Proofs of Statements

We provide proofs for important results in this electronic companion.

EC.1. Section 2 Results

COROLLARY 2 *For any β such that SDP(β) has an optimal solution, there exists an optimal solution ϕ^* of SDP(β) such that $\phi^*(\tilde{\beta}) = z_{IP}(\tilde{\beta})$ for all $\tilde{\beta}$ satisfying $S(\tilde{\beta}) \neq \emptyset$.*

Proof: Let ϕ^* be an extension of z_{IP} satisfying the claim of Proposition 1. Because the value function of a mixed-integer program is superadditive and nondecreasing (Jeroslow 1979), ϕ^* is superadditive and nondecreasing. The function ϕ^* is an extension of z_{IP}, so $\phi^*(a^j) \geq c_j$ for all $j \in \{1, ..., n\}$, which proves feasibility. Optimality is immediate. □

PROPOSITION 2 *Suppose $A \in \mathbb{Z}_+^{m\times n}$ and $b \in \mathbb{Z}_+^m$. The truncated value function $z_{IP} \in \mathbb{R}^{|\widehat{\mathcal{B}}(0,b)|}$ is in the polyhedron $\Phi(b)$.*

One can prove Proposition 2 directly by verifying z_{IP} satisfies all of the constraints that define $\Phi(b)$.

THEOREM 4 *Suppose $A \in \mathbb{Z}_+^{m\times n}$ and $b \in \mathbb{Z}_+^m$. For each $\beta \in \widehat{\mathcal{B}}(0,b)$, SDP2($\beta$) is a strong dual to IP(β).*

Proof: We embed the feasible region $\Phi(\beta)$ in $\mathbb{R}^{|\widehat{\mathcal{B}}(0,b)|}$. Let $\Phi_{emb}(\beta) = \{\phi \in \mathbb{R}^{|\widehat{\mathcal{B}}(0,b)|} \mid \phi$ satisfies (2a) – (2d) with respect to $\beta\}$. Observe that $\Phi(b) \subseteq \Phi_{emb}(\beta)$, for all $\beta \in \widehat{\mathcal{B}}(0,b)$, and for any $\phi \in \Phi(\beta)$, there exists $\phi_{emb} \in \Phi_{emb}(\beta)$ such that $\phi_{emb}(\beta') = \phi(\beta')$, for all $\beta' \in \widehat{\mathcal{B}}(0,\beta)$. Furthermore, because A and β are nonnegative, IP(β) is feasible, and by Assumption **A2**, IP(β) is bounded.

Define $\phi^* \in \mathbb{R}^{|\widehat{\mathcal{B}}(0,b)|}$ such that $\phi^*(\beta') = z_{IP}(\beta')$, for all $\beta' \in \widehat{\mathcal{B}}(0,b)$. Then $\phi^* \in \Phi(b)$, which implies that $\phi^* \in \Phi_{emb}(\beta)$. Define the truncated vector $\tilde{\phi} \in \mathbb{R}^{|\widehat{\mathcal{B}}(0,\beta)|}$ such that $\tilde{\phi}(\beta') = \phi^*(\beta')$, for all $\beta' \in \widehat{\mathcal{B}}(0,\beta)$. Then $\tilde{\phi}(\beta) = \phi^*(\beta) = z_{IP}(\beta)$ and $\tilde{\phi}$ satisfies (2a)-(2d) with respect to β, which implies that $\tilde{\phi} \in \Phi(\beta)$. Hence, $\tilde{\phi}$ is an optimal solution for SDP2(β). □

COROLLARY 3 *Suppose $A \in \mathbb{Z}_+^n$ and $b \in \mathbb{Z}_+^m$. For any $\phi \in \Phi(b), \phi(\beta) \geq z_{IP}(\beta)$, for all $\beta \in \widehat{\mathcal{B}}(0,b)$.*

The proof of Corollary 3 follows from Theorem 4 and is omitted.

Proposition 3

(I) For any $\beta \in \mathcal{B}(\tilde{b}, b)$, there exists a $\tilde{\beta} \in D^+$ with $\tilde{\beta} \geq \beta$ such that $z_{IP}(\tilde{\beta}) = z_{IP}(\beta)$.

(II) For any $\beta \in \mathcal{B}(\tilde{b}, b)$, there exists a $\tilde{\beta} \in D^-$ with $\tilde{\beta} \leq \beta$ such that $z_{IP}(\tilde{\beta}) = z_{IP}(\beta)$.

Proof: We prove only (I) as (II) is analogous. If $\beta \in D^+$, the proof is trivial. Suppose $\beta \in \mathcal{B}(\tilde{b}, b)$ is not level-set maximal. Let $s^0 = 0$ and $\beta^0 = \beta$. Perform the following improvement procedure: for each $i \in \{1, ..., m\}$, let $s^i = \max\{s \in \mathbb{Z}_+ \mid \beta^{i-1} + s \cdot e_i \in \mathcal{B}(\tilde{b}, b), z_{IP}(\beta^{i-1} + s \cdot e_i) = z_{IP}(\beta)\}$ and $\beta^i = \beta^{i-1} + s^i \cdot e_i$. At the end of each improvement procedure (which consists of a finite number of iterations), $z_{IP}(\beta^m) = z_{IP}(\beta^0) = z_{IP}(\beta)$, and either $\beta^m = \beta^0$ or $\beta^m \neq \beta^0$.

Suppose $\beta^m \neq \beta^0$. Then $\beta^m \geq \beta^0$ and $\beta^m_i \geq \beta^0_i + 1$ for some $i \in \{1, ..., m\}$. Because $\mathcal{B}(\tilde{b}, b)$ is a finite set, this case can only occur finitely often. Suppose $\beta^m = \beta^0$. By definition, β^m is level-set maximal because there does not exist $i \in \{1, ..., m\}$ such that $z_{IP}(\beta^m + e_i) = z_{IP}(\beta^m)$. Hence, we conclude that after finitely many applications of the improvement procedure, $\beta^m = \beta^0$ and β^m is level-set maximal with $z_{IP}(\beta) = z_{IP}(\beta^m)$ and $\beta^m \geq \beta$. □

Theorem 5 *Level-set-minimal and level-set-maximal vectors are sufficient to bound the gap function. That is,*

(I) $\inf_{\beta \in \mathcal{B}(\tilde{b}, b)} \Gamma(\beta) = \inf_{\beta \in D^-} \Gamma(\beta)$.

(II) $\sup_{\beta \in \mathcal{B}(\tilde{b}, b)} \Gamma(\beta) = \sup_{\beta \in D^+} \Gamma(\beta)$.

(III) $\inf_{\beta \in \mathcal{B}^+(\tilde{b}, b)} \gamma(\beta) = \inf_{\beta \in D^+ \cap \mathcal{B}^+(\tilde{b}, b)} \gamma(\beta)$.

(IV) $\sup_{\beta \in \mathcal{B}^+(\tilde{b}, b)} \gamma(\beta) = \sup_{\beta \in D^- \cap \mathcal{B}^+(\tilde{b}, b)} \gamma(\beta)$.

