Proofs and Additional Results

EC.1. Proofs in Sections 3 and 4:

Proof for Proposition 1: Consider firm X's strategy to be $\tilde{x}_i = -x_i$.[9] Then the game is equivalent to the following definition: the strategy sets are $\tilde{x}_i \in [-1, 0]$ and $y_i \in [0, 1]$ for all i. The payoff functions are

$$f_X(\tilde{\boldsymbol{x}}, \boldsymbol{y}) = \sum_{i=1}^{n} -\frac{\tilde{x}_i}{\beta_i} \left(\alpha^X - \sum_{j=1}^{n} g_{ij} \tilde{x}_j - \log(-\tilde{x}_i) + \log(1 + \tilde{x}_i - y_i) \right)$$

when $\tilde{x}_i = 0$[10] or $x_i + y_i = -\tilde{x}_i + y_i < 1$ for all i; If $-\tilde{x}_i + y_i \geq 1$ and $\tilde{x}_i < 0$ for some i then $f_X = -\infty$. Similarly

$$f_Y(\tilde{\boldsymbol{x}}, \boldsymbol{y}) = \sum_{i=1}^{n} \frac{y_i}{\beta_i} \left(\alpha^Y + \sum_{j=1}^{n} g_{ij} y_j - \log(y_i) + \log(1 + \tilde{x}_i - y_i) \right)$$

when $y_i = 0$ or $-\tilde{x}_i + y_i < 1$ for all i; If $-\tilde{x}_i + y_i \geq 1$ and $y_i > 0$ for some i then $f_Y = -\infty$. Clearly, f_X and f_Y map $[-1, 0]^n \times [0, 1]^n$ to $\mathbb{R} \cup \{-\infty\}$. Next we show that the game satisfies Assumption (A1) to (A4) preceding Theorem 4 in Milgrom and Roberts (1990) and is thus a supermodular game.

A1: complete lattice The strategy set $S_X = [-1, 0]^n$ and $S_Y = [0, 1]^n$ are complete lattices equipped with the usual component-wise partial order.

A2: order upper semi-continuity We only show this property for f_X and that for f_Y follows similarly. For fixed $\boldsymbol{y}$, f_X is infinitely differentiable if $\tilde{x}_i \in (y_i - 1, 0)$ for all i. When $\tilde{x}_i \to 0$, the corresponding term in f_X tends to 0. When $\tilde{x}_i \to y_i - 1$, $f_X \to -\infty$. The point that needs special treatment is when $y_i = 1$ for some i. In this case $\tilde{x}_i \equiv 1 - y_i = 0$ and the ith term in $\lim_{\tilde{\boldsymbol{x}}_n \to \tilde{\boldsymbol{x}}} f_X$ can either be $-\infty$ or 0 depending on the choice of the converging sequence. But in both cases, it is easy to verify that $\lim_{\tilde{\boldsymbol{x}}_n \to \tilde{\boldsymbol{x}}} f_X \le f_X(\tilde{\boldsymbol{x}}, \boldsymbol{y})$ by our definition of f_X. Therefore, f_X is order upper semi-continuous and has a finite upper bound.

A3: supermodularity and A4: inreasing difference We show a stronger result that f_X is supermodular in $(\tilde{\boldsymbol{x}}, \boldsymbol{y})$, i.e.,

$$f_X(\tilde{\boldsymbol{x}}_1, \boldsymbol{y}_1) + f_X(\tilde{\boldsymbol{x}}_2, \boldsymbol{y}_2) \le f_X(\tilde{\boldsymbol{x}}_1 \wedge \tilde{\boldsymbol{x}}_2, \boldsymbol{y}_1 \wedge \boldsymbol{y}_2) + f_X(\tilde{\boldsymbol{x}}_1 \vee \tilde{\boldsymbol{x}}_2, \boldsymbol{y}_1 \vee \boldsymbol{y}_2). \tag{EC.1}$$

Since the property for f_Y can be obtained similarly, this implies both A3 and A4. Consider $f_X(\tilde{\boldsymbol{x}}_1, \boldsymbol{y}_1)$ and $f_X(\tilde{\boldsymbol{x}}_2, \boldsymbol{y}_2)$. If either of them is $-\infty$, then (EC.1) automatically holds. If both of them are finite, then[11] $-\tilde{\boldsymbol{x}}_1 + \boldsymbol{y}_1 < 1$ and $-\tilde{\boldsymbol{x}}_2 + \boldsymbol{y}_2 < 1$, then $-\tilde{\boldsymbol{x}}_1 \wedge \tilde{\boldsymbol{x}}_2 + \boldsymbol{y}_1 \wedge \boldsymbol{y}_2 < 1$ and $-\tilde{\boldsymbol{x}}_1 \vee \tilde{\boldsymbol{x}}_2 + \boldsymbol{y}_1 \vee \boldsymbol{y}_2 < 1$. Moreover, f_X is finite and infinitely differentiable inside the hypercube formed by $(\tilde{\boldsymbol{x}}_1, \boldsymbol{y}_1)$, $(\tilde{\boldsymbol{x}}_2, \boldsymbol{y}_2)$, $(\tilde{\boldsymbol{x}}_1 \wedge \tilde{\boldsymbol{x}}_2, \boldsymbol{y}_1 \wedge \boldsymbol{y}_2)$ and $(\tilde{\boldsymbol{x}}_1 \vee \tilde{\boldsymbol{x}}_2, \boldsymbol{y}_1 \vee \boldsymbol{y}_2)$. Since

$$\frac{\partial f_X}{\partial \tilde{x}_i \partial y_j} = \frac{1 - y_i}{\beta_i (1 + \tilde{x}_i - y_i)^2} \mathbb{I}_{\{i=j\}} \ge 0, \quad \frac{\partial f_X}{\partial \tilde{x}_i \partial \tilde{x}_j} = \frac{g_{ij}}{\beta_i} + \frac{g_{ji}}{\beta_j} \ge 0 \quad \text{for } i \ne j, \tag{EC.2}$$

by finding a path integral, the standard result in the lattice theory implies (EC.1).

After verifying the assumptions, the game is a supermodular game; from Theorem 5 in Milgrom and Roberts (1990), there exists pure-strategy Nash equilibria.

We prove the following lemma. To simplify the notation, we only focus on the case of symmetric products and show results for $\bar{y}(x)$. The claims carry over to asymmetric firms.

LEMMA EC.1 **(Property of h, h^{-1} and $\bar{y}(x)$:).**

1. *The function h maps $\mathbb{R}_{>0}$ to $\mathbb{R}$; h^{-1} maps $\mathbb{R}$ to $\mathbb{R}_{>0}$. Both h and h^{-1} are increasing. $h'(x) = 1/x + 1$. $(h^{-1})'(x) = h^{-1}(x)/(1 + h^{-1}(x))$. $h^{-1}(x) > x$ for $x < 1$ and $h^{-1}(x) < x$ for $x > 1$.*

2. *For* $0 < x < \frac{3-\sqrt{5}}{4}$, $h(x/(1-2x))$ *is concave; for* $\frac{3-\sqrt{5}}{4} < x < 1/2$, $h(x/(1-2x))$ *is convex.*
3. $\bar{y}(0) = 1$, $y^*(0) < 1$.
4. *We have*

$$\bar{y}'(x) = -1 - \frac{1 - \frac{\gamma x}{h^{-1}(\alpha-1+\gamma x)+1}}{h^{-1}(\alpha - 1 + \gamma x)}. \tag{EC.3}$$

   Moreover, $\bar{y}'(x) \lesseqqgtr -1$ *if and only if* $x \lesseqqgtr (\exp(\alpha)+1)/\gamma$.

5. *When* $x < (\exp(\alpha)+1)/\gamma$, $\bar{y}'(x)$ *is increasing in* x. *Moreover,* $\bar{y}''(x) = 0$ *has one solution for* $x \geq 0$.

Proof of Lemma EC.1: Part 1 is straightforward by direct calculation. For part 2, note that

$$\begin{aligned}\frac{dh(x/(1-2x))}{dx} &= \frac{1-x}{x(1-2x)^2}\\ \frac{d^2h(x/(1-2x))}{dx^2} &= \frac{-4x^2+6x-1}{x^2(1-2x)^3}.\end{aligned}$$

It is easy to see that $\frac{d^2h(x/(1-2x))}{dx^2} < 0$ for $0 < x < \frac{3-\sqrt{5}}{4}$ and $\frac{d^2h(x/(1-2x))}{dx^2} > 0$ for $\frac{3-\sqrt{5}}{4} < x < 1/2$.

Part 3 is straightforward by Equation (7). For Part 4, the derivative of $\bar{y}(x)$ can be calculated directly. Moreover, we have

$$\begin{aligned}\bar{y}'(x) > -1 &\iff 1 - \frac{\gamma x}{h^{-1}(\alpha - 1 + \gamma x) + 1} < 0\\ &\iff h^{-1}(\alpha - 1 + \gamma x) < \gamma x - 1\\ &\iff \alpha - 1 + \gamma x < h(\gamma x - 1)\\ &\iff x > \frac{1+e^{\alpha}}{\gamma}.\end{aligned} \tag{EC.4}$$

Therefore, $\bar{y}'(x) \lesseqqgtr -1$ if and only if $x \lesseqqgtr (\exp(\alpha)+1)/\gamma$.

To show part 5, note that

$$\bar{y}''(x) = -\frac{d\frac{h^{-1}(\alpha-1+\gamma x)+1-\gamma x}{h^{-1}(\alpha+1-\gamma x)(h^{-1}(\alpha+1-\gamma x)+1)}}{dx} = \frac{\gamma h^{-1} + \gamma(h^{-1}+1-\gamma x)(2h^{-1}+1)\frac{h^{-1}}{h^{-1}+1}}{(h^{-1}(h^{-1}+1))^2}, \tag{EC.5}$$

where we simply omit the parentheses and let $h^{-1} \triangleq h^{-1}(\alpha - 1 + \gamma x)$ as there is no confusion. When $x < (\exp(\alpha)+1)/\gamma$, $\bar{y}'(x) < -1$ and $h^{-1}(\alpha - 1 + \gamma x) + 1 - \gamma x > 0$ by (EC.4). Therefore, $\bar{y}''(x) > 0$ and $\bar{y}'(x)$ is increasing in x. For the second claim, by (EC.5)

$$\bar{y}''(x) \lesseqgtr 0 \iff \frac{h^{-1}+1}{2h^{-1}+1} \lesseqgtr (\gamma x - 1 - h^{-1})$$

Note that the LHS is decreasing in h^{-1}, and thus x, while the RHS is increasing in x because its derivative in x is $\gamma/(h^{-1}+1)$. Therefore, there is only one solution to $\bar{y}''(x) = 0$, which may or may not fall in $[0,1]$.

Proof of Proposition 2: We first show that $\gamma = h^{-1}(\alpha - 1 + \gamma) + 1$ has a unique solution for $\gamma \geq 0$. Define $f_0(\gamma) \triangleq \gamma - h^{-1}(\alpha - 1 + \gamma) - 1$. By Lemma EC.1,

$$\frac{f_0(\gamma)}{d\gamma} = 1 - \frac{h^{-1}(\alpha - 1 + \gamma)}{h^{-1}(\alpha - 1 + \gamma) + 1} = \frac{1}{h^{-1}(\alpha - 1 + \gamma) + 1} > 0. \tag{EC.6}$$

Moreover, $f_0(0) = -h^{-1}(\alpha - 1) - 1 < 0$ and $f_0(+\infty) > 0$ because

$$f_0(+\infty) > 0 \iff \lim_{x\to+\infty}(x - h^{-1}(\alpha - 1 + x) - 1) > 0 \iff \lim_{x\to+\infty}(h(x-1) - (x-1) - \alpha) > 0.$$

Since $f_0(\cdot)$ is a continuous function, $f_0(\gamma) = 0$ has a unique solution $\gamma_1 \in (0, +\infty)$.

Next we show that for $\gamma < \gamma_1$ and $x \in [0,1]$, $\bar{y}'(x) < -1$ so $\bar{y}(x)$ has an inverse $x^*(y)$. According to (EC.3), we have

$$\bar{y}(x) < 1 \iff \gamma x - h^{-1}(\alpha - 1 + \gamma x) - 1 < 0.$$

Note that the LHS of the last inequality is $f(\gamma x)$. The inequality holds because $\gamma x \leq \gamma < \gamma_1$, $f_0'(\gamma x) > 0$, (by (EC.6)) and $f_0(\gamma_1) = 0$ (by the definition of γ_1). Applying the same argument to $\bar{x}'(y)$, we conclude that the inverse of $\bar{x}(y)$ coincides with $y^*(x)$.

We proceed to prove the existence of a single symmetric equilibrium. It suffices to show that $\bar{y}(x)$ (the inverse of $x^*(y)$) and $y^*(x)$ has one and only one intersection for $x \in [0,1]$. Since $\bar{x}'(y) < -1$ (by symmetry), it follows that $(y^*)' \in (0,-1)$. This implies that $(\bar{y} - y^*)' < 0$ and there is at most one intersection of $\bar{y}(x)$ and $y^*(x)$. Because $\bar{y}(x)$ is a continuous and decreasing function, it must intersect $y = x$. By symmetry, $y^*(x)$ passes the same intersection. Therefore, $\bar{y}(x)$ and $y^*(x)$ has one and only one intersection, which corresponds to the unique symmetric Nash equilibrium.

Proof of Proposition 3 : This proposition is a straightforward corollary to Proposition 5. Hence, we refer the readers to the proof of Proposition 5.

