\sum_{n=0}^{N}(t_{n+1}-t_n)\,\eta_n\mathbb{P}(\mathcal{G}) + \lambda\sum_{n=0}^{N}(t_{n+1}-t_n)\left(\rho^d + n\left(\frac{1}{\theta}-1\right)\right)\mathbb{P}(\mathcal{G}^c) \\
&= \lambda\sum_{n=0}^{N}(t_{n+1}-t_n)\left(\rho^d + n\left(\frac{1}{\theta}-1\right)\right)\mathbb{P}(\mathcal{G}^c) \\
&= \lambda\left[\rho^d T + \sum_{n=0}^{N}(t_{n+1}-t_n)\,n\left(\frac{1}{\theta}-1\right)\right]\mathbb{P}(\mathcal{G}^c) \\
&= \lambda\left[\rho^d T + \sum_{n=0}^{N-1} T\left(\theta^n - \theta^{n+1}\right)n\left(\frac{1}{\theta}-1\right) + TN\theta^N\left(\frac{1}{\theta}-1\right)\right]\mathbb{P}(\mathcal{G}^c) \\
&= \lambda\left[\rho^d T + T\left(\frac{1}{\theta}-1\right)\sum_{n=1}^{N}\theta^n\right]\mathbb{P}(\mathcal{G}^c)
\end{aligned}
$$

$$\leq \lambda \left[\rho^d T + T\left(\frac{1}{\theta}-1\right)\sum_{n=1}^{\infty}\theta^n \right] \mathbb{P}\left(\mathcal{G}^c\right)$$
$$= \left(\rho^d+1\right)\lambda T \mathbb{P}\left(\mathcal{G}^c\right),$$

where the second equality holds since $\sum_{n=0}^{N}\left(t_{n+1}-t_n\right)\eta_n = 0$, the fourth equality follows from the definition of $t_n = T\left(1-\theta^n\right)$, the second inequality follows since $\theta \in (0,1)$; and, the last equality follows from the property that $\sum_{n=1}^{\infty}\theta^n = \frac{1}{1-\theta}-1 = \frac{\theta}{1-\theta}$.

Next, we prove Part 2 (i.e., the bound for A_2).

First, we establish an upper bound of $\mathsf{E}[(\hat{\rho}_n-\rho^d)^2]$. We have:

$$\begin{aligned}
\mathsf{E}\left[\left(\hat{\rho}_n-\rho^d\right)^2\right] &= \mathsf{E}\left[\left(\hat{\rho}_n-\rho^d\right)^2 \mathbf{1}\{\mathcal{G}\}\right] + \mathsf{E}\left[\left(\hat{\rho}_n-\rho^d\right)^2 \mathbf{1}\{\mathcal{G}^c\}\right] \\
&= \mathsf{E}\left[\left(-\eta_n - \sum_{m=0}^{n-1}\frac{\hat{\Delta}_m}{\lambda(T-t_{m+1})}\right)^2 \mathbf{1}\{\mathcal{G}\}\right] + \mathsf{E}\left[\left(\hat{\rho}_n-\rho^d\right)^2 \mathbf{1}\{\mathcal{G}^c\}\right] \\
&= \underbrace{\mathsf{E}\left[\left(\eta_n + \sum_{m=0}^{n-1}\frac{\hat{\Delta}_m}{\lambda(T-t_{m+1})}\right)^2\right]}_{B_1} + \underbrace{\mathsf{E}\left[\left\{\left(\hat{\rho}_n-\rho^d\right)^2 - \left(\eta_n + \sum_{m=0}^{n-1}\frac{\hat{\Delta}_m}{\lambda(T-t_{m+1})}\right)^2\right\}\mathbf{1}\{\mathcal{G}^c\}\right]}_{B_2}.
\end{aligned}$$

For B_1, we have

$$\begin{aligned}
B_1 &= \eta_n^2 + \sum_{m=0}^{n-1}\mathsf{E}\left[\left(\frac{\hat{\Delta}_m}{\lambda(T-t_{m+1})}\right)^2\right] \\
&\quad + 2\sum_{m=0}^{n-2}\sum_{m'=m+1}^{n-1}\mathsf{E}\left[\frac{\hat{\Delta}_m}{\lambda(T-t_{m+1})}\frac{\hat{\Delta}_{m'}}{\lambda(T-t_{m'+1})}\right] + 2\eta_n\sum_{m=0}^{n-1}\mathsf{E}\left[\frac{\hat{\Delta}_m}{\lambda(T-t_{m+1})}\right] \\
&= \eta_n^2 + \sum_{m=0}^{n-1}\mathsf{E}\left[\left(\frac{\hat{\Delta}_m}{\lambda(T-t_{m+1})}\right)^2\right] \\
&= \eta_n^2 + \sum_{m=0}^{n-1}\mathsf{E}\left[\frac{\left(t_{m+1}-t_m\right)\hat{\rho}_m}{\lambda\left(T-t_{m+1}\right)^2}\right] \\
&\leq \eta_n^2 + \sum_{m=0}^{n-1}\frac{t_{m+1}-t_m}{\lambda\left(T-t_{m+1}\right)^2},
\end{aligned}$$

where the second equality follows since $\mathsf{E}[\hat{\Delta}_m]=0$ together with the property that for any m, m' with $m<m'$,

$$\mathsf{E}\left[\hat{\Delta}_m\Delta_{m'}\right] = \mathsf{E}\left[\hat{\Delta}_m\mathsf{E}\left[\hat{\Delta}_{m'}\middle|\mathcal{F}_{t_{m+1}-}\right]\right]. = \mathsf{E}\left[\hat{\Delta}_m\cdot 0\right] = 0.$$

Next, we bound B_2. We discuss the following two cases: $\hat{\rho}_n \leq \rho^d$ and $\hat{\rho}_n > \rho^d$.