Proof: We prove only (I) as the other proofs are similar. Suppose $\tilde{\beta} \in \mathcal{B}(\tilde{b}, b) \backslash D^-$. By Proposition 3, there exists $\bar{\beta} \in D^-$ such that $z_{IP}(\bar{\beta}) = z_{IP}(\tilde{\beta})$ and $\bar{\beta} \leq \tilde{\beta}$. By the monotonicity of z_{LP}, $\Gamma(\tilde{\beta}) \geq \Gamma(\bar{\beta}) \geq \inf_{\beta \in D^-} \Gamma(\beta)$. Hence, $\inf_{\beta \in \mathcal{B}(\tilde{b}, b)} \Gamma(\beta) \geq \inf_{\beta \in D^-} \Gamma(\beta)$. Finally, we have that $D^- \subseteq \mathcal{B}(\tilde{b}, b)$, which implies $\inf_{\beta \in D^-} \Gamma(\beta) \geq \inf_{\beta \in \mathcal{B}(\tilde{b}, b)} \Gamma(\beta)$ and proves the equality. □

Theorem 6 *The absolute gap function defined over rational vectors, $\Gamma : \mathcal{B}(\tilde{b}, b) \cap \mathbb{Q}^m \to \mathbb{R}$, is the minimum of finitely many Gomory functions.*

Proof: It is known that $-z_{IP}(\beta) = \min\{-c^T x \mid Ax \leq \beta, x \in \mathbb{Z}^n_+\}$ is a Gomory function when defined over rational vectors (Blair and Jeroslow 1982). Also, $z_{LP}(\beta) = \min\{(v^k)^T\beta \mid k \in \mathcal{K}\}$. For each $k \in \mathcal{K}$, let $L_k : \mathcal{B}(\tilde{b}, b) \cap \mathbb{Q}^m \to \mathbb{R}$, where $L_k(\beta) = (v^k)^T\beta - z_{IP}(\beta)$. Notice that for each $k \in \mathcal{K}, L_k$ is a Gomory function, as it is the sum of two Gomory functions. Further, $\Gamma(\beta) = \min\{(v^k)^T\beta \mid k \in \mathcal{K}\} - z_{IP}(\beta) = \min\{L_k(\beta) \mid k \in \mathcal{K}\}$, for all $\beta \in \mathcal{B}(\tilde{b}, b) \cap \mathbb{Q}^m$. □

PROPOSITION 4 *For any $\beta \in \mathcal{B}(\tilde{b}, b)$, if $\hat{x} \in opt_{LP}(\beta)$, then for all x such that $0 \leq x \leq \hat{x}$, $x \in opt_{LP}(\tilde{\beta})$, for all $\tilde{\beta} \in \Lambda(\beta, x, \hat{x})$.*

Proof: From Assumption **A2**, $|z_{LP}(\beta')| < \infty$, for all $\beta' \in \mathcal{B}(\tilde{\beta}, \beta)$. Hence, $\text{opt}_{LP}(\beta')$ and $\text{opt}_{DLP}(\beta')$ are nonempty, for all $\beta' \in \mathcal{B}(\tilde{\beta}, \beta)$. Let $v \in \text{opt}_{DLP}(\beta)$ and $\hat{x} \in \text{opt}_{LP}(\beta)$. Consider $x \in \mathbb{R}^n$ such that $0 \leq x \leq \hat{x}$. For any $\beta' \in \mathbb{R}^m$, let $M^<_{\beta'} = \{i \in \{1, ..., m\} \mid a_i^T\hat{x} < \beta'_i\}$. By linear programming complementary slackness:

(I) $v_i = 0$ for all $i \in M^<_\beta$.

(II) $\hat{x}_j > 0 \Rightarrow (a^j)^T v = c_j$.

We show that v also satisfies these two conditions when x replaces $\hat{x}$ and $\beta - A(\hat{x} - x)$ replaces β in (I) and (II). For condition (II), notice $\{j : x_j > 0\} \subseteq \{j : \hat{x}_j > 0\}$, since $x_j \leq \hat{x}_j$. This implies $(a^j)^T v = c_j$, for all j such that $x_j > 0$, which proves condition (II). Furthermore, $M^<_{\beta - A(\hat{x} - x)} = M^<_\beta$, since $a_i^T\hat{x} < \beta_i$ if and only if $a_i^T x < \beta_i - a_i^T(\hat{x} - x)$. This implies condition (I) holds, so $v \in \text{opt}_{DLP}(\beta - A(\hat{x} - x))$. By assumption, $v \in \text{opt}_{DLP}(\beta)$, thus, $\beta^T v = c^T\hat{x}$. Hence, complementary slackness implies that $0 = (\beta_i - a_i^T\hat{x})v_i = (\beta_i - a_i^T(\hat{x} - x) - a_i^T x)v_i$. The vector x is feasible for $\text{LP}(\beta - A(\hat{x} - x))$. To see this, first note that $x \in \mathbb{R}^n_+$. Also $\beta - A\hat{x} \geq 0$, which implies $Ax \leq Ax + \beta - A\hat{x} = \beta - A(\hat{x} - x)$. By complementary slackness (primal solution x, dual solution v), $x \in \text{opt}_{LP}(\beta - A(\hat{x} - x))$. By monotonicity, $x \in \text{opt}_{LP}(\tilde{\beta})$, for any $\tilde{\beta} \in \Lambda(\beta, x, \hat{x})$. □

COROLLARY 4 *Let $\beta, x, \hat{x}$ satisfy the conditions in Proposition 4. Suppose further that $x \in \mathbb{Z}^n_+$. Then for any $\tilde{\beta} \in \Lambda(\beta, x, \hat{x})$, $\Gamma(\tilde{\beta}) = 0$.*

Proof: By Proposition 4, $x \in \text{opt}_{LP}(\tilde{\beta})$. Now, $x \in \mathbb{Z}^n_+$, thus $x \in S(\tilde{\beta})$ and $c^T x \leq z_{IP}(\tilde{\beta})$. Therefore, $c^T x = z_{LP}(\tilde{\beta}) \geq z_{IP}(\tilde{\beta}) \geq c^T x$. Thus, $z_{LP}(\tilde{\beta}) = z_{IP}(\tilde{\beta})$, and $\Gamma(\tilde{\beta}) = 0$. □

LEMMA EC.1. (Nemhauser and Wolsey 1988) *Suppose* $\hat{x} \in opt_{IP}(\beta)$, *for some* $\beta \in \mathcal{B}(\tilde{b}, b)$ *and* ϕ^* *is an optimal solution to SDP*(β)*. Then* $\phi^*(Ax) = c^T x$ *and* $\phi^*(Ax) + \phi^*(\beta - Ax) = \phi^*(\beta)$, *for all* $x \in \mathbb{Z}^n_+$ *such that* $x \leq \hat{x}$.