Proof of Corollary 1: From Definition 1, it is clear that in any Nash equilibrium (x^*, y^*), we have

$$x^* + y^* = 1 - \frac{x^*}{h^{-1}(\alpha - 1 + \gamma x^*)}.$$

From Lemma EC.1, by the property of $h^{-1}(\cdot)$ we have that

$$1 - x^* - y^* = O(\gamma^{-1}).$$

To compare the prices in equilibrium, we can write down the difference in prices charged by firm X minus that by firm Y (after scaling)

$$\frac{\gamma}{2}(x^* - y^*) - (\log(x^*) - \log(y^*)).$$

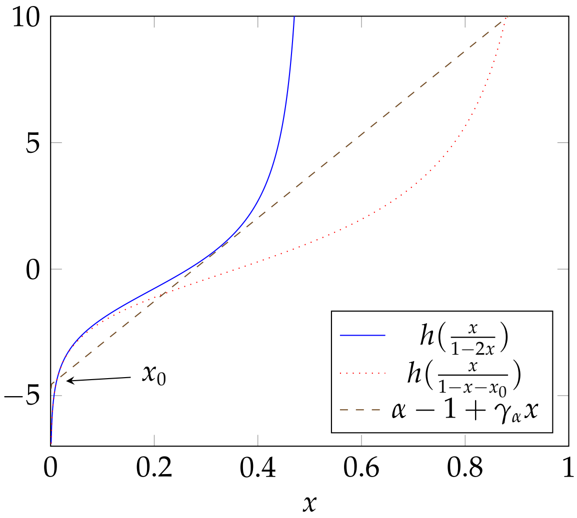


Figure EC.1 Illustration of $h(x/(1-2x))$ and $h(x/(1-x-x_0))$.

Since $x^* - y^* > 1 - 2\delta$, the first term is linear in γ. However, by the equilibrium condition

$$x^* + y^* = 1 - \frac{x^*}{h^{-1}(\alpha - 1 + \gamma x^*)} = 1 - \frac{y^*}{h^{-1}(\alpha - 1 + \gamma y^*)}$$
$$\Rightarrow \frac{x^*}{h^{-1}(\alpha - 1 + \gamma x^*)} = \frac{y^*}{h^{-1}(\alpha - 1 + \gamma y^*)}.$$

Note that $h^{-1}(\alpha - 1 + \gamma x^*) = \Omega(\gamma)$, $h^{-1}(\alpha - 1 + \gamma y^*) \geq h^{-1}(\alpha - 1)$. Therefore, $y^* = O(\gamma^{-1})$ and $-\log(y^*)$ grows slower than a linear term in γ. Therefore, the difference is positive for a sufficiently γ and we have completed the proof.

Proof of Proposition 4 : We will use the following steps to establish the result.

i. Define α_0. Show that for given $\alpha < \alpha_0$, we can find γ_α so that for $\gamma < \gamma_\alpha$,

$$h\left(\frac{x}{1-2x}\right) = \alpha - 1 + \gamma x \tag{EC.7}$$

has only one solution x_0, which is between 0 and $\tilde{x} \triangleq \frac{3-\sqrt{5}}{4} \approx 0.19$. Note that (EC.7) is a necessary condition for any symmetric Nash equilibria, which is derived from the first-order condition (8).

ii. For a sufficiently small[12] α, we can find ϵ so that for $\gamma \in (\gamma_\alpha - \epsilon, \gamma_\alpha)$,

$$h\left(\frac{x}{1-x-x_0}\right) = \alpha - 1 + \gamma x \tag{EC.8}$$

has three solutions, where x_0 is the solution to (EC.7) (which depends on α and γ) and thus automatically one of the solutions to (EC.8).

iii. Combining step i and ii, show that there does not exist any symmetric equilibrium when α is sufficiently small and $\gamma \in (\gamma_\alpha - \epsilon, \gamma_\alpha)$.

iv. Under the same condition as step iii, show that there exist two asymmetric equilibria.

Next we start proving the result according to the above steps.

Step i: Define

$$\alpha_0 = 1 - \frac{1-\tilde{x}}{(1-2\tilde{x})^2} + h\left(\frac{\tilde{x}}{1-2\tilde{x}}\right) \approx -1.98,$$

The definition of α_0 is interpreted as follows. By part 2 of Lemma EC.1, when $x < \tilde{x}$ ($x > \tilde{x}$), $h(x/(1-2x))$ is concave (convex). If we start from the point $(\tilde{x}, h(\tilde{x}/(1-2\tilde{x})))$ and introduce a tangent to $h(x/(1-2x))$ at $\tilde{x}$, then the tangent has the following expression:

$$\frac{y - h(\tilde{x}/(1-2\tilde{x}))}{x-\tilde{x}} = \left.\frac{d(h(x/(1-2x)))}{dx}\right|_{x=\tilde{x}} = \frac{1-\tilde{x}}{\tilde{x}(1-2\tilde{x})^2}.$$

This tangent intersects y-axis at $(0, \alpha_0 - 1)$. Equivalently, if we introduce a tangent to the convex segment ($x \in [\tilde{x}, 1/2)$) of $h(x/(1-2x))$ from $(0, \alpha_0 - 1)$, then the point of tangency is $(\tilde{x}, h(\tilde{x}/(1-2\tilde{x})))$. Moreover, such tangent is unique, because $h(x/(1-2x))$ is an increasing convex function for $x \geq \tilde{x}$.

For $\alpha < \alpha_0$, we introduce a tangent to the convex segment of $h(x/(1-2x))$ from $(0, \alpha)$. This tangent is unique. Let γ_α denote the slope of the tangent, which, together with the point of tangency x', solves

$$\gamma_\alpha = \left.\frac{d(h(x/(1-2x)))}{dx}\right|_{x=x'} = \frac{1-x'}{x'(1-2x')^2}; \tag{EC.9}$$

$$\gamma_\alpha = \frac{h(x'/(1-2x')) - \alpha}{x'};$$

$$x' > \tilde{x}.$$

The first equation states that the slope must be equal to the derivative at the point of tangency; the second equation states that the line must pass $(0, \alpha)$; the third inequality states that the point of tangency is at the convex segment of $h(x/(1-2x))$. It is clear that the expression of the tangent is $y = \alpha - 1 + \gamma_\alpha x$, which is the same as the RHS of (EC.7). Since $h(x/(1-2x))$ is increasing and convex for $x > \tilde{x}$, it follows that as $\alpha < \alpha_0$ decreases, the point of tangency x' and the slope γ_α both increase. This fact will be used in step ii of the proof.

Next we show that if $\gamma < \gamma_\alpha$, then there is one solution to (EC.7), which is between 0 and $\tilde{x}$. First notice that the tangent $y = \alpha - 1 + \gamma_\alpha x$ is always below the convex segment. Thus, $\alpha - 1 + \gamma x < \alpha - 1 + \gamma_\alpha x \leq h(x/(1-2x))$ for $x \geq \tilde{x}$, and there is no solution to (EC.7) that is greater than or equal to $\tilde{x}$. For $x \in [0, \tilde{x})$, $h(x/(1-2x))$ is increasing and concave; moreover, $\lim_{x \searrow 0} h(x/(1-2x)) = -\infty$ and $h(\tilde{x}/(1-2\tilde{x})) > \alpha - 1 + \gamma\tilde{x}$. Therefore, by the continuity of both $h(\cdot)$ and $y = \alpha - 1 + \gamma x$, there is one solution x_0 to (EC.7) that is between 0 and $\tilde{x}$. Hence we have completed step i. The idea of the proof can be illustrated graphically by Figure EC.2.

Step ii: We will show that for $\alpha < -15.2$ and $\gamma = \gamma_\alpha$ as defined in step i, there are three solutions to $\alpha - 1 + \gamma x = h(x/(1-x-x_0))$; that is, there are three intersections of the dashed line and the red curve in Figure EC.1.[13] This is sufficient for the claim in step ii because all functions we are considering are continuous and we can find a sufficiently small ϵ so that for $\gamma \in (\gamma_\alpha - \epsilon, \gamma_\alpha)$ there are still three solutions to (EC.8).[14]

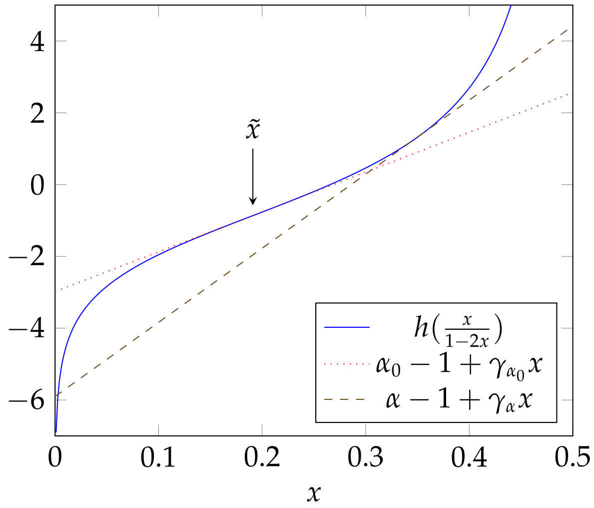


Figure EC.2 Illustration of step i of the proof for Proposition 4. It is clear from the figure that when $\gamma < \gamma_\alpha$, $y = \alpha - 1 + \gamma x$ and $h(x/(1-2x))$ have only one intersection, which is between 0 and $\tilde{x}$.

We first show that if $\gamma_\alpha < \frac{dh(x/(1-x-x_0))}{dx}|_{x=x_0}$, then there are three solutions to $\alpha - 1 + \gamma_\alpha x = h(x/(1-x-x_0))$. By the definition of x_0, $h(x/(1-x-x_0))$ intersects $\alpha - 1 + \gamma_\alpha x$ at x_0. It suffices to show that there are two more intersections of $h(x/(1-x-x_0))$ and $\alpha - 1 + \gamma_\alpha x$ to the right of x_0. To prove that, we derive the structure of $h(x/(1-x-x_0))$. The derivative of $h\left(\frac{x}{1-x-x_0}\right)$ is $\frac{(1-x_0)^2}{x(1-x-x_0)^2}$. We can show that $h\left(\frac{x}{1-x-x_0}\right)$ is concave for $0 < x < (1-x_0)/3$ and convex for $(1-x_0)/3 < x < 1-x_0$. A continuous function consisting of a concave segment and a convex segment can have at most three intersections with a straight line. Therefore, we only need to show that there are at least two solutions to $h(x/(1-x-x_0)) = \alpha - 1 + \gamma_\alpha x$ to the right of x_0. Since $\gamma_\alpha < \frac{dh(x/(1-x-x_0))}{dx}|_{x=x_0}$, it follows that $h(x/(1-x-x_0)) > \alpha - 1 + \gamma_\alpha x$ for $x \searrow x_0$. Moreover, since $\lim_{x \to 1-x_0} h(x/(1-x-x_0)) - (\alpha - 1 + \gamma_\alpha x) = +\infty$, it suffices to show that there exists $x' \in (x_0, 1-x_0)$ such that $h(x'/(1-x'-x_0)) < \alpha - 1 + \gamma_\alpha x'$. Such x' is constructed as follows. Suppose the point of tangency of $\alpha - 1 + \gamma_\alpha x$ and $h(x/(1-2x))$ is x'. Since $x' > \tilde{x} > x_0$ by step i, it follows that

$$\alpha - 1 + \gamma_\alpha x' = h\left(\frac{x'}{1-2x'}\right) > h\left(\frac{x'}{1-x'-x_0}\right).$$

This completes the proof for the claim in this paragraph.

Next we show that for $\alpha < -15.2$, we must have $\gamma_\alpha < \frac{dh(x/(1-x-x_0))}{dx}|_{x=x_0}$. By the analysis in step i, the point of tangency x' satisfies

$$\alpha = 1 - \frac{1-x'}{(1-2x')^2} + h(\frac{x'}{1-2x'}).$$

As previously shown in step i, x' is decreasing in α. Therefore, it can be calculated that when $\alpha \le -15.2$, we have $x' \ge 0.41$. Because the line $\alpha - 1 + \gamma_\alpha x$ passes $(x_0, h(x/(1-x-x_0)))$, we have the following relationship between x' and x_0 by (EC.9):

$$h\left(\frac{x_0}{1-2x_0}\right) = \frac{1-x'}{x'(1-2x')^2}(x_0 - x') + h\left(\frac{x'}{1-2x'}\right). \tag{EC.10}$$

The RHS of the above equation can be interpreted as follows: if we introduce a tangent to $h(x/(1-2x))$ at x', then the RHS is the value of y as the tangent intersects $x = x_0$. Because $h(x/(1-2x))$ is convex for $x \ge \tilde{x}$, the RHS decreases in x' for a fixed x_0. Therefore, since $x' \in [0.41, 0.5)$ and $x_0 < \tilde{x} < 0.2$, it follows that

$$h\left(\frac{x_0}{1-2x_0}\right) \le \frac{1-0.41}{0.41 \times (1-0.82)^2}(0.2-0.41) + h\left(\frac{0.41}{1-0.82}\right) < -5.$$

This implies that $-\log(x_0/(1-2x_0)) \ge 5$ and $x_0 < 0.008$. Therefore,

$$\begin{aligned} \frac{dh(x/(1-x-x_0))}{dx}|_{x=x_0} &= \frac{(1-x_0)^2}{x_0(1-2x_0)^2} \ge \frac{1}{x_0} > -18\log\left(\frac{x_0}{1-2x_0}\right) \\ &= 18\left(\frac{1-x'}{x'(1-2x')^2}(x'-x_0) - h\left(\frac{x'}{1-2x'}\right) + \frac{x_0}{1-2x_0}\right) \end{aligned} \tag{EC.11}$$

$$
\begin{aligned}
&\geq 18\left(\frac{1-x'}{x'(1-2x')^2}(x'-x_0)-\frac{x'}{1-2x'}-\log\left(\frac{x'}{1-2x'}\right)\right)\\
&\geq 18\left(\frac{1-x'}{x'(1-2x')^2}(x'-x_0)-\frac{2x'}{1-2x'}\right)\\
&\geq 3\frac{1-x'}{x'(1-2x')^2}(x'-x_0)+3\left(\frac{5(1-x')}{x'(1-2x')^2}(x'-x_0)-\frac{12x'}{1-2x'}\right)\\
&\geq 3\frac{1-x'}{x'(1-2x')^2}(x'-x_0) \qquad \text{(EC.12)}\\
&\geq \frac{1-x'}{x'(1-2x')^2}=\gamma_\alpha.
\end{aligned}
$$

Here the second inequality is because $\exp(1/18x) > 1/x - 2$ for $x < 0.008$ by numerical computation. Equation (EC.11) is due to (EC.10). Inequality EC.12 is because of the following facts: for $0.5 > x' > 0.41$ and $x_0 < 0.008$, we have

$$
\frac{5(1-x')}{x'(1-2x')^2}(x'-x_0)\geq\frac{5(x'-x_0)}{(1-2x')^2}\geq\frac{2}{(1-2x')^2}\geq\frac{10}{1-2x'}\geq\frac{12x'}{1-2x'}.
$$

Hence we have completed step ii.

