Missing Proofs

EC.1. Individual Fairness Guarantees

In this section we show that the Nash equilibria of Trading Post are (approximately) proportional. We start with a lemma.

Lemma EC.1. *In any Nash equilibrium allocation* $\mathbf{x}$ *of the Trading Post mechanism and every agent* i*, the price of every item* j *that agent* i *values at* $\mathbf{x}$ *(i.e., every item* j *such that* $u_i(\mathbf{x'}_i) > u_i(\mathbf{x}_i)$ *for* $x'_{i,j} = 1$ *and* $x'_{i,j'} = x_{i,j'}$ *for all* $j' \neq j$*) is strictly positive.*

Proof Assume that there exists some agent i and an item j whose price is zero, and thus it remains unallocated, despite the fact that $u_i(\mathbf{x'}_i) > u_i(\mathbf{x}_i)$ for $x'_{i,j} = 1$ and $x'_{i,j'} = x_{i,j'}$ for all $j' \neq j$. Since the price of item j is zero, then agent i can claim all of this item using an arbitrarily small amount of spending on it. However, to spend on item j the agent needs to reduce its spending on some of the other items. We will argue that there always exists a way for i to reduce the spending on other items and use that spending on j such that the agent's utility strictly increases, which contradicts the assumption that $\mathbf{x}$ is a Nash equilibrium.

Let $\delta = u_i(\mathbf{x'}_i) - u_i(\mathbf{x}_i) > 0$ be the gain in agent i's utility if all of item j were to be added to the agent's existing bundle in $\mathbf{x}$. By concavity of agent i's valuation function we get $u_i(\alpha_i \cdot \mathbf{x}_i) \geq \alpha_i u_i(\mathbf{x}_i)$ for any $\alpha_i \in [0,1]$. By appropriately reducing its current spending to maintain the proportions of its current bundle, agent i can unilaterally reach allocation $\alpha_i \cdot \mathbf{x}_i$ for any $\alpha_i \in [0,1]$. It is easy to verify that no matter how small the value of δ is, there exists an α_i close enough to 1 such that $(1-\alpha_i)u_i(\mathbf{x}_i) < \delta$ and thus $u_i(\mathbf{x}_i) - u_i(\alpha_i \cdot \mathbf{x}_i) < \delta$. Therefore, agent i can reduce its spending to reach an allocation $\alpha_i \cdot \mathbf{x}_i$ without losing more than δ in the process and can then use this spending to claim all of item j. Therefore, the final allocation of agent i is $\mathbf{y}_i$ such that $y_{i,j} = 1$ and $y_{i,j'} = \alpha_i x_{i,j'}$ for all $j' \neq j$. Note that the only difference between bundle $\alpha_i \cdot \mathbf{x}_i$ and bundle $\mathbf{y}_i$ is the addition of all of item j in the latter, which is also the only difference between bundle $\mathbf{x}_i$ and bundle $\mathbf{x'}_i$. Therefore, since $\mathbf{x}_i \gg \alpha_i \cdot \mathbf{x}_i$, and by concavity of agent i's valuation function, we get that $u_i(\mathbf{y}_i) - u_i(\alpha_i \cdot \mathbf{x}_i) \geq u_i(\mathbf{x'}_i) - u_i(\mathbf{x}_i) = \delta$. This implies that $u_i(\mathbf{y}_i) > u_i(\mathbf{x}_i)$ and contradicts the assumption that $\mathbf{x}$ is a Nash equilibrium. □

THEOREM 6 (restated): *For each $\Delta \geq 0$, in each Nash equilibrium of $TP(\Delta)$ with non-negative, non-decreasing, and concave utilities, each agent i gets a value of $\frac{B_i}{\mathcal{B}}(1-\epsilon_i)$ of its maximum possible utility (taken over all bundles), where $\epsilon_i = \frac{\Delta \cdot (m-1)}{B_i}$.*

Proof. Let $\tilde{\mathbf{b}}$ be a pure Nash equilibrium and consider any fixed agent i.

Consider first the case of $\Delta = 0$. If i is satiated, then it gets its proportionality value. Otherwise, by Lemma 1, every agent who is not satiated at $\tilde{\mathbf{b}}$ is spending only on items that at least one other agent is also spending on. Let W be the set of items that agent i cannot be satiated without. By Lemma EC.1, the price of every good in W is positive, which combined with Lemma 1 implies every such good has at least two agents submitting a non-zero bid on it. For every good j, let $D_j = \sum_{k \neq i} \tilde{b}_{k,j}$ be the sum of bids from the other agents at j. Note that $D_j > 0$ for all $j \in W$.

Define strategy $\mathbf{y} = (y_1, \ldots, y_m)$ for agent i by $y_j = \frac{B_i \cdot D_j}{\mathcal{B} - B_i}$. Note that $y_j > 0$ for all $j \in W$. It can be verified that $\sum_{j=1}^m y_j = B_i$ and that the fraction received by agent i from every good $j \in W$ is:

$$\frac{y_j}{y_j + D_j} = \frac{\left(\frac{B_i \cdot D_j}{\mathcal{B} - B_i}\right)}{\left(\frac{B_i \cdot D_j}{\mathcal{B} - B_i}\right) + D_j} = \frac{B_i \cdot D_j}{B_i \cdot D_j + D_j \cdot (\mathcal{B} - B_i)} = \frac{B_i}{\mathcal{B}}.$$

Thus $\mathbf{y}$ is a safe strategy that guarantees agent i a fraction of at least $B_i/\mathcal{B}$ of every good in W. Since $\tilde{\mathbf{b}}$ is a pure Nash equilibrium, strategy $\mathbf{y}$ is not an improvement; together with concavity of the utilities, we get: $u_i(\tilde{\mathbf{b}}) \geq u_i(\mathbf{y}, \tilde{b}_{-i}) \geq u_i\left(\frac{B_i}{\mathcal{B}} \cdot \mathbf{1}_W\right) \geq \frac{B_i}{\mathcal{B}} \cdot u_i(\mathbf{1}_W) = u_i(\mathbf{1})$, where $\mathbf{1}_W$ is the vector of m dimensions with entries of 1 for coordinates in W and 0 otherwise, at which agent i is achieving its maximum possible value.

