Online Appendix for "Preservation of Additive Convexity and Its Applications in Stochastic Optimization Problems"

Xiting Gong, Tong Wang

This appendix consists of four parts. In Appendix A, we provide the proofs for Theorems 1 and 2 in §2 and Theorem 4 in §3.3. In Appendices B and C, we provide the technical details for the two applications in §3.1 and for a serial inventory system with disposal, respectively. Finally, we provide our detailed numerical setting and results in Appendix D.

Appendix A: Proofs for Theorems 1, 2, and 4

Proof of Theorem 1. We first state without a proof the following well-known result originated in Karush (1959), which is frequently used in establishing additive convexity of value functions in many stochastic optimization problems, and in particular, serial inventory systems.

LEMMA EC.1. *Suppose that $f(\cdot)$ is a convex function on a convex set $I \subset \Re$. Then, there exist an increasing convex function $g(\cdot)$ on I and a decreasing convex function $h(\cdot)$ on I such that, for any $x \in I$ and $y \in I$ with $x \le y$,*

$$\min_{x \le \theta \le y} f(\theta) = g(x) + h(y).$$

In particular, if $f(\cdot)$ attains its unconstrained minimizer on I at S, then $g(x) = f(\max\{S, x\}) - f(S)$ and $h(x) = f(\min\{S, y\})$.

By Lemma 1, we work with problem (4) and apply Lemma EC.1 to prove Theorem 1. Since $f(\cdot)$ is additively convex, there exist convex functions $f_0(\cdot), \ldots, f_n(\cdot)$ such that $f(\mathbf{x}) = \sum_{i=0}^n f_i(x_i)$ for any $\mathbf{x} \in V$. Then, the equation (4) can be rewritten as

$$\begin{aligned} g(\mathbf{x}) &= \min_{\substack{y_0 \le y_1 \le \cdots \le y_k \le x_{k+1} \\ y_i \ge x_i,\, i=0,\ldots,k}} \left\{ \sum_{i=0}^{k} f_i(y_i) + \sum_{i=k+1}^{n} f_i(x_i) \right\} \\ &= \min_{\substack{y_1 \le \cdots \le y_k \le x_{k+1} \\ y_i \ge x_i,\, i=1,\ldots,k}} \left\{ \min_{x_0 \le y_0 \le y_1} f_0(y_0) + \sum_{i=1}^{k} f_i(y_i) \right\} + \sum_{i=k+1}^{n} f_i(x_i). \end{aligned} \tag{EC.1}$$

Applying Lemma EC.1, we obtain that there exist an increasing convex function $g_0(\cdot)$ and a decreasing convex function $h_0(\cdot)$ such that

$$\min_{x_0 \le y_0 \le y_1} f_0(y_0) = g_0(x_0) + h_0(y_1). \tag{EC.2}$$

Define ξ_0 as a minimizer of $f_0(\cdot)$ on I. Then,

$$g_0(x_0)=f_0(\max\{x_0,\xi_0\})-f_0(\xi_0), \quad \text{and} \quad h_0(y_1)=f_0(\min\{y_1,\xi_0\}).$$

After plugging (EC.2) into (EC.1), we obtain

$$\begin{aligned} g(\mathbf{x}) &= g_0(x_0)+\min_{\substack{y_1\le\cdots\le y_k\le x_{k+1}\\ y_i\ge x_i,\, i=1,\ldots,k}}\left\{h_0(y_1)+\sum_{i=1}^{k} f_i(y_i)\right\}+\sum_{i=k+1}^{n} f_i(x_i)\\ &= g_0(x_0)+\min_{\substack{y_2\le\cdots\le y_k\le x_{k+1}\\ y_i\ge x_i,\, i=2,\ldots,k}}\left\{\min_{x_1\le y_1\le y_2}\{h_0(y_1)+f_1(y_1)\}+\sum_{i=2}^{k} f_i(y_i)\right\}+\sum_{i=k+1}^{n} f_i(x_i). \end{aligned} \tag{EC.3}$$

Since both $h_0(\cdot)$ and $f_1(\cdot)$ are convex, by applying Lemma EC.1 again, we obtain that there exist an increasing convex function $g_1(\cdot)$ and a decreasing convex function $h_1(\cdot)$ such that

$$\min_{x_1\le y_1\le y_2}\{h_0(y_1)+f_1(y_1)\}=g_1(x_1)+h_1(y_2). \tag{EC.4}$$

Define ξ_1 as a minimizer of $f_0(y_1)+f_1(y_1)$ on I. Since $f_0(y_1)$ and $f_1(y_1)$ are both increasing in y_1 when $y_1\ge\xi_0$, it follows that $\xi_1\le\xi_0$. In addition, ξ_1 is also a minimizer of $h_0(y_1)+f_1(y_1)=f_0(\min\{y_1,\xi_0\})+f_1(y_1)$. Then, we have

$$\begin{aligned} g_1(x_1) =& h_0(\max\{x_1,\xi_1\})+f_1(\max\{x_1,\xi_1\})-h_0(\xi_1)-f_1(\xi_1)\\ =& f_0(\min\{\max\{x_1,\xi_1\},\xi_0\})+f_1(\max\{x_1,\xi_1\})-f_0(\xi_1)-f_1(\xi_1),\\ h_1(y_2) =& h_0(\min\{y_2,\xi_1\})+f_1(\min\{y_2,\xi_1\})=\sum_{i=0}^{1} f_i(\min\{y_2,\xi_1\}). \end{aligned}$$

After plugging (EC.4) into (EC.3), we obtain

$$g(\mathbf{x})=\sum_{i=0}^{1} g_i(x_i)+\min_{\substack{y_2\le\cdots\le y_k\le x_{k+1}\\ y_i\ge x_i,\, i=2,\ldots,k}}\left\{h_1(y_2)+\sum_{i=2}^{k} f_i(y_i)\right\}+\sum_{i=k+1}^{n} f_i(x_i). \tag{EC.5}$$

For $i=2,\ldots,k$, define ξ_i as a minimizer of $f_0(y_i)+f_1(y_i)+\cdots+f_i(y_i)$ on I. Then, $\xi_{k-1}\le\xi_{k-2}\le\cdots\le\xi_1$ since $f_i(y_i)$ is increasing in y_i for all $i=2,\ldots,k$. Repeating the above procedure sequentially on $y_2,\ldots,y_k$, we obtain, for $i=2,\ldots,k$, an increasing convex function $g_i(\cdot)$ and a decreasing convex function $h_i(\cdot)$ with the following forms:

$$\begin{aligned} g_i(x_i) &= \sum_{j=0}^{i-1} f_j(\min\{\max\{\xi_i,x_i\},\xi_{i-1}\})+f_i(\max\{\xi_i,x_i\})-\sum_{j=0}^{i} f_j(\xi_i), \\ h_i(y_{i+1}) &= \sum_{j=0}^{i} f_j(\min\{y_{i+1},\xi_i\}). \end{aligned} \tag{EC.6}$$

In addition, we have

$$g(\mathbf{x}) = \sum_{i=0}^{k} g_i(x_i) + h_k(x_{k+1}) + \sum_{i=k+1}^{n} f_i(x_i). \tag{EC.7}$$

Therefore, $g(\mathbf{x})$ is additively convex in $\mathbf{x}$ and increasing in $(x_1, \ldots, x_k)$.