Case 1: $\hat{\rho}_n \le \rho^d$.

We have

$$\left(\hat{\rho}_n - \rho^d\right)^2 - \left(\eta_n + \sum_{m=0}^{n-1} \frac{\hat{\Delta}_m}{\lambda(T - t_{m+1})}\right)^2 \le \left(\hat{\rho}_n - \rho^d\right)^2 \le \left(\rho^d\right)^2.$$

Case 2: $\hat{\rho}_n > \rho^d$.

We have:

$$\begin{aligned}
\hat{\rho}_n &= \max_{s\in\{0,\cdots,n\}} \left\{ \left(\rho^d - \eta_s - \sum_{m=0}^{s-1} \frac{\hat{\Delta}_m}{\lambda(T - t_{m+1})}\right)^+ \right\} \\
&\le \max_{s\in\{0,\cdots,n\}} \left\{ \rho^d - \eta_s - \sum_{m=0}^{s-1} \frac{\hat{\Delta}_m}{\lambda(T - t_{m+1})} \right\},
\end{aligned}$$

where the first equality follows from the definition of $\hat{\rho}_n$ and the second equality holds due to the following reasons: If $y = \max\left\{x_1^+, x_2^+, \cdots, x_K^+\right\}$, where $(\cdot)^+$ is a projection to $[0,1]$, and $y > 0$, then there must exist at least one $k\{1,\ldots,K\}$ such that $x_k > 0$. Moreover, $y \le \max\{x_1, x_2, \cdots, x_K\}$.

We can further bound:

$$\begin{aligned}
\hat{\rho}_n - \rho^d &\le \max_{s\in\{0,\cdots,n\}} \left\{ -\eta_s - \sum_{m=0}^{s-1} \frac{\hat{\Delta}_m}{\lambda(T - t_{m+1})} \right\} \\
&\le \max_{s\in\{0,\cdots,n\}} \left\{ -\eta_s + \sum_{m=0}^{s-1} \frac{(t_{m+1} - t_m)\hat{\rho}_m}{T - t_{m+1}} \right\} \\
&\le \max_{s\in\{0,\cdots,n\}} \left\{ -\eta_s + \sum_{m=0}^{s-1} \frac{t_{m+1} - t_m}{T - t_{m+1}} \right\} \\
&\le -\eta_N + \sum_{m=0}^{N-1} \frac{t_{m+1} - t_m}{T - t_{m+1}} \\
&= \frac{\left(\ln\left(x_0 + 1\right)\right)^\alpha}{\left(\lambda T\right)^{1/2}} \sum_{m=0}^{N-1} \frac{T^{1/2}\left(t_{m+1} - t_m\right)^{1/2}}{T - t_{m+1}} + \sum_{m=0}^{N-1} \frac{t_{m+1} - t_m}{T - t_{m+1}} \\
&\le \frac{\left(\ln\left(x_0 + 1\right)\right)^\alpha}{\left(\lambda T\right)^{1/2}} \sum_{m=0}^{N-1} \frac{1}{\theta^{m/2+1}} + \sum_{m=0}^{N-1} \frac{1}{\theta} \\
&\le N \frac{\left(\ln\left(x_0 + 1\right)\right)^\alpha}{\left(\lambda T\right)^{1/2}} \theta^{-\frac{N+1}{2}} + N\theta^{-1} \\
&\le N\left(\ln\left(x_0 + 1\right)\right)^\alpha \frac{\left(x_0 + 1\right)^{1/2}}{\left(\lambda T\right)^{1/2}} + N\left(x_0 + 1\right)^{\frac{1}{N+2}}.
\end{aligned}$$

Put the results for the two cases together, we get:

$$\left(\hat{\rho}_n - \rho^d\right)^2 - \left(\eta_n + \sum_{m=0}^{n-1} \frac{\hat{\Delta}_m}{\lambda(T - t_{m+1})}\right)^2 \leq \left(\max\left\{\rho^d, N\left(\ln\left(x_0+1\right)\right)^\alpha \frac{\left(x_0+1\right)^{1/2}}{\left(\lambda T\right)^{1/2}} + N\left(x_0+1\right)^{\frac{1}{N+2}}\right\}\right)^2.$$

So, we can upper bound A_2 with

$$\lambda \sum_{n=0}^{N} \left(t_{n+1}-t_n\right)\eta_n^2 + \lambda \sum_{n=0}^{N}\sum_{m=0}^{n-1} \left(t_{n+1}-t_n\right) \frac{t_{m+1}-t_m}{\lambda\left(T-t_{m+1}\right)^2} + \lambda T \left(\max\left\{\rho^d, N\left(\ln\left(x_0+1\right)\right)^\alpha \frac{\left(x_0+1\right)^{1/2}}{\left(\lambda T\right)^{1/2}} + N\left(x_0+1\right)^{\frac{1}{N+2}}\right\}\right)^2 \mathbb{P}\left(\mathcal{G}^c\right). \tag{EC.4}$$