For $\Delta > 0$, let $S_i = \{j \in [m] \mid \widetilde{b}_{k,j} = 0, \forall k \neq i\}$ be the set of items on which no agent in $N \setminus \{i\}$ is bidding on in the equilibrium. Note that $|S_i| \leq m - 1$ since there must be at least one item with non-zero bids from the other agents. Also w.l.o.g. we can assume the budgets are normalized so that $B_\ell \geq 1$ for all agents ℓ. Let $B_i' = B_i - \Delta \cdot |S_i|$ and $D_j = \sum_{k \neq i} \tilde{b}_{k,j}$ be the sum of bids from the other agents at good j. For all items $j \notin S_i$ we have $D_j > 0$.

Define strategy profile $\mathbf{z} = (z_1, \ldots, z_m)$ for agent i:

$$z_j = \begin{cases} \Delta, & \text{if } j \in S_i \\ \frac{B_i' \cdot D_j}{\mathcal{B} - B_i'}, & \text{otherwise} \end{cases}$$

We show first that the strategy $\mathbf{z}$ is feasible, by not exceeding i's budget:

$$\sum_{j=1}^{m} z_j = \Delta \cdot |S_i| + \sum_{j \notin S_i} z_j = \Delta \cdot |S_i| + \sum_{j \notin S_i} \frac{B_i' \cdot D_j}{\mathcal{B} - B_i'} = \Delta \cdot |S_i| + \left(\frac{B_i - \Delta \cdot |S_i|}{\mathcal{B} - B_i + \Delta \cdot |S_i|} \right) \cdot \left(\sum_{j \notin S_i} D_j \right)$$
$$\leq \Delta \cdot |S_i| + \left(\frac{B_i - \Delta \cdot |S_i|}{\mathcal{B} - B_i + \Delta \cdot |S_i|} \right) \cdot (\mathcal{B} - B_i) \leq B_i$$

Clearly for each item $j \in S_i$, agent i receives a fraction of at least $B_i / \mathcal{B}$ since agent i is the only agent bidding on the item. From the items $j \notin S_i$, agent i gets a fraction of:

$$\frac{z_j}{z_j + D_j} = \frac{\left(\frac{B_i' \cdot D_j}{\mathcal{B} - B_i'}\right)}{\left(\frac{B_i' \cdot D_j}{\mathcal{B} - B_i'}\right) + D_j} = \frac{B_i'}{\mathcal{B}} = \frac{B_i - \Delta \cdot |S_i|}{\mathcal{B}} \geq \frac{B_i - \Delta \cdot (m-1)}{\mathcal{B}} = \frac{B_i}{\mathcal{B}} \left(1 - \frac{\Delta \cdot (m-1)}{B_i} \right)$$

By taking $\epsilon_i = \frac{\Delta \cdot (m-1)}{B_i}$ we obtain the required statement. □

EC.2. Equilibrium Existence for Concave Valuations

Theorem 8 (restated): The Trading Post game with no minimum bid has exact pure Nash equilibria for all concave, continuous, and strictly increasing utilities.

Proof. Our main tool for proving the existence of exact pure Nash equilibria is a result for discontinuous games due to Reny (1999). First, we define the *better-reply secure* property of a game with strategy space $S = S_1 \times \ldots \times S_n$ and utilities u_i.

Definition EC.1. Agent i can *secure* a payoff of $\alpha \in \mathbb{R}$ at $s \in S$ if there exists $\bar{s}_i \in S_i$, such that $u_i(\bar{s}_i, s_{-i}^{'}) \geq \alpha$ for all $s_{-i}^{'}$ close enough to s_{-i}.

Definition EC.2. A game $G = (S_i, u_i)_{i=1}^{n}$ is *better-reply secure* if whenever (s^*, u^*) is in the closure of the graph of its vector payoff function and s^* is not a Nash equilibrium, then some agent i can secure a payoff strictly above u_i^* at s^*.

Lemma EC.2 **(Reny, 1999)**. *If each S_i is a nonempty, compact, convex subset of a metric space, and each $u_i(s_1, \ldots, s_n)$ is quasi-concave in s_i, then the game $G = (S_i, u_i)_{i=1}^{n}$ has at least one pure Nash equilibrium if in addition G is better-reply secure.*

The strategy spaces in Trading Post are nonempty, convex, compact subsets of the Euclidean space. Moreover, by Lemma EC.7, the utilities of the agents are quasi-concave in an agent's own bidding strategy. We show the game is also better-reply secure.

First note that Trading Post has discontinuities at strategy profiles where everyone bids zero on some item, captured by the set

$$\mathcal{D} = \{\mathbf{b} \in S \mid \exists\, j \in M \text{ such that } b_{i,j} = 0, \forall i \in N\}.$$

Let $(\mathbf{b}^*, \mathbf{u}^*)$ be a point in the closure of the graph of the vector payoff function. Then there exists a sequence of strategy profiles $(\mathbf{b}^K)_{K \geq 1}$ such that $\mathbf{u}^* = \lim_{K \to \infty}(u_1(\mathbf{b}^K), \ldots, u_n(\mathbf{b}^K))$, where $\mathbf{b}^* = \lim_{K \to \infty} b^K$. If $\mathbf{b}^* \notin \mathcal{D}$, then better-reply security holds at $(\mathbf{b}^*, \mathbf{u}^*)$ by continuity. Thus we consider $\mathbf{b}^* \in \mathcal{D}$. Let $\mathbf{x}^K$ denote the allocation obtained at bids $\mathbf{b}^K$ and $\bar{\mathbf{x}}$ the allocation obtained in the limit: $\bar{x}_{i,j} = \lim_{K \to \infty} \frac{b_{i,j}^K}{\sum_{\ell \in N} b_{\ell,j}^K}$.