Finally, we prove that $\partial f(\mathbf{x})/\partial x_i \le a$ implies $\partial g(\mathbf{x})/\partial x_i \le a$ for $i = 1, \ldots, n$. For $i = 1, \ldots, k$, $\partial g(\mathbf{x})/\partial x_i = \partial g_i(x_i)/\partial x_i$ by (EC.7). Note that ξ_{i-1} is a minimizer of $\sum_{j=0}^{i-1} f_j(\cdot)$. Then, the desired result follows directly from (EC.6) and the fact that $\sum_{j=0}^{i-1} f_j(\min\{\max\{\xi_i, x_i\}, \xi_{i-1}\})$ is decreasing in x_i. When $i = k+1, \ldots, n$, the desired result directly follows from (EC.7) and the fact that $h_k(\cdot)$ is a decreasing function. **Q.E.D.**

Proof of Theorem 2. We prove Theorem 2 by applying Theorem 1. As preparation, we first introduce two auxiliary functions. Define

$$\widetilde{V} = \{\mathbf{u} \in (-I) \times \Re^n \mid u_0 \le u_1 \le \cdots \le u_n\},$$

where $(-I) := \{x \in \Re \mid -x \in I\}$. For any $\mathbf{u} = (u_0, u_1, \ldots, u_n) \in \widetilde{V}$, we define

$$\tilde{f}(\mathbf{u}) = f(-u_n, -u_{n-1}, \ldots, -u_0), \quad \text{and} \quad \tilde{h}(\mathbf{u}) = h(-u_n, -u_{n-1}, \ldots, -u_0).$$

Note that for any $\mathbf{u} = (u_0, u_1, \ldots, u_n) \in V$, we have $(-u_n, -u_{n-1}, \ldots, -u_0) \in V$. Thus, both $\tilde{f}(\mathbf{u})$ and $\tilde{h}(\mathbf{u})$ are well defined on $\widetilde{V}$.

With these two auxiliary functions, when $k \ge 1$, we can rewrite the optimization problem (5) as

$$\begin{aligned} \tilde{h}(\mathbf{u}) &= h(-u_n, -u_{n-1}, \ldots, -u_0) \\ &= \min_{\mathbf{y} \in \mathbb{C}(-u_n, -u_{n-1}, \ldots, -u_0)} f(-u_n, -u_{n-1}, \ldots, -u_{n-k+1}, \mathbf{y}) \\ &= \min_{\mathbf{y} \in \mathbb{C}(-u_n, -u_{n-1}, \ldots, -u_0)} \tilde{f}(-y_n, -y_{n-1}, \ldots, -y_k, u_{n-k+1}, \ldots, u_n), \end{aligned} \tag{EC.8}$$

We next transform the decision vector $\mathbf{y}$ in (EC.8) by $\mathbf{v} := (v_0, v_1, \ldots, v_{n-k})$, where $v_j = -y_{n-j}$ for $j = 0, \ldots, n-k$. Then, after some simple algebra we can rewrite (EC.8) as

$$\tilde{h}(\mathbf{u}) = \min_{\mathbf{v} \in \widetilde{\mathbb{C}}(\mathbf{u})} \tilde{f}(\mathbf{v}, u_{n-k+1}, \ldots, u_n), \tag{EC.9}$$

where

$$\widetilde{\mathbb{C}}(\mathbf{u}) = \{\mathbf{v} \in \Re^{n-k+1} \mid (\mathbf{v}, u_{n-k+1}, \ldots, u_n) \in \widetilde{V},\ v_0 - u_0 \ge v_1 - u_1 \ge \cdots \ge v_{n-k} - u_{n-k} \ge 0\}.$$

Since $f(\mathbf{x})$ is additively convex in $\mathbf{x}$ on V and decreasing in $x_k, \ldots, x_{n-1}$, it follows that $\tilde{f}(\mathbf{u})$ is additively convex in $\mathbf{u}$ on $\widetilde{V}$ and increasing in $u_1, u_2, \ldots, u_{n-k}$. After applying Theorem 1 on the optimization problem (EC.9), we obtain that $\tilde{h}(\mathbf{u})$ is additively convex in $\mathbf{u}$ on $\widetilde{V}$ and increasing in $u_1,\ldots,u_{n-k}$. Consequently, $h(\mathbf{x}) = \tilde{h}(-x_n, -x_{n-1},\ldots,-x_0)$ is additively convex in $\mathbf{x}$ on V and decreasing in $x_k,\ldots,x_{n-1}$. In addition, if there exists a constant b such that $\partial f(\mathbf{x})/\partial x_m \ge b$, it follows that $\partial \tilde{f}(\mathbf{u})/\partial u_{n-m} \le -b$. By applying Theorem 1, we obtain that $\partial \tilde{h}(\mathbf{u})/\partial u_{n-m} \le -b$ and therefore $\partial h(\mathbf{x})/\partial x_m \ge b$. **Q.E.D.**

Proof of Theorem 4. Since $x_j^{t_0} \le \tilde{D}_j^{t_0}$ for all $j \ne i$, it follows that $x_j^t \le \tilde{D}_j^t$ for all $j \ne i$ and $t = t_0,\ldots,T$. Note from Yu et al. (2015) that parallel allocation of capacities to demands is optimal in each period. After parallel allocating capacities to demands in all classes but class i, the system generates the total profit $\sum_{j\ne i}(r_j - u_j)x_j^t$, and the system state becomes

$$(0,\ldots,0,x_i^t,0,\ldots,0,\tilde{D}_1^t - x_1^t,\ldots,\tilde{D}_{i-1}^t - x_{i-1}^t,\tilde{D}_i^t,\tilde{D}_{i+1}^t - x_{i+1}^t,\ldots,\tilde{D}_N^t - x_N^t). \tag{EC.10}$$

Since the demands in classes $1,\ldots,i-1$ cannot be satisfied by the class-i capacity, the unsatisfied demands $\tilde{D}_1^t - x_1^t,\ldots,\tilde{D}_{i-1}^t - x_{i-1}^t$ after the parallel allocation and all newly arrived demands in classes $1,\ldots,i-1$ will be fully backlogged until the end of the sales horizon, incurring the following expected total backlogging cost:

$$\sum_{j=1}^{i-1} g_j\Big((T+1-t)(\tilde{D}_j^t - x_j^t) + \sum_{s=t}^{T}(T+1-s)\mathbb{E}[D_j^t]\Big).$$

For $j = 1,\ldots,i-1$, define $\Theta_j^t(d) = g_j((T+1-t)d - \sum_{s=t}^T (T+1-s)\mathbb{E}[D_j^t])$. Then, $\Theta_1^t(\cdot),\ldots,\Theta_{i-1}^t(\cdot)$ are linear functions, and we can rewrite the above cost as $-\sum_{j=1}^{i-1}\Theta_j^t(x_j^t - \tilde{D}_j^t)$.

We next consider the remaining part of the value function $\Theta^t(\mathbf{X}^t,\tilde{\mathbf{D}}^t)$ associated with classes $i,\ldots,N$. From the system state (EC.10), the effective state variables are the class-i capacity x_i^t and the outstanding demands $(\tilde{D}_i^t,\tilde{D}_{i+1}^t - x_{i+1}^t,\ldots,\tilde{D}_N^t - x_N^t)$ in classes $i,\ldots,N$. To apply Theorem 1, we transform these state variables to $\hat{\mathbf{x}}^t := (\hat{x}_0^t,\hat{x}_i^t,x_{i+1}^t,\ldots,\hat{x}_N^t)$, where $\hat{x}_0^t = -x_i^t$, $\hat{x}_i^t = \hat{x}_0^t + \tilde{D}_i^t$, and $\hat{x}_j^t = \hat{x}_{j-1}^t + \tilde{D}_j^t - x_j^t$, $j = i+1,\ldots,N$. The state space for $\hat{\mathbf{x}}^t$ is

$$V := \{\mathbf{x} \in (-\infty,0] \times \Re^{N-i+1} \mid x_0 \le x_i \le x_{i+1} \le \cdots \le x_N\}.$$

Define $V_t(\hat{\mathbf{x}})$ as the optimal expected total *cost* from period t onward, given that the system state is $\hat{\mathbf{x}}$ at the beginning of period t. For convenience, define $\bar{\mathbf{x}}^t := (\bar{x}_0^t,\bar{x}_i^t,\ldots,\bar{x}_N^t)$, where $\bar{x}_0^t = \hat{x}_0^t$ and $\bar{x}_j^t = \hat{x}_j^t + D_i^t + \cdots + D_j^t$, $j = i,\ldots,N$. Then, $\bar{\mathbf{x}}^t \in V$ and it is the system state after demand $\mathbf{D}_t$ realizes. Define $\bar{V}_t(\bar{\mathbf{x}})$ as the optimal expected total *cost* from period t onward, given that the system state is $\bar{\mathbf{x}}$ after demand $\mathbf{D}^t$ realizes in period t. Then,

$$V_t(\hat{\mathbf{x}}) = \mathbb{E}_{\mathbf{D}^t}\left\{\bar{V}_t(\hat{x}_0,\hat{x}_i + D_i^t,\hat{x}_{i+1} + \sum_{j=i}^{i+1} D_j^t,\ldots,\hat{x}_N + \sum_{j=i}^{N} D_j^t)\right\}. \tag{EC.11}$$