Now, we bound the two terms in (EC.4). For the first term in (EC.4), we have:

$$\begin{aligned}
\lambda \sum_{n=0}^{N} \left(t_{n+1}-t_n\right)\eta_n^2 &= \left(\ln\left(x_0+1\right)\right)^{2\alpha} \sum_{n=0}^{N} \left(t_{n+1}-t_n\right) \left[\frac{1}{\left(t_{n+1}-t_n\right)^{1/2}} \mathbf{1}\left\{n<N\right\} - \sum_{m=0}^{n-1} \frac{\left(t_{m+1}-t_m\right)^{1/2}}{T-t_{m+1}}\right]^2 \\
&= \left(\ln\left(x_0+1\right)\right)^{2\alpha} \sum_{n=0}^{N} \left(\theta^n - \theta^{n+1}\right) \left[\frac{1}{\left(1-\theta\right)^{1/2}\theta^{n/2}} \mathbf{1}\left\{n<N\right\} - \sum_{m=0}^{n-1} \frac{\left(1-\theta\right)^{1/2}}{\theta^{(m+2)/2}}\right]^2 \\
&= \left(\ln\left(x_0+1\right)\right)^{2\alpha} \sum_{n=0}^{N} \left[\mathbf{1}\left\{n<N\right\} - \frac{\left(1-\theta\right)\left(1-\theta^{n/2}\right)}{\left(1-\theta^{1/2}\right)\theta^{1/2}}\right]^2 \\
&= \left(\ln\left(x_0+1\right)\right)^{2\alpha} \sum_{n=0}^{N} \left[\mathbf{1}\left\{n<N\right\} + \frac{\left(1-\theta\right)^2\left(1-\theta^{n/2}\right)^2}{\left(1-\theta^{1/2}\right)^2\theta} \right. \\
&\qquad \left. -2\frac{\left(1-\theta\right)\left(1-\theta^{n/2}\right)}{\left(1-\theta^{1/2}\right)\theta^{1/2}} \mathbf{1}\left\{n<N\right\}\right] \\
&\leq \left(\ln\left(x_0+1\right)\right)^{2\alpha} \sum_{n=0}^{N} \left[1 + \frac{\left(1+\theta^{1/2}\right)^2}{\theta}\right] \\
&\leq 5\left(\ln\left(x_0+1\right)\right)^{2\alpha} \frac{N+1}{\theta},
\end{aligned}$$

where the first inequality follows since $1-\theta = (1-\theta^{1/2})(1+\theta^{1/2})$ and $(1-\theta^{n/2})^2 \leq 1$ and the last inequality follows since $1+\theta^{1/2} \leq 2$ and $1 \leq 1/\theta$.

For the second term in (EC.4), we have:

$$\begin{aligned}
\sum_{n=0}^{N}\sum_{m=0}^{n-1} \left(t_{n+1}-t_n\right) \frac{t_{m+1}-t_m}{\left(T-t_{m+1}\right)^2} &= \sum_{m=0}^{N-1} \frac{t_{m+1}-t_m}{\left(T-t_{m+1}\right)^2} \left[\sum_{n=m+1}^{N} \left(t_{n+1}-t_n\right)\right] \\
&= \sum_{m=0}^{N-1} \frac{t_{m+1}-t_m}{T-t_{m+1}}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{m=0}^{N-1} \frac{1-\theta}{\theta} \\
&\leq \frac{N}{\theta}.
\end{aligned}
$$

Combining the bounds for all the first two terms in (EC.4), we can bound A_2 with

$$
\begin{aligned}
&5\left(\ln\left(x_0+1\right)\right)^{2\alpha} \frac{N+1}{\theta} + \frac{N}{\theta} \\
&\quad + \lambda T \left(\max\left\{ \rho^d, N\left(\ln\left(x_0+1\right)\right)^{\alpha} \frac{\left(x_0+1\right)^{1/2}}{\left(\lambda T\right)^{1/2}} + N\left(x_0+1\right)^{\frac{1}{N+2}} \right\} \right)^2 \mathbb{P}\left(\mathcal{G}^c\right).
\end{aligned}
$$

This completes the proof of Lemma EC.1. ■

Proof of Lemma EC.2.

Conditional on $\mathcal{G}$, we have:

$$
\rho_n = \rho^d - \eta_n - \sum_{m=0}^{n-1} \frac{\hat{\Delta}_m}{\lambda(T-t_{m+1})}, \ \forall\ n=0,\cdots,N \quad \text{and} \quad \hat{\rho}_N \in (0,1).
$$