Step iii: Observe that a necessary condition for a symmetric equilibrium (x^*, x^*) is that

$$
h\left(\frac{x^*}{1-2x^*}\right)=\alpha-1+\gamma x^*.
$$

By step i, the equation only has one solution $x_0 < \tilde{x}$. We will show that x_0 is not the best response to x_0; that is, x_0 is a local maximizer but not a global maximizer. Note that by step ii, the first-order condition, $\alpha - 1 + \gamma x - h(x/(1-x-x_0)) = 0$, has three solutions. Both the smallest solution, which is x_0, and the largest solution, denoted $\hat{x}$, correspond to local maxima. We only need to show that $\hat{x}$ generates a higher profit than x_0, so $\hat{x}$ is the global maximizer. We will show this for $\alpha < -15.2$ and $\gamma = \gamma_\alpha$. Then again, by continuity, the same claim holds for $\gamma \in (\gamma_\alpha - \epsilon, \gamma_\alpha)$ for a sufficiently small ϵ.

If we substitute the first-order condition, $\alpha = h(\frac{x}{1-x-x_0}) + 1 - \gamma_\alpha x$, into the profit (7), then it suffices to show that

$$\hat{x} + \hat{x}^2\left(\frac{1}{1-\hat{x}-x_0} - \frac{\gamma_\alpha}{2}\right) > x_0 + x_0^2\left(\frac{1}{1-2x_0} - \frac{\gamma_\alpha}{2}\right). \tag{EC.13}$$

By step ii, $x_0 < 0.008$ and $\hat{x}$ is to the right of $\tilde{x}$. If we can show $\frac{1}{1-\hat{x}-x_0} \geq \frac{\gamma_\alpha}{2}$, then

$$\begin{aligned}\hat{x} + \hat{x}^2\left(\frac{1}{1-\hat{x}-x_0} - \frac{\gamma_\alpha}{2}\right) &\geq \hat{x} > \tilde{x} = \frac{3-\sqrt{5}}{4}\\ &> 0.008 + \frac{0.008^2}{1-2\times 0.008} > x_0 + x_0^2\left(\frac{1}{1-2x_0} - \frac{\gamma_\alpha}{2}\right),\end{aligned}$$

and step iii is proved.

Now we prove $\frac{1}{1-\hat{x}-x_0} \geq \frac{\gamma_\alpha}{2}$ by contradiction. The fact that $\alpha - 1 + \gamma_\alpha x$ passes $(\hat{x}, h(\hat{x}/(1-x_0-\hat{x})))$ and is the tangent at $(x', h(x'/(1-2x')))$ implies that

$$h\left(\frac{\hat{x}}{1-\hat{x}-x_0}\right) = \frac{1-x'}{x'(1-2x')^2}(\hat{x}-x') + h\left(\frac{x'}{1-2x'}\right) \tag{EC.14}$$

$$\iff \hat{x}\left(\frac{1}{1-\hat{x}-x_0} - \frac{1-x'}{x'(1-2x')^2}\right) + \frac{1-x'}{(1-2x')^2} = h\left(\frac{x'}{1-2x'}\right) - \log\frac{\hat{x}}{1-\hat{x}-x_0}. \tag{EC.15}$$

Since $\hat{x} > x'$ by their definitions, it follows from (EC.14) that

$$h\left(\frac{\hat{x}}{1-\hat{x}-x_0}\right) > h\left(\frac{x'}{1-2x'}\right) \Rightarrow \frac{\hat{x}}{1-\hat{x}-x_0} > \frac{x'}{1-2x'} \iff \hat{x} - x' > \frac{x'(x'-x_0)}{1-x'}. \tag{EC.16}$$

Because $x' \geq 0.4$ and $x_0 < 0.008$, we have $\hat{x} > 0.5$ and thus

$$\begin{aligned}&\frac{3\hat{x}}{2(1-\hat{x}-x_0)} \geq h(\frac{\hat{x}}{1-\hat{x}-x_0}) \geq \frac{(1-x')(\hat{x}-x')}{x'(1-2x')^2} > \frac{x'-x_0}{(1-2x')^2} \geq \frac{0.402}{(1-2x')^2}\\ \Rightarrow &\frac{2(1-\hat{x}-x_0)}{1.5} < \frac{(1-2x')^2}{0.402} \Rightarrow \hat{x} > 1 - x_0 - 4(1-2x')^2.\end{aligned} \tag{EC.17}$$

Here the first inequality is because of the fact that $2\log x < x$ for $x \geq 1$ and that $\frac{\hat{x}}{1-\hat{x}-x_0} > \frac{0.5}{1-0.5} = 1$; the second inequality is because of (EC.14); the third inequality is due to (EC.16). By the proof of step ii, (EC.12) in particular, we have established that $\frac{1}{x_0} > 3\frac{(1-x')(x'-x_0)}{x'(1-2x')^2} > 3 \times 0.402(1-2x')^{-2}$, which implies that $x_0 < 0.9(1-2x')^2$. Therefore, combined with (EC.17), we have

$$\hat{x} > 1 - 4.9(1-2x')^2 > 1-(1-2x')^{1.01} \tag{EC.18}$$

for $x' \geq 0.41$. If

$$\frac{1}{1-\hat{x}-x_0} < \frac{\gamma_\alpha}{2} = \frac{1-x'}{2x'(1-2x')^2}, \tag{EC.19}$$

then the LHS of (EC.15) is

$$\begin{aligned}\hat{x}\left(\frac{1}{1-\hat{x}-x_0} - \frac{1-x'}{x'(1-2x')^2}\right) + \frac{1-x'}{(1-2x')^2} &\leq -\hat{x}\frac{1-x'}{2x'(1-2x')^2} + \frac{1-x'}{(1-2x')^2} \\ &\leq \frac{1-x'}{(1-2x')^2}\left(1 - \frac{1-(1-2x')^{1.01}}{2x'}\right) \\ &= -\frac{(1-x')(1-(1-2x')^{0.01})}{2x'(1-2x')} < 0.\end{aligned}$$

Here the first inequality is by (EC.19) and the second inequality is by (EC.18). However, the RHS of (EC.15) is

$$\begin{aligned} h\left(\frac{x'}{1-2x'}\right) - \log\frac{\hat{x}}{1-\hat{x}-x_0} &\geq h\left(\frac{x'}{1-2x'}\right) - \log\frac{1}{1-\hat{x}-x_0} \\ &\geq h\left(\frac{x'}{1-2x'}\right) - \log\left(\frac{1-x'}{2x'(1-2x')^2}\right) \geq 0,\end{aligned}$$

where the second inequality is by (EC.19) and the third inequality is by explicitly computing the function for $x' \geq 0.41$. This leads to a contradiction and thus we have proved step iii.

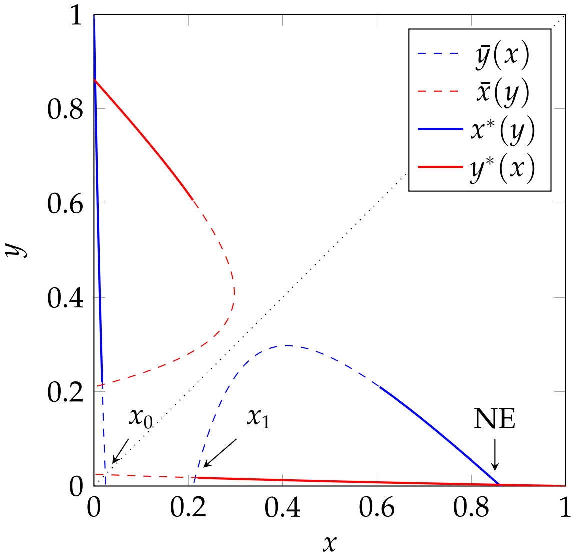


Figure EC.3 The curves of $\bar{y}(x)$, $\bar{x}(y)$ and the best responses for step iv of the proof of Proposition 4. The best response functions (e.g. $x^*(y)$) only overlap partially on the corresponding "bar" functions (e.g. $\bar{y}(x)$) because the latter are non-monotone and do not have inverses. Nash equilibria must be intersections of $\bar{y}(x)$ and $\bar{x}(y)$, but the converse is not true. First, if at the intersection one of the "bar" functions is increasing (say $\bar{y}(x)$ is increasing at x_1), then it corresponds to a local minimum rather than a local maximum (x_1 is a local minimizer of revenue when firm Y plays $\bar{y}(x_1)$). This can be derived by observing the sign of the first-order condition (5). Second, even if both $\bar{x}(y)$ and $\bar{y}(x)$ are increasing at the intersection (such as x_0), it may only correspond to a local maximum but not a global maximum. It is shown in step iii that x_0 is not the best response if the competitor plays x_0.

Step iv: Having established that symmetric equilibria do not exist, we will proceed to show that there exist two asymmetric equilibria. More precisely, we will show that $\bar{y}(x)$ and $y^*(x)$ intersect at some $x_1 < x_0$, and that $\bar{y}(x) = y^*(x_1)$ has only one solution $x = x_1$. The first claim states that $y = y^*(x_1)$ is the best response to $x = x_1$; the second claim states that x_1 is the only local maximum of firm X when Y plays $y^*(x_1)$, and therefore the best response.

To show the first claim, we will first prove that $y^*(x)$ is continuous for $x \in [0, x_0]$. By part 5 of Lemma EC.1, $\bar{y}'(x)$ crosses 0 from negative to positive for $x \geq 0$. Therefore, $\bar{y}(x)$ can be decreasing, then increasing, and then decreasing as $x \in [0, +\infty]$ increases. Now by step ii, $\bar{y}(x) = x_0$ has three solutions for $x \in (0,1)$. This implies that $\bar{y}(x)$ must be decreasing, then increasing, and then decreasing for $x \in [0,1]$, as illustrated in Figure EC.3 (we restrict the image of $\bar{y}(x)$ to $[0,1]$ in that figure). The same argument applies to $\bar{x}(y)$ by symmetry, and therefore for $x < x_0$, $\bar{x}(y) = x$ has three solutions. To show $y^*(x)$ is continuous for $x \in [0, x_0]$, it suffices to show that from the three local maxima satisfying $\bar{x}(y) = x$, the global maximum is the largest one: the maximizer is continuous if it doesn't switch between different local maxima. The proof for this is the same as step iii, when we show that the largest local maximum is the global maximum for $x = x_0$. In particular, in (EC.13), the term $\frac{1}{1-\hat{x}-x_0} - \frac{\gamma_a}{2}$ is still positive for $x < x_0$. By continuity, the fact $y^*(0) < 1 = \bar{y}(0)$ (Lemma EC.1) and $y^*(x_0) > x_0 = \bar{y}(x_0)$ (step iii of the proof) implies that $\bar{y}$ and y^* intersect for some $x_1 < x_0$.

Next we prove the second claim. Since $\bar{y}(x) = x_0$ has three solutions by step ii and $x = x_0$ is the smallest solution, it follows from the structure of $\bar{y}(x)$ (it is decreasing, then increasing, then decreasing) that $\bar{y}(x)$ is strictly decreasing in the interval for $x \in [0, x_0]$. Therefore, for $x \in [0, x_0]$, there is only one solution to $\bar{y}(x) = y^*(x_1)$, which is $x = x_1$. To show there is no solution for $x \in (x_0, \bar{y}(x_1))$, note that $\bar{y}(x) = x$ has only one solution $x = x_0$ by step i. Furthermore, $\bar{y}(x) < x$ when $x > x_0$ (see Figure EC.3 for illustration). This implies that for $x \in (x_0, \bar{y}(x_1))$, we have $\bar{y}(x_1) = y^*(x_1) > x > \bar{y}(x)$; therefore $\bar{y}(x) =$

$y^*(x_1)$ has no solution for $x \in (x_0, \bar{y}(x_1))$. For $x \geq \bar{y}(x_1)$, first note that $\bar{y}(x)$ is a decreasing function for $x > y^*(x_0)$. This is because $y^*(x_0)$ is the largest of the three local maxima to $\bar{y}(x) = x_0$ and must be on the second decreasing segment of $\bar{y}(x)$. By the structure of $\bar{y}(x)$, it is decreasing for $x > y^*(x_0)$. Since we have shown that $\bar{y}(x_1) = y^*(x_1) > y^*(x_0)$, it follows that $\bar{y}(x) < \bar{y}(y^*(x_0)) = x_0 = \bar{y}(x_0) < \bar{y}(x_1)$ for $x \in [\bar{y}(x_1), 1]$, where the last inequality is by the fact that $\bar{y}(x)$ is decreasing for $x < x_0$. Therefore, there is no solution to $\bar{y}(x) = y^*(x_1)$ for x in that interval. This completes step iv and the proof.