With the system state $\bar{\mathbf{x}}$, the firm needs to make decisions $y^t_{ii}, y^t_{i,i+1}, \ldots, y^t_{i,N}$ subject to the constraints $0 \le y^t_{ij} \le \bar{x}_j - \bar{x}_{j-1}$, $j = i, \ldots, N$ and $\sum_{j=i}^N y^t_{ij} + \bar{x}_0 \le 0$. We further transform these decision variables to $\bar{\mathbf{y}} := (\bar{y}_0, \bar{y}_i \ldots, \bar{y}_{N-1})$, where $\bar{y}_j = \bar{x}_j + \sum_{i'=j+1}^N y^t_{i,i'}$, $j = 0, i, \ldots, N-1$. Then, the decision constraints for $\bar{\mathbf{y}}$ is given by

$$\mathbb{C}(\bar{\mathbf{x}}) = \{\bar{\mathbf{y}} \in (-\infty, 0] \times \Re^{N-i} \mid (\bar{\mathbf{y}}, \bar{x}_N) \in V,\ \bar{y}_0 - \bar{x}_0 \ge \bar{y}_i - \bar{x}_i \ge \cdots \ge \bar{y}_{N-1} - \bar{x}_{N-1} \ge 0\}.$$

With the system state $\bar{\mathbf{x}}$, the value functions $V_t(\hat{\mathbf{x}})$ and $\bar{V}_t(\bar{\mathbf{x}})$, and the new decision variables $\bar{\mathbf{y}}$, after simple algebra we can rewrite the optimality equation (11) as follows: for $t = t_0, \ldots, T$,

$$\begin{aligned}\bar{V}_t(\bar{\mathbf{x}}) = \min_{\bar{\mathbf{y}} \in \mathbb{C}(\bar{\mathbf{x}})} \left\{ -\alpha_{ii}\bar{y}_0 + \sum_{j=i}^{N-1} (\alpha_{i,j} - \alpha_{i,j+1})\bar{y}_j + V_{t+1}(\bar{\mathbf{y}}, \bar{x}_N) \right\} \\ + (\alpha_{ii} - g_i)\bar{x}_0 + \sum_{j=i}^{N-1} (\alpha_{i,j+1} - \alpha_{i,j} + g_j - g_{j+1})\bar{x}_j + g_N \bar{x}_N. \qquad \text{(EC.12)}\end{aligned}$$

The boundary condition is given by $V_{T+1}(\cdot) \equiv 0$.

We next prove by induction on t that $V_t(\hat{\mathbf{x}})$ is additively convex in $\hat{\mathbf{x}}$ on V and $V_t(\hat{\mathbf{x}}) + (\alpha_{i,j} - \alpha_{i,j+1})\hat{x}_j$ is increasing in $\hat{x}_j$, $j = 1, \ldots, N-1$. Since $V_{T+1}(\cdot) = 0$ and α_{ij} is decreasing in j, these results clearly hold for period $T+1$. Suppose that they hold for period $t+1$. Then, it follows that the minimand in (EC.12) is additively convex in $(\bar{\mathbf{y}}, \bar{x}_N)$ and increasing in $(\bar{y}_i, \ldots, \bar{y}_{N-1})$. Applying Theorem 1 on the minimization problem (EC.12) with $k = N - i$ and $n = N - i + 1$, and noting that $g_i > g_{i+1} > \cdots > g_N$, we obtain that $\bar{V}_t(\bar{\mathbf{x}})$ is additively convex in $\bar{\mathbf{x}}$ on V and $\bar{V}_t(\bar{\mathbf{x}}) + (\alpha_{i,j} - \alpha_{i,j+1})\bar{x}_j$ is increasing in $\bar{x}_j$, $j = i, \ldots, N-1$. By (EC.11), we have $V_t(\hat{\mathbf{x}})$ is additively convex in $\hat{\mathbf{x}}$ on V and $V_t(\hat{\mathbf{x}}) + (\alpha_{i,j} - \alpha_{i,j+1})\hat{x}_j$ is increasing in $\hat{x}_j$, $j = i, \ldots, N-1$. Thus, the results hold for period t. The induction proof is complete.

Then, for $t = t_0, \ldots, T$ and $\hat{\mathbf{x}} \in V$, we have $V_t(\hat{\mathbf{x}}) = V_{t,0}(\hat{x}_0) + V_{t,i}(\hat{x}_i) + \cdots + V_{t,N}(\hat{x}_N)$, where $V_{t,0}(\cdot), V_{t,i}(\cdot) \ldots, V_{t,N}(\cdot)$ are convex functions. Define

$$\Theta^t_{i0}(x) = -V_{t,0}(-x), \quad \text{and} \quad \Theta^t_{i'}(x) = -V_{t,i'}(-x),\ i' = i, \ldots, N. \qquad \text{(EC.13)}$$

Then, $\Theta^t_{i0}(\cdot)$, $\Theta^t_i(x), \ldots, \Theta^t_N(x)$ are concave functions and $\Theta^t_j(x) + (\alpha_{i,j} - \alpha_{i,j+1})x$ is increasing in x for $j = i, \ldots, N-1$. One can easily verify that

$$\begin{aligned}\Theta^t(\mathbf{X}^t, \tilde{\mathbf{D}}^t) &= \sum_{j=1}^{i-1} \Theta^t_j(x^t_j - \tilde{D}^t_j) - V_t(-x^t_i, -x^t_i + \tilde{D}^t_i, \sum_{j=i}^{i+1}(-x^t_j + \tilde{D}^t_j), \ldots, \sum_{j=i}^{N}(-x^t_j + \tilde{D}^t_j)) + \sum_{j \ne i}(r_j - u_j)x^t_j \\ &= \sum_{j=1}^{i-1} \Theta^t_j(x^t_j - \tilde{D}^t_j) + \Theta^t_{i0}(x^t_i) + \sum_{i'=i}^{N} \Theta^t_{i'}(\sum_{j=i}^{i'}(x^t_j - \tilde{D}^t_j)) + \sum_{j \ne i}(r_j - u_j)x^t_j.\end{aligned}$$

Therefore, the decomposition result on $\Theta^t(\mathbf{X}^t, \tilde{\mathbf{D}}^t)$ holds.

Following Definition 1 in Yu et al. (2015) and the decomposition result on $\Theta^t(\mathbf{X}^t, \tilde{\mathbf{D}}^t)$, we can express the optimal protection level p^t_{ij} as follows: for $j = i+1, \dots, N$,

$$p^t_{ij} = \arg\max_{p \geq 0} \{-a_{ij} p + \Theta^{t+1}_{i0}(p) + \sum_{i'=i}^{j-1} \Theta^{t+1}_{i'}(p)\}.$$

Since $\Theta^{t+1}_j(x) + (\alpha_{i,j} - \alpha_{i,j+1})x$ is increasing in x for $j = i+1, \dots, N-1$, one can easily verify that $0 \leq p^t_{i,i+1} \leq p^t_{i,i+2} \leq \cdots \leq p^t_{i,N}$. The expression of the optimal upgrading quantity y^{t*}_{ij} follows directly from the remark after Definition 1 in Yu et al. (2015).

Finally, we provide the detailed procedure and formulas for computing $\Theta^t_{i0}(\cdot)$ and $\{\Theta^t_j(\cdot)\}_{j=i}^N$. By their definitions in (EC.13), it suffices to compute $V_{t,0}(\cdot)$ and $\{V_{t,j}(\cdot)\}_{j=i}^N$. The calculation is proceeded recursively and backwardly on periods $T, T-1, \dots, t+1, t$, with the boundary condition $V_{T+1,0}(\cdot) = V_{T+1,i}(\cdot) = \cdots = V_{T+1,N}(\cdot) = 0$. Suppose we have computed the functions $V_{t+1,0}(\cdot)$ and $\{V_{t+1,j}(\cdot)\}_{j=i}^N$. We describe below how to compute $V_{t,0}(\cdot)$ and $\{V_{t,j}(\cdot)\}_{j=i}^N$.