PROPOSITION 5 *Let* $\beta \in \mathcal{B}(\tilde{b}, b)$*. For all* $j \in \{1, ..., n\}$ *such that there exists* $x^I \in opt_{IP}(\beta)$ *and* $x^L \in opt_{LP}(\beta)$ *with* $x^I_j \geq 1$ *and* $x^L_j \geq 1$, *we have* $z_{LP}(\beta - a^j) = z_{LP}(\beta) - c_j$, *and* $z_{IP}(\beta - a^j) = z_{IP}(\beta) - c_j$.

Proof: Let e_j be the j^{th} standard basis vector in $\mathbb{R}^n$, which implies $e_j \in \mathbb{Z}^n_+$. By hypothesis, $e_j \leq x^I$ and $e_j \leq x^L$. Recall that z_{IP} can be extended to an optimal solution ϕ^* of SDP(β) (Corollary 2) so that $\phi^*(\beta) = z_{IP}(\beta)$, for all β such that $S(\beta) \neq \emptyset$. By Lemma $EC.1$, $z_{IP}(\beta - Ae_j) = z_{IP}(\beta - a^j) = \phi^*(\beta - a^j) = \phi^*(\beta) - \phi^*(a^j) = z_{IP}(\beta) - z_{IP}(a^j)$.

Observe that $\beta - A(x^L - e_j) = a^j + \beta - Ax^L \geq a^j$. Thus, $a^j \in \Lambda(\beta, e_j, x^L)$. By Proposition 4, $e_j \in \text{opt}_{LP}(a^j)$. This implies that $z_{LP}(a^j) = c_j$. Because $e_j \in S(a^j), z_{IP}(a^j) = c_j$. Therefore, $z_{IP}(\beta - a^j) = z_{IP}(\beta) - c_j$.

By the superadditivity of z_{LP}, $z_{LP}(\beta - a^j) \leq z_{LP}(\beta) - z_{LP}(a^j) = z_{LP}(\beta) - c_j$. Observe that $x^L - e_j \in P(\beta - a^j)$. Therefore, $z_{LP}(\beta - a^j) \geq c^T(x^L - e_j) = z_{LP}(\beta) - c_j$. This implies $z_{LP}(\beta - a^j) = z_{LP}(\beta) - c_j$. □

THEOREM 7 *Under the conditions of Proposition* 5, $\Gamma(\beta - a^j) = \Gamma(\beta)$*. If, in addition,* $z_{LP}(\beta) > c_j$ *and* $z_{IP}(\beta) \geq c_j$, *then* $\gamma(\beta - a^j) = (z_{IP}(\beta) - c_j)/(z_{LP}(\beta) - c_j)$.

We omit the proof of Theorem 7 as it follows from Proposition 5.

THEOREM 8 *Let* $\beta \in \widehat{\mathcal{B}}(0, b)$ *and suppose* $A \in \mathbb{Z}^{m \times n}_+$ *and* $b \in \mathbb{Z}^m_+$*. Suppose there exists an* $x \in opt_{IP}(\beta)$ *such that* $x_j \geq 1$ *for some* $j \in \{1, ..., n\}$*. Then for any optimal solution* ϕ^* *to SDP2*(β), $\phi^*(\beta - a^j) = z_{IP}(\beta - a^j)$.

Proof: By an argument similar to that shown in the proof of Proposition 5, $z_{IP}(\beta - a^j) = z_{IP}(\beta) - z_{IP}(a^j)$. Because $(\beta - a^j) + a^j = \beta \in \widehat{\mathcal{B}}(0, b)$ and ϕ^* is an optimal solution to SDP2(β),

$$\phi^*(\beta - a^j) + \phi^*(a^j) \leq \phi^*(\beta)$$

$$= z_{IP}(\beta)$$

$$\iff \phi^*(\beta - a^j) \le z_{IP}(\beta) - \phi^*(a^j).$$

The first inequality comes from the superadditive constraints in SDP2(β), and the equality is due to SDP2(β) being a strong dual (Theorem 4). By Corollary 3, $\phi^*(a^j) \ge z_{IP}(a^j)$, which implies that

$$\phi^*(\beta - a^j) \le z_{IP}(\beta) - z_{IP}(a^j)$$

$$= z_{IP}(\beta - a^j).$$

By Corollary 3, $\phi^*(\beta - a^j) \ge z_{IP}(\beta - a^j)$, which proves the desired equality. □

EC.2. Section 3 Results

PROPOSITION 6 *The optimal objective value of* (3) *is* $\mathbb{E}_{\xi^{(1)}}[\Gamma(\xi^{(1)})]$.

Proof: Let $(\tilde{\phi}, \tilde{\psi}) = (z_{IP}, \Gamma)$, the truncated value function and the truncated gap function. Observe that $\tilde{\phi} \in \Phi(b)$ (Proposition 2). Consider $\beta \in \widehat{\mathcal{B}}(0,b)$. Then $\tilde{\psi}(\beta) = \Gamma(\beta) = z_{LP}(\beta) - z_{IP}(\beta) \le \beta^T v^k - \tilde{\phi}(\beta)$, for all $k \in \mathcal{K}$. Hence the solution is feasible.

Let $(\bar{\phi}, \bar{\psi})$ be a feasible solution for (3). Because $\bar{\phi}$ satisfies (3c), by Corollary 3, $\bar{\phi}(\beta) \ge z_{IP}(\beta)$, for all $\beta \in \widehat{\mathcal{B}}(0,b)$. By (3a), for each $\beta \in \widehat{\mathcal{B}}(0,b)$, $\bar{\psi}(\beta) \le \beta^T v^k - \bar{\phi}(\beta) \le \beta^T v^k - z_{IP}(\beta)$, for all $k \in \mathcal{K}$. This implies that $\bar{\psi}(\beta) \le z_{LP}(\beta) - z_{IP}(\beta) = \tilde{\psi}(\beta)$ and $\sum_{\beta \in \widehat{\mathcal{B}}(0,b)} \mu^{(1)}(\beta)\bar{\psi}(\beta) \le \sum_{\beta \in \widehat{\mathcal{B}}(0,b)} \mu^{(1)}(\beta)\tilde{\psi}(\beta)$.

Thus, $(\tilde{\phi}, \tilde{\psi})$ is an optimal solution to (3), and the optimal objective value is $\mathbb{E}_{\xi^{(1)}}[\Gamma(\xi^{(1)})]$. □

COROLLARY 5 *Suppose* $\mu^{(1)}(\beta) > 0$ *for all* $\beta \in \widehat{\mathcal{B}}(0,b)$. *Then, any feasible solution* $(\bar{\phi}, \bar{\psi})$ *in which* $\bar{\phi} \ne z_{IP}$ *is not optimal for* (3).

One can prove Corollary 5 by noticing that every component of an optimal solution ϕ^* of (3) must be as small as possible. We omit a formal proof.