Consider the set of items on which no agent bids at $\mathbf{b}^*$: $J = \{j \in M \mid b_{i,j}^* = 0, \forall i \in N\}$. Let ℓ be an item in J. Then there exists an agent i and a value $0 < \alpha < 1 - \frac{B_i}{\mathcal{B}}$, where $\mathcal{B} = \sum_{i \in N} B_i$ such that agent i gets a fraction of at most $f = \frac{B_i}{\mathcal{B}} + \alpha < 1$ from item ℓ in every K^{th} term of the sequence, possibly except the first K_α terms.

Consider the set of items that agent i bids on in the limit: $L_i = \{\, j \in M \mid b_{i,j}^* > 0 \,\}$. Given $0 < \epsilon < \min_{i,j} b_{i,j}^*$, construct an alternative profile $\mathbf{b}_i'(\epsilon)$ of agent i, at which the bid on item ℓ is $b_{i,\ell}'(\epsilon) = \epsilon$, the bid on items $j \in L_i$ is $b_{i,j}'(\epsilon) = b_{i,j}^* - \epsilon/|L_i|$, and the bid on items outside $L_i \cup \{\ell\}$ is $b_{i,j}'(\epsilon) = b_{i,j}^* = 0$. The strategy $\mathbf{b}_i'(\epsilon)$ guarantees that agent i gains the entire item ℓ, while along the sequence $(\mathbf{b}^K)_{K \geq K_\alpha}$ the agent was receiving a fraction of at most $f < 1$ from ℓ. Moreover, by playing $b_{i,j}'(\epsilon)$, agent i loses a fraction of at most ϵ/w^* from every other item $j \in L_i$, where $w^* = \min_{j \in L_i} \sum_{k=1}^n b_{k,j}^*$.

Define a new allocation $\widetilde{x}_i$ for agent i, given by $\widetilde{x}_{i,j} = \frac{b_{i,j}^*}{\sum_{k \in N} b_{k,j}^*}$, for all $j \in L_i$, and $\widetilde{x}_{i,j} = 1$ for all $j \notin L_i$. Note that for agent i, allocation $\widetilde{x}_i$ strictly dominates $\bar{\mathbf{x}}_i$. Since the utilities are strictly increasing, this implies that $u_i(\widetilde{x}_i) > u_i^*$. By continuity of the utilities, there exists small enough

$\epsilon > 0$ such that $u_i(\mathbf{b}_i'(\epsilon), \mathbf{b}_{-i}^*) > u_i^*$, since for very small ϵ the allocation at bids $(\mathbf{b}_i'(\epsilon), \mathbf{b}_{-i}^*)$ is close to $\widetilde{x}_i$. Note also that $u_i^* > u_i(\mathbf{b}^*)$, since agent i loses item ℓ at $\mathbf{b}^*$.

Moreover, by continuity of the utilities at $(\mathbf{b}_i'(\epsilon), \mathbf{b}_{-i}^*)$, the strategy $\mathbf{b}_i'(\epsilon)$ continues to guarantee agent i a payoff strictly above u_i^* for any small enough perturbation of magnitude bounded by $\delta(\epsilon)$ of the strategies of the others around $\mathbf{b}_{-i}^*$. Thus the game is better-reply secure, which by Lemma EC.2 implies the existence of a pure Nash equilibrium. □

Theorem 7 (restated): The Trading Post game with no minimum bid has exact pure Nash equilibria for all CES utilities with perfect competition and $\rho \in (-\infty, 1]$.

Proof. The proof for $\rho = 0$ follows from the existence of Nash equilibria in Fisher markets with Cobb-Douglas utilities (Brânzei et al. 2014), for which the induced allocations are also proportional. Thus we assume $\rho \neq 0$. In this case, the proof is the same as that of Theorem 8 for valuations that are concave and strictly increasing, except the way we find an agent that can improve strictly its utility at points with discontinuities is by using perfect competition.

EC.3. Equilibrium Existence for Leontief Valuations

Theorem 9 (restated) : *The Trading Post mechanism with Leontief valuations and perfect competition has exact pure Nash equilibria if and only if the corresponding Fisher market has market equilibrium prices that are strictly positive everywhere. When this happens, the Nash equilibrium utilities in Trading Post are unique and the price of anarchy is* 1.

Proof. Suppose $(\mathbf{v}, \mathbf{B})$ are valuations and budgets for which the Fisher market equilibrium prices are strictly positive everywhere (recall also that the valuations satisfy perfect competition). Then we show that some pure Nash equilibrium exists in the Trading Post game with the same valuations and budgets, and which induces the same allocation.

Let $(\mathbf{p}, \mathbf{x})$ be the Fisher market equilibrium prices and allocation. Define matrix of bids $\mathbf{b}$ by $b_{i,j} = p_j \cdot x_{i,j}$, for all $i \in N, j \in M$. We claim that $\mathbf{b}$ is a pure Nash equilibrium in the Trading Post game. From the conditions in the theorem statement, we have that for each item j, there are two agents $i \neq i'$ such that $b_{i,j} \cdot b_{i',j} > 0$. Also the utility of each agent i in the market equilibrium is the same as that of strategy profile $\mathbf{b}$ and can be written as follows:

$$u_i(\mathbf{x}_i) = \min_{j \in M : v_{i,j} > 0} \left\{ \frac{x_{i,j}}{v_{i,j}} \right\}$$

By definition of the market equilibrium, each agent i gets each item in its demand set in the same fraction f_i, that is

$$f_i = \frac{x_{i,j}}{v_{i,j}}, \text{for } j \in M \text{ with } v_{i,j} > 0$$

Then agent i can only improve its utility by taking weight from some item(s) and shifting it towards others in its demand. However, since all the items are received in the same fraction f_i, it follows that agent i can only decrease its utility by such deviations. Thus the profile $\mathbf{b}$ is a pure Nash equilibrium of Trading Post.