First, we compute the convex functions $\bar{V}_{t,0}(\cdot)$ and $\{\bar{V}_{t,j}(\cdot)\}_{j=i}^N$ such that the additively convex function $\bar{V}_t(\bar{x}_0, \bar{x}_i, \dots, \bar{x}_N) := \bar{V}_{t,0}(\bar{x}_0) + \sum_{j=i}^N \bar{V}_{t,j}(\bar{x}_j)$. To this end, we apply the formulas in the proof of Theorem 1. Define $f_{t,0}(x) = -\alpha_{ii} x + V_{t+1,0}(x)$, and

$$f_{t,j}(x) = (\alpha_{i,j} - \alpha_{i,j+1})x + V_{t+1,j}(x), \quad \text{for } j = i, \dots, N-1.$$

One can easily prove by induction on t that $f_{t,0}(x)$ is decreasing in x. In addition, $f_{t,j}(x)$ is increasing in x for $j = i, \dots, N-1$. Define $\xi_{t,0} = \arg\min_{x \leq 0} f_{t,0}(x)$ and $\xi_{t,j} = \arg\min_{x \leq 0}\{f_{t,0}(x) + f_{t,i}(x) + \cdots + f_{t,j}(x)\}$, for $j = i, \dots, N-1$. Then, $0 = \xi_{t,0} \leq \xi_{t,i} \leq \cdots \leq \xi_{t,N-1}$. Further, one can easily verify that $p^t_{ij} = -\xi_{t,j}$, for $j = i+1, \dots, N$. Thus, the protection levels $\{p^t_{ij}\}_{j=i+1}^N$ can also be computed through the computation of $\{\xi_{t,j}\}_{j=i}^{N-1}$.

By applying the formula of $g(\mathbf{x})$ in the proof of Theorem 1 on (EC.12), we obtain that $\bar{V}_{t,0}(x) = (\alpha_{ii} - g_i)x$, and

$$\begin{aligned}
\bar{V}_{t,i}(x) =& f_{t,0}(\min\{\max\{\xi_i, x\}, 0\}) + f_{t,i}(\max\{\xi_i, x\}) - f_{t,0}(\xi_i) - f_{t,i}(\xi_i) + (\alpha_{i,i+1} - \alpha_{i,i} + g_i - g_{i+1})x, \\
\bar{V}_{t,j}(x) =& f_{t,0}(\min\{\max\{\xi_j, x_j\}, \xi_{j-1}\}) + \sum_{i'=i}^{j-1} f_{t,i'}(\min\{\max\{\xi_j, x_j\}, \xi_{j-1}\}) + f_{t,j}(\max\{\xi_j, x_j\}) \\
& - f_{t,0}(\xi_j) - \sum_{i'=i}^{j} f_{t,i'}(\xi_j) + (\alpha_{i,j+1} - \alpha_{i,j} + g_j - g_{j+1})x_j \quad \text{for } j = i+1, \dots, N-1, \\
\bar{V}_{t,N}(x) =& f_{t,0}(\min\{\xi_{N-1}, x\}) + \sum_{i'=i}^{N-1} f_{t,i'}(\min\{\xi_{N-1}, x\}) + V_{t+1,N}(x) + g_N x.
\end{aligned}$$

Second, we compute the convex functions $V_{t,0}(\cdot)$ and $\{V_{t,j}(\cdot)\}_{j=i}^{N}$. From identity (EC.11), we have

$$V_{t,0}(x) = (\alpha_{ii} - g_i)x, \quad \text{and} \quad V_{t,j}(x) = \mathbb{E}_{\mathbf{D}^t}[\bar{V}_{t,j}(x + \sum_{i'=i}^{j} D_{i'}^t)], \; j = i, \ldots, N.$$

This completes the computation of $V_{t,0}(\cdot)$ and $\{V_{t,j}(\cdot)\}_{j=i}^{N}$. **Q.E.D.**

Appendix B: Technical Details for Section 3.1

Appendix B.1: Stochastic Inventory Control with Remanufacturing

In this appendix, we revisit the stochastic inventory control problem for a hybrid manufacturing/remanufacturing inventory system with multiple types of returns studied by Zhou et al. (2011), and apply our two preservation results (i.e., Theorems 1 & 2) to prove an additive-convexity result established in Zhou et al. (2011).

Consider a periodic-review inventory system which produces a single product over a planning horizon of T periods, indexed by $1, 2, \ldots, T$. In each period t, the manufacturer receives random customer demand D_t for the serviceable product and K types of random customer returns of cores with varying physical conditions, denoted by $R_{t,1}, \ldots, R_{t,K}$. The demands and returns in different periods are independent, but they could have an arbitrary joint probability distribution within the same period. A serviceable product can be either manufactured from raw materials with ample supply or remanufactured from each type of cores, with identical zero lead times. The unit manufacturing cost is p, and the unit remanufacturing cost of a type-k core is r_k, $k = 1, \ldots, K$, with $r_1 \le r_2 \le \cdots \le r_K < p$. There are a unit holding cost s and a unit disposal cost u for cores, regardless of their types. We assume $p + u - r_K \ge 0$ to ensure that producing a serviceable product via remanufacturing a type-K core is less costly than via manufacturing and meanwhile disposing a type-K core (since otherwise the manufacturer will never remanufacture but dispose all type-K cores). Unsatisfied demand in each period is backlogged. The expected holding and backlogging cost function of the serviceable products in period t is denoted by $G_t(\cdot)$, which is assumed to be convex. With the one-period discount factor $\alpha \in (0, 1]$, the manufacturer's objective is to find the optimal manufacturing, remanufacturing, and disposal policy in each period that minimizes its expected total discounted cost over the planning horizon.

For each period t, the system state can be represented by the vector $\mathbf{x}_t = (x_{t,0}, \ldots, x_{t,K})$, where $x_{t,0}$ is the starting inventory level of the serviceable product and $x_{n,k}$ is the aggregate starting inventory level of the serviceable product and type-1 to type-k cores, $k = 1, \ldots, K$. Since unsatisfied demand in each period is backlogged, the state space is given by

$$V := \{\mathbf{x} \in \Re^{K+1} \mid x_0 \le x_1 \le \cdots \le x_K\}.$$

Given the system state $\mathbf{x}_t$, denote $V_t(\mathbf{x}_t)$ as the optimal expected total discounted cost from period t onward, $t=1,\ldots,T$. For $t=1,\ldots,T$, the optimality equation can be written as

$$\begin{aligned} V_t(\mathbf{x}_t) = \min_{\mathbf{w}_t,\mathbf{y}_t} \Big\{ & \sum_{k=1}^{K} (r_k w_{t,k} + s(y_{t,k} - y_{t,k-1})) + p\big(y_{t,0} - x_{t,0} - \sum_{k=1}^{K} w_{t,k}\big) \\ & + \sum_{k=1}^{K} u(x_{t,k} - x_{t,k-1} - (y_{t,k} - y_{t,k-1}) - w_{t,k}) + G_t(y_{t,0}) \\ & + \alpha \mathbb{E}\big[V_{t+1}\big(y_{t,0} - D_t, y_{t,1} - D_t + R_{t,1}, \ldots, y_{t,K} + \sum_{k=1}^{K} R_{t,k} - D_t\big)\big]\Big\}, \end{aligned} \tag{EC.14}$$

subject to the decision constraints

$$\begin{aligned} & 0 \le w_{t,k} \le x_{t,k} - x_{t,k-1} - (y_{t,k} - y_{t,k-1}), \quad k = 1,\ldots,K, \\ & y_{t,k} \ge y_{t,k-1}, \quad k=1,\ldots,K, \\ & \sum_{k=1}^{K} w_{t,k} \le y_{t,0} - x_{t,0}. \end{aligned}$$

And the boundary condition is given by $V_{t+1}(\cdot) \equiv 0$.