COROLLARY 6 *Let* $(\tilde{\phi}, \tilde{\psi})$ *be an optimal solution to* (3) *and* δ_{EAD} *the optimal objective value. Then the variance of the absolute gap over* $\widehat{\mathcal{B}}(0,b)$ *can be calculated as follows:*

$$\sigma_\Gamma^2 = \sum_{\beta \in \widehat{\mathcal{B}}(0,b)} \mu^{(1)}(\beta)(\tilde{\psi}(\beta))^2 - \delta_{EAD}^2.$$

Proof: For any $\beta \in \widehat{\mathcal{B}}(0,b)$ such that $\mu^{(1)}(\beta) > 0, \tilde{\phi}(\beta) = z_{IP}(\beta)$; otherwise, $(\tilde{\phi},\tilde{\psi})$ is not an optimal solution. It follows that for all such β, $\tilde{\psi}(\beta) = \Gamma(\beta)$. Hence, $\sum_{\beta\in\widehat{\mathcal{B}}(0,b)} \mu^{(1)}(\beta)(\tilde{\psi}(\beta))^2 = \sum_{\beta:\mu^{(1)}(\beta)>0} \mu^{(1)}(\beta)(\tilde{\psi}(\beta))^2 = \sum_{\beta:\mu^{(1)}(\beta)>0} \mu^{(1)}(\beta)(\Gamma(\beta))^2 = \mathbb{E}_{\xi^{(1)}}\left[(\Gamma(\xi^{(1)}))^2\right]$. Therefore, $\sum_{\beta\in\widehat{\mathcal{B}}(0,b)} \mu^{(1)}(\beta)(\tilde{\psi}(\beta))^2 - \delta_{EAD}^2 = \mathbb{E}_{\xi^{(1)}}\left[(\Gamma(\xi^{(1)}))^2\right] - \mathbb{E}_{\xi^{(1)}}\left[\Gamma(\xi^{(1)})\right]^2 = \sigma_\Gamma^2$. □

PROPOSITION 7 *The optimal objective value of* (4) *is* Δ_{SAD}.

Proof: Let $\tilde{\beta} \in \arg\max\{\Gamma(\beta) \mid \beta \in \widehat{\mathcal{B}}(0,b)\}$. Let $\tilde{\phi} = z_{IP}$, $\tilde{\psi}(\tilde{\beta}) = \Gamma(\tilde{\beta})$, and $\tilde{\psi}(\beta) = 0$ for all $\beta \in \widehat{\mathcal{B}}(0,b)\backslash\tilde{\beta}$. It is immediate that $\tilde{\phi} \in \Phi(b)$ (Proposition 2). By construction, $\tilde{\psi}$ satisfies (4b). Now, $\tilde{\psi}(\tilde{\beta}) = \Gamma(\tilde{\beta})$, and $\tilde{\psi}(\beta) = 0 \le \Gamma(\beta)$, for all $\beta \in \widehat{\mathcal{B}}(0,b)\backslash\tilde{\beta}$. Thus, $\tilde{\psi}(\beta) \le \Gamma(\beta) = z_{LP}(\beta) - z_{IP}(\beta) \le \beta^T v^k - \tilde{\phi}(\beta)$, for all $k \in \mathcal{K}$.

Let $(\bar{\phi},\bar{\psi})$ be feasible for (4), where $\bar{\psi}(\beta) = 0$ for all $\beta \in \widehat{\mathcal{B}}(0,b)\backslash\bar{\beta}$, for some $\bar{\beta} \in \widehat{\mathcal{B}}(0,b)$. Because $\bar{\phi}$ satisfies (4d), by Corollary 3, $\bar{\phi}(\beta) \ge z_{IP}(\beta)$ for all $\beta \in \widehat{\mathcal{B}}(0,b)$. By (4a), $\bar{\psi}(\bar{\beta}) \le z_{LP}(\bar{\beta}) - \bar{\phi}(\bar{\beta}) \le z_{LP}(\bar{\beta}) - z_{IP}(\bar{\beta}) = \Gamma(\bar{\beta})$. Because $\tilde{\beta} \in \arg\max\{\Gamma(\beta) \mid \beta \in \widehat{\mathcal{B}}(0,b)\}, \bar{\psi}(\bar{\beta}) = \Gamma(\bar{\beta}) \le \Gamma(\tilde{\beta}) = \tilde{\psi}(\tilde{\beta})$, which implies $\sum_{\beta\in\widehat{\mathcal{B}}(0,b)} \bar{\psi}(\beta) \le \sum_{\beta\in\widehat{\mathcal{B}}(0,b)} \tilde{\psi}(\beta) = \max\{\Gamma(\beta) \mid \beta \in \widehat{\mathcal{B}}(0,b)\} = \Delta_{SAD}$. □

PROPOSITION 8 *The optimal objective value of* (5) *is* Δ_{IAD}.

Proof: For each $\beta \in \widehat{\mathcal{B}}(0,b)$, let $\tilde{\phi}(\beta) = z_{IP}(\beta)$. Let $\tilde{\psi} = \Delta_{IAD}$. It is immediate that $\tilde{\phi} \in \Phi(b)$ (Proposition 2). Further, $\tilde{\psi} = \Delta_{IAD} \le \Gamma(\beta) \le \beta^T v^k - z_{IP}(\beta) = \beta^T v^k - \tilde{\phi}(\beta)$, for all $\beta \in \widehat{\mathcal{B}}(0,b)\backslash\{0\}$, $k \in \mathcal{K}$. Thus, $(\tilde{\phi},\tilde{\psi})$ is feasible for (5).

Let $(\bar{\phi},\bar{\psi})$ be feasible for (5). Recall that for all $\beta \in \widehat{\mathcal{B}}(0,b)$, by Corollary 3, $z_{IP}(\beta) \le \bar{\phi}(\beta)$. By (5a), for all $\beta \in \widehat{\mathcal{B}}(0,b)\backslash\{0\}$ and $k \in \mathcal{K}$, $\bar{\psi} \le \beta^T v^k - \bar{\phi}(\beta)$, which implies $\bar{\psi} \le z_{LP}(\beta) - z_{IP}(\beta) = \Gamma(\beta)$. Hence, $\bar{\psi} \le \Delta_{IAD} = \tilde{\psi}$. This proves that $(\tilde{\phi},\tilde{\psi})$ is optimal for (5). □

EC.3. Section 4 Results

PROPOSITION 9 *The optimal objective value of* (6) *is* $\delta_{EAH} = \mathbb{E}_{\xi^{(p)}}[\Gamma(\xi^{(p)})]$.

Proof: The proof is similar to that of Proposition 6; however, we present a proof to illustrate the new details introduced by the use of $\widehat{\mathcal{B}}_p(0,b)$. Let $(\tilde{\phi},\tilde{\psi}) = (z_{IP},\Gamma)$, where $\tilde{\psi} \in \mathbb{R}^{|\widehat{\mathcal{B}}_p(0,b)|}$. Observe that

$\tilde{\phi} \in \Phi(b)$ (Proposition 2). Consider $\beta \in \widehat{\mathcal{B}}_p(0,b)$. Then $\tilde{\psi}(\beta) = \Gamma(\beta) = z_{LP}(\beta) - z_{IP}(\beta) = z_{LP}(\beta) - z_{IP}(\lfloor\beta\rfloor) \le \beta^T v^k - \tilde{\phi}(\lfloor\beta\rfloor)$, for all $k \in \mathcal{K}$. Hence the solution is feasible.