Therefore, we can bound

$$
\begin{aligned}
x_0 - \sum_{n=0}^{N-1} \hat{S}_n &= \lambda T \rho^d - \sum_{n=0}^{N-1} \left(\lambda\left(t_{n+1}-t_n\right) \hat{\rho}_n + \hat{\Delta}_n \right) \\
&= \lambda T \rho^d - \sum_{n=0}^{N-1} \left[\lambda\left(t_{n+1}-t_n\right) \left(\rho^d - \eta_n - \sum_{m=0}^{n-1} \frac{\hat{\Delta}_m}{\lambda(T-t_{m+1})} \right) + \hat{\Delta}_n \right] \\
&= \lambda T \rho^d - \lambda t_N \rho^d + \lambda \sum_{n=0}^{N-1} \left(t_{n+1}-t_n\right) \eta_n + \sum_{n=0}^{N-1} \sum_{m=0}^{n-1} \left(t_{n+1}-t_n\right) \frac{\hat{\Delta}_m}{T-t_{m+1}} - \sum_{n=0}^{N-1} \hat{\Delta}_n \\
&= \lambda T \rho^d - \lambda t_N \rho^d - \lambda\left(T-t_N\right) \eta_N + \sum_{n=0}^{N-1} \sum_{m=0}^{n-1} \left(t_{n+1}-t_n\right) \frac{\hat{\Delta}_m}{T-t_{m+1}} - \sum_{n=0}^{N-1} \hat{\Delta}_n \\
&= \lambda T \rho^d - \lambda t_N \rho^d - \lambda\left(T-t_N\right) \eta_N + \sum_{m=0}^{N-2} \frac{\hat{\Delta}_m}{T-t_{m+1}} \left[\sum_{n=m+1}^{N-1} \left(t_{n+1}-t_n\right) \right] - \sum_{n=0}^{N-1} \hat{\Delta}_n \\
&= \lambda T \rho^d - \lambda t_N \rho^d - \lambda\left(T-t_N\right) \eta_N + \sum_{m=0}^{N-2} \frac{t_N - t_{m+1}}{T-t_{m+1}} \hat{\Delta}_m - \sum_{n=0}^{N-1} \hat{\Delta}_n \\
&= \lambda\left(T-t_N\right) \rho^d - \lambda\left(T-t_N\right) \eta_N - \left(T-t_N\right) \sum_{n=0}^{N-1} \frac{\hat{\Delta}_n}{T-t_{n+1}} \\
&= \lambda\left(T-t_N\right) \left(\rho^d - \eta_N - \sum_{n=0}^{N-1} \frac{\hat{\Delta}_n}{\lambda\left(T-t_{n+1}\right)} \right) \\
&= \lambda\left(T-t_N\right) \hat{\rho}_N
\end{aligned}
$$

$$= \lambda T \left(x_0+1\right)^{\frac{-N}{N+2}} \hat{\rho}_N \ > \ 0. \ \blacksquare$$

Proof of Lemma EC.3.

We have:

$$\mathsf{E}\left[\left(\sum_{n=0}^{N} \hat{S}_n - x_0\right)^+\right] = \underbrace{\mathsf{E}\left[\left(\sum_{n=0}^{N} \hat{S}_n - x_0\right)^+ \mathbf{1}\{\mathcal{G}\}\right]}_{C_1} + \underbrace{\mathsf{E}\left[\left(\sum_{n=0}^{N} \hat{S}_n - x_0\right)^+ \mathbf{1}\{\mathcal{G}^c\}\right]}_{C_2}.$$

First, we establish an upper bound of C_1. Note that

$$\begin{aligned}
C_1 &= \mathsf{E}\left[\left(\sum_{n=0}^{N} \hat{S}_n - x_0\right)^+ \middle| \mathcal{G}\right] \mathbb{P}(\mathcal{G}) \\
&= \mathsf{E}\left[\left(\hat{S}_N - X_{t_N-}\right)^+ \middle| \mathcal{G}\right] \mathbb{P}(\mathcal{G}) \\
&= \mathsf{E}\left[\left(\hat{S}_N - \lambda\left(T-t_N\right)\hat{\rho}_N\right)^+ \middle| \mathcal{G}\right] \mathbb{P}(\mathcal{G}) \\
&= \mathsf{E}\left[\left(\hat{S}_N - \lambda\left(T-t_N\right)\hat{\rho}_N\right)^+\right] \\
&\le \frac{1}{2}\lambda^{\frac{1}{2}}\left(T-t_N\right)^{\frac{1}{2}} \\
&= \frac{1}{2}\left(\lambda T\right)^{\frac{1}{2}}\left(x_0+1\right)^{-\frac{N}{2(N+2)}},
\end{aligned}$$

where the second equality follow from Lemma EC.2 and the inequality follows since $\hat{\rho}_N \in [0,1]$ combined with the fact that if a random variable X has mean μ and standard deviation σ, then $\mathsf{E}[(X-\mu)^+] \le \frac{\sigma}{2}$ (see equation (18) in Gallego and Van Ryzin (1994)).

Next, we establish an upper bound of C_2. We denote by $\tilde{Z}$ the total number of customers arriving to the system over $[0,T]$. Then, $\tilde{Z}$ is a Poisson random variable with parameter λT. We have:

$$\begin{aligned}
C_2 &\le \mathsf{E}\left[\left(\tilde{Z} - x_0\right)^+ \mathbf{1}\{\mathcal{G}^c\}\right] \\
&\le \mathsf{E}\left[\tilde{Z}\mathbf{1}\{\mathcal{G}^c\}\right] \\
&\le \mathsf{E}[\tilde{Z}^2]^{\frac{1}{2}}\,\mathsf{E}[\mathbf{1}\{\mathcal{G}^c\}]^{\frac{1}{2}} \\
&= \left(\lambda^2T^2+\lambda T\right)^{\frac{1}{2}}\mathbb{P}(\mathcal{G}^c)^{\frac{1}{2}} \\
&\le \left(\lambda T+1\right)\mathbb{P}(\mathcal{G}^c)^{\frac{1}{2}}.
\end{aligned}$$

The first inequality follows since $\sum_{n=0}^{N}\hat{S}_n \le \tilde{Z}$ and the third inequality follows from Cauchy-Schwarz inequality.