Proof of Proposition 5 : We will divide the proof into the following steps.

i. For any $\epsilon > 0$, we can find γ_1 sufficiently large so that for $\gamma > \gamma_1$, we have $\bar{y}'(x) < -1 + \epsilon$ and $\bar{x}'(y) < -1 + \epsilon$. As a result, both $\bar{y}(x)$ and $\bar{x}(y)$ are decreasing and have inverses.

ii. We can find a sufficiently large $\gamma_2 \geq \gamma_1$ so that for $\gamma > \gamma_2$, there are at most three Nash equilibria.

iii. Show that for any $\delta \in (0, 1/4)$, we can find a sufficiently large $\gamma_3 > \gamma_2$ so that for $\gamma > \gamma_3$, there exist two Nash equilibria (x_1^*, y_1^*) and (x_3^*, y_3^*), where $x_1^*, y_3^* \in (1 - \delta, 1)$ and $x_3^*, y_1^* \in (0, \delta)$.

iv. Show that there exists another equilibrium (x_2^*, y_2^*) such that $x_1^* > x_2^* > x_3^*$ and $y_1^* < y_2^* < y_3^*$. The equilibria (x_1^*, y_1^*) and (x_3^*, y_3^*) are stable, while (x_2^*, y_2^*) is unstable.

Step i: We first derive a lower bound for $h^{-1}(x)$. First observe that $h^{-1}(x) \lesseqqgtr x$ if and only if $x \gtreqqless 1$. Moreover, when $x > 1$, $h(x - \log(x)) = x - \log(x) + \log(x - \log(x)) < x$. Hence $h^{-1}(x) \geq x - \log(\max\{x, 1\})$. By this lower bound, we have

$$\begin{aligned}\bar{y}'(x) < -1 + \epsilon &\iff \frac{\gamma x - 1 - h^{-1}}{h^{-1}(h^{-1} + 1)} < \epsilon \\ &\Longleftarrow \frac{\log(\max\{\alpha^X - 1 + \gamma x, 1\}) - \alpha^X}{(\alpha^X - 1 + \gamma x)^+ (\alpha^X + \gamma x)^+} < \epsilon,\end{aligned}$$

where $h^{-1}(\alpha^X - 1 + \gamma x)$ in the numerator is replaced by its lower bound we have just derived and $h^{-1}(\alpha^X - 1 + \gamma x)$ in the denominator is replaced by the lower bound $\alpha^X - 1 + \gamma x$. It is clear to see that the LHS of the last inequality is $O(\log(\gamma)/\gamma^2)$ as $\gamma \to \infty$ for a given $x > 0$. Similarly, $\bar{x}'(y) < -1 + \epsilon$ can be reduced to an inequality in which α^X is replaced by α^Y. Therefore, we can choose γ_1 sufficiently large, so that $\bar{y}'(x) < -1 + \epsilon$ and $\bar{x}'(y) < -1 + \epsilon$. As a result, $\bar{y}(x)$ and $\bar{x}(y)$ are both decreasing; their inverses coincide with the best responses.

Step ii: As Nash equilibria are solutions to $\bar{y}(x) = y^*(x)$, we will show that there are at most three solutions to the equation. It suffices to show that there are at most two solutions to $\bar{y}'(x) = (y^*)'(x)$. To prove this, we derive the structures of $\bar{y}'(x)$ and $(y^*)'(x)$ in more detail.

Combining step i of the proof and part 4 of Lemma EC.1, we have

$$\bar{y}'(x) \in \begin{cases} (-\infty, -1] & x \leq (e^{\alpha^X} + 1)/\gamma \\ (-1, -1+\epsilon) & x > (e^{\alpha^X} + 1)/\gamma \end{cases}, \quad \bar{x}'(y) \in \begin{cases} (-\infty, -1] & y \leq (e^{\alpha^Y} + 1)/\gamma \\ (-1, -1+\epsilon) & y > (e^{\alpha^Y} + 1)/\gamma \end{cases} \tag{EC.20}$$

By the property of function inverses, it is equivalent to

$$(x^*)'(y) \in \begin{cases} (-\frac{1}{1-\epsilon}, -1) & x^*(y) < (e^{\alpha^X} + 1)/\gamma \\ [-1, 0) & x^*(y) \geq (e^{\alpha^X} + 1)/\gamma \end{cases},$$

$$(y^*)'(x) \in \begin{cases} (-\frac{1}{1-\epsilon}, -1) & y^*(x) < (e^{\alpha^Y} + 1)/\gamma \\ [-1, 0) & y^*(x) \geq (e^{\alpha^Y} + 1)/\gamma \end{cases}. \tag{EC.21}$$

By the fact that $y^*(x)$ is decreasing and is the inverse of $\bar{x}(y)$, $y^*(x) \lesseqgtr (e^{\alpha^Y}+1)/\gamma$ is equivalent to $x \gtreqless \bar{x}((e^{\alpha^Y}+1)/\gamma)$, where

$$\bar{x}\left(\frac{e^{\alpha^Y}+1}{\gamma}\right) = 1 - \frac{e^{\alpha^Y}+1}{\gamma} - \frac{e^{\alpha^Y}+1}{\gamma h^{-1}(\alpha^Y + e^{\alpha^Y})}.$$

We will show that $(e^{\alpha^X}+1)/\gamma < \bar{x}((e^{\alpha^Y}+1)/\gamma)$ for a sufficiently large γ, which implies that the intervals of $\bar{y}'(x) \in (-\infty, -1]$ and $(y^*)'(x) \in (-\frac{1}{1-\epsilon}, -1)$ do not intersect; moreover, the solutions to $\bar{y}''(x) = 0$ and $(y^*)''(x) = 0$ both fall into $\big((e^{\alpha^X}+1)/\gamma, \bar{x}((e^{\alpha^Y}+1)/\gamma)\big)$.[15] By (EC.5), the solution x_{s1} to $\bar{y}''(x) = 0$ satisfies

$$\gamma x_{s1} - 1 - h^{-1}(\alpha^X - 1 + \gamma x_{s1}) = \frac{h^{-1}+1}{2h^{-1}+1} \in (0,1),$$

which implies that

$$\frac{e^{\alpha^X}+1}{\gamma} < x_{s1} < \frac{e^{\alpha^X+1}+2}{\gamma}.$$

The left inequality already implies that $x_{s1} > (e^{\alpha^X}+1)/\gamma$. To show $x_{s1} < \bar{x}\left(\frac{e^{\alpha^X}+1}{\gamma}\right)$ for a sufficiently large γ, note that

$$\gamma > e^{\alpha^X+1} + e^{\alpha^Y} + 3 + \frac{e^{\alpha^Y}+1}{h^{-1}(\alpha^Y + e^{\alpha^Y})} \implies \frac{e^{\alpha^X+1}+2}{\gamma} < \bar{x}\left(\frac{e^{\alpha^X}+1}{\gamma}\right) \implies x_{s1} < \bar{x}\left(\frac{e^{\alpha^X}+1}{\gamma}\right).$$

Similarly, by the property of function inverses, the solution x_{s2} to $(y^*)''(x) = 0$ satisfies

$$\gamma y^*(x_{s2}) - 1 - h^{-1}(\alpha^Y - 1 + \gamma y^*(x_{s2})) \in (0,1),$$

which implies that

$$\frac{e^{\alpha^Y}+1}{\gamma} < y^*(x_{s2}) < \frac{e^{\alpha^Y+1}+2}{\gamma}.$$

The left inequality already implies that $x_{s2} < \bar{x}(e^{\alpha^Y}+1)/\gamma$. To show $x_{s2} > \frac{e^{\alpha^X}+1}{\gamma}$ for a sufficiently large γ, note that

$$\gamma > e^{\alpha^Y+1} + e^{\alpha^X} + 3 + \frac{e^{\alpha^X}+1}{h^{-1}(\alpha^X + e^{\alpha^X})} \implies \frac{e^{\alpha^Y+1}+2}{\gamma} < \bar{x}\left(\frac{e^{\alpha^X}+1}{\gamma}\right) \implies x_{s2} > \frac{e^{\alpha^X}+1}{\gamma}.$$

Therefore, we let

$$\gamma > \gamma_2 \triangleq \max\left\{\gamma_1, e^{\alpha^X+1} + e^{\alpha^Y+1} + 3 + \frac{e^{\alpha^X}+1}{h^{-1}(\alpha^X)} + \frac{e^{\alpha^Y}+1}{h^{-1}(\alpha^Y)}\right\}$$

which is sufficiently large and partition the interval $[0,1]$ into three sub-intervals, $[0,(e^{\alpha^X}+1)/\gamma]$, $((e^{\alpha^X}+1)/\gamma, \bar{x}(e^{\alpha^Y}+1/\gamma)]$, and $(\bar{x}(e^{\alpha^Y}+1/\gamma),1]$. By the previous analysis, the solutions to $\bar{y}''(x)=0$ and $(y^*)''(x)=0$ fall into the mid interval. By (EC.20) and (EC.21), we have

$$\bar{y}'(x)\begin{cases} \leq -1 \text{ and increasing} & x \in [0,(e^{\alpha^X}+1)/\gamma] \\ > -1 & x \in ((e^{\alpha^X}+1)/\gamma, \bar{x}(e^{\alpha^Y}+1/\gamma)] \\ > -1 \text{ and decreasing} & x \in (\bar{x}(e^{\alpha^Y}+1/\gamma),1] \end{cases}$$

$$(y^*)'(x)\begin{cases} < -1 \text{ and decreasing} & x \in [0,(e^{\alpha^X}+1)/\gamma] \\ \leq -1 & x \in ((e^{\alpha^X}+1)/\gamma, \bar{x}(e^{\alpha^Y}+1/\gamma)] \\ > -1 \text{ and increasing} & x \in (\bar{x}(e^{\alpha^Y}+1/\gamma),1] \end{cases}$$

Hence there exist at most two solutions, in the first and third intervals respectively, to $\bar{y}'^{*}(x) = (y^*)'(x)$. This completes step ii.

Step iii: We first show the following claim: For a sufficiently large γ, if we let $x_2 = (\exp(\alpha^X)+1)/\gamma$, i.e., the boundary between the first and second intervals defined above, then $\bar{y}(x_2) < y^*(x_2)$. This is because

$$\bar{y}(x_2) < y^*(x_2) \iff \bar{x}(\bar{y}(x_2)) > x_2 \iff 1 - \bar{y}(x_2) - \frac{\bar{y}(x_2)}{h^{-1}(\alpha^Y - 1 + \gamma\bar{y}(x_2))} > x_2$$
$$\iff \frac{x_2}{h^{-1}(\alpha^X - 1 + \gamma x_2)} > \frac{\bar{y}(x_2)}{h^{-1}(\alpha^Y - 1 + \gamma\bar{y}(x_2))}$$

$$
\begin{aligned}
&\overset{\text{(I)}}{\iff} \frac{e^{\alpha^X}+1}{\gamma e^{\alpha^X}} > \frac{1}{h^{-1}(\alpha^Y-1+\gamma\bar{y}(x_2))}\left(1-\frac{(e^{\alpha^X}+1)^2}{\gamma e^{\alpha^X}}\right)\\
&\iff h^{-1}(\alpha^Y-1+\gamma\bar{y}(x_2)) > \frac{\gamma e^{\alpha^X}}{e^{\alpha^X}+1} - e^{\alpha^X} - 1\\
&\iff \alpha^Y-1+\gamma\bar{y}(x_2) > h\left(\frac{\gamma e^{\alpha^X}}{e^{\alpha^X}+1} - e^{\alpha^X} - 1\right)\\
&\overset{\text{(II)}}{\Longleftarrow} \alpha^Y+\gamma\bar{y}(x_2) > \frac{\gamma e^{\alpha^X}}{e^{\alpha^X}+1} - e^{\alpha^X} + \log\gamma\\
&\iff \frac{\gamma}{e^{\alpha^X}+1} > 2-\alpha^Y+e^{-\alpha^X}+\log\gamma\\
&\overset{\text{(III)}}{\Longleftarrow} \gamma > 2|(e^{\alpha^X}+1)(2-\alpha^Y)|+2+2e^{-\alpha^X}+6(e^{\alpha^X}+1)^2.
\end{aligned}
$$

Here (I) follows from the fact that $h^{-1}(\alpha^X-1+\gamma x_2)=\exp(\alpha^X)$ and that

$$\bar{y}(x_2)=1-x_2-\frac{x_2}{h^{-1}(\alpha^X-1+\gamma x_2)}=1-\frac{(e^{\alpha^X}+1)^2}{\gamma e^{\alpha^X}};$$

(II) follows from the definition of $h(\cdot)$ and the fact that $\frac{\gamma e^{\alpha^X}}{e^{\alpha^X}+1}-e^{\alpha^X}-1<\gamma$; (III) holds because $\gamma>2|(e^{\alpha^X}+1)(2-\alpha^Y)|+2+2e^{-\alpha^X}+6(e^{\alpha^X}+1)^2$ implies $\frac{\gamma}{2}\geq(e^{\alpha^X}+1)(2-\alpha^Y)+1+e^{-\alpha^X}$. Moreover, we have $\frac{\gamma}{2}\geq(e^{\alpha^X}+1)\log\gamma$ because

$$\left(\frac{\gamma}{\log\gamma}\right)'=\frac{\log(\gamma)-1}{(\log\gamma)^2}>0$$

for $\gamma>6(e^{\alpha^X}+1)^2$ and that

$$\left.\frac{\gamma}{\log\gamma}\right|_{\gamma=6(e^{\alpha^X}+1)^2}=\frac{6(e^{\alpha^X}+1)^2}{\log(6(e^{\alpha^X}+1)^2)}\geq 2(e^{\alpha^X}+1)\frac{3}{2\log\sqrt{6}}\geq 2(e^{\alpha^X}+1)\Rightarrow\frac{\gamma}{\log\gamma}\geq 2(e^{\alpha^X}+1)$$

for all $\gamma>6(e^{\alpha^X}+1)^2$. Therefore, summing up the two inequalities gives (III). Hence when we choose γ according to the last inequality (III), we always have $\bar{y}(x_2)<y^*(x_2)$.