In equation (EC.14), the decision variables are $\mathbf{w}_t = (w_{t,1},\ldots,w_{t,K})$ and $\mathbf{y}_t = (y_{t,0},\ldots,y_{t,K})$, where $w_{t,k}$ is the remanufacturing quantity of type-k cores, $y_{t,0}$ is the inventory level of the serviceable product after manufacturing and remanufacturing decisions but before the demand D_t is realized, and $y_{t,k}$ is the aggregate inventory level of the serviceable product and type-1 to type-k cores after all decisions are made but before demand and returns occur. Then, the manufacturing quantity is $y_{t,0} - x_{t,0} - \sum_{k=1}^{K} w_{t,k}$, and the disposal quantity of type-k cores is $x_{t,k} - x_{t,k-1} - (y_{t,k} - y_{t,k-1}) - w_{t,k}$, $k=1,\ldots,K$. The decision constraints above ensure that the manufacturing, remanufacturing and disposal quantities are all nonnegative; and the ending inventory levels of all types of cores are nonnegative. Inside the brackets, the first term is the total remanufacturing and holding cost of cores; the second term is the total manufacturing cost; the next two terms are the total disposal cost and the expected inventory holding and backlogging cost of the serviceable product; and the last term is the optimal expected discounted cost-to-go from period $n+1$ onward.

The dynamic programming problem (EC.14) appears very complex: the value function $V_t(\mathbf{x}_t)$ has a state space V of $K+1$ dimensions and there are in total $2K+1$ decision variables in each period. It turns out that, however, this problem has a very simple optimal policy structure since the value function $V_t(\mathbf{x}_t)$ is additively convex for each period n. Zhou et al. (2011) established the following important result.

THEOREM EC.1 **(Zhou et al., 2011)**. *For $t=1,2,\ldots,T+1$,* *(a)* $V_t(\mathbf{x}_t)$ *is additively convex in* $\mathbf{x}_t$ *on* V*;*

(b) $-(r_{k+1}-r_k) \le \partial V_t(\mathbf{x}_t)/\partial x_{t,k} \le 0$*, for* $k=1,\dots,K-1$*, and* $\partial V_t(\mathbf{x}_t)/\partial x_{t,K} \ge -(p-r_K)$*.*

To prove this result, Zhou et al. (2011) fully characterize the optimal policy in period t by inductively assuming Theorem EC.1 holds for period $t+1$, and based on that verify parts (a) and (b) for period t by discussing case-by-case the forms of $V_t(\mathbf{x}_t)$ in different regions and their boundaries in the whole state space V. Since the optimal policy divides the state space into multiple regions, the resulting proof is very complex and tedious even for the case with $K=2$. We show how to apply our preservation results to prove Theorem EC.1 as follows.

As preparation to prove Theorem EC.1, we first replace the decision vector $\mathbf{w}_t=(w_{t,1},\dots,w_{t,K})$ by $\mathbf{z}_t=(z_{t,1},\dots,z_{t,K})$ where $z_{t,k}=x_{t,k}+y_{t,0}-x_{t,0}-\sum_{i=1}^k w_{t,i}$ for $k=1,\dots,K$. Here, $z_{t,k}$ represents the aggregate inventory level of the serviceable product and type-1 to type-k cores after the manufacturing and remanufacturing decisions but before the disposal decisions. It is easy to calculate that

$$w_{t,1}=y_{t,0}-x_{t,0}-(z_{t,1}-x_{t,1}),$$

and for $k=2,\dots,K$,

$$w_{t,k}=z_{t,k-1}-x_{t,k-1}-(z_{t,k}-x_{t,k}).$$

Thus, choosing the decision vector $\mathbf{z}_t$ is equivalent to choosing $\mathbf{w}_t$. With the decision vector $\mathbf{z}_t$ and after some simple algebra, we can rewrite the optimality equation (EC.14) as follows

$$\begin{aligned}V_t(\mathbf{x}_t)=\min_{\mathbf{y}_t,\mathbf{z}_t}\Big\{&\sum_{k=1}^{K-1}(r_{k+1}-r_k)z_{t,k}+(p+u-r_K)z_{t,K}+(r_1-s)y_{t,0}+G_t(y_{t,0})+(s-u)y_{t,K}\\&+\alpha\mathbb{E}\big[V_{t+1}\big(y_{t,0}-D_t,y_{t,1}-D_t+R_{t,1},\dots,y_{t,K}+\sum_{k=1}^K R_{t,k}-D_t\big)\big]\Big\}\\&-r_1x_{t,0}-\sum_{k=1}^{K-1}(r_{k+1}-r_k)x_{t,k}-(p-r_K)x_{t,K},\end{aligned}\tag{EC.15}$$

subject to the constraints

$$\begin{aligned}&0\le y_{t,1}-y_{t,0}\le z_{t,1}-y_{t,0}\le x_{t,1}-x_{t,0},\\&0\le y_{t,k}-y_{t,k-1}\le z_{t,k}-z_{t,k-1}\le x_{t,k}-x_{t,k-1},\ \text{for } k=2,\dots,K,\\&x_{t,K}\le z_{t,K}.\end{aligned}$$

We next divide problem (EC.15) into two layers of optimization problems, where we first optimize the manufacturing and remanufacturing decisions in the first layer and then optimize the disposal decisions in the second layer. Specifically, we define, for any $(y_{t,0},\mathbf{z}_t)\in V$,

$$\hat V_t(y_{t,0},\mathbf{z}_t)=\min_{\substack{\mathbf{y}_t\in V\\0\ge y_{t,1}-z_{t,1}\ge\cdots\ge y_{t,K}-z_{t,K}}} f_t(\mathbf{y}_t),\tag{EC.16}$$

where

$$f_t(\mathbf{y}_t) = (r_1 - s)y_{t,0} + G_t(y_{t,0}) + (s-u)y_{t,K} \\ +\alpha\mathbb{E}\big[V_{t+1}\big(y_{t,0}-D_t, y_{t,1}-D_t+R_{t,1},\ldots,y_{t,K}+\sum_{k=1}^{K}R_{t,k}-D_t\big)\big]. \tag{EC.17}$$

Then, the optimality equation (EC.15) can be rewritten as

$$V_t(\mathbf{x}_t) = \min_{\substack{(y_{t,0},\mathbf{z}_t)\in V \\ y_{t,0}-x_{t,0}\geq z_{t,1}-x_{t,1}\geq\cdots\geq z_{t,K}-x_{t,K}\geq 0}} g_t(y_{t,0},\mathbf{z}_t) - r_1 x_{t,0} - \sum_{k=1}^{K-1}(r_{k+1}-r_k)x_{t,k} - (p-r_K)x_{t,K}, \tag{EC.18}$$

where

$$g_t(y_{t,0},\mathbf{z}_t) = \sum_{k=1}^{K-1}(r_{k+1}-r_k)z_{t,k} + (p+u-r_K)z_{t,K} + \hat{V}_t(y_{t,0},\mathbf{z}_t). \tag{EC.19}$$

We now prove Theorem EC.1 by induction on n. Since $V_{t+1}(\mathbf{x}) \equiv 0$, the theorem obviously holds for period $T+1$. Now assume inductively that the theorem holds for period $t+1$. In what follows, we prove that it also holds for period t. By the inductive assumption, $V_{t+1}(\mathbf{x})$ is additively convex in $\mathbf{x}$ and satisfies the properties in Theorem EC.1(b). Then, it follows from (EC.17) that $f_t(\mathbf{y}_t)$ is additively convex in $\mathbf{y}_t$ and satisfies

$$-\alpha(r_{k+1}-r_k) \leq \partial f_t(\mathbf{y}_t)/\partial y_{t,k} \leq 0, \text{ for } k=1,\ldots,K-1,$$

and $\partial f_t(\mathbf{y}_t)/\partial y_{t,K} \geq s-u-\alpha(p-r_K)$. By applying Theorem 2 on problem (EC.16), we obtain that $\hat{V}_t(y_{t,0},\mathbf{z}_t)$ is additively convex in $(y_{t,0},\mathbf{z}_t)$ and satisfies

$$-\alpha(r_{k+1}-r_k) \leq \partial \hat{V}_t(y_{t,0},\mathbf{z}_t)/\partial z_{t,k} \leq 0, \text{ for } k=1,\ldots,K-1,$$