Let $(\bar{\phi}, \bar{\psi})$ be a feasible solution for (6). Because $\bar{\phi}$ satisfies (6c), by Corollary 3, $\bar{\phi}(\beta) \ge z_{IP}(\beta)$, for all $\beta \in \widehat{\mathcal{B}}(0,b)$. By (6a), for each $\beta \in \widehat{\mathcal{B}}_p(0,b)$, $\bar{\psi}(\beta) \le \beta^T v^k - \bar{\phi}(\lfloor\beta\rfloor) \le \beta^T v^k - z_{IP}(\beta)$, for all $k \in \mathcal{K}$. This implies that $\bar{\psi}(\beta) \le z_{LP}(\beta) - z_{IP}(\beta) = \tilde{\psi}(\beta)$ and $\sum_{\beta \in \widehat{\mathcal{B}}(0,b)} \mu^{(p)}(\beta)\bar{\psi}(\beta) \le \sum_{\beta \in \widehat{\mathcal{B}}(0,b)} \mu^{(p)}(\beta)\tilde{\psi}(\beta)$.

Thus, $(\tilde{\phi}, \tilde{\psi})$ is an optimal solution to (6), and the optimal objective value is $\mathbb{E}_{\xi^{(p)}}[\Gamma(\xi^{(p)})]$. □

Corollary 7 *Suppose $\mu^{(p)}(\beta) > 0$ for all $\beta \in \widehat{\mathcal{B}}_p(0,b)$. Then, any feasible solution $(\bar{\phi}, \bar{\psi})$ in which $\bar{\phi} \ne z_{IP}$ is not optimal for (6).*

We omit the proof of Corollary 7 due to reasoning similar to that of Corollary 5.

Corollary 8 *Let $(\tilde{\phi}, \tilde{\psi})$ be an optimal solution to (6) and δ_{EAH} the optimal objective value. Then the variance of the absolute gap over $\widehat{\mathcal{B}}_p(0,b)$ can be calculated as follows:*

$$\sigma_\Gamma^2 = \sum_{\beta \in \widehat{\mathcal{B}}_p(0,b)} \mu^{(p)}(\beta)(\tilde{\psi}(\beta))^2 - \delta_{EAH}^2.$$

The proof of Corollary 8 is similar to that of Corollary 6 and is omitted.

Lemma EC.2. *For any $\beta \in \mathcal{B}(0,b)$:*

(I) $||\beta - \lfloor\beta\rceil^{(p)}||_\infty \le 1/p$.

(II) $z_{IP}(\lfloor\beta\rceil^{(p)}) = z_{IP}(\beta)$.

(III) $\Gamma(\beta) \ge \Gamma(\lfloor\beta\rceil^{(p)})$.

Proof: Let $\beta \in \mathcal{B}(0,b)$. For any $i \in \{1,...,m\}$, $|\beta_i - (\lfloor\beta\rceil^{(p)})_i| = \beta_i - (\lfloor\beta\rceil^{(p)})_i = \frac{\alpha_i - \lfloor\alpha_i\rfloor}{p} \le 1/p$, where $\beta = \alpha/p$. This proves (I). Also, $\lfloor\beta\rfloor = \lfloor(\lfloor\beta\rceil^{(p)})\rfloor$, and (II) follows. The monotonicity of z_{LP} and (II) imply (III). □

Proposition 10 *(Mangasarian and Shiau 1987) The linear programming value function z_{LP} is Lipschitz continuous over $\mathcal{B}(0,b)$.*

Proof: Mangasarian and Shiau (1987) show that over the set of right-hand sides for which the linear program is feasible, the optimal solutions follow a Lipschitzian relationship. That is, for any

β^1, β^2 such that $z_{LP}(\beta^1)$ and $z_{LP}(\beta^2)$ are finite, there exist optimal solutions $x^1 \in \text{opt}_{LP}(\beta^1), x^2 \in \text{opt}_{LP}(\beta^2)$ such that $||x^1 - x^2||_S \leq M'||\beta^1 - \beta^2||_R$, where $||\cdot||_S$ is a norm on $\mathbb{R}^n$ and $||\cdot||_R$ is a norm on $\mathbb{R}^m$. LP(β) is feasible and bounded for all $\beta \in \mathcal{B}(0,b)$, and the objective function is linear. Hence z_{LP} is Lipschitz continuous over $\mathcal{B}(0,b)$. □

THEOREM 9 *Suppose that for all $\beta \in \widehat{\mathcal{B}}_p(0,b), \mu^{(p)}(\beta) > 0$. Let $(\tilde{\phi}, \tilde{\psi})$ be an optimal solution to (6), and define the extension $\tilde{\psi}_p : \mathcal{B}(0,b) \to \mathbb{R}$ by $\tilde{\psi}_p(\beta) = \tilde{\psi}_p(\lfloor\beta\rfloor^{(p)})$. Then the following results hold:*

$$||\Gamma - \tilde{\psi}_p||_\infty \leq M/p, \tag{EC.1a}$$

$$\mathbb{E}_\xi|\Gamma(\xi) - \tilde{\psi}_p(\xi)| = \mathbb{E}_\xi(\Gamma(\xi) - \tilde{\psi}_p(\xi)) \leq M/p. \tag{EC.1b}$$

Proof: Let $\beta \in \mathcal{B}(0,b)$. By Lemma *EC.*2 ((II) and (III)) and Corollary 7, $|\Gamma(\beta) - \tilde{\psi}_p(\beta)| = |\Gamma(\beta) - \tilde{\psi}(\lfloor\beta\rfloor^{(p)})| = |\Gamma(\beta) - \Gamma(\lfloor\beta\rfloor^{(p)})| = \Gamma(\beta) - \Gamma(\lfloor\beta\rfloor^{(p)}) = z_{LP}(\beta) - z_{LP}(\lfloor\beta\rfloor^{(p)})$. By the definition of the Lipschitz constant M and Lemma *EC.*2 (I), $z_{LP}(\beta) - z_{LP}(\lfloor\beta\rfloor^{(p)}) \leq M||\beta - \lfloor\beta\rfloor^{(p)}||_\infty \leq M/p$. This proves (EC.1a). (EC.1b) follows from

$$\begin{aligned}
\mathbb{E}_\xi|\Gamma(\xi) - \tilde{\psi}_p(\xi)| &= \int_{\beta \in \mathcal{B}(0,b)} |\Gamma(\beta) - \tilde{\psi}_p(\beta)|\mu(\beta)d\beta \\
&= \int_{\beta \in \mathcal{B}(0,b)} (\Gamma(\beta) - \tilde{\psi}_p(\beta))\mu(\beta)d\beta \\
&\leq M/p \int_{\beta \in \mathcal{B}(0,b)} \mu(\beta)d\beta \\
&= M/p.
\end{aligned}$$
□

THEOREM 10 *Suppose the probability mass function $\mu^{(p)}$ approximates the probability density function μ as follows: $\mu^{(p)}(b^t) = \int_{\mathcal{B}^t} \mu(\beta)d\beta$, for all $t \in \{1, ..., T^{(p)}\}$. Then, the difference between the expectation of the absolute gap function over $\mathcal{B}(0,b)$ and the optimal objective value of (6) is bounded. Specifically, $|\mathbb{E}_\xi[\Gamma(\xi)] - \delta_{EAH}| \leq M/p$.*