Because $\bar{y}(x_2)<y^*(x_2)$ and $1=\bar{y}(0)>y^*(0)$ (by Lemma EC.1), there must be at least one solution to $\bar{y}(x)=y^*(x)$ in $(0,x_2)$, which corresponds to a Nash equilibrium. Moreover, we can choose $\gamma>(e^{\alpha^X}+1)/\delta$ so that the solution is less than $x_2=(\exp(\alpha^X)+$

$1)/\gamma < \delta$. It establishes the existence of (x_3^*, y_3^*), where $x_3^* < \delta$ and $y_3^* > 1-\delta$. Symmetrically, we can establish the existence of (x_1^*, y_1^*). From the proof, we can pick

$$\gamma > \gamma_3 = \max\Big\{\gamma_2, \frac{(e^{\alpha^X}+1)(e^{\alpha^Y}+1)}{\delta}, 2|(e^{\alpha^X}+1)(2-\alpha^Y)| + 2 + 2e^{-\alpha^X} + 6(e^{\alpha^X}+1)^2,$$
$$2|(e^{\alpha^Y}+1)(2-\alpha^X)| + 2 + 2e^{-\alpha^Y} + 6(e^{\alpha^Y}+1)^2\Big\}.$$

Step iv: By step ii, $\bar{y}'(x) - (y^*)'(x)$ first increases (in the first interval) then decreases (in the third interval) in x and $\bar{y}'(x) - (y^*)'(x) = 0$ has at most two solutions. By step iii, $\bar{y}(x) - y^*(x) = 0$ has two solutions x_1^* (in the third interval) and x_3^* (in the first interval) when $\gamma > \gamma_3$. These two observations, combined with the fact that $\bar{y}(0) - y^*(0) > 0$ and $\bar{y}(1) - y^*(1) < 0$, imply that $\bar{y}'(x_1^*) - (y^*)'(x_1^*) < 0$ and that $\bar{y}'(x_3^*) - (y^*)'(x_3^*) < 0$. As a result, $\bar{y}(x_1^* - \epsilon) - y^*(x_1^* - \epsilon) > 0$ and $\bar{y}(x_3^* + \epsilon) - y^*(x_3^* + \epsilon) < 0$ for some small $\epsilon > 0$. Because $\bar{y}(x)$ and $y^*(x)$ are both continuous,[16] there is another solution, x_2^* to $\bar{y}(x) - y^*(x) = 0$ between x_1^* and x_3^*. Furthermore, $\bar{y}'(x_2^*) - (y^*)'(x_2^*) > 0$. This implies that (x_1^*, y_1^*) and (x_3^*, y_3^*) are stable, whereas (x_2^*, y_2^*) is not.

Proof of Proposition 6 We show that if

$$\alpha^X > \alpha_1 \triangleq \max\left\{1 + h\left(\frac{\gamma h^{-1}(\alpha^Y - 1 + \gamma)}{\exp(\alpha^Y)+1}\right), \gamma + \log(\gamma - 1)\right\},$$

then there is one and only one Nash equilibrium. For such α^X, it follows from (EC.3) that

$$\frac{\gamma x}{h^{-1}(\alpha^X - 1 + \gamma x) + 1} < \frac{\gamma}{h^{-1}(\alpha^X - 1) + 1} < \frac{\gamma}{h^{-1}(\gamma + \log(\gamma - 1) - 1) + 1} = 1$$
$$\Rightarrow \bar{y}'(x) < -1 \tag{EC.22}$$

Moreover,

$$\bar{y}'(x) = -1 - \frac{1 - \frac{\gamma x}{h^{-1}(\alpha^X - 1 + \gamma x) + 1}}{h^{-1}(\alpha^X - 1 + \gamma x)}$$
$$> -1 - \frac{1}{h^{-1}(\alpha^X - 1)} > -1 - \frac{1}{h^{-1}(\alpha_1 - 1)} \geq -1 - \frac{\exp(\alpha^Y) + 1}{\gamma h^{-1}(\alpha^Y - 1 + \gamma)}$$

These bounds on $\bar{y}'(x)$, combined with the fact that $\bar{y}(0)=1$, imply that

$$\bar{y}(x) > 1-\left(1+\frac{\exp(\alpha^Y)+1}{\gamma h^{-1}(\alpha^Y-1+\gamma)}\right)x \geq 1-x-\frac{\exp(\alpha^Y)+1}{\gamma h^{-1}(\alpha^Y-1+\gamma)}. \tag{EC.23}$$

On the other hand, it follows from (EC.4) that for $1 \geq y > (\exp(\alpha^Y)+1)/\gamma$, $\bar{x}'(y) > -1$ and

$$\bar{x}(y) = 1-y-\frac{y}{h^{-1}(\alpha^Y-1+\gamma y)} < 1-y-\frac{\exp(\alpha^Y)+1}{\gamma h^{-1}(\alpha^Y-1+\gamma)}. \tag{EC.24}$$

For any Nash equilibrium (x^*,y^*), we must have $\bar{x}(y^*)=x^*$ and $\bar{y}(x^*)=y^*$. It follows from (EC.23) that $x^*+y^*>1-\frac{\exp(\alpha^Y)+1}{\gamma h^{-1}(\alpha^Y-1+\gamma)}$ and from (EC.24) that $y^* \leq (\exp(\alpha^Y)+1)/\gamma$. So we only focus on this region for Nash equilibria. Because $\bar{y}'(x) < -1$, the inverse of $\bar{y}(x)$ is $x^*(y)$; moreover, by (EC.22), $(x^*)'(y) > -1$. On the other hand, it follows from (EC.4) that $\bar{x}'(y) \leq -1$ for $y \in [0,(\exp(\alpha^Y)+1)/\gamma]$. These observations imply that $\bar{x}(y)=x^*(y)$ has at most one solution for $y \in [0,(\exp(\alpha^Y)+1)/\gamma]$, and thus there is exactly one Nash equilibrium since we have established the existence.

EC.2. Proofs in Sections 5 and 6:

Proof of Proposition 7: The idea of the proof is based on Milgrom and Roberts (1990). Suppose there are at least two distinct pure-strategy Nash equilibria. By the proof of Proposition 1, the transformed game is a supermodular game. Therefore, we can find the largest equilibrium $(\bar{\boldsymbol{x}},\bar{\boldsymbol{y}})$ and smallest equilibrium $(\underline{\boldsymbol{x}},\underline{\boldsymbol{y}})$ in the transformed game[17], satisfying $\bar{\boldsymbol{x}} \geq \underline{\boldsymbol{x}}$ and $\bar{\boldsymbol{y}} \geq \underline{\boldsymbol{y}}$. Without loss of generality, let j be such that $|\bar{x}_j-\underline{x}_j| = \max_i \left\{\max\left\{|\bar{x}_i-\underline{x}_i|,|\bar{y}_i-\underline{y}_i|\right\}\right\}$. Because $(\bar{\boldsymbol{x}},\bar{\boldsymbol{y}})$ and $(\underline{\boldsymbol{x}},\underline{\boldsymbol{y}})$ are Nash equilibria, we must have

$$\frac{\partial f_X}{\partial x_j}(\bar{\boldsymbol{x}},\bar{\boldsymbol{y}}) = \frac{\partial f_X}{\partial x_j}(\underline{\boldsymbol{x}},\underline{\boldsymbol{y}}) = 0.$$

We parameterize the path from $(\underline{\boldsymbol{x}},\underline{\boldsymbol{y}})$ to $(\bar{\boldsymbol{x}},\bar{\boldsymbol{y}})$ by $t \in [0,1]$: $(\underline{\boldsymbol{x}} + t(\bar{\boldsymbol{x}} - \underline{\boldsymbol{x}}), \underline{\boldsymbol{y}} + t(\bar{\boldsymbol{y}} - \underline{\boldsymbol{y}}))$. Therefore,

$$\begin{aligned}0 &= \frac{\partial f_X}{\partial x_j}(\bar{\boldsymbol{x}},\bar{\boldsymbol{y}}) - \frac{\partial f_X}{\partial x_j}(\underline{\boldsymbol{x}},\underline{\boldsymbol{y}}) \\ &= \int_0^1 \sum_{i=1}^n \Bigg((\bar{x}_i - \underline{x}_i)\frac{\partial^2 f_X}{\partial x_j \partial x_i}(\underline{\boldsymbol{x}} + t(\bar{\boldsymbol{x}} - \underline{\boldsymbol{x}}), \underline{\boldsymbol{y}} + t(\bar{\boldsymbol{y}} - \underline{\boldsymbol{y}})) \\ &\quad + (\bar{y}_i - \underline{y}_i)\frac{\partial^2 f_X}{\partial x_j \partial y_i}(\underline{\boldsymbol{x}} + t(\bar{\boldsymbol{x}} - \underline{\boldsymbol{x}}), \underline{\boldsymbol{y}} + t(\bar{\boldsymbol{y}} - \underline{\boldsymbol{y}}))\Bigg) dt \end{aligned} \tag{EC.25}$$

Equation (EC.2) provides the expressions of the second-order derivatives. In particular, $\partial f_X / \partial x_i \partial x_j \geq 0$ and $\partial f_X / \partial x_j \partial y_i = 0$ for $i \neq j$, $\partial f_X / \partial x_j \partial y_j \geq 0$. In addition, we have

$$\frac{\partial^2 f_X}{\partial x_i \partial x_i} = \frac{2g_{ii}}{\beta_i} + \frac{(1-y_i)^2}{\beta_i x_i (1 + x_i - y_i)^2}.$$

Therefore, we denote $(\boldsymbol{x}(t),\boldsymbol{y}(t)) = (\underline{\boldsymbol{x}} + t(\bar{\boldsymbol{x}} - \underline{\boldsymbol{x}}), \underline{\boldsymbol{y}} + t(\bar{\boldsymbol{y}} - \underline{\boldsymbol{y}}))$ and have

$$\begin{aligned}(\text{EC.25}) &\leq (\bar{x}_j - \underline{x}_j)\int_0^1 \sum_{i=1}^n \frac{\partial^2 f_X}{\partial x_j \partial x_i}(\boldsymbol{x}(t),\boldsymbol{y}(t)) + \frac{\partial^2 f_X}{\partial x_j \partial y_j}(\boldsymbol{x}(t),\boldsymbol{y}(t)) dt \\ &= (\bar{x}_j - \underline{x}_j)\int_0^1 \sum_{i=1}^n \left(\frac{g_{ij}}{\beta_i} + \frac{g_{ji}}{\beta_j}\right) + \frac{(1-y_j(t))^2}{\beta_j x_j(t)(1 + x_j(t) - y_j(t))^2} + \frac{1 - y_j(t)}{\beta_j (1 + x_j(t) - y_j(t))^2} dt.\end{aligned}$$

If we can show that when all g_{ij} are sufficiently small, the intergrand is negative for all points on the path $(\boldsymbol{x}(t),\boldsymbol{y}(t))$, i.e.,

$$\sum_{i=1}^n \left(\frac{g_{ij}}{\beta_i} + \frac{g_{ji}}{\beta_j}\right) + \frac{(1-y_j(t))^2}{\beta_j x_j(t)(1 + x_j(t) - y_j(t))^2} + \frac{1 - y_j(t)}{\beta_j (1 + x_j(t) - y_j(t))^2} < 0, \tag{EC.26}$$

then it leads to a contradiction and we can prove the proposition.

Fix some $\delta > 0$. Suppose $\sum_{i=1}^n (g_{ij}/\beta_i + g_{ji}/\beta_j) < \delta$ (this is satisfied when g_{ij} are sufficiently small). We first show (EC.26) holds for $t = 0$ and $t = 1$, i.e., $(\underline{\boldsymbol{x}},\underline{\boldsymbol{y}})$ and $(\bar{\boldsymbol{x}},\bar{\boldsymbol{y}})$. For

$t = 0$ and $t = 1$, the two points are Nash equilibria. Therefore, the first-order condition holds (we omit (t) for simplicity)

$$-(\alpha_X - 1) + \sum_{i=1}^{n} \left(\frac{g_{ij}}{\beta_i} + \frac{g_{ji}}{\beta_j} \right) x_j + h\left(\frac{-x_j}{1 + x_j - y_j} \right) = 0$$
$$\alpha_Y - 1 + \sum_{i=1}^{n} \left(\frac{g_{ij}}{\beta_i} + \frac{g_{ji}}{\beta_j} \right) y_j - h\left(\frac{y_j}{1 + x_j - y_j} \right) = 0.$$

Because $h(\cdot)$ is an increasing function and has a well-defined inverse from Lemma EC.1, we have

$$h^{-1}(\alpha_X - 1 + \delta) \geq \frac{-x_j}{1 + x_j - y_j} \geq h^{-1}(\alpha_X - 1)$$
$$h^{-1}(\alpha_Y - 1 + \delta) \geq \frac{y_j}{1 + x_j - y_j} \geq h^{-1}(\alpha_Y - 1).$$

Summing up the inequalities, we draw a conclusion that $y_j - x_j < 1 - c_0$ for a constant $1 > c_0 > 0$ that only depends on α_X, α_Y and δ. This implies that $-x_j < 1 - c_0$, $1 - y_j > c_0$ and moreover

$$\frac{1 - y_j}{-x_j} > \frac{c_0 - x_j}{-x_j} > 1 + \frac{c_0}{1 - c_0}.$$

Plugging those inequalities into (EC.26), we obtain that for $t = 0$ and 1,

$$\begin{aligned}
&\sum_{i=1}^{n} \left(\frac{g_{ij}}{\beta_i} + \frac{g_{ji}}{\beta_j} \right) + \frac{(1 - y_j(t))^2}{\beta_j x_j(t)(1 + x_j(t) - y_j(t))^2)} + \frac{1 - y_j(t)}{\beta_j(1 + x_j(t) - y_j(t))^2} \\
\leq &\sum_{i=1}^{n} \left(\frac{g_{ij}}{\beta_i} + \frac{g_{ji}}{\beta_j} \right) - \frac{c_0}{1 - c_0} \frac{1 - y_j(t)}{\beta_j(1 + x_j(t) - y_j(t))^2} \\
\leq &\sum_{i=1}^{n} \left(\frac{g_{ij}}{\beta_i} + \frac{g_{ji}}{\beta_j} \right) - \frac{c_0^2}{\beta_j(1 - c_0)}.
\end{aligned}$$

When g_{ij} are sufficiently small, the above quantity is negative. For $t \in (0, 1)$, the inequalities $y_j(t) - x_j(t) < 1 - c_0$, $-x_j(t) < 1 - c_0$, and $1 - y_j(t) > c_0$ still hold for the same c_0

because of the convex combination. Therefore, (EC.26) holds by the same argument when g_{ij} are sufficiently small. This proves the proposition. From the proof, we can also see that if we choose $\delta = 1$ and $c_0 = (1 + h^{-1}(\alpha^X) + h^{-1}(\alpha^Y))^{-1}$, then the condition is met if

$$\sum_{i=1}^{n} \left(\frac{g_{ij}}{\beta_i} + \frac{g_{ji}}{\beta_j} \right) \le \min\{1, \frac{c_0^2}{\beta_j(1-c_0)}\}$$

for all j.