and $\partial \hat{V}_t(y_{t,0},\mathbf{z}_t)/\partial z_{t,K} \geq \min\{s-u-\alpha(p-r_K),0\}$. Then, it follows from (EC.19) that $g_t(y_{t,0},\mathbf{z}_t)$ is additively convex in $(y_{t,0},\mathbf{z}_t)$ and satisfies

$$(1-\alpha)(r_{k+1}-r_k) \leq \partial g_t(y_{t,0},\mathbf{z}_t)/\partial z_{t,k} \leq r_{k+1}-r_k, \text{ for } k=1,\ldots,K-1,$$

and $\partial g_t(y_{t,0},\mathbf{z}_t)/\partial z_{t,K} \geq \min\{s+(1-\alpha)(p-r_K), p+u-r_K\}$. Since $0\leq\alpha\leq 1$, $p > r_K \geq \cdots \geq r_1$, $s\geq 0$, and $p+u-r_K \geq 0$, it follows that $g_t(y_{t,0},\mathbf{z}_t)$ is increasing in $z_{t,1},\ldots,z_{t,K}$. Finally, after applying Theorem 1 on (EC.18), we obtain that $V_t(\mathbf{x}_t)$ is additively convex in $\mathbf{x}_t$ and satisfies that $-(r_{k+1}-r_k) \leq \partial V_t(\mathbf{x}_t)/\partial x_{t,k} \leq 0$, for $k=1,\ldots,K-1$, and $V_t(\mathbf{x}_t)/\partial x_{t,K} \geq -(p-r_K)$. Thus, the theorem holds for period t, completing the induction proof.

Finally, we remark that our preservation results can also be applied to prove two other additive-convexity results in the literature for stochastic inventory systems with remanufacturing. First, Decroix (2006) studied a multi-echelon inventory system with remanufacturing and showed that the value functions are additively convex if the remanufactured products flow into the most upstream stage *or* if they flow into a downstream stage but without core disposal. Second, Zhou and Yu (2011) extended Simpson (1978)'s model to include the acquisition effort of cores and showed that the value functions are additively convex. For both systems, we can apply our preservation results to prove the additive-convexity results easily. We expect that there are other stochastic inventory systems with remanufacturing having additively convex value functions and our preservation results would be useful in identifying such systems.

Appendix B.2: Dynamic Inventory Rationing with Multiple Demand Classes

In this appendix, we revisit the dynamic inventory rationing problem with backlogging and multiple demand classes studied by Topkis (1968) and Bao et al. (2018), and apply our first preservation result (i.e., Theorem 1) to prove an additive-convexity result established in Bao et al. (2018).

Consider a single-product single-period inventory rationing problem with n distinct classes of demands. The period is divided into k intervals, indexed by $t=k,k-1,\dots,1$. At the beginning of each interval t, the firm first reviews its on-hand stock level z_t, and for each $j=1,\dots,n$, the amount b_t^j of backlogged class-j demand from previous intervals. Then, the firm observes, for each $j=1,\dots,n$, the newly arrived class-j demand d_t^j in this interval; and thus the total outstanding class-j demand in interval t is $B_t^j := b_t^j + d_t^j$. After that, the firm decides how much outstanding demand from each class to satisfy with the on-hand inventory. Let u_t^j denote the amount of unsatisfied outstanding class-j demand at the end of interval t. Then, $B_t^j - u_t^j$ is the amount of class-j demand the firm satisfies in interval t. At the end of the interval, the leftover inventory is carried over to the next interval; and any unsatisfied demand in each class is backlogged to the next interval. The demands for classes $1,\dots,n$ in each interval can have an arbitrary joint probability distribution, while the demands in different intervals are independent. Let $h_t(\cdot)$ denote the function of holding cost in interval t, which is assumed to be convex on $\Re^+$. For $j=1,\dots,n$, let p_t^j denote the unit backlogging cost for class-j demand in interval t, with $0\le p_t^1\le p_t^2\le\cdots\le p_t^n$ for each t. The firm's objective is to find the optimal inventory rationing policy that minimizes the expected total cost over the planning horizon. For convenience, we denote $\mathbf{d}_t=(d_t^1,\dots,d_t^n)^\intercal$, $\mathbf{u}_t=(u_t^1,\dots,u_t^n)^\intercal$, $\mathbf{B}_t=(B_t^1,\dots,B_t^n)^\intercal$, and $\mathbf{p}_t=(p_t^1,\dots,p_t^n)$; and we represent $\mathbf{0}$ and $\mathbf{1}$ as the vectors of all 0's and all 1's with length n, respectively. We will also suppress the subscript t unless confusion would otherwise arise.

We next formulate the firm's optimization problem as a dynamic program. For each interval t, the system state is given by an initial inventory level z and a vector of unsatisfied outstanding demands

$\mathbf{B} = (B_1, \dots, B_n)$. Clearly, we have $z \geq 0$ and $\mathbf{B} \geq \mathbf{0}$. Given the system state $(z, \mathbf{B})$, denote $f_t(z, \mathbf{B})$ as the optimal expected total cost from interval t onward. Then, for $t = 1, \dots, k$, the optimality equation can be written as

$$f_t(z, \mathbf{B}) = \min_{\substack{\mathbf{0} \leq \mathbf{u} \leq \mathbf{B} \\ \mathbf{1} \cdot (\mathbf{B} - \mathbf{u}) \leq z}} \{\mathbf{p}_t \cdot \mathbf{u} + h_t(z - \mathbf{1} \cdot (\mathbf{B} - \mathbf{u})) + \mathbb{E}[f_{t-1}(z - \mathbf{1} \cdot (\mathbf{B} - \mathbf{u}), \mathbf{u} + \mathbf{d}_{t-1})]\}, \qquad \text{(EC.20)}$$

where $\mathbf{d}_0 = \mathbf{0}$ (since no demand arrives in interval 0). The boundary condition is given by

$$f_0(z, \mathbf{B}) = v_1(z) + v_2(z - \mathbf{1} \cdot \mathbf{B}),$$

where $v_1(\cdot)$ is convex on $\Re^+$ and $v_2(\cdot)$ is convex on $\Re$.

In equation (EC.20), the decision variables are $\mathbf{u} = (u_1, \dots, u_n)$. Then, the ending on-hand inventory level after decisions is $z - \mathbf{1} \cdot (\mathbf{B} - \mathbf{u}) = z - \sum_{j=1}^n (B_j - u_j)$. The decision constraints simply ensure that the ending on-hand inventory level and the amounts of satisfied and backlogged demands in classes 1 to n are all nonnegative. Inside the brackets, the first two terms are the total backlogging and holding cost incurred in interval t, respectively; and the last term is the optimal expected total cost-to-go from interval $t-1$ onward.

Topkis (1968) showed that the optimal rationing policy for each interval t is a *rationing level policy*, which is completely characterized by n critical rationing levels. However, since the dynamic programming problem involves an $(n+1)$-dimensional state space, Topkis (1968) raised the difficulty of calculating the rationing levels due to the curse of dimensionality. Bao et al. (2018) revisited this problem and proved that after a state transformation the value functions are additively convex. Specifically, they transformed the state variables $(z, \mathbf{B})$ to $\mathbf{x} := (x^0, x^1, \dots, x^n)$, where $x^0 = -z$ and $x^j = -z + \sum_{i=n-j+1}^n B^i$, for $j = 1, \dots, n$. Since $z \geq 0$ and $\mathbf{B} \geq \mathbf{0}$, the state space of the transformed state $\mathbf{x}$ becomes

$$V := \{\mathbf{x} \in (-\infty, 0] \times \Re^n \mid x^0 \leq x^1 \leq \cdots \leq x^n\}.$$

For any $\mathbf{x} \in V$, define $V_t(\mathbf{x})$ as the optimal expected total cost from interval t onward, given that the transformed system state is $\mathbf{x}$ at the beginning of interval t. Then, they established the following important additive-convexity result.