Proof: Observe:

$$|\mathbb{E}_\xi[\Gamma(\xi)] - \delta_{EAH}| = \left| \int_{\mathcal{B}(0,b)} \mu(\beta)\Gamma(\beta)d\beta - \sum_{\beta \in \widehat{\mathcal{B}}_p} \mu^{(p)}(\beta)\Gamma(\beta) \right|$$

$$
\begin{aligned}
&= \left| \int\limits_{\mathcal{B}(0,b)} \mu(\beta)\Gamma(\beta)d\beta - \sum_{t\in T^{(p)}} \mu^{(p)}(\beta)\Gamma(\beta) \right| \\
&= \left| \sum_{t\in T^{(p)}} \int\limits_{\mathcal{B}^t} \mu(\beta)\Gamma(\beta)d\beta - \sum_{t\in T^{(p)}} \int\limits_{\mathcal{B}^t} d\beta\Gamma(b^t) \right| \\
&= \left| \sum_{t\in T^{(p)}} \int\limits_{\mathcal{B}^t} \mu(\beta)(\Gamma(\beta)-\Gamma(b^t))d\beta \right| \\
&\le \sum_{t\in T^{(p)}} \int\limits_{\mathcal{B}^t} \left|\mu(\beta)(\Gamma(\beta)-\Gamma(b^t))\right| d\beta \\
&\le \sum_{t\in T^{(p)}} \int\limits_{\mathcal{B}^t} \mu(\beta)M/p d\beta \\
&= M/p \int\limits_{\mathcal{B}(0,b)} \mu(\beta)d\beta \\
&= M/p. \qquad \square
\end{aligned}
$$

LEMMA EC.3. *Let $l \in \{1,...,L\}$. For any $\epsilon > 0$, there exists $\beta \in (\mathcal{B}^l)^\circ$ such that $z_{LP}(d^l) - z_{LP}(\beta) < \epsilon$, where $(\mathcal{B}^l)^\circ$ denotes the interior of $\mathcal{B}^l$.*

Proof: Fix $\epsilon > 0$. Notice that if $z_{LP}(d^l) = 0$, then $z_{LP}(\beta) = 0$ for all $\beta \in (\mathcal{B}^l)^\circ$. Now assume $z_{LP}(d^l) \neq 0$, and let $\delta = \frac{1}{2}\min\left\{\frac{\epsilon}{z_{LP}(d^l)}, \frac{1}{\max\limits_{i\in\{1,...,m\}} d_i^l}\right\}$. Define $\beta = (1-\delta)d^l$. Thus, $\beta \in (\mathcal{B}^l)^\circ$. Choose $x^d \in \text{opt}_{LP}(d^l)$ and let $x^\beta = (1-\delta)x^d$. Then, $Ax^\beta = (1-\delta)Ax^d \le (1-\delta)d^l = \beta$. Thus, $x^\beta \in P(\beta)$. It follows that $z_{LP}(\beta) \ge c^T x^\beta = (1-\delta)c^T x^d = z_{LP}(d^l) - \delta z_{LP}(d^l) \ge z_{LP}(d^l) - \frac{\epsilon}{2z_{LP}(d^l)} z_{LP}(d^l) = z_{LP}(d^l) - \frac{\epsilon}{2} > z_{LP}(d^l) - \epsilon$. Further, $z_{LP}(d^l) \ge z_{LP}(\beta)$ because z_{LP} is increasing. $\square$

PROPOSITION 11 *For any unit hyper-cube $\mathcal{B}^l$, $\sup\limits_{\beta\in\mathcal{B}^l} \Gamma(\beta) = z_{LP}(d^l) - z_{IP}(b^l)$.*

Proof: For any $\beta \in \mathcal{B}^l$, $z_{LP}(\beta) \le z_{LP}(d^l)$ and $z_{IP}(b^l) \le z_{IP}(\beta)$, which implies $z_{LP}(\beta) - z_{IP}(\beta) \le z_{LP}(d^l) - z_{IP}(b^l)$. Fix $\epsilon > 0$. By Lemma $EC.3$, there exists $\beta^* \in (\mathcal{B}^l)^\circ$ such that $z_{LP}(d^l) - z_{LP}(\beta^*) < \epsilon$. Because $A \in \mathbb{Z}^{m\times n}$ and $\lfloor \beta^* \rfloor = b^l$, we have that $z_{IP}(\beta^*) = z_{IP}(b^l)$. Therefore, $[z_{LP}(d^l) - z_{IP}(b^l)] - [z_{LP}(\beta^*) - z_{IP}(\beta^*)] < \epsilon$. Therefore, $z_{LP}(d^l) - z_{IP}(b^l) = \sup\limits_{\beta\in\mathcal{B}^l}[z_{LP}(\beta) - z_{IP}(\beta)] = \sup\limits_{\beta\in\mathcal{B}^l}\Gamma(\beta)$. $\square$

COROLLARY 9 $\Delta_{SAH} = \sup\limits_{\beta\in\mathcal{B}(0,b)} \Gamma(\beta) = \sup\limits_{l\in\{1,...,L\}} z_{LP}(d^l) - z_{IP}(b^l)$.

We omit the proof of Corollary 9 as it follows from Proposition 11.

Proposition 12 *The optimal solution of* (8) *is* Δ_{SAH}.

Proof: Let $\tilde{\beta} \in \arg\max\{z_{LP}(r(\beta)) - z_{IP}(\beta) \mid \beta \in \widehat{\mathcal{B}}(0,b)\}$ so that $z_{LP}(r(\tilde{\beta})) - z_{IP}(\tilde{\beta}) = \sup\{z_{LP}(d^\ell) - z_{IP}(b^\ell) = \Delta_{SAH}$, by Corollary 9. Let $\tilde{\phi} = z_{IP}$, $\tilde{\psi}(\tilde{\beta}) = z_{LP}(r(\tilde{\beta})) - z_{IP}(\tilde{\beta})$, and $\tilde{\psi}(\beta) = 0$, for all $\beta \in \widehat{\mathcal{B}}(0,b)\backslash\tilde{\beta}$. It is immediate that $\tilde{\phi} \in \Phi(b)$ (Proposition 2). By construction, $\tilde{\psi}$ satisfies (8b). For all $\beta \in \widehat{\mathcal{B}}(0,b)\backslash\tilde{\beta}$ and $k \in \mathcal{K}$, $\tilde{\psi}(\beta) = 0 \le z_{LP}(r(\beta)) - z_{IP}(\beta) \le r(\beta)^T v^k - \tilde{\phi}(\beta)$. Additionally, $\tilde{\psi}(\tilde{\beta}) = z_{LP}(r(\tilde{\beta})) - z_{IP}(\tilde{\beta}) \le r(\tilde{\beta})^T v^k - \tilde{\phi}(\tilde{\beta})$, for all $k \in \mathcal{K}$. Hence, $(\tilde{\phi}, \tilde{\psi})$ is feasible for (8).