Proof of Proposition EC.2 : We will use Kakutani's fixed-point theorem to establish the existence of such a stationary point. The correspondence we are using, however, is different from the best-response correspondence in the standard proof. Let Ω be the following compact and convex subset of $\mathbb{R}^{2n}$: $[0,\delta]^n \times [1-\delta, 1]^n$. We are going to construct a correspondence $f : \Omega \mapsto \Omega$ as follows.

For given $(x_1^{(t)}, \dots, x_n^{(t)}, y_1^{(t)}, \dots, y_n^{(t)}) \in \Omega$, consider the following equations for $(x_i^{(t+1)}, y_i^{(t+1)})$:

$$\alpha^X - 1 + \sum_{j \neq i}^{n} \left(g_{ij} + \frac{\beta_i g_{ji}}{\beta_j} \right) x_j^{(t)} + 2 g_{ii} x_i^{(t+1)} = \log \frac{x_i^{(t+1)}}{1 - x_i^{(t+1)} - y_i^{(t+1)}} + \frac{x_i^{(t+1)}}{1 - x_i^{(t+1)} - y_i^{(t+1)}} \tag{EC.27}$$

$$\alpha^Y - 1 + \sum_{j \neq i}^{n} \left(g_{ij} + \frac{\beta_i g_{ji}}{\beta_j} \right) y_j^{(t)} + 2 g_{ii} y_i^{(t+1)} = \log \frac{y_i^{(t+1)}}{1 - x_i^{(t+1)} - y_i^{(t+1)}} + \frac{y_i^{(t+1)}}{1 - x_i^{(t+1)} - y_i^{(t+1)}} \tag{EC.28}$$

In other words, compared to (5) and (6), the above equations characterize the Nash equilibrium when firm X and Y are competing on the market share of customer i while the choice probabilities of the other $n-1$ customers are set exogenously (given by $x_j^{(t)}$ and $y_j^{(t)}$). Therefore, regarding $\sum_{j \neq i}^{n} \left(g_{ij} + \frac{\beta_i g_{ji}}{\beta_j} \right) x_j^{(t)}$ and $\sum_{j \neq i}^{n} \left(g_{ij} + \frac{\beta_i g_{ji}}{\beta_j} \right) y_j^{(t)}$ as part of the exogenous quality of the products, we can apply Proposition 3 and Proposition 6 to show the existence and uniqueness of $(x_i^{(t+1)}, y_i^{(t+1)}) \in [0,\delta] \times [1-\delta, 1]$ when g_{ij} are sufficiently large. Applying the same argument for $i = 1, \dots, n$, we can naturally define the correspondence f, namely, $f(x_1^{(t)}, \dots, x_n^{(t)}, y_1^{(t)}, \dots, y_n^{(t)}) = (x_1^{(t+1)}, \dots, x_n^{(t+1)}, y_1^{(t+1)}, \dots, y_n^{(t+1)})$.

Next we apply Kakutani's fixed-point theorem. The correspondence f is clearly non-empty, compact and convex. Moreover, since both sides of (EC.27) and (EC.28) are continuously differentiable, f is upper hemi-continuous. Therefore, Kakutani's fixed-point theorem implies that we can find $\{x_i^*\}_{i=1}^n \in [0,\delta]^n$ and $\{y_i^*\}_{i=1}^n \in [1-\delta, 1]^n$ such that $f(x_1^*, \dots, x_n^*, y_1^*, \dots, y_n^*) = (x_1^*, \dots, x_n^*, y_1^*, \dots, y_n^*)$. Clearly this is a solution to (5) and (6) inside Ω.

Proof of Proposition 8: Consider the first-order condition (5) for symmetric equilibria:

$$\alpha - 1 + \sum_{j=1}^{n} \left(g_{ij} + \frac{\beta_i g_{ji}}{\beta_j} \right) x_j^* = \log \frac{x_i^*}{1 - 2x_i^*} + \frac{x_i^*}{1 - 2x_i^*}.$$

If $\beta_j g_{ij} + \beta_i g_{ji}$, or equivalently $g_{ij} + \beta_i g_{ji} / \beta_j$, is decreasing in i for all $j = 1, 2, \dots, n$, then the right-hand side $h(x_i^*/(1-2x_i^*))$ is also decreasing in i. By the monotonicity of $h(\cdot)$ and $x/(1-2x)$, we can prove the proposition.

Proof of Proposition 9: We provide some additional analysis in Section EC.3.7.

To prove the first part, similar to Proposition EC.2, we will use Kakutani's fixed-point theorem to establish the existence of such an equilibrium. Let Ω be the following compact and convex subset of $\mathbb{R}^4$: $[1-\delta, 1] \times [0, \delta] \times [0, \delta] \times [1-\delta, 1]$. We are going to construct a correspondence $f : \Omega \mapsto \Omega$ as follows.

For given $(x_2, y_2) \in [0,\delta] \times [1-\delta, 1]$, consider the following equations for (x, y):

$$\log \frac{x}{1-x-y} + \frac{x}{1-x-y} = \alpha^X - 1 + (\gamma_{12} + \gamma_{21}) n_2 x_2 + 2 n_1 \gamma_1 x$$
$$\log \frac{y}{1-x-y} + \frac{y}{1-x-y} = \alpha^Y - 1 + (\gamma_{12} + \gamma_{21}) n_2 y_2 + 2 n_1 \gamma_1 y.$$

We first show that there exists one and only one solution $(\hat{x},\hat{y}) \in [1-\delta,1] \times [0,\delta]$ for any $(x_2,y_2) \in [0,\delta] \times [1-\delta,1]$, given that γ_1 and γ_2 are sufficiently large. This result follows from Proposition 5. Let $\hat{\alpha}^X = \alpha^X + (\gamma_{12}+\gamma_{21})n_2x_2$ and $\hat{\alpha}^Y = \alpha^Y + (\gamma_{12}+\gamma_{21})n_2y_2$ be the new quality parameters. Clearly $(\hat{\alpha}^X,\hat{\alpha}^Y) \in \Delta \triangleq [\alpha^X, \alpha^X + (\gamma_{12}+\gamma_{21})n_2] \times [\alpha^X, \alpha^X + (\gamma_{12}+\gamma_{21})n_2]$ is in a compact set. Therefore, we can choose $2n_1\gamma_1 \geq \max_{(\hat{\alpha}^X,\hat{\alpha}^Y)\in\Delta}\{\Gamma\}$, where Γ is the threshold of γ derived in Proposition 5, which is continuous in $\hat{\alpha}^X$ and $\hat{\alpha}^Y$. The maximum of a continuous function over a compact set is finite, and thus for γ_1 sufficiently large, there exists a unique solution, denoted $(\hat{x}_1,\hat{y}_1)$.

Let the first two entries of $f(x_1,y_1,x_2,y_2)$ be this solution $f_1 = \hat{x}_1$ and $f_2 = \hat{y}_1$. Similarly, for γ_2 sufficiently large, there is a unique solution (x,y) to

$$\log\frac{x}{1-x-y} + \frac{x}{1-x-y} = \alpha^X - 1 + (\gamma_{12}+\gamma_{21})n_1x_1 + 2n_2\gamma_2 x$$
$$\log\frac{y}{1-x-y} + \frac{y}{1-x-y} = \alpha^Y - 1 + (\gamma_{12}+\gamma_{21})n_1y_1 + 2n_2\gamma_2 y,$$

denoted $\hat{x}_2$ and $\hat{y}_2$. Let the last two entries of f be $f_3 = \hat{x}_2$ and $f_4 = \hat{y}_2$. This defines the correspondence f. Clearly, f is nonempty and convex. Moreover, since the equations (EC.31) to (EC.34) are continuously differentiable on both sides, f is upper hemi-continuous. Therefore, applying Kakutani's fixed-point theorem, we can always find a solution to (EC.31) and (EC.34) inside Ω. It is clear that the fixed point falls into the interior of Ω.

Next we show that the fixed point of f, denoted $(x_1^*,y_1^*,x_2^*,y_2^*)$, is indeed a Nash equilibrium. It suffices to show that for $(y_1^*,y_2^*) \in (0,\delta) \times (1-\delta,1)$,

$$\log\frac{x_1}{1-x_1-y_1^*} + \frac{x_1}{1-x_1-y_1^*} = \alpha^X - 1 + (\gamma_{12}+\gamma_{21})n_2x_2 + 2n_1\gamma_1x_1$$
$$\log\frac{x_2}{1-x_2-y_2^*} + \frac{x_2}{1-x_2-y_2^*} = \alpha^X - 1 + (\gamma_{12}+\gamma_{21})n_1x_1 + 2n_2\gamma_2x_2$$

does not have other solutions (x_1, x_2) in the domain when γ_1 and γ_2 are sufficiently large. It suffices to show that

$$h\left(\frac{x}{1-x-y_1^*}\right) = \hat{\alpha}^X - 1 + \gamma x \tag{EC.29}$$

has a unique solution of x for all $y_1^* \in (0, \delta]$ and $\hat{\alpha}^X \in [\alpha^X, \alpha^X + (\gamma_{12} + \gamma_{21})n_2]$ when γ is sufficiently large. The uniqueness of x_2 is similar.

The proof for this claim is similar to the analysis in the proof of Proposition 4. Note that $dh(\frac{x}{1-x-y_1^*})/dx = \frac{(1-y_1^*)^2}{x(1-x-y_1^*)^2}$ and $h(\frac{x}{1-x-y_1^*})$ is concave for $0 < x < (1-y_1^*)/3$ and convex for $(1-y_1^*)/3 < x < 1-y_1^*$. Because $x > 0$, the solution to (EC.29) satisfies $\frac{x}{1-x-y_1^*} > h^{-1}(\hat{\alpha}^X - 1)$, implying $x > \frac{3h^{-1}(\alpha^X-1)}{4(h^{-1}(\alpha^X-1)+1)} \triangleq \hat{x}$ as $y_1^* \leq \delta < 1/4$. Note that $\hat{x}$ only depends on the parameter α^X. If $\hat{x} \geq (1-y_1^*)/3$, then $\hat{\alpha}^X - 1 + \gamma x$ does not intersect $h\left(\frac{x}{1-x-y_1^*}\right)$ in its concave segment. Hence the solution to (EC.29) is unique. If $\hat{x} < (1-y^*)/3 < 1/4$, then we choose

$$\gamma > \frac{1}{\hat{x}(3/4-\hat{x})^2} \geq \frac{dh\left(\frac{x}{1-x-y_1^*}\right)}{dx} \tag{EC.30}$$

for all $y_1^* \in (0, \delta]$ and $0 < x < (1-y_1^*)/3$. This implies that the slope of $\hat{\alpha}^X - 1 + \gamma x$ is larger than the derivative of $h\left(\frac{x}{1-x-y_1^*}\right)$ at their possible intersections in the concave segment. Because $h(x/(1-x-y_1^*)) \to -\infty$ as $x \to 0$, there are no such intersections. Similar to the previous case, there is at most one solution to (EC.29). Therefore, given (y_1^*, y_2^*), the solution (x_1^*, x_2^*) obtained from the fixed point is the unique solution to the first order condition of firm X, which must coincide with the best response. Thus, the fixed point must be a Nash equilibrium.

We prove the second part of Proposition 9 by contradiction. Suppose the claim does not hold. Without loss of generality, suppose $x_2^* - x_1^* \geq \epsilon_0 \geq |y_1^* - y_2^*|$ for some $\epsilon_0 > 0$. By (EC.31) and (EC.32), we have

$$h\left(\frac{x_1^*}{1-x_1^*-y_1^*}\right) - h\left(\frac{x_2^*}{1-x_2^*-y_2^*}\right) \geq (\gamma_{12}+\gamma_{21}-2\gamma_1)n_1\epsilon_0$$

Because $\gamma_{12}+\gamma_{21}-2\gamma_1 \geq 0$ and $h(\cdot)$ is a strictly increasing function, we have $\frac{x_1^*}{1-x_1^*-y_1^*} - \frac{x_2^*}{1-x_2^*-y_2^*} \geq 0$. Because $x_1^* < x_2^*$, we must have $1-x_1^*-y_1^* < 1-x_2^*-y_2^*$, i.e., $y_1^*-y_2^* > \epsilon_0$. It contradicts the fact that $|y_1^*-y_2^*| \leq \epsilon_0$. Therefore, we must have $x_1^* = x_2^*$ and $y_1^* = y_2^*$.

Proof of Proposition 10. The result follows the same steps as in the proof of Proposition 1. After applying the transformation $\tilde{\boldsymbol{x}}_t = -\boldsymbol{x}_t$ and substituting $\boldsymbol{x}$ into $p_{i,t}^X$, we have that the payoff function of firm X is

$$f_X(\{\tilde{\boldsymbol{x}}\}_{t=1}^{\infty}, \{\boldsymbol{y}\}_{t=1}^{\infty}) = \sum_{t=1}^{\infty} r^t \sum_{i=1}^{n} -\frac{\tilde{x}_{i,t}}{\beta_i}\left(\alpha^X - \sum_{j=1}^{n} g_{ij}\tilde{x}_{j,t-1} - \log(-\tilde{x}_{i,t}) + \log(1+\tilde{x}_{i,t}-y_{i,t})\right),$$

when $\tilde{x}_{i,t} = 0$ or $x_{i,t}+y_{i,t} = -\tilde{x}_{i,t}+y_{i,t} < 1$ for all i and t; If $-\tilde{x}_i + y_i \geq 1$ and $\tilde{x}_i < 0$ for some i then $f_X = -\infty$. The payoff function of firm Y can be defined similarly. It is not hard to show A1 to A3 in the proof of Proposition 1 continue to hold in the infinite-horizon case. Therefore, pure-strategy OLNE always exists.