For the other direction, if a bid profile $\mathbf{b}$ is a pure Nash equilibrium in the Trading Post game, then consider the market allocation and prices $(\mathbf{x}, \mathbf{p})$, where $p_j = \sum_{k=1}^n b_{k,j}$ and the induced allocation $\mathbf{x}$ be given by $x_{i,j} = \frac{b_{i,j}}{\sum_{k=1}^n b_{k,j}}$. From the perfect competition requirement, clearly $x_{i,j}$ is always well defined. Then at $(\mathbf{x}, \mathbf{p})$ all the goods are allocated, all the money is spent, and each agent gets an optimal bundle from its desired goods. To see the latter, note again that an agent receives all the items in the same fractions at $(\mathbf{x}, \mathbf{p})$ and since all the prices are positive (since on each item there are at least two non-zero bids), then an agent cannot decrease its spending on any item(s). Thus $(\mathbf{x}, \mathbf{p})$ is a market equilibrium and since the market equilibrium utilities are unique, it follows that the same is true for the pure Nash equilibrium of Trading Post.

From the correspondence between the Nash equilibria of Trading Post and the market equilibria of the Fisher mechanism, we obtain that on such instances the price of anarchy is 1. □

EC.4. Beyond Pure Nash Equilibria

In this section we show that, in the Trading Post game, Nash equilibrium (NE) where no agent is satiated has to be pure. For this we will show that no matter how other agents play at a NE, there is a *unique pure* best response strategy for non-satiated agents. At Nash equilibrium, since strategy of an agent is a probability distribution over pure strategies that are in her best response, this will imply that the Nash equilibrium has to be pure.

Now on let us fix a Nash equilibrium $\boldsymbol{\sigma} = (\boldsymbol{\sigma}_1, \ldots, \boldsymbol{\sigma}_n)$ where no agent is satiated, *i.e.*, $u_i(\boldsymbol{\sigma}) < u_i(\mathbf{1})$, $\forall i \in N$. To show uniqueness of best response we will show that fixing mixed strategy/bid profile for all other agents, agent i's utility is a strictly concave function of its own bid profile. Let M_i be the set of goods that agent i *cares for*, formally $M_i = \{j \mid \exists \mathbf{x} \in \mathbb{R}^m, x_j > 0, u_i(\mathbf{x}) > u_i(x_1, \ldots, x_{(j-1)}, 0, x_{j+1}, \ldots, x_m)\}$. Clearly, at $\boldsymbol{\sigma}_i$, agent i will play a bid profile $\mathbf{s}_i$ with non-zero probability only if $s_{ij} = 0$, $\forall j \notin M_i$ (do not waste money) (see Lemma 1). Therefore, it is without loss of generality to let the set of pure bid profiles of agent i to be $S_i = \{(s_{ij})_{j \in M_i} \mid \sum_{j \in M_i} = B_i\}$, i.e., set of all possible (non-wasteful) ways in which she can split her budget across goods. Let $\boldsymbol{\sigma}_{-i} = (\boldsymbol{\sigma}_1, \ldots, \boldsymbol{\sigma}_{i-1}, \boldsymbol{\sigma}_{i+1}, \ldots, \boldsymbol{\sigma}_n)$ be a mixed-strategy profile of all the agents except i, and let agents i's payoff function w.r.t. $\boldsymbol{\sigma}_{-i}$ be denoted by $u_i^{\boldsymbol{\sigma}_{-i}} : S_i \rightarrow \mathbb{R}$.

Next we derive the result for both additive and Leontief separately, and later extend it to concave under a mild assumption.

EC.4.1. Perfect Substitutes (Additive valuations)

For the additive utilities, $M_i = \{j \in M \mid v_{ij} > 0\}$, and function $u_i^{\boldsymbol{\sigma}_{-i}}$ is as follows.

$$u_i^{\boldsymbol{\sigma}_{-i}}(\mathbf{s}_i) = \sum_{\mathbf{s}_{-i} \in S_{-i}} (\Pi_{k \neq i} \boldsymbol{\sigma}_k(\mathbf{s}_k)) \sum_{j \in M_i} v_{ij} \frac{s_{ij}}{s_{ij} + \sum_{k \neq i} s_{kj}}, \ \forall \mathbf{s}_i \in S_i \tag{EC.1}$$

Next we will show that function $u_i^{\boldsymbol{\sigma}_{-i}}$ is strictly concave on the entire domain of $\mathbb{R}^{|M_i|}$, therefore it is strictly concave on S_i as well. For this, we will show that Hessian of $u_i^{\boldsymbol{\sigma}_{-i}}$ is negative definite.

LEMMA EC.3. *Given a mixed-strategy profile $\boldsymbol{\sigma}_{-i}$ of all agents $k \neq i$ such that every good $j \in M_i$ is bought by at least one of them with non-zero probability, payoff function of agent i, namely $u_i^{\boldsymbol{\sigma}_{-i}}$, is strictly concave.*

Proof. We first show that $u_i^{\boldsymbol{\sigma}_{-i}}$ is strictly concave on $\mathbb{R}_+{}^{|M_i|}$. Take the derivative of $u_i^{\boldsymbol{\sigma}_{-i}}$ with respect to s_{ij} for each j,

$$\frac{\partial u_i^{\boldsymbol{\sigma}_{-i}}}{\partial s_{ij}} = \sum_{\mathbf{s}_{-i} \in S_{-i}} (\Pi_{k \neq i} \boldsymbol{\sigma}_k(\mathbf{s}_k)) v_{ij} \frac{\sum_{k \neq i} s_{kj}}{(s_{ij} + \sum_{k \neq i} s_{kj})^2}$$

Differentiating the above w.r.t. s_{ig}s for each good g, we get:

$$\frac{\partial^2 u_i^{\boldsymbol{\sigma}_{-i}}}{\partial s_{ij}\partial s_{ig}} = 0,\ \forall g \neq j; \qquad \frac{\partial^2 u_i^{\boldsymbol{\sigma}_{-i}}}{\partial s_{ij}^2} = -2\sum_{\mathbf{s}_{-i}\in S_{-i}} (\Pi_{k\neq i}\boldsymbol{\sigma}_i(\mathbf{s}_k))v_{ij}\frac{\sum_{k\neq i} s_{kj}}{(s_{ij}+\sum_{k\neq i} s_{kj})^3}$$