THEOREM EC.2 **(Bao et al., 2018)**. *For $t = 1, \dots, k$, $V_t(\mathbf{x})$ is additively convex in $\mathbf{x}$ on V and increasing in $(x^1, x^2, \dots, x^{n-1})$.*

In what follows, we apply Theorem 1 to prove Theorem EC.2. To start with, we change the decision variables $\mathbf{u}$ in problem (EC.20) to $\mathbf{y} := (y^0, y^1, \dots, y^{n-1})$, where $y^0 = -z + \mathbf{1} \cdot (\mathbf{B} - \mathbf{u})$ and $y^j = y^{j-1} + u^{n+1-j}$ for $j = 1, \dots, n-1$. With the transformed state variables $\mathbf{x}$, the value function

$V_t(\mathbf{x})$ and the new decision variables $\mathbf{y}$, after some simple algebra we can rewrite the optimality equation (EC.20) as follows: for $t=1,\ldots,k$,

$$V_t(\mathbf{x}) = \min_{\mathbf{y}\in\mathbb{C}(\mathbf{x})}\Big\{h_t(-y^0) - p_t^n y^0 + \sum_{i=1}^{n-1}(p_t^{n-i+1} - p_t^{n-i})y^i + p_t^1 x^n + \mathbb{E}\big[V_{t-1}((\mathbf{y},x^n) + (0, d_{t-1}^n, \sum_{i=n-1}^{n} d_{t-1}^i, \ldots, \sum_{i=1}^{n} d_{t-1}^i))\big]\Big\}, \qquad \text{(EC.21)}$$

where

$$\mathbb{C}(\mathbf{x}) = \{\mathbf{y}\in(-\infty,0]\times\Re^{n-1} \mid (\mathbf{y},x^n)\in V,\ y^0 - x^0 \geq y^1 - x^1 \geq \cdots \geq y^{n-1} - x^{n-1} \geq 0\}.$$

In addition, the boundary condition is given by $V_0(\mathbf{x}) = v_1(-x^0) + v_2(-x^n)$.

We now prove Theorem EC.2 by induction on t. Since $V_0(\mathbf{x}) = v_1(-x^0) + v_2(-x^n)$, with both $v_1(\cdot)$ and $v_2(\cdot)$ being convex functions, the theorem is clearly true when $t=0$. Now we assume inductively that the theorem holds for interval $t-1$, i.e., $V_{t-1}(\mathbf{x})$ is additively convex in $\mathbf{x}$ on V and increasing in $(x^1,\ldots,x^{n-1})$. Since $p_t^1 \leq p_t^2 \leq \cdots \leq p_t^n$, it follows that the minimand in (EC.21) is additively convex in $(\mathbf{y},x^n)$ and increasing in $(y^1,\ldots,y^{n-1})$. Hence, by applying Theorem 1 on the optimization problem (EC.21) with $k=n-1$, we obtain that $V_t(\mathbf{x})$ is additively convex in $\mathbf{x}$ and increasing in $(x^1,\ldots,x^{n-1})$. Therefore, the theorem holds for interval t, completing the induction proof.

Appendix C: Serial Inventory System with Disposal

In this appendix, we apply Theorem 2 to a serial inventory system with inventory disposal, and prove that its value functions are additively convex under a sufficient condition on its cost parameters for prioritized optimal disposal decisions.

Consider a serial inventory system modified from the one studied in §3.2 as follows: the system is not allowed to expedite inventory, but can dispose inventory at different stages at the beginning of each period. Specifically, we assume that the system receives a unit revenue r_i for disposed inventory at stage i, $i=1,\ldots,n$. The other settings and notation are the same as those in §3.2. We assume that $c_n > \alpha r_n$ to avoid arbitrage through procurement and disposal at stage n.

Same as §3.2, we use $\mathbf{x}_t = (x_{t,0},\ldots,x_{t,n})$ to represent the system state in period t and V is the state space. Define $\mathbf{y}_t = (y_{t,1},\ldots,y_{t,n})$, where $y_{t,i}$ is the echelon inventory level at stage i after the disposal decisions but before the ordering decisions, $i=1,\ldots,n$. Then, the disposal quantity at stage 1 is $x_1 - y_1$ and that at stage i is $(x_{t,i} - x_{t,i-1}) - (y_{t,i} - y_{t,i-1})$, $i=2,\ldots,n$. Since the disposal quantity at each stage is nonnegative and at most the inventory level at that stage, the decision constraints for $\mathbf{y}_t$ are given by $\mathbf{y}_t \in \mathbb{C}_1(\mathbf{x}_t)$, where

$$\mathbb{C}_1(\mathbf{x}) = \{(y_1, \dots, y_n) \in \mathbb{R}^n \mid x_0 \le y_1 \le \cdots \le y_n,\ y_n - x_n \le \cdots \le y_1 - x_1 \le 0\}.$$

Further, define $\hat{\mathbf{y}}_t = (\hat{y}_{t,0}, \dots, \hat{y}_{t,n})$, where $\hat{y}_{t,i}$ is the echelon inventory position at stage i after all the decisions are made, $i = 0, \dots, n$. Then, the order quantity from stage 1 to stage zero is $\hat{y}_{t,0} - x_{t,0}$ and that from stage $i+1$ to stage i is $\hat{y}_{t,i} - y_{t,i}$, $i = 1, \dots, n$. The decision constraints for $\hat{\mathbf{y}}_t$ are given by $\hat{\mathbf{y}}_t \in \mathbb{C}_2(\mathbf{y}_t)$, where

$$\mathbb{C}_2(x_0, \mathbf{y}) = \{(\hat{y}_0, \dots, \hat{y}_n) \in \Re^{n+1} \mid x_0 \le \hat{y}_0 \le y_1 \le \hat{y}_1 \le \cdots \le y_n \le \hat{y}_n\}.$$

Define $V_t(\mathbf{x})$ as the optimal expected total discounted cost from period t onward, given system state $\mathbf{x}$ at the beginning of period t. For $t = 1, \dots, T$, the optimality equation can be written as

$$V_t(\mathbf{x}) = \min_{\mathbf{y} \in \mathbb{C}_1(\mathbf{x})} \Big\{ \mathbb{E}[h_0(x_0 - D_t)^+ + b(D_t - x_0)^+] - (c_0 + h_1)x_0 - \sum_{i=1}^{n}(c_i + r_{i+1} - r_i - h_i + h_{i+1})y_i + g_t(x_0, \mathbf{y}) \Big\} + \sum_{i=1}^{n}(r_{i+1} - r_i)x_i, \qquad \text{(EC.22)}$$

where $h_{n+1} = r_{n+1} = 0$ and

$$g_t(x_0, \mathbf{y}) = \min_{\hat{\mathbf{y}} \in \mathbb{C}_2(x_0, \mathbf{y})} \Big\{ \sum_{i=0}^{n} c_i \hat{y}_i + \alpha \mathbb{E}[V_{t+1}(\hat{y}_0 - D_t, \dots, \hat{y}_n - D_t)] \Big\}. \qquad \text{(EC.23)}$$

The boundary condition is $V_{T+1}(\mathbf{x}) \equiv -\sum_{i=1}^{n} r_i(x_i - x_{i-1})$. That is, we assume that all the inventories at stages 1 to n are disposed at the end of the planning horizon. This assumption is nonessential and can be easily relaxed.

In what follows, we apply our second preservation result (i.e., Theorem 2) to establish the additive convexity of the value function $V_t(\mathbf{x})$ for each period t under a sufficient condition. We first present our sufficient condition and main result as follows.

ASSUMPTION EC.1. $c_i + r_{i+1} - r_i - h_i + h_{i+1} \ge (c_i + \alpha(r_{i+1} - r_i))^+$, $i = 1, \dots, n-1$.