Putting the bounds for C_1 and C_2 together completes the proof of Lemma EC.3. ■

Proof of Lemma EC.4.

Recall that $\mathcal{G} \triangleq \cap_{n=1}^{N}(\mathcal{G}_{n0} \cap \mathcal{G}_{n1})$, where

$$\mathcal{G}_{n0} \triangleq \left\{ \eta_{n-1} - \eta_n > \frac{\hat{\Delta}_{n-1}}{\lambda(T-t_n)} \right\},$$
$$\mathcal{G}_{n1} \triangleq \left\{ \rho^d - \eta_n - \sum_{m=0}^{n-1} \frac{\hat{\Delta}_m}{\lambda(T-t_{m+1})} \in (0,1) \right\}.$$

Thus, we can bound the probability of bad event as follows:

$$\mathbb{P}(\mathcal{G}^c) \le \sum_{n=1}^{N} \mathbb{P}(\mathcal{G}_{n0}^c) + \sum_{n=1}^{N} \mathbb{P}(\mathcal{G}_{n1}^c).$$

First, we establish an upper bound of $\mathbb{P}(\mathcal{G}_{n0}^c)$ for $n = 1, \cdots, N$. We have

$$\begin{aligned}
\mathbb{P}(\mathcal{G}_{n0}^c) &= \mathbb{P}\left(\frac{\hat{\Delta}_{n-1}}{\lambda(T-t_n)} \ge \eta_{n-1} - \eta_n \right) \\
&\le \exp\left(-0.1\lambda(T-t_n)(\eta_{n-1}-\eta_n) \min\left\{ \frac{\lambda(T-t_n)(\eta_{n-1}-\eta_n)}{\lambda(t_n - t_{n-1})}, 1 \right\} \right) \\
&= \exp\left(-0.1\lambda(T-t_n)(\eta_{n-1}-\eta_n) \min\left\{ \frac{T-t_n}{t_n - t_{n-1}}(\eta_{n-1}-\eta_n), 1 \right\} \right) \\
&= \exp\left(-0.1\lambda T\theta^n(\eta_{n-1}-\eta_n) \min\left\{ \frac{\theta}{1-\theta}(\eta_{n-1}-\eta_n), 1 \right\} \right) \\
&\le \exp\left(-0.1(\lambda T)^{\frac{1}{2}}(\ln(x_0+1))^\alpha \frac{\left(1-\theta^{1/2}\right)\theta^{(n-1)/2}}{(1-\theta)^{1/2}} \min\left\{ \frac{(\ln(x_0+1))^\alpha}{(\lambda T)^{1/2}} \frac{1-\theta^{1/2}}{\theta^{(n-1)/2}(1-\theta)^{3/2}}, 1 \right\} \right) \\
&= \exp\left(-0.1(\ln(x_0+1))^{2\alpha} \min\left\{ \frac{\left(1-\theta^{1/2}\right)^2}{(1-\theta)^2}, \frac{(\lambda T)^{1/2}}{(\ln(x_0+1))^\alpha} \frac{\left(1-\theta^{1/2}\right)\theta^{(n-1)/2}}{(1-\theta)^{1/2}} \right\} \right) \\
&\le \exp\left(-0.1(\ln(x_0+1))^{2\alpha} \min\left\{ \left(1-\theta^{1/2}\right)^2, \frac{(\lambda T)^{1/2}}{(\ln(x_0+1))^\alpha} \left(1-\theta^{1/2}\right)\theta^{(N-1)/2} \right\} \right),
\end{aligned}$$

where the first inequality follows from Lemma EC.5; the second inequality follows since

$$\begin{aligned}
\eta_{n-1} - \eta_n &= \frac{(\ln(x_0+1))^\alpha}{(\lambda T)^{1/2}} \cdot \left[\frac{T^{1/2}}{(t_n - t_{n-1})^{1/2}} - \frac{T^{1/2}}{(t_{n+1}-t_n)} \mathbf{1}\{n < N\} + \frac{T^{1/2}(t_n - t_{n-1})^{1/2}}{T - t_n} \right] \\
&\ge \frac{(\ln(x_0+1))^\alpha}{(\lambda T)^{1/2}} \cdot \left[\frac{T^{1/2}}{(t_n - t_{n-1})^{1/2}} - \frac{T^{1/2}}{(t_{n+1}-t_n)} + \frac{T^{1/2}(t_n - t_{n-1})^{1/2}}{T - t_n} \right] \\
&= \frac{(\ln(x_0+1))^\alpha}{\lambda^{1/2}} \frac{1-\theta^{1/2}}{T^{1/2}\theta^{(n+1)/2}(1-\theta)^{1/2}},
\end{aligned}$$

for all $n=1,\dots,N$; and, the last inequality follows since $\theta\in[0,1]$ and $n\le N$.

Second, we establish an upper bound of $\mathbb{P}(\mathcal{G}_{n1}^c)$. We can write $\mathcal{G}_{n1}^c = E_{n1}\cup E_{n2}$ where

$$E_{n1} \triangleq \left\{\rho^d-\eta_n-\sum_{m=0}^{n-1}\frac{\hat{\Delta}_m}{\lambda(T-t_{m+1})}\le 0\right\},$$
$$E_{n2} \triangleq \left\{\rho^d-\eta_n-\sum_{m=0}^{n-1}\frac{\hat{\Delta}_m}{\lambda(T-t_{m+1})}\ge 1\right\}.$$

The two events E_1 and E_2 are independent. So, $\mathbb{P}(\mathcal{G}_1^c)=\mathbb{P}(E_1)+\mathbb{P}(E_2)$. We start with deriving an upper bound for $\mathbb{P}(E_1)$.