Proof of Proposition 11. If $\lim_{t\to\infty} \boldsymbol{x}_t = \boldsymbol{x}_\infty$ and $\lim_{t\to\infty} \boldsymbol{y}_t = \boldsymbol{y}_\infty$, then for all $\epsilon > 0$, there exists $T_\epsilon > 0$ such that for $t > T_\epsilon$, $\|\boldsymbol{x}_t - \boldsymbol{x}_\infty\|_2 < \epsilon$ and $\|\boldsymbol{y}_t - \boldsymbol{y}_\infty\|_2 < \epsilon$. Consider firm X's revenue generated starting from period T given $\boldsymbol{y}_t$:

$$\sum_{t=T}^{\infty} r^t \sum_{i=1}^{n} x_{i,t} p_{i,t}^X = \sum_{t=T}^{\infty} r^t \sum_{i=1}^{n} \frac{x_{i,t}}{\beta_i}\left(\alpha^X + \sum_{j=1}^{n} g_{ij} x_{j,t-1} - \log\left(\frac{x_{i,t}}{1-x_{i,t}-y_{i,t}}\right)\right).$$

Because $\{\boldsymbol{x}_t\}_{t=0}^{\infty}$ and $\{\boldsymbol{y}_t\}_{t=0}^{\infty}$ are OLNE, $\{\boldsymbol{x}_t\}_{t=T}^{\infty}$ maximizes the above quantity. It implies that $\boldsymbol{x}_t$ maximizes

$$\sum_{i=1}^{n} \frac{x_{i,t}}{\beta_i}\left(\alpha^X + \sum_{j=1}^{n}\left(g_{ij}x_{j,t-1} + r\frac{\beta_i g_{ji}}{\beta_j}x_{j,t+1}\right) - \log\left(\frac{x_{i,t}}{1-x_{i,t}-y_{i,t}}\right)\right).$$

By the continuity of the payoff function and the conditions that $\|\boldsymbol{x}_t - \boldsymbol{x}_\infty\|_2 < \epsilon$ and $\|\boldsymbol{y}_t - \boldsymbol{y}_\infty\|_2 < \epsilon$ for a sufficiently small ϵ, we have

$$\begin{aligned}&\sum_{i=1}^{n} \frac{x_{i,\infty}}{\beta_i}\left(\alpha^X + \sum_{j=1}^{n}\left(g_{ij}x_{j,\infty} + r\frac{\beta_i g_{ji}}{\beta_j}x_{j,\infty}\right) - \log\left(\frac{x_{i,\infty}}{1-x_{i,\infty}-y_{i,\infty}}\right)\right)\\ &\geq \max_{\boldsymbol{x}}\left\{\sum_{i=1}^{n} \frac{x_i}{\beta_i}\left(\alpha^X + \sum_{j=1}^{n}\left(g_{ij}x_{j,\infty} + r\frac{\beta_i g_{ji}}{\beta_j}x_{j,\infty}\right) - \log\left(\frac{x_i}{1-x_i-y_{i,\infty}}\right)\right)\right\} - \epsilon_1\end{aligned}$$

for all $\epsilon_1 > 0$. Because ϵ_1 is an arbitrary positive number, we have that

$$x_{i,\infty} = \underset{0\leq x\leq 1-y_{i,\infty}}{\arg\max}\left\{\frac{x_i}{\beta_i}\left(\alpha^X + \sum_{j=1}^{n}\left(g_{ij} + r\frac{\beta_i g_{ji}}{\beta_j}\right)x_{j,\infty} - \log\left(\frac{x_i}{1-x_i-y_{i,\infty}}\right)\right)\right\}.$$

The first-order condition is thus

$$\alpha^X - 1 + \sum_{j=1}^{n}\left(g_{ij} + r\frac{\beta_i g_{ji}}{\beta_j}\right)x_{j,\infty} - \log\frac{x_{i,\infty}}{1-x_{i,\infty}-y_{i,\infty}} - \frac{x_{i,\infty}}{1-x_{i,\infty}-y_{i,\infty}} = 0.$$

Similarly, we have

$$\alpha^Y - 1 + \sum_{j=1}^{n}\left(g_{ij} + r\frac{\beta_i g_{ji}}{\beta_j}\right)y_{j,\infty} - \log\frac{y_{i,\infty}}{1-x_{i,\infty}-y_{i,\infty}} - \frac{y_{i,\infty}}{1-x_{i,\infty}-y_{i,\infty}} = 0.$$

Compared to (5) and (6), it is clear that they match the static game with $\tilde{\alpha}^X = \alpha^X$, $\tilde{\alpha}^Y = \alpha^Y$ and $\sum_{j=1}^{n}\left(\tilde{g}_{ij} + \frac{\tilde{\beta}_i\tilde{g}_{ji}}{\tilde{\beta}_j}\right) = \sum_{j=1}^{n}\left(g_{ij} + \frac{r\beta_i g_{ji}}{\beta_j}\right)$. Thus we have completed the proof.

Proof of Proposition 12. To show that X retains the leading position, we only need to show that $x_{i,1} - y_{i,1} > (1-\epsilon)\delta$ for all i. The proof for $t = 2$ is similar by induction and replacing $(1-\epsilon)\delta$ by δ. Suppose $\{\boldsymbol{x}_t\}_{t=1}^2$ and $\{\boldsymbol{y}_t\}_{t=1}^2$ form an OLNE. By the first-order condition for $x_{i,1}$, we have

$$\alpha^X - 1 + \sum_{j=1}^n \left(g_{ij} x_{j,0} + r \frac{\beta_i g_{ji}}{\beta_j} x_{j,2} \right) - \log \frac{x_{i,1}}{1 - x_{i,1} - y_{i,1}} - \frac{x_{i,1}}{1 - x_{i,1} - y_{i,1}} = 0.$$

Rearranging the terms, we have

$$y_{i,1} = 1 - x_{i,1} - \frac{x_{i,1}}{h^{-1}\left(\alpha^X - 1 + \sum_{j=1}^n \left(g_{ij} x_{j,0} + r \frac{\beta_i g_{ji}}{\beta_j} x_{j,2} \right)\right)}.$$

Symmetrically, we have the first-order condition for $y_{i,1}$

$$x_{i,1} = 1 - y_{i,1} - \frac{y_{i,1}}{h^{-1}\left(\alpha^Y - 1 + \sum_{j=1}^n \left(g_{ij} y_{j,0} + r \frac{\beta_i g_{ji}}{\beta_j} y_{j,2} \right)\right)}.$$

For simplicity, we let

$$\begin{aligned} h^{-1}(x,i,1) &\triangleq h^{-1}\left(\alpha^X - 1 + \sum_{j=1}^n \left(g_{ij} x_{j,0} + r \frac{\beta_i g_{ji}}{\beta_j} x_{j,2} \right)\right) \\ h^{-1}(y,i,1) &\triangleq h^{-1}\left(\alpha^Y - 1 + \sum_{j=1}^n \left(g_{ij} y_{j,0} + r \frac{\beta_i g_{ji}}{\beta_j} y_{j,2} \right)\right). \end{aligned}$$

Solving $x_{i,1}$ and $y_{i,1}$ from both equations, we have

$$\begin{aligned} x_{i,1} &= \frac{h^{-1}(x,i,1)}{1 + h^{-1}(x,i,1) + h^{-1}(y,i,1)} \\ y_{i,1} &= \frac{h^{-1}(y,i,1)}{1 + h^{-1}(x,i,1) + h^{-1}(y,i,1)}. \end{aligned}$$

Since $x_{i,0} - y_{i,0} > \delta$ for all i, using the expressions of $x_{i,1}$ and $y_{i,1}$, we have

$$x_{i,1} - y_{i,1} = \frac{h^{-1}(x,i,1) - h^{-1}(y,i,1)}{1 + h^{-1}(x,i,1) + h^{-1}(y,i,1)}$$

Using the facts that $h^{-1}(x) \leq \max\{1, x\}$ and $x_{j,1} + y_{j,1} \leq 1$, we have that

$$1 + h^{-1}(x,i,1) + h^{-1}(y,i,1) \leq \max\left\{3, \alpha^X + \alpha^Y + \sum_{j=1} g_{ij} + \sum_{j=1}^n r \frac{\beta_i g_{ji}}{\beta_j}\right\}.$$

When g_{ij}s are sufficiently large and r is sufficiently small, the above quantity is less than equal to $(1+\epsilon_1)\sum_{j=1}^n g_{ij}$, for a small $\epsilon_1 > 0$. On the other hand, because $x_{j,0} - y_{j,0} > \delta$, for sufficiently large g_{ij},

$$\begin{aligned} h^{-1}(x,i,1) &\geq \alpha^X - 1 + \sum_{j=1}^n \left(g_{ij}x_{j,0} + r\frac{\beta_i g_{ji}}{\beta_j}x_{j,2} \right) \\ &\quad - \log\left(\alpha^X - 1 + \sum_{j=1}^n \left(g_{ij}x_{j,0} + r\frac{\beta_i g_{ji}}{\beta_j}x_{j,2} \right) \right) \\ h^{-1}(y,i,1) &\leq \max\left\{ 1, \alpha^Y - 1 + \sum_{j=1}^n \left(g_{ij}y_{j,0} + r\frac{\beta_i g_{ji}}{\beta_j}y_{j,2} \right) \right\} \end{aligned}$$

which implies that

$$\begin{aligned} h^{-1}(x,i,1) - h^{-1}(y,i,1) &\geq \alpha^X - \max\{2, \alpha^Y\} + \sum_{j=1}^n g_{ij}(x_{j,0} - y_{j,0}) + \sum_{j=1}^n r\frac{\beta_i g_{ji}}{\beta_j}(x_{j,2} - y_{j,2}) \\ &\quad - \log\left(\alpha^X - 1 + \sum_{j=1}^n \left(g_{ij}x_{j,0} + r\frac{\beta_i g_{ji}}{\beta_j}x_{j,2} \right) \right) \\ &\geq (1-\epsilon_2)\sum_{j=1}^n g_{ij}\delta, \end{aligned}$$

for any small $\epsilon_2 > 0$ when g_{ij} are large and r is small. Combining the above inequalities, we have

$$x_{i,1} - y_{i,1} \geq \frac{(1-\epsilon_2)\sum_{j=1}^n g_{ij}\delta}{(1+\epsilon_1)\sum_{j=1}^n g_{ij}} \geq (1-\epsilon)\delta$$

when ϵ_1 and ϵ_2 are small. This proves the result.

Next we show that the revenue of firm X is higher. The difference in revenue can be expressed as

$$\sum_{t=1}^2 r^t \sum_{i=1}^n (x_{i,t}p_{i,t}^X - y_{i,t}p_{i,t}^Y)$$

$$
\begin{aligned}
&= \sum_{t=1}^{2} r^t \sum_{i=1}^{n} \frac{x_{i,t}}{\beta_i} \left(\alpha^X + \sum_{j=1}^{n} g_{ij} x_{j,t-1} - \log\left(\frac{x_{i,t}}{1 - x_{i,t} - y_{i,t}} \right) \right) \\
&\quad - \sum_{t=1}^{2} r^t \sum_{i=1}^{n} \frac{y_{i,t}}{\beta_i} \left(\alpha^Y + \sum_{j=1}^{n} g_{ij} y_{j,t-1} - \log\left(\frac{y_{i,t}}{1 - x_{i,t} - y_{i,t}} \right) \right) \\
&= \sum_{t=1}^{2} r^t \sum_{i=1}^{n} \frac{1}{\beta_i} \Big(x_{i,t}\alpha^X - y_{i,t}\alpha^Y + \sum_{j=1}^{n} g_{ij} (x_{i,t}x_{j,t-1} - y_{i,t}y_{j,t-1}) + (x_{i,t} - y_{i,t}) \log(1 - x_{i,t} - y_{i,t}) \\
&\quad - x_{i,t} \log(x_{i,t}) + y_{i,t} \log(y_{i,t}) \Big).
\end{aligned}
$$

Note that by the first part of the proof, $x_{i,t} > y_{i,t} + (1-\epsilon)\delta$ for all i and t; moreover, $x_{i,t}\log(x_{i,t}) \leq 0$ and $y_{i,t}\log(y_{i,t}) > -e^{-1}$ for $y_{i,t} \in (0,1)$. As g_{ij} becomes sufficiently large, the above expression is always positive. This proves the claim.

EC.3. Additional Results

EC.3.1. Bertrand Versus Cournot Competition

In this study, we consider Cournot competition instead of Bertrand competition. In other words, the firms do not compete in the price space but in the quantity space. There are three reasons. First, the modeling choice between Bertrand and Cournot competitions is usually viewed in the literature as a way to reflect firms' monopolistic power, or equivalently, the intensity of head-to-head competition. For example, Singh and Vives (1984) establish that Cournot competition is associated with an environment with higher firm profits and lower consumer surplus than Bertrand competition, thereby is regarded as more monopolistic than Bertrand competition. We choose Cournot competition because market dominance usually arises in the network context.

Second, Cournot competition seems to provide more precise description of certain industries. As Cabral (2017, Section 8.3, pp. 200-201) articulates, "[i]f output and capacity are difficult to adjust, then the Cournot model describes duopoly competition better." He also argues that "[m]ost real-world industries seem closer to the case when capacity is difficult to adjust." Thus, Cournot competition is the right model because "capacity or output decisions are normally the long-run variable, prices being set in the short run." He gives a number of examples such as wheat, cement, steel, cars, computers, video-game consoles. While we shall not take it too far to overrule the legitimacy of Bertrand competition, the Cournot model adopted in our paper does reflect some industry practice as argued by Cabral (2017).