THEOREM EC.3. *Under Assumption EC.1, for $t = 1, \dots, T$, the value function $V_t(\mathbf{x})$ is additively convex in $\mathbf{x}$ and $V_t(\mathbf{x}) + \sum_{i=1}^{n-1}(r_i - r_{i+1})x_i$ is decreasing in $x_1, \dots, x_{n-1}$.*

We next apply Theorem 2 to prove Theorem EC.3 by induction on t. Once can directly check from the boundary condition that the theorem holds for period $T+1$. Now suppose the theorem holds for period $t+1$, and we prove below that it holds for period t. By the inductive assumption,

$V_{t+1}(\mathbf{x}) = \sum_{i=0}^{n} V_{t+1}^{i}(x_i)$ for any $\mathbf{x} \in V$, where $V_{t+1}^{0}(\cdot), \ldots, V_{t+1}^{n}(\cdot)$ are convex functions and $V_{t+1}^{i}(x_i) + (r_i - r_{i+1})x_i$ is decreasing in x_i for $i = 1, \ldots, n-1$. Then, we can rewrite (EC.23) as

$$g_t(x_0, \mathbf{y}) = \min_{x_0 \le \hat{y}_0 \le y_1} \{c_0 \hat{y}_0 + \alpha \mathbb{E}[V_{t+1}^{0}(\hat{y}_0 - D_t)]\} + \sum_{i=1}^{n-1} \min_{y_i \le \hat{y}_i \le y_{i+1}} \{c_i \hat{y}_i + \alpha \mathbb{E}[V_{t+1}^{i}(\hat{y}_i - D_t)]\} + \min_{\hat{y}_n \ge y_n} \{c_n \hat{y}_n + \alpha \mathbb{E}[V_{t+1}^{n}(\hat{y}_n - D_t)]\}. \quad \text{(EC.24)}$$

By applying Karush's Lemma (see Lemma EC.1) on (EC.24), one can verify that $g_t(x_0, \mathbf{y})$ is additively convex in $(x_0, \mathbf{y})$ and $g_t(x_0, \mathbf{y}) - (c_i + \alpha(r_{i+1} - r_i))^+ y_i$ is decreasing in y_i for $i = 1, \ldots, n-1$. By Assumption EC.1, the minimand in (EC.22) for period t is additively convex in $(x_0, \mathbf{y})$ and decreasing in $y_1, \ldots, y_{n-1}$. Applying Theorem 2 with $k = 1$, we obtain that the term $\min_{\mathbf{y} \in \mathbb{C}_1(\mathbf{x})}\{\cdot\}$ in (EC.22) is additively convex in $\mathbf{x}$ and decreasing in $x_1, \ldots, x_{n-1}$. Consequently, the theorem holds for period t. This completes the proof of Theorem EC.3.

We now provide some remarks on Theorem EC.3 and Assumption EC.1. First, problem (EC.23) corresponds to a serial inventory system with orders having one period of lead time. For this problem, the additive-convexity property is preserved under the optimal policy. Second, problem (EC.22) corresponds to an optimal disposal problem with n resources. By Theorem 2, the additive-convexity property is also preserved *if* the optimal policy disposes inventories at more upstream stages with higher priorities. Assumption 2 provides a sufficient condition on the cost parameters for these priorities, and requires the following two inequalities for each $i = 1, \ldots, n-1$:

$$r_{i+1} - h_i \ge r_i - h_{i+1} - c_i, \quad \text{(EC.25)}$$

$$r_{i+1} + (\alpha r_i - h_i) \ge r_i + (\alpha r_{i+1} - h_{i+1}). \quad \text{(EC.26)}$$

These inequalities can be interpreted as follows. Suppose there are one unit at stage i and one unit at stage $i+1$. The inequality (EC.25) requires that, if we need to dispose one unit now and keep one unit at stage i in the next period, then it is more profitable to dispose the unit at stage $i+1$ than dispose the unit at stage i (while shipping the unit from stage $i+1$ to stage i). The inequality (EC.26) requires that, if we need to dispose the two units in two consecutive periods, then it is more profitable to dispose the unit at stage $i+1$ first. When both inequalities are satisfied for $i = 1, \ldots, n-1$, the optimal policy assigns higher priorities on disposing inventories at more upstream stages, and the additive-convexity property is preserved under the optimal disposal policy.

We note that Angelus (2011) studied a similar serial system where inventories at all stages can be disposed under Markovian-modulated demands. Angelus (2011) proposed a heuristic called *disposal* *saturation policy* to manage this system, which requires a prioritized disposal policy (i.e., inventory at each stage cannot be disposed *unless* all inventories at its upstream stages are disposed). Angelus (2011) did not address the issue of when this heuristic is optimal. Partly inspired by this study, we consider a slightly different system without allowing inventory disposal at stage 0. For our system, Theorem EC.3 implies that the disposal saturation policy is optimal under Assumption 2. We expect a similar result to hold for Angelus (2011)'s system and leave the detailed investigation to the interested reader.

Appendix D: Numerical Study in Section 3.3

We consider problems with $N = 4$ products and $T \in \{3, 10, 20\}$ periods. The unit backlogging costs of the four products are $g_1 = 1$, $g_2 = 0.9$, $g_3 = 0.8$, and $g_4 = 0.7$. The profit margins α_{ij}, $1 \le i \le j \le N$, are given by the matrix

$$(\alpha_{ij})_{4\times 4} = \begin{pmatrix} 16 & 14 & 12 & 10 \\ 0 & 15 & 13 & 11 \\ 0 & 0 & 14 & 12 \\ 0 & 0 & 0 & 13 \end{pmatrix}.$$

When $T = 3$, the demands for each class i in different periods follow the same uniform distribution with support $[0, \bar{D}^i]$; and the maximum demands $(\bar{D}^1, \ldots, \bar{D}^4)$ are chosen from the six scenarios in Table EC.1. In each scenario, product i's initial capacity X_i is chosen from $\{0, T, 2T, \ldots, \bar{D}^i T/2\}$ to ensure the robustness of our results. We drop the instances where at most one of the initial capacities X_1, X_2, X_3 is positive where APSR is optimal. That is, we only test the instances where APSR may be suboptimal. In total, we test 1866 instances with $T = 3$ under different initial capacities and maximum demands. When $T = 10$ and 20, for tractability of the optimal policy, we suppose that the demands for each class i in different periods follow the same uniform distribution with support $[0, \bar{D}^i/2]$. In addition, we test the initial capacities by choosing X_i from $\{T/2, T/2 + 5, \ldots, \bar{D}^i T/4\}$. In total, we test 720 instances with $T = 10$, and 5670 instances with $T = 20$ under different initial capacities and maximum demands.

Table EC.1 Scenarios of maximum demands of four products when $T = 3$

Scenario	1	2	3	4	5	6
$\bar{D}^1$	8	8	8	8	8	8
$\bar{D}^2$	6	6	4	4	10	10
$\bar{D}^3$	4	10	10	6	6	4
$\bar{D}^4$	10	4	6	10	4	6

Following Yu et al. (2015), given an initial capacity $\mathbf{X}$, we define the performance measure as

$$\Delta_{opt} = \left| \frac{\Pi_{APSR}(\mathbf{X}) - \Pi(\mathbf{X})}{\Pi(\mathbf{X})} \right| \times 100\%,$$

where $\Pi_{APSR}(\mathbf{X})$ and $\Pi(\mathbf{X})$ are the expected total profits under APSR and the optimal policy, respectively. Then, Δ_{opt} measures the optimality gap in percentage by using APSR rather than the optimal policy. For each instance, we use backward induction to evaluate $\Pi(\mathbf{X})$ for the optimal policy; and for APSR, we first use backward induction to calculate the protection levels and then Monte Carlo simulation to evaluate $\Pi_{APSR}(\mathbf{X})$. All the numerical experiments are conducted using Matlab 2015b and run on a Unix server with 3.06GHx Quad-Core Intel Xeon X5667 processor. The longest running time to evaluate $\Pi(\mathbf{X})$ for an instance is about 82 hours, whereas that to evaluate $\Pi_{APSR}(\mathbf{X})$ is about 5 minutes. Thus, APSR is much more efficient in computation than the optimal policy.

The statistics for the Δ_{opt} values are summarized in Table EC.2. It can be seen that APSR performs consistently very well, with an average optimality gap of 0.169% and the maximum optimality gap of 8.959% over a total of 8256 instances with $N = 4$ and $T \in \{3, 10, 20\}$.

Table EC.2 The optimality gap Δ_{opt} of APSR compared with the optimal policy

	Mean	Std.	Median	90%-percentile	95%-percentile	99%-percentile	Max.
T=3	0.022%	0.033%	0.013%	0.050%	0.076%	0.175%	0.536%
T=10	0.105%	0.329%	0.319%	0.243%	0.474%	1.400%	4.017%
T=20	0.381%	0.871%	0.075%	1.099%	2.015%	4.207%	8.959%

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