Consider $\mu\in(0,\lambda)$ and suppose that $\eta_n<\rho^d$ and $\theta<\frac{2}{\rho^d-\eta_n}$[6]. We have:

$$\begin{aligned}
\mathbb{P}(E_{n1}) &= \mathbb{P}\left(\sum_{m=0}^{n-1}\frac{\hat{\Delta}_m}{\lambda(T-t_{m+1})}\ge\rho^d-\eta_n\right)\\
&= \mathbb{P}\left(\sum_{m=0}^{n-1}\frac{(T-t_n)\hat{\Delta}_m}{\lambda(T-t_{m+1})}\ge (T-t_n)(\rho^d-\eta_n)\right)\\
&= \mathbb{P}\left(\exp\left(\frac{\mu}{\lambda}\sum_{m=0}^{n-1}\frac{T-t_n}{T-t_{m+1}}\hat{\Delta}_m\right)\ge\exp\left(\mu(T-t_n)\left(\rho^d-\eta_n\right)\right)\right)\\
&\le \mathsf{E}\left[\exp\left(\frac{\mu}{\lambda}\sum_{m=0}^{n-1}\frac{T-t_n}{T-t_{m+1}}\hat{\Delta}_m-\mu(T-t_n)\left(\rho^d-\eta_n\right)\right)\right]\\
&\le \exp\left(\frac{\mu^2}{\lambda}\sum_{m=0}^{n-1}\frac{(T-t_n)^2}{(T-t_{m+1})^2}(t_{m+1}-t_m)-\mu(T-t_n)(\rho^d-\eta_n)\right)\\
&= \exp\left(\frac{\mu^2T\theta^{2n}}{\lambda}\sum_{m=0}^{n-1}\frac{1-\theta}{\theta^{m+2}}-\mu T\theta^n(\rho^d-\eta_n)\right)\\
&= \exp\left(\frac{\mu^2T\theta^{2n}}{\lambda}\frac{1}{\theta}\left(\frac{1}{\theta^n}-1\right)-\mu T\theta^n(\rho^d-\eta_n)\right)\\
&\le \exp\left(\mu^2\frac{T\theta^{n-1}}{\lambda}-\mu T\theta^n(\rho^d-\eta_n)\right),
\end{aligned}$$

where the first inequality follows from Markov inequality, and the second inequality follows by repeated application of moment generating function of Poisson random variable together with the fact that $e^{\tilde{\mu}}\le 1+\tilde{\mu}+\tilde{\mu}^2$ for $\tilde{\mu}\in[0,1]$, similar to the argument in the proof of Theorem 3.

The last bound for $\mathbb{P}(E_{n1})$ is minimized at

$$\mu=\frac{\theta\lambda(\rho^d-\eta_n)}{2},$$

[6] This condition is satisfied in the asymptotic regime that we will analyze in the later part of this proof. As we will show momentarily, $\lim_{k\to\infty}\theta^{(k)}=0$ and $\lim_{k\to\infty}\eta_n^{(k)}=0$ for all $n=1,\dots,N$ where N satisfies the condition stated in Theorem 4. Therefore, this condition is guaranteed to hold for sufficiently large k.

which is in $(0,\lambda)$ since $\theta < \frac{2}{\rho^d - \eta_n}$, and yields the following bound:

$$\mathbb{P}(E_{n1}) \le \exp\left(-\frac{\lambda T \theta^{n+1}(\rho^d - \eta_n)^2}{4}\right) \le \exp\left(-\frac{\lambda T \theta^{N+1}(\rho^d - \eta_1)^2}{4}\right).$$

The last inequality holds since $\theta \in [0,1]$ and $0 < \rho^d - \eta_0 \le \rho^d - \eta_1 \le \rho^d - \eta_2 \le \cdots \le \rho^d - \eta_N$.

We can derive a bound for $\mathbb{P}(E_{n2})$ using a similar argument as above (i.e., as in the proof of Theorem 3). Specifically, assuming $\eta_n < \rho^d$ and $\theta < \frac{2}{1-\rho^d+\eta_n}$, for $\mu \in (0,\lambda)$, we have:

$$\begin{aligned}\mathbb{P}(E_{n2}) &= \mathbb{P}\left(\sum_{m=0}^{n-1} \frac{-\hat{\Delta}_m}{\lambda(T - t_{m+1})} \ge 1 - (\rho^d - \eta_n)\right)\\ &\le \exp\left(\mu^2 \frac{T\theta^{n-1}}{\lambda} - \mu T \theta^n (1 - \rho^d + \eta_n)\right).\end{aligned}$$

The last bound is minimized at

$$\mu = \frac{\theta\lambda(1-\rho^d+\eta_n)}{2},$$

which is in $(0,\lambda)$ since $\theta < \frac{2}{1-\rho^d+\eta_n}$, and yields the following bound:

$$\mathbb{P}(E_{n2}) \le \exp\left(-\frac{\lambda T \theta^{n+1}(1-\rho^d + \eta_n)^2}{4}\right) \le \exp\left(-\frac{\lambda T \theta^{N+1}(1-\rho^d + \eta_N)^2}{4}\right).$$