Let $(\bar{\phi}, \bar{\psi})$ be feasible for (8). Then $\bar{\psi}(\beta) = 0$ for all $\beta \in \widehat{\mathcal{B}}(0,b)\backslash\bar{\beta}$ for some $\bar{\beta} \in \widehat{\mathcal{B}}(0,b)$. Because $\bar{\phi} \in \Phi(b)$, by Corollary 3, $\bar{\phi}(\beta) \ge z_{IP}(\beta)$ for all $\beta \in \widehat{\mathcal{B}}(0,b)$. Let $k^* \in \arg\min\{r(\bar{\beta})^T v^k \mid k \in \mathcal{K}\}$. By (8a), $\bar{\psi}(\bar{\beta}) \le r(\bar{\beta})^T v^{k^*} - \bar{\phi}(\bar{\beta}) \le z_{LP}(r(\bar{\beta})) - z_{IP}(\bar{\beta}) \le z_{LP}(r(\tilde{\beta})) - z_{IP}(\tilde{\beta}) = \tilde{\psi}(\tilde{\beta})$. Hence, $\sum_{\beta \in \widehat{\mathcal{B}}(0,b)} \tilde{\psi}(\beta) \ge \sum_{\beta \in \widehat{\mathcal{B}}(0,b)} \bar{\psi}(\beta)$. This proves $(\tilde{\phi}, \tilde{\psi})$ is optimal for (8). Further, $\delta_{SAH} = \sum_{\beta \in \widehat{\mathcal{B}}(0,b)} \tilde{\psi}(\beta) = \tilde{\psi}(\tilde{\beta}) = \Delta_{SAH}$. □

Proposition 13 *The infimum of the absolute gap function over* $\mathcal{B}(0,b)\backslash\mathcal{B}(0,1)$ *can be computed by considering only the integral points in* $\mathcal{B}(0,b)$. *Specifically,* $\inf_{\beta \in \mathcal{B}(0,b)\backslash\mathcal{B}(0,1)} \Gamma(\beta) = \min_{\beta \in \widehat{\mathcal{B}}(0,b)\backslash\{0\}} \Gamma(\beta)$. *Further,* (5) *can be used to compute* $\Delta_{IAH} = \inf_{\beta \in \mathcal{B}(0,b)\backslash\mathcal{B}(0,1)} \Gamma(\beta)$.

Proof: Let $\beta \in \mathcal{B}(0,b)\backslash\mathcal{B}(0,1)$, then $\Gamma(\beta) = z_{LP}(\beta) - z_{IP}(\beta) = z_{LP}(\beta) - z_{IP}(\lfloor\beta\rfloor) \ge z_{LP}(\lfloor\beta\rfloor) - z_{IP}(\lfloor\beta\rfloor) = \Gamma(\lfloor\beta\rfloor)$. Because $\beta \in \mathcal{B}(0,b)\backslash\mathcal{B}(0,1)$, $\lfloor\beta\rfloor \in \widehat{\mathcal{B}}(0,b)\backslash\{0\}$. In addition, for any $\beta' \in \widehat{\mathcal{B}}(0,1)$ and $\epsilon > 0$, using arguments similar to those in Lemma $EC.3$, from the continuity of z_{LP} and the fact that $z_{IP}(\tilde{\beta}) = z_{IP}(\beta')$, there exists $\tilde{\beta} \in (\mathcal{B}(\beta', \beta'+1))^\circ \subset \mathcal{B}(0,b)\backslash\mathcal{B}(0,1)$ such that $\Gamma(\tilde{\beta}) - \Gamma(\beta') < \epsilon$. Hence, $\Delta_{IAH} = \inf_{\beta \in \mathcal{B}(0,b)\backslash\mathcal{B}(0,1)} \Gamma(\beta) = \min_{\beta \in \widehat{\mathcal{B}}(0,b)\backslash\{0\}} \Gamma(\beta) = \Delta_{IAD}$, by Proposition 8. □

EC.4. Section 5 Results

Proposition 14 *The optimal objective value of* (9) *is* $\mathbb{E}_{\xi^{(1)}}(\gamma(\xi^{(1)}))$.

Proof: Let $(\tilde{\phi}, \tilde{\psi}) = (z_{IP}, \gamma)$. It is immediate that $\tilde{\phi} \in \Phi(b)$ (Proposition 2). Also, for each $\beta \in \widehat{\mathcal{B}}^+(0,b)$, $\gamma(\beta) \cdot \beta^T v^k \ge \gamma(\beta) \cdot z_{LP}(\beta) = z_{IP}(\beta) = \tilde{\phi}(\beta)$, for all $k \in \mathcal{K}$. Hence, $(\tilde{\phi}, \tilde{\psi})$ is feasible for (9).

Let $(\bar{\phi}, \bar{\psi})$ be feasible for (9). Because $\bar{\phi} \in \Phi(b)$, by Corollary 3, $\bar{\phi}(\beta) \ge z_{IP}(\beta)$, for all $\beta \in \widehat{\mathcal{B}}^+(0,b)$. By (9b), $\bar{\psi}(\beta) \ge \frac{\bar{\phi}(\beta)}{\beta^T v^k}$, for all $k \in \mathcal{K}$, which implies $\bar{\psi}(\beta) \ge \frac{\bar{\phi}(\beta)}{z_{LP}(\beta)} \ge \gamma(\beta) = \tilde{\psi}(\beta)$. Therefore,

$\sum_{\beta\in\widehat{\mathcal{B}}^+(0,b)} \mu^{(1)}(\beta)\bar{\psi}(\beta) \geq \sum_{\beta\in\widehat{\mathcal{B}}^+(0,b)} \mu^{(1)}(\beta)\tilde{\psi}(\beta)$, and $(\tilde{\phi},\tilde{\psi})$ is optimal. Thus, the optimal objective value is $\sum_{\beta\in\widehat{\mathcal{B}}^+(0,b)} \mu^{(1)}(\beta)\tilde{\psi}(\beta) = \sum_{\beta\in\widehat{\mathcal{B}}^+(0,b)} \mu^{(1)}(\beta)\gamma(\beta) = \mathbb{E}_{\xi^{(1)}}(\gamma(\beta))$. □

Corollary 10 *Suppose $\mu^{(1)}(\beta)>0$ for all $\beta\in\widehat{\mathcal{B}}^+(0,b)$. Then, any feasible solution $(\bar{\phi},\bar{\psi})$ in which $\bar{\phi}\neq z_{IP}$ is not optimal for* (9)*.*

We omit the proof of Corollary 10 due to reasoning similar to that of Corollary 5.

The variance of the relative gap function, denoted by σ_γ^2, indicates the extent to which this approximation error varies over $\widehat{\mathcal{B}}^+(0,b)$. The variance can be calculated from optimal solutions of (9).