Since bids are non-negative and at least one agent other than i is bidding on good j with non-zero probability as per $\boldsymbol{\sigma}_{-i}$, the second term in above expression is strictly negative for all $j \in M_i$. Therefore, the hessian is diagonal matrix with negative entries in the diagonal, and hence negative definite. Thus, function $u_i^{\boldsymbol{\sigma}_{-i}}$ is strictly concave on $\mathbb{R}^{|M_i|}$. Therefore it remains strictly concave on convex subset $S_i \subset \mathbb{R}^{|M_i|}$ as well. $\square$

Strict concavity of $u_i^{\boldsymbol{\sigma}_{-i}}$ established in Lemma EC.3 implies that every agent has a unique best response against $\boldsymbol{\sigma}_{-i}$. At Nash equilibrium since only strategies in best response can have non-zero probability, it follows that all Nash equilibria have to be pure. Thus we get the next theorem.

Theorem EC.1. *In a market with linear utilities, every Nash equilibrium of the corresponding Trading Post game, where no agent is satiated, is pure.*

Proof. At any given strategy profile, if only agent i is biding on a good j with non-zero probability, then it can not be Nash equilibrium since then i can reduce its bid on good j to very small amount while still getting it fully, and use this saved money to buy other goods (Lemma 1). Therefore, for every agent i, and every good $j \in M_i$, there is someone other than i is bidding on j at $\boldsymbol{\sigma}_{-i}$. Then Lemma EC.3 implies that $\boldsymbol{\sigma}_i$ is pure for every agent i. $\square$

We observe that the equilibria of Trading Post with linear utilities are not necessarily unique.

Example EC.1. Consider a market with four agents and two goods. Agents 1 and 2 only want the first and second good, respectively, while agents 3 and 4 like both goods equally. Then the bid profiles $\mathbf{s}_1 = (1,0), \mathbf{s}_2(0,1), \mathbf{s}_3 = (1-\epsilon,\epsilon), \mathbf{s}_4 = (\epsilon, 1-\epsilon)$ are in NE for any $0 \le \epsilon \le 1$.

EC.4.2. Perfect Complements (Leontief utilities)

In case of Leontief utilities, the payoff of agents i is $\min_j \frac{x_{ij}}{v_{ij}}$, where x_{ij} is the amount of good j agent i gets. Then, the set of goods i cares for is $M_i = \{j \mid v_{ij} > 0\}$. Like the linear case, for Leontief valuations too we will show uniqueness of best response, however the approach is different because the utility function $u_i^{\boldsymbol{\sigma}_{-i}}$ (defined below) is no more strictly concave.

$$u_i^{\boldsymbol{\sigma}_{-i}}(\mathbf{s}_i) = \sum_{\mathbf{s}_{-i} \in S_{-i}} (\Pi_{k \neq i} \boldsymbol{\sigma}_i(\mathbf{s}_k)) \min_j \frac{1}{v_{ij}} \frac{s_{ij}}{s_{ij} + \sum_{k \neq i} s_{kj}}, \quad \forall \mathbf{s}_i \in S_i \tag{EC.2}$$

We can show that $u_i^{\boldsymbol{\sigma}_{-i}}$ is concave using the concavity of $\frac{s_{ij}}{s_{ij}+\sum_{k\neq i} s_{kj}}$ w.r.t. s_{ij}, however it is not strictly concave in general. Instead we will show that function $u_i^{\boldsymbol{\sigma}_{-i}}$ has a unique optimum over all the possible strategies of agent i in game $\mathcal{TP}(\Delta)$. This will suffice to show that all equilibria are pure.

The Leontief function ensures that, at Nash equilibrium $\boldsymbol{\sigma}$, for every agent $i \in N$ if $\sigma_i(\mathbf{s}_i) > 0$ for some $\mathbf{s}_i \in S_i$, then $s_{ij} \geq \Delta$ for every good $j \in M_i$, since otherwise $u_i^{\boldsymbol{\sigma}_{-i}}(\mathbf{s}_i) = 0$. Therefore, if there is a good j that only one agent wants, i.e., $j \in M_i$ for exactly one agent i, then we can give it to her at the minimum bid Δ. Let M' be the set of goods liked by atleast two agents. Since every good in M' is liked by at least two agents, relevant bid profiles $\mathbf{s}_{-i}$ are only those where every good in M' is bid on by some agent k other than i since bidding zero fetches zero amount of the good. To capture this we define *valid* strategy profiles.

Definition EC.3. Profile $\mathbf{s}_{-i}$ is *valid* if $\forall j \in M'$, $\sum_{k\neq i} s_{kj} > 0$. Similarly, mixed-profiles $\boldsymbol{\sigma}_{-i}$ is said to be *valid* if each pure profile $\mathbf{s}_{-i}$ with $P(\mathbf{s}_{-i}) = \Pi_{k\neq i}\boldsymbol{\sigma}_k(\mathbf{s}_k) > 0$ is valid.

Lemma EC.4. *If $\boldsymbol{\sigma}$ is a Nash equilibrium of game $\mathcal{TP}(\Delta)$ for any $\Delta > 0$, then profile $\boldsymbol{\sigma}_{-i}$ is valid for each $i \in \mathcal{A}$.*

Proof For some i if $\boldsymbol{\sigma}_{-i}$ is not valid, then there exists $\mathbf{s}_{-i}$ with $P(\mathbf{s}_{-i}) > 0$ and good $j \in M'$ such that no one is bidding on it at $\mathbf{s}_{-i}$. Let $k \neq i$ is an agent with $v_{kj} > 0$. Clearly she gets zero utility whenever she plays $\mathbf{s}_k$. Instead if she replaces $\mathbf{s}_k$ with $\mathbf{t}$ where $t_j = \Delta$, $t_{j'} = s_{kj'} - \Delta$ where $s_{kj'} \geq 2\Delta$ (assuming Δ to be small there is such a good), and $t_d = s_{kd}, \forall d \neq j, j'$, will give her strictly better utility. □

Due to Lemma EC.4 it suffice to consider only valid strategy profiles, both mixed as well as pure.