We conclude that as long as $\theta < \min\left\{\frac{2}{\rho^d-\eta_n}, \frac{2}{1-\rho^d+\eta_n}\right\}$ for all $n = 1,\dots,N$, we have:

$$\begin{aligned}\mathbb{P}(\mathcal{G}^c) &\le \sum_{n=1}^{N} \mathbb{P}(\mathcal{G}^c_{n0}) + \sum_{n=1}^{N} \mathbb{P}(\mathcal{G}^c_{n1})\\ &\le N \exp\left(-0.1\left(\ln(x_0+1)\right)^{2\alpha} \min\left\{\left(1-\theta^{1/2}\right)^2, \frac{(\lambda T)^{1/2}}{\left(\ln(x_0+1)\right)^{\alpha}}\left(1-\theta^{1/2}\right)\theta^{(N-1)/2}\right\}\right)\\ &\quad + N\exp\left(-\frac{\lambda T\theta^{N+1}(\rho^d-\eta_1)^2}{4}\right) + N\exp\left(-\frac{\lambda T\theta^{N+1}(1-\rho^d+\eta_N)^2}{4}\right).\end{aligned}$$

Now, observe that

$$\lim_{k\to\infty} \theta^{(k)} = 0, \quad \lim_{k\to\infty} \frac{(\lambda T^{(k)})^{1/2}(\theta^{(k)})^{(N+1)/2}}{(\ln(x_0^{(k)}+1))^{\alpha}} = \infty, \quad \lim_{k\to\infty} \eta_n^{(k)} = 0$$

for all $n = 1,\dots,N$ where N satisfies the condition in Theorem 4. The first one is obvious. The second one follows since the numerator is on the order of

$$k^{\frac{1}{2}\frac{1}{N+2}} \ \ge\ (\ln(k))^{2\alpha}$$

(by the condition on N in Theorem 4). As for the last limit, it is the consequence of the first two limits. To see this, note that

$$\begin{aligned}
\eta_n &= \frac{(\ln(x_0+1))^\alpha}{(\lambda T)^{1/2}} \cdot \left[\frac{T^{1/2}}{(t_{n+1}-t_n)^{1/2}} \mathbf{1}\{n<N\} - \sum_{m=0}^{n-1} \frac{T^{1/2}(t_{m+1}-t_m)^{1/2}}{T-t_{m+1}} \right] \\
&= \frac{(\ln(x_0+1))^\alpha}{(\lambda T)^{1/2}} \cdot \left[\frac{1}{\theta^{n/2}(1-\theta)^{1/2}} \mathbf{1}\{n<N\} - \sum_{m=0}^{n-1} \frac{(1-\theta)^{1/2}}{\theta^{(m+2)/2}} \right] \\
&= \frac{(\ln(x_0+1))^\alpha}{(\lambda T)^{1/2}} \cdot \left[\frac{1}{\theta^{n/2}(1-\theta)^{1/2}} \mathbf{1}\{n<N\} - \frac{(1-\theta)^{1/2}}{\theta} \left(\frac{1}{\theta^{n/2}}-1\right)\left(\frac{1}{\theta^{1/2}}-1\right)^{-1} \right] \\
&= \frac{(\ln(x_0+1))^\alpha}{(\lambda T)^{1/2}} \cdot \left[\frac{1}{\theta^{n/2}(1-\theta)^{1/2}} \mathbf{1}\{n<N\} - \frac{(1-\theta)^{1/2}}{1-\theta^{1/2}} \frac{1-\theta^{n/2}}{\theta^{(n+1)/2}} \right].
\end{aligned}$$

It is not difficult to see that the first two limits above imply $\lim_{k\to\infty} \eta_n^{(k)} = 0$.

Let $\hat\beta \triangleq \min\{\rho^d, 1-\rho^d\}/2$. (Note that $\hat\beta \in (0,1)$.) Based on our limit results above, we conclude there exists a constant $K>0$ such that, for all $k>K$, we have:

$$\begin{aligned}
\frac{(\lambda T^{(k)})^{1/2}}{(\ln(x_0^{(k)}+1))^\alpha}\left(1-(\theta^{(k)})^{1/2}\right)(\theta^{(k)})^{(N-1)/2} &\geq (1-(\theta^{(k)})^{1/2})^2, \\
(1-(\theta^{(k)})^{1/2})^2 &\geq 1/2, \\
\lambda T^{(k)}(\theta^{(k)})^{N+1} &\geq \frac{0.2}{\hat\beta^2}\cdot(\ln(x_0^{(k)}+1))^{2\alpha}, \\
\rho^d - \eta_1^{(k)} &\geq \hat\beta, \\
1-\rho^d+\eta_N^{(k)} &\geq \hat\beta,
\end{aligned}$$

which yields the final bound for the probability of bad event

$$\mathbb{P}(\mathcal{G}^c) \leq 3N\exp\left(-0.05\left(\ln(x_0^{(k)}+1)\right)^{2\alpha}\right).$$

C: Useful Auxiliary Results

LEMMA EC.5. *Let $\Lambda \in (0,\lambda]$ be a random variable. Let X be a random variable such that, conditional on Λ, X follows a Poisson distribution with parameter Λ. Then, for any $\epsilon>0$,*

$$\mathbb{P}(X-\Lambda \geq \epsilon) \leq \exp\left(-0.1\epsilon\min\left\{\frac{\epsilon}{\lambda},1\right\}\right).$$

Proof of Lemma EC.5.