Corollary 11 *Let $(\tilde{\phi},\tilde{\psi})$ be an optimal solution to* (9) *and δ_{ERD} the optimal objective value. Then the variance of the relative gap over $\widehat{\mathcal{B}}(0,b)$ can be calculated as follows:*

$$\sigma_\gamma^2 = \sum_{\beta\in\widehat{\mathcal{B}}^+(0,b)} \mu^{(1)}(\beta)(\tilde{\psi}(\beta))^2 - \delta_{ERD}^2.$$

The proof of Corollary 11 is similar to that of Corollary 6 and is omitted.

Proposition 15 *The optimal objective value of* (10) *is Δ_{SRD}.*

The proof is similar to that of Proposition 8 and is omitted.

Proposition 16 *The optimal objective value of* (11) *is $1-\Delta_{IRD}$.*

The proof is similar to that of Proposition 7 and is omitted.

EC.5. Section 6 Results

Proposition 17 *The supremum of the relative gap function over $\mathcal{B}(0,b)^+$ can be computed by only considering the integral points. Specifically, $\sup_{\beta\in\mathcal{B}^+(0,b)}\gamma(\beta) = \max_{\beta\in\widehat{\mathcal{B}}^+(0,b)}\gamma(\beta)$. Further,* (10) *can be used to compute $\Delta_{SRH} = \sup_{\beta\in\mathcal{B}^+(0,b)}\gamma(\beta)$.*

Proof: For any $\beta\in\mathcal{B}^+(0,b)$, $\gamma(\beta)=\frac{z_{IP}(\beta)}{z_{LP}(\beta)} = \frac{z_{IP}(\lfloor\beta\rfloor)}{z_{LP}(\beta)} \leq \frac{z_{IP}(\lfloor\beta\rfloor)}{z_{LP}(\lfloor\beta\rfloor)}$, and because $\beta\in\mathcal{B}^+(0,b)$, $\lfloor\beta\rfloor\in\widehat{\mathcal{B}}^+(0,b)$. Hence, $\Delta_{SRH} = \sup_{\beta\in\mathcal{B}^+(0,b)}\gamma(\beta) = \max_{\beta\in\widehat{\mathcal{B}}^+(0,b)}\gamma(\beta) = \Delta_{SRD}$, by Proposition 12. □

Proposition 18 *For any unit hyper-cube $\mathcal{B}^l$, $\inf_\beta\{\gamma(\beta) \mid \beta\in\mathcal{B}^l\} = z_{IP}(b^l)/z_{LP}(d^l)$.*

Proof: For any $\beta \in \mathcal{B}^l$, $z_{LP}(\beta) \leq z_{LP}(d^l)$ and $z_{IP}(b^l) \leq z_{IP}(\beta)$, which implies $z_{IP}(\beta)/z_{LP}(\beta) \geq z_{IP}(b^l)/z_{LP}(d^l)$. Fix $\epsilon > 0$. By Lemma $EC.3$, there exists $\beta^* \in (\mathcal{B}^l)^\circ$ such that $z_{LP}(d^l) - z_{LP}(\beta^*) \leq \frac{\epsilon z_{LP}(d^l) z_{LP}(b^l)}{z_{IP}(b^l)+1}$. Then,

$$\begin{aligned}
\frac{z_{IP}(\beta^*)}{z_{LP}(\beta^*)} - \frac{z_{IP}(b^l)}{z_{LP}(d^l)} &= \frac{z_{IP}(b^l)}{z_{LP}(d^l)} \cdot \frac{z_{LP}(d^l) - z_{LP}(\beta^*)}{z_{LP}(\beta^*)} \\
&\leq \frac{z_{IP}(b^l)}{z_{LP}(d^l)} \cdot \frac{z_{LP}(d^l) - z_{LP}(\beta^*)}{z_{LP}(b^l)} \\
&\leq \frac{z_{IP}(b^l)}{z_{LP}(d^l) z_{LP}(b^l)} \cdot \frac{z_{LP}(d^l) z_{LP}(b^l)}{z_{IP}(b^l)+1} \cdot \epsilon \\
&< \epsilon.
\end{aligned}$$

By definition of infimum, $z_{IP}(b^l)/z_{LP}(d^l) = \inf_{\beta \in \mathcal{B}^l} \gamma(\beta)$. □

COROLLARY 12 $\Delta_{IRH} = \inf_{\beta \in \mathcal{B}^+(0,b)} \gamma(\beta) = \inf_{l \in \{1,\ldots,L\}} z_{IP}(b^l)/z_{LP}(d^l)$.

Corollary 12 follows from Proposition 18, and we omit the proof.

PROPOSITION 19 $\delta_{IRH} = 1 - \Delta_{IRH}$.

Proof: Let $\tilde{\beta} \in \arg\min_{\beta \in \widehat{\mathcal{B}}^+(0,b)} \{z_{IP}(\beta)/z_{LP}(r(\beta))\}$, $\tilde{\psi}(\beta) = 1 - \Delta_{IRH}$, if $\beta = \tilde{\beta}$ and 0 otherwise, and let $\tilde{\phi} = z_{IP}$. It is immediate that $\tilde{\phi} \in \Phi(b)$ (Proposition 2). Observe that $(1 - \tilde{\psi}(\tilde{\beta})) r(\tilde{\beta})^T v^k = \Delta_{IRH} r(\tilde{\beta})^T v^k \geq \Delta_{IRH} z_{LP}(r(\tilde{\beta})) = z_{IP}(\tilde{\beta}) \geq \tilde{\phi}(\tilde{\beta})$, for all $k \in \mathcal{K}$. Further, for all $\beta \in \widehat{\mathcal{B}}^+(0,b) \backslash \tilde{\beta}$, $(1 - \tilde{\psi}(\beta)) r(\beta)^T v^k \geq z_{LP}(r(\beta)) \geq z_{IP}(\beta) = \tilde{\phi}(\beta)$, for all $k \in \mathcal{K}$. Thus, $(\tilde{\phi}, \tilde{\psi})$ is feasible.

Let $(\bar{\phi}, \bar{\psi})$ be feasible for (12). For all $\beta \in \widehat{\mathcal{B}}^+(0,b), k \in \mathcal{K}$, $(1 - \bar{\phi}(\beta)) r(\beta)^T v^k \geq \bar{\phi}(\beta) \geq z_{IP}(\beta)$, because $\bar{\phi} \in \Phi(b)$. By (12a), $1 - \bar{\phi}(\beta) \geq \frac{\bar{\phi}(\beta)}{r(\beta)^T v^k} \geq \frac{z_{IP}(\beta)}{r(\beta)^T v^k}$, which implies $1 - \bar{\psi}(\beta) \geq \gamma(\beta) \geq \Delta_{IRH}$. Because $\bar{\psi}(\beta) > 0$ for at most one $\beta \in \widehat{\mathcal{B}}^+(0,b)$, $\sum_{\beta \in \widehat{\mathcal{B}}^+(0,b)} \bar{\psi}(\beta) \leq 1 - \Delta_{IRH} = \sum_{\beta \in \widehat{\mathcal{B}}^+(0,b)} \tilde{\psi}(\beta)$. Hence, $(\tilde{\phi}, \tilde{\psi})$ is an optimal solution. □