LEMMA EC.5. *Given a valid strategy profile $\mathbf{s}_{-i}$ of $\mathcal{TP}(\Delta)$ for $\Delta > 0$ and good $j \in M_i$, consider function $f_j = \frac{1}{v_{ij}} \frac{s_{ij}}{s_{ij}+\sum_{k\neq i} s_{kj}}$. f_j seen as a function of s_{ij} is strictly concave if $j \in M'$ and concave if $j \notin M'$, and as a function of $(s_{i1}, \ldots, s_{in})$ it is concave.*

Proof. Taking double derivative of f_j we get,

$$\frac{\partial^2 f_j}{\partial s_{ij}^2} = \frac{-2}{v_{ij}} \frac{\sum_{k\neq i} s_{kj}}{(s_{ij}+\sum_{k\neq i} s_{kj})^3}$$
$$\forall g \neq j,\ \frac{\partial^2 f_j}{\partial s_{ij} \partial s_{ig}} = 0 \quad \text{and} \quad \frac{\partial^2 f_j}{\partial s_{ig}^2} = 0$$

The first equality implies strict concavity with respect to s_{ij}. This is due to the fact that $\sum_{k\neq i} s_{kj} \geq \Delta > 0$, given that there is another agent who would want good j, i.e., $j \in M'$. Both equalities together imply that the Hessian of f_j is negative semi-definite, and therefore f_j is concave with respect to $\mathbf{s}_i$. □

Next we show uniqueness of best response for agent i against any given $\boldsymbol{\sigma}_{-i}$. For simplicity, we will abuse notation and use $u_i(\mathbf{s}_i, \mathbf{s}_{-i})$ to denote $u_i(\frac{s_{i,1}}{\sum_k s_{k,1}}, \ldots, \frac{s_{i,n}}{\sum_k s_{k,n}})$

LEMMA EC.6. *Given a valid mixed-strategy profile $\boldsymbol{\sigma}_{-i}$ of all agents $k \neq i$ in Trading Post game $\mathcal{TP}(\Delta)$ for any $\Delta > 0$, payoff function of agent i, namely $u_i^{\boldsymbol{\sigma}_{-i}}$, has a unique optimum.*

Proof. First it is easy to see using Lemma EC.5 that each of the term inside summation of $u_i^{\boldsymbol{\sigma}_{-i}}$ is a concave function w.r.t. $\mathbf{s}_i$, since minimum of a set of concave functions is a concave function. And by the same argument $u_i^{\boldsymbol{\sigma}_{-i}}$ itself is concave because it is summation of a set of concave functions.

To the contrary suppose there are two optimum $\mathbf{s}_i, \mathbf{s}'_i \in S_i$, $\mathbf{s}_i \neq \mathbf{s}'_i$. Clearly $\sum_j s_{ij} = \sum_j s'_{ij} = B_i$, $s_{ij}, s'_{ij} \geq \Delta$, $\forall j \in M_i$, $s_{ij} = s'_{ij} = \Delta$, $\forall j \in M_i \setminus M'$, and $s_{ij} = s'_{ij} = 0, \forall j \notin M_i$ (since i is not *satiated*). Therefore, there exists a good $j \in M_i \cap M'$ where bid at $\mathbf{s}$ is more than that at $\mathbf{s}'$. Let $j^* \in \{j \mid s_{ij} > s'_{ij}\}$. We have $j^* \in M'$ and $s_{ij^*} > \Delta$.

CLAIM EC.1. *Given $\mathbf{s}$, if there is a good j with $s_{ij} > \Delta > 0$ and $u_i(\mathbf{s}_i, \mathbf{s}_{-i}) < \frac{1}{v_{ij}} \frac{s_{ij}}{\sum_{k\neq i} s_{kj} + s_{ij}}$, then $\mathbf{s}_i$ is not a best response to $\mathbf{s}_{-i}$.*

Proof. It is easy to see that there exists δ, such that for $t_j = s_{ij} - n\delta \geq \Delta$, $t_k = s_{ik} + \delta$, $\forall g \neq j$, we have $u_i(\mathbf{s}) < u_i(\mathbf{t}, \mathbf{s}_{-i})$. □

$\forall \mathbf{s}_{-i}$ with $P(\mathbf{s}_{-i}) > 0$ if we have $u_i(\mathbf{s}_i, \mathbf{s}_{-i}) < \frac{1}{v_{ij^*}} \frac{s_{ij^*}}{\sum_{k \neq i} s_{kj} + s_{ij^*}}$, then from the above claim it follows that $\mathbf{s}_i$ is not an optimal solution of $u_i^{\boldsymbol{\sigma}_{-i}}$.

Otherwise for some $\mathbf{s}_{-i}$ we have $u_i(\mathbf{s}_i, \mathbf{s}_{-i}) = \frac{1}{v_{ij^*}} \frac{s_{ij^*}}{\sum_{k \neq i} s_{kj} + s_{ij^*}}$. Due to concavity of $u_i^{\boldsymbol{\sigma}_{-i}}$ we have that entire line-segment from $\mathbf{s}_i$ to $\mathbf{s}'_i$ is optimal. Call this line segment $\mathcal{L}$. On this line-segment bid on good j^* is strictly changing (decreasing). Since $u_i(\mathbf{s}_i, \mathbf{s}_{-i}) = \min_j \frac{1}{v_{ij}} \frac{s_{ij}}{\sum_{k \neq i} s_{kj} + s_{ij}}$ is governed by the utility obtained from good j^*, it is strictly concave on $\mathcal{L}$ at $\mathbf{s}_i$ (due to Lemma EC.5).