For any $s>0$, we have

$$\mathbb{P}(X-\Lambda\geq\epsilon) = \mathbb{P}(\exp(s(X-\Lambda)) \geq \exp(s\epsilon))$$

$$
\begin{aligned}
&\leq \mathsf{E}\left[\exp\left(s\left(X-\Lambda\right)\right)\right]\exp\left(-s\epsilon\right) \\
&= \mathsf{E}\left[\mathsf{E}\left[\exp\left(s\left(X-\Lambda\right)\right)|\Lambda\right]\right]\exp\left(-s\epsilon\right) \\
&= \mathsf{E}\left[\exp\left(\Lambda\left(e^s-1-s\right)\right)\right]\exp\left(-s\epsilon\right) \\
&\leq \exp\left(\lambda\left(e^s-1-s\right)\right)\exp\left(-s\epsilon\right) \\
&= \exp\left(\lambda\left(e^s-1-s\right)-s\epsilon\right),
\end{aligned}
$$

where the first inequality follows from Markov's inequality, the second equality follows from the law of total expectation, the second inequality follows from the property that for $x\geq 0$, $e^x-1-x\geq 0$ and the condition that $\Lambda\leq\lambda$.

We take $s=\min\left\{\frac{\epsilon}{\lambda},1\right\}$. Consider first the scenario where $\epsilon\leq\lambda$. We have

$$
\begin{aligned}
\lambda\left(e^s-1-s\right)-s\epsilon &= \lambda\left(e^{\epsilon/\lambda}-1-\frac{\epsilon}{\lambda}\right)-\frac{\epsilon^2}{\lambda} \\
&\leq 0.9\frac{\epsilon^2}{\lambda}-\frac{\epsilon^2}{\lambda} \\
&= -0.1\frac{\epsilon^2}{\lambda},
\end{aligned}
$$

where the inequality follows since $e^x-1-x\leq 0.9x^2$ for $x\in[-1,1]$.

Consider now the scenario where $\epsilon>\lambda$. We have

$$
\begin{aligned}
\lambda\left(e^s-1-s\right)-s\epsilon &= \lambda\left(e-2\right)-\epsilon \\
&\leq 0.9\lambda-\epsilon \\
&\leq 0.9\epsilon-\epsilon \\
&= -0.1\epsilon.
\end{aligned}
$$

The two scenarios above jointly imply

$$
\mathbb{P}\left(X-\Lambda\geq\epsilon\right)\leq\exp\left(-0.1\epsilon\min\left\{\frac{\epsilon}{\lambda},1\right\}\right).
$$

■

D: Additional Simulations

We consider both logit demand model $\bar{F}(p)=2/(1+e^{ap})$ and exponential demand model $\bar{F}(p)=e^{-ap}$. We use $\lambda=1$ and $T=1$. We test the performance of four policies (similar to those in Tables 1-3) under different parameter values. The results can be seen in Table EC.1 and EC.2.

a	$x_0/(\lambda T)$	k	FP	MD $N=1$	MD $N=2$	MD $N=3$
0.5	0.1	200	14.26	9.90	9.06	8.58
		300	11.10	7.89	6.79	6.60
		400	9.98	6.69	5.99	5.59
		500	8.45	5.86	5.19	4.88
		1000	6.21	3.80	3.41	3.07
0.5	0.3	200	7.83	6.09	5.97	5.92
		300	6.46	4.73	4.60	4.66
		400	5.61	4.20	3.97	3.90
		500	5.07	3.58	3.52	3.48
		1000	3.38	2.40	2.34	2.27
1	0.1	200	13.89	9.95	9.25	8.57
		300	10.80	8.04	7.13	6.59
		400	9.63	6.63	6.03	5.73
		500	8.55	5.84	5.06	4.83
		1000	6.41	3.94	3.28	3.23
1	0.3	200	7.88	6.15	5.95	5.77
		300	6.15	4.91	4.73	4.73
		400	5.39	4.19	4.12	4.03
		500	4.98	3.60	3.48	3.42
		1000	3.48	2.41	2.32	2.22

Table EC.1 Percentage losses of different policies under logit demand.

a	$x_0/(\lambda T)$	k	FP	MD $N=1$	MD $N=2$	MD $N=3$
0.5	0.1	200	14.07	11.64	10.57	10.07
		300	12.01	9.00	8.43	7.96
		400	10.66	7.52	6.89	6.60
		500	9.00	6.90	6.11	5.98
		1000	6.49	4.58	3.96	3.69
0.5	0.3	200	8.51	7.23	6.92	6.72
		300	7.05	5.55	5.53	5.45
		400	6.21	4.84	4.65	4.54
		500	5.26	4.31	4.08	3.91
		1000	3.82	2.83	2.73	2.63
1	0.1	200	14.45	11.61	10.59	10.27
		300	11.99	8.93	8.32	7.75
		400	10.60	8.02	6.81	6.77
		500	9.16	6.84	6.04	5.80
		1000	6.72	4.48	3.81	3.52
1	0.3	200	8.51	7.25	6.95	6.73
		300	7.26	5.72	5.64	5.56
		400	6.13	4.84	4.76	4.74
		500	5.30	4.20	4.14	4.04
		1000	3.79	2.94	2.74	2.68

Table EC.2 Percentage losses of different policies under exponential demand.