Furthermore, at a point if a set of functions are concave and one of them is strictly concave then their summation is strictly concave. Therefore, $u_i^{\boldsymbol{\sigma}_{-i}}$ is strictly concave at $\mathbf{s}_i$ on $\mathcal{L}$ and hence either $\mathbf{s}_i$ is not optimum or other points on $\mathcal{L}$ are not optimum. In either case we get a contradiction. □

Since at Nash equilibrium only optimal strategies can have non-zero probability, the next theorem follows using Lemmas EC.4 and EC.6.

Theorem EC.2. *For market with Leontief utilities every Nash equilibrium of the corresponding* $\Delta > 0$ *Trading Post game* $\mathcal{TP}(\Delta)$*, where no agent is* satiated*, is pure.*

The Trading Post game with Leontief valuations does not always have a unique pure Nash equilibrium, as can be seen from the next example.

Example EC.2. Let there be an instance of Trading Post with agents $N = \{1, 2\}$, items $M = \{1, 2\}$, and Leontief valuations $v_{i,j} = 1$, $\forall i, j \in \{1, 2\}$. Then every strategy profile of the form $v_{1,1} = v_{2,1} = a$, with $a \in (0, 1)$ is a pure Nash equilibrium of Trading Post, since both agents get the items in the optimal ratios.

EC.4.3. Concave Valuations

As stated in preliminaries section, the valuation function of each agent i namely $u_i : \mathbb{R}^m_+ \to \mathbb{R}$ is concave, non-negative, and non-decreasing in general, where $m = |M|$ is the number of goods in the market.

Enough Competition. For every good j, there are two agents $k \neq i$, such that $\frac{\partial u_i}{\partial x_{ij}}(\mathbf{x})$ and $\frac{\partial u_k}{\partial x_{kj}}(\mathbf{y})$ are infinity at bundles $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m_+$ if $x_j = 0$ and $y_j = 0$ respectively.

In Trading Post mechanism, since strategy of an agent is to specify money bid on each good, for $\mathbf{s}_i \in S_i$ and $\mathbf{s}_{-i} \in S_{-i}$ let us define

$$u_i(\mathbf{s}_i, \mathbf{s}_{-i}) = u_i\left(\frac{s_{i,1}}{s_{i,1} + \sum_{k\neq i} s_{k,1}}, \ldots, \frac{s_{i,m}}{s_{i,m} + \sum_{k\neq i} s_{k,m}}\right) \tag{EC.3}$$

Then, function $u_i^{\boldsymbol{\sigma}_{-i}}$ can be written as follows, where $P(\mathbf{s}_{-i}) = \Pi_{k\neq i}\boldsymbol{\sigma}_k(s_k)$.

$$u_i^{\boldsymbol{\sigma}_{-i}}(\mathbf{s}_i) = \sum_{\mathbf{s}_{-i}\in S_{-i}} P(\mathbf{s}_{-i}) u_i(\mathbf{s}_i, \mathbf{s}_{-i}),\ \forall \mathbf{s}_i \in S_i \tag{EC.4}$$

Recall the definition of set of goods M_i that agent i care for. That is, at NE i plays $\mathbf{s}_i$ with non-zero probability only if $s_{ij} = 0$ for all $j \notin M_i$. Next we will show that function $u_i^{\boldsymbol{\sigma}_{-i}}$ is strictly concave on the entire domain of $\mathbb{R}^{|M_i|}$ given that $\boldsymbol{\sigma}_{-i}$ is a NE, therefore it is strictly concave on S_i as well. For this, first we show that u_i is (strictly) concave in $\mathbf{s}_i$.

Lemma EC.7. *For a fixed $\mathbf{s}_{-i} \in S_{-i}$, function u_i is concave in $\mathbf{s}_i$, and is strictly concave if $\mathbf{s}_{-i}$ satisfies $\sum_{k\neq i} s_{ij} > 0, \forall j \in M_i$.*

Proof. We will show a general claim, and then apply it to our setting. Let $[m]$ denote the set $\{1, \ldots, m\}$.

Claim EC.2. *Given concave, non-decreasing function $f : \mathbb{R}^m_+ \to \mathbb{R}$, and m single variate concave functions $g_i : \mathbb{R}_+ \to \mathbb{R}_+$ for $i \in [m]$, composition function $f(g_1(x_1), \ldots, g_m(x_m))$ is concave. Further if either f is strictly concave, or all g_i's are strictly concave and $f(\mathbf{x} + \boldsymbol{\delta}) > f(\mathbf{x})$ for $\boldsymbol{\delta} > 0$, then $f(g_1(x_1), \ldots, g_m(x_m))$ is strictly concave.*

Proof. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m_+$, and $\mathbf{x}'$ and $\mathbf{y}'$ be such that $x'_i = g_i(x_i)$ and $y'_i = g_i(y_i)$ for all $i \in [m]$. Let $\lambda \in (0, 1)$ be a constant and $\mathbf{z} = \lambda\mathbf{x} + (1-\lambda)\mathbf{y}$, then by concavity of g_i's we get $g_i(z_i) \geq \lambda x'_i + (1-\lambda)y'_i, \forall i \in [m]$. Using this we get,

$$\begin{aligned}
&\lambda f(g_1(x_1), \ldots, g_m(x_m)) + (1-\lambda) f(g_1(y_1), \ldots, g_m(y_m)) \\
&\qquad = \lambda f(\mathbf{x}') + (1-\lambda) f(\mathbf{y}') \\
&\qquad \leq f(\lambda\mathbf{x}' + (1-\lambda)\mathbf{y}') \qquad (\because\ f \text{ is concave}) \\
&\qquad \leq f(g_1(z_1), \ldots, g_m(z_m)) \quad (\because\ g_i\text{s are concave and } f \text{ is non-decreasing})
\end{aligned}$$

Above, the second inequality becomes strict if f is strictly concave, and the third inequality becomes strict if g_i's are strictly concave and $f(\mathbf{x}+\boldsymbol{\delta}) > f(\mathbf{x})$ if $\boldsymbol{\delta} > 0$. $\square$

For our purpose, set $f = u_i$ which is concave and non-decreasing. Given $\mathbf{s}_{-i}$ let $D_j = \sum_{k\neq i} s_{ij}$ and set $g_j(s_{ij}) = \frac{s_{ij}}{s_{ij}+D_j}$, $\forall j \in M$. Clearly, where $\frac{0}{0}$ is considered as 0.

$$\frac{\partial g_j}{\partial s_{ij}} = \frac{D_j}{(s_{ij}+D_j)^2} \geq 0, \qquad \frac{\partial^2 g_j}{\partial s_{ij}^2} = \frac{-2D_j}{(s_{ij}+D_j)^3} \leq 0$$

Thus, g_j is concave, and is strictly concave if $D_j > 0$. Clearly, $u_i(.,\mathbf{s}_{-i}) = f(g_1(s_{i1}),\ldots,g_m(s_{im}))$ is concave by the above claim. Further more, if for every good $j \in M_i$, $D_j > 0$ then no matter how agent i bids, she has to share every good that she cares for with some other agent and hence can not achieve utility of $u_i(\mathbf{1})$. In that case, we have that $f(\mathbf{x}+\boldsymbol{\delta}) > f(\mathbf{x})$ if $\boldsymbol{\delta} > 0$, and the strict concavity follows using the above claim. $\square$

Using the property of u_i established in the above lemma, next we will show that Hessian of $u_i^{\boldsymbol{\sigma}_{-i}}$ is negative definite almost always.

Lemma EC.8. *Given a mixed-strategy profile $\boldsymbol{\sigma}_{-i}$ of all agents $k \neq i$ such that there is an $\mathbf{s}'_{-i} \in S_{-i}$ played with positive probability where $\sum_{k\neq i} s'_{k,j} > 0$, $\forall j$, the payoff function of agent i, namely $u_i^{\boldsymbol{\sigma}_{-i}}$, is strictly concave.*

Proof. Fix any $\mathbf{s}_i \in S_i$, and let H be the Hessian of $u_i^{\boldsymbol{\sigma}_{-i}}$ at $\mathbf{s}_i$. By definition of strictly concave functions it suffices to show that H is negative definite. Let $H(\mathbf{s}_{-i})$ be the Hessian of $u_i(.,\mathbf{s}_{-i})$ at $\mathbf{s}_i$ for each $\mathbf{s}_{-i} \in S_{-i}$. By definition of $u_i^{\boldsymbol{\sigma}_{-i}}$ from (EC.4) Hessian of $u_i^{\boldsymbol{\sigma}_{-i}}$ at $\mathbf{s}_i$ is $H = \sum_{\mathbf{s}_{-i}\in S_{-i}} P(\mathbf{s}_{-i})H(\mathbf{s}_{-i})$.

By Lemma EC.7 since each of these $u_i(.,\mathbf{s}_{-i})$ is concave, $H(\mathbf{s}_{-i})$ is negative semi-definite. Furthermore, the one with $\sum_{k\neq i} s'_{k,j} > 0$, $\forall j$ and $P(\mathbf{s}'_{-i}) > 0$ due to the hypothesis gives strictly concave $u_i(.,\mathbf{s}'_{-i})$ (Lemma EC.7), and hence $H(\mathbf{s}'_{-i})$ is negative definite. For any $\mathbf{x} \in \mathbb{R}^m$, we have

$$\mathbf{x}^T H \mathbf{x} = \mathbf{x}^T \left(\sum_{\mathbf{s}_{-i}\in S_{-i}} \alpha(\mathbf{s}_{-i}) H(\mathbf{s}_{-i}) \right) \mathbf{x} = \sum_{\mathbf{s}_{-i}\in S_{-i}} \alpha(\mathbf{s}_{-i})(\mathbf{x}^T H(\mathbf{s}_{-i})\mathbf{x}) < 0$$

The last inequality follows from the fact that $\mathbf{x}^T H(\mathbf{s}_{-i})\mathbf{x} \leq 0$, $\forall \mathbf{s}_{-i} \in S_{-i}$, and $\mathbf{x}^T H(\mathbf{s}'_{-i})\mathbf{x} < 0$. By definition of negative definite matrices, the proof follows. $\square$

Using the strict concavity of $u_i^{\boldsymbol{\sigma}_{-i}}$ established in Lemma EC.8, next we show the main result.

THEOREM EC.3. *In a market with concave utilities and enough competition, every Nash equilibrium of the corresponding Trading Post game, where no agent is satiated, is pure.*

Proof. Let $\boldsymbol{\sigma} = (\boldsymbol{\sigma}_1, \ldots, \boldsymbol{\sigma}_n)$ be a Nash equilibrium profile, and to the contrary suppose it is mixed. Let i be an agent who is playing a mixed strategy. At Nash equilibrium since only strategies in best response can have non-zero probability, there are $\mathbf{s}_i \neq \mathbf{s}'_i \in S_i$ such that $u_i^{\boldsymbol{\sigma}_{-i}}$ is maximized at both.

Due to enough competition for every good j there is an agent $k \neq i$ who always bids on good j, i.e., $\forall \mathbf{s}_k \in S_k$ with $\boldsymbol{\sigma}_k(\mathbf{s}_k) > 0$, we have $s_{kj} > 0$. Therefore, at $\boldsymbol{\sigma}_{-i}$ every $\mathbf{s}_{-i}$ with non-zero probability satisfies $\sum_{k \neq i} s_{kj} > 0, \ \forall j$. Using this Lemma EC.7 implies $u_i^{\boldsymbol{\sigma}_{-i}}$ is strictly concave and there by has a unique maximum. A contradiction to both $\mathbf{s}_i$ and $\mathbf{s}'_i$ being maximizer of $u_i^{\boldsymbol{\sigma}_{-i}}$. $\square$