Perhaps more importantly, in the network context, there is a delicate technical issue for not considering Bertrand competition. Price competition is usually modeled in two stages in the network literature: firms set prices in the first stage and customers determine their quantities in the purchasing subgame (second stage). In the second stage, when the choice is modeled by MNL, it is well documented that there exist multiple equilibria (Brock and Durlauf 2001, Du et al. 2016) for given prices. Therefore, without an explicit rule to choose one particular equilibrium, the game of the first stage is not well defined. The previous papers studying price competition with network effects either use a linear-quadratic framework so that there is a unique equilibrium in the second stage (Candogan et al. 2012, Chen et al. 2018), or only focus on symmetric equilibria so that one particular equilibrium is chosen in the second stage (Tan and Zhou 2017). Therefore, in our setup, we cannot define Bertrand competition properly and Cournot competition is more reasonable.

EC.3.2. Positive Prices in the Nash Equilibrium

Here we show that for homogeneous customers, only positive prices may arise from a Nash equilibrium. Although the MNL model may induce negative prices, it is never optimal to do so for the competing firms.

PROPOSITION EC.1. *Suppose Assumptions 1 and 2 hold. Then the equilibrium prices of both firms are positive.*

Proof of Proposition EC.1. Suppose x^* and y^* are the equilibrium strategies of firm X and Y. The equilibrium prices are then

$$p^X = \frac{1}{\beta}\left(\alpha^X + \frac{\gamma x^*}{2} - \log\left(\frac{x^*}{1-x^*-y^*}\right)\right)$$
$$p^Y = \frac{1}{\beta}\left(\alpha^Y + \frac{\gamma y^*}{2} - \log\left(\frac{y^*}{1-x^*-y^*}\right)\right)$$

by Equation 3. Because x^* and y^* form a Nash equilibrium, we must have

$$x^* = \underset{0\le x\le 1-y^*}{\arg\max}\, x\left(\alpha^X + \frac{\gamma x}{2} - \log\left(\frac{x}{1-x-y^*}\right)\right)$$
$$y^* = \underset{0\le y\le 1-x^*}{\arg\max}\, y\left(\alpha^y + \frac{\gamma y}{2} - \log\left(\frac{y}{1-x^*-y}\right)\right)$$

If $p^X \le 0$, then the revenue $nx^*p^X \le 0$ and we can show that x^* is not the best response. Indeed, since $y^* < 1$, firm X can always choose x sufficiently small, so that $-\log\left(\frac{x}{1-x-y^*}\right) > -\alpha^X - \frac{\gamma x}{2}$. Such choice of x makes positive revenues. Therefore, x^* cannot be an equilibrium strategy and we must have $p^X > 0$. Similarly, we have $p^Y > 0$.

EC.3.3. Existence of Solutions to (1) and (2)

Here we establish the existence of solutions $(x_1,\dots,x_n,y_1,\dots,y_n)$ to (1) and (2) given p_i^X and p_i^Y for all i. Consider the following $2n$-dimensional nonlinear dynamic system

$$x_{i,t+1} = \frac{\exp\left\{\alpha^X - \beta_i p_i^X + \sum_{j=1}^n g_{ij}x_{j,t}\right\}}{1+\exp\left\{\alpha^X - \beta_i p_i^X + \sum_{j=1}^n g_{ij}x_{j,t}\right\} + \exp\left\{\alpha^Y - \beta_i p_i^Y + \sum_{j=1}^n g_{ij}y_{j,t}\right\}}$$

$$y_{i,t+1} = \frac{\exp\left\{\alpha^Y - \beta_i p_i^Y + \sum_{j=1}^n g_{ij}y_{j,t}\right\}}{1+\exp\left\{\alpha^X - \beta_i p_i^X + \sum_{j=1}^n g_{ij}x_{j,t}\right\} + \exp\left\{\alpha^Y - \beta_i p_i^Y + \sum_{j=1}^n g_{ij}y_{j,t}\right\}},$$

for $i=1,\dots,n$. This dynamic system maps $(x_{1,t},\dots,x_{n,t},y_{1,t},\dots,y_{n,t}) \in [0,1]^{2n}$ to a point in $[0,1]^{2n}$ itself. Clearly the mapping is continuous and $[0,1]^{2n}$ is a compact convex subset of the Euclidean space $\mathbb{R}^{2n}$. by Brouwer's fixed-point theorem, there exists a fixed point. The fixed point is the solution to (1) and (2).

EC.3.4. Uniform Price for All Customers

Although we consider Cournot competition and the firms do not set prices directly, the prices associated with the Nash equilibrium are usually personalized. That is, different customers may experience different unit prices. In this section, we formulate the game when the firms adopt uniform pricing.

For given $(y_1,\dots,y_n)$, firm X's strategy $(x_1,\dots,x_n)$ satisfy (1). Now we impose the additional requirement that $p_i^X \equiv p^X$ for all i. That is, $(x_1,\dots,x_n)$ satisfy

$$x_i = \frac{\exp\left\{\alpha^X - \beta_i p^X + \sum_{j=1}^n g_{ij}x_j\right\}}{1+\exp\left\{\alpha^X - \beta_i p^X + \sum_{j=1}^n g_{ij}x_j\right\} + \exp\left\{\alpha^Y - \beta_i p^Y + \sum_{j=1}^n g_{ij}y_j\right\}}.$$

Combining with (3), the constraint of uniform pricing becomes

$$\frac{1}{\beta_i}\left(\alpha^X + \sum_{j=1}^n g_{ij}x_j - \log\left(\frac{x_i}{1-x_i-y_i}\right)\right) \text{ are equal for all } i.$$

Therefore, the strategy and payoff function of firm X given the opponent's strategy $(y_1,\dots,y_n)$ are defined below

$$\max_{\{x_i\}_{i=1}^n} \sum_{i=1}^n \frac{x_i}{\beta_i}\left(\alpha^X+\sum_{j=1}^n g_{ij}x_j-\log(x_i)+\log(1-x_i-y_i)\right)$$

$$\begin{aligned} \text{subject to } & 0\le x_i\le 1-y_i, \\ & \frac{1}{\beta_1}\left(\alpha^X+\sum_{j=1}^n g_{1j}x_j-\log\left(\frac{x_1}{1-x_1-y_1}\right)\right)=\dots \\ & =\frac{1}{\beta_n}\left(\alpha^X+\sum_{j=1}^n g_{nj}x_j-\log\left(\frac{x_n}{1-x_n-y_n}\right)\right) \end{aligned}$$

Similar to the last section, it is easy to see that for any given p^X and $(y_1,\dots,y_n)$, there always exist such $(x_1,\dots,x_n)$. So the game is well defined.

However, we suspect that there may not exist pure-strategy Nash equilibrium for this game. There are two primary methods showing the existence of pure-strategy Nash equilibria: Kakutani's fixed-point theorem and supermodular game. As we have shown, the best-response function is not continuous with or without the additional constraint and thus may not have a closed graph. Thus Kakutani's fixed-point theorem may not be applied. Moreover, because of the additional constraint, the strategy set is not a lattice any more and the game may not be supermodular even after transformations. Nevertheless, for homogeneous customers (Section 4) the theoretical results apply to uniform pricing as well because the resulting prices are always equal for all customers.

EC.3.5. Strong Network Effects for Heterogeneous Customers

The case for strong network effects, however, is more complex. In this case, the multiplicity of Nash equilibria is expected to grow exponentially when customers are heterogeneous. A general treatment for the types of equilibria might be unavailable. Nevertheless, we are able to derive a slightly weaker result:

PROPOSITION EC.2. *Given* α^X, α^Y, β_i *for all* i *and* $\delta \in (0, 1/4)$, *when the network effects* g_{ij} *are sufficiently large for all* i *and* j, *there exist* $\{x_i^*\}_{i=1}^n \in [0, \delta]^n$ *and* $\{y_i^*\}_{i=1}^n \in [1-\delta, 1]^n$ *satisfying the Nash equilibrium first-order conditions* (5) *and* (6) *simultaneously.*

Although $\{x_i^*\}_{i=1}^n$ and $\{y_i^*\}_{i=1}^n$ are not guaranteed to be a Nash equilibrium due to the non-concavity of the payoff functions, the proposition confirms the close relationship between strong network effects and market dominance. To establish the result, we apply Kakutani's fixed-point theorem. However, the standard best-response correspondence fails to meet the conditions of convexity and upper hemi-continuity that are required for the theorem, because the payoff functions are non-concave. We design a different correspondence based on the following idea: when the choice probabilities of the other $n-1$ customers are fixed, the equilibrium analysis of an individual customer can be reduced to the results established in Section 4, especially the existence and uniqueness. Therefore, the correspondence can be defined by simultaneously updating the Nash equilibrium of n individual customers while fixing the choice probabilities of others. It can be shown to satisfy the conditions of Kakutani's theorem.

EC.3.6. Network Structures: Connectivity and Imbalance

In this section, we investigate the impact of network structure on the duopoly competition. We follow the setup in Section 5.1 and study symmetric Nash equilibria. That is, $\alpha^X = \alpha^Y = \alpha$, $x_i^* = y_i^*$ for all i as the unique Nash equilibrium because the networks structures are sufficiently weak. Therefore, we can isolate the effect of the network from asymmetric equilibria.

We first focus on the connectivity of the network, i.e., $g_{ij} \equiv g$. By the reason explained in Section 4, the firm may still use targeted marketing and sell different amount to different customers. This makes comparison of equilibria impossible, because it is hard to nail down a particular equilibrium. To make comparisons, we impose Assumption 2, i.e., the consumption $x_i^* \equiv x^*$.

PROPOSITION EC.3. *Suppose Assumptions 1 to 3 hold. When g increases, x_i^* increases for all i.*

The proof is straightforward using the first-order condition as in the proof of Proposition 8.

Next we focus on the network imbalance. For the balanced network, we assume $g_{ij} \equiv g$. For the imbalanced network, we fix the total network effects and let customer $i = 1$ be the only hub. That is, $g_{1j} = g_{j1} = n^2 g/(2n+1)$ for all j while $g_{ij} = 0$ otherwise. Let the equilibrium consumption of the balanced network be $\bar{x}^*$ for all customer. For the imbalanced network, let the equilibrium quantity of customer one be x_1^* and that of other customers be x_2^*. The next proposition states that imbalance helps the firm gain market share when the market is large and the product is not very popular either because of low quality or weak network effects.

PROPOSITION EC.4. *Suppose $x^* < 1/4$. Consider the following regime: fix α and ng and increase n. If n is sufficiently large, then $x_1^* > x_2^* > x^*$.*

In the asymptotic regime we fix ng so that the average network effect of an individual is constant. The result shows that the consumption of the leaf node in the imbalanced network is larger than that of the balanced network, not to mention the star in the imbalanced network.

F ollowing the proof of Proposition 8, by the first-order conditions, we have the following equations:

$$\begin{aligned} \alpha - 1 + 2ngx^* &= \log \frac{x^*}{1-2x^*} + \frac{x^*}{1-2x*} \\ \alpha - 1 + 2n\frac{ng}{2n+1}(x_1^* + (n-1)x_2^*) &= \log \frac{x_1^*}{1-2x_1^*} + \frac{x_1^*}{1-2x_1^*} \\ \alpha - 1 + 2n\frac{ng}{2n+1}x_1^* &= \log \frac{x_2^*}{1-2x_2^*} + \frac{x_2^*}{1-2x_2^*} \end{aligned}$$

As n increases, x^* doesn't change because α and ng are fixed. At the same time, x_1^* has a constant lower bound, which is solved from

$$\alpha - 1 + 2\frac{ng}{3}x_1^* = \log \frac{x_1^*}{1-2x_1^*} + \frac{x_1^*}{1-2x_1^*}.$$

Similarly, x_2^* is also lower bounded as n increases, because the left-hand side of the third equation is lower bounded by $\alpha - 1 + 2ng/3x_1^*$. Therefore, as n increases the term $x_1^* + (n-1)x_2^*$ tends to infinity, and as a result, $\log(x_1^*/(1-2x_1^*)) + x_1^*/(1-2x_1^*)$ tends to infinity. Because of the property of $\log(x/(1-2x)) + x/(1-2x)$ (e.g., by Figure EC.2), x_1^* tends to $1/2$. This implies that $nx_1^*/(2n+1) > x^*$ when n is sufficiently large. Comparing the left-hand sides of the first and the third equations, we have $x_2^* > x^*$. It is clear that $x_1^* > x_2^*$.

EC.3.7. Additional Analysis for Section 5.2:

We focus on the best responses. Similar to Section 4, we only focus on the case that firm X (Y) will choose a price so that x_i (y_i) is the choice probability of all customers in community i for $i = 1, 2$. According to (5), the best response of firm X, (x_1, x_2), satisfies

$$\log \frac{x_1}{1 - x_1 - y_1} + \frac{x_1}{1 - x_1 - y_1} = \alpha^X - 1 + (\gamma_{12} + \gamma_{21}) n_2 x_2 + 2 n_1 \gamma_1 x_1, \tag{EC.31}$$

$$\log \frac{x_2}{1 - x_2 - y_2} + \frac{x_2}{1 - x_2 - y_2} = \alpha^X - 1 + (\gamma_{12} + \gamma_{21}) n_1 x_1 + 2 n_2 \gamma_2 x_2, \tag{EC.32}$$

for given (y_1, y_2). The best response of Y can be derived similarly.

$$\log \frac{y_1}{1 - x_1 - y_1} + \frac{y_1}{1 - x_1 - y_1} = \alpha^Y - 1 + (\gamma_{12} + \gamma_{21}) n_2 y_2 + 2 n_1 \gamma_1 y_1, \tag{EC.33}$$

$$\log \frac{y_2}{1 - x_2 - y_2} + \frac{y_2}{1 - x_2 - y_2} = \alpha^Y - 1 + (\gamma_{12} + \gamma_{21}) n_1 y_1 + 2 n_2 \gamma_2 y_2. \tag{EC.34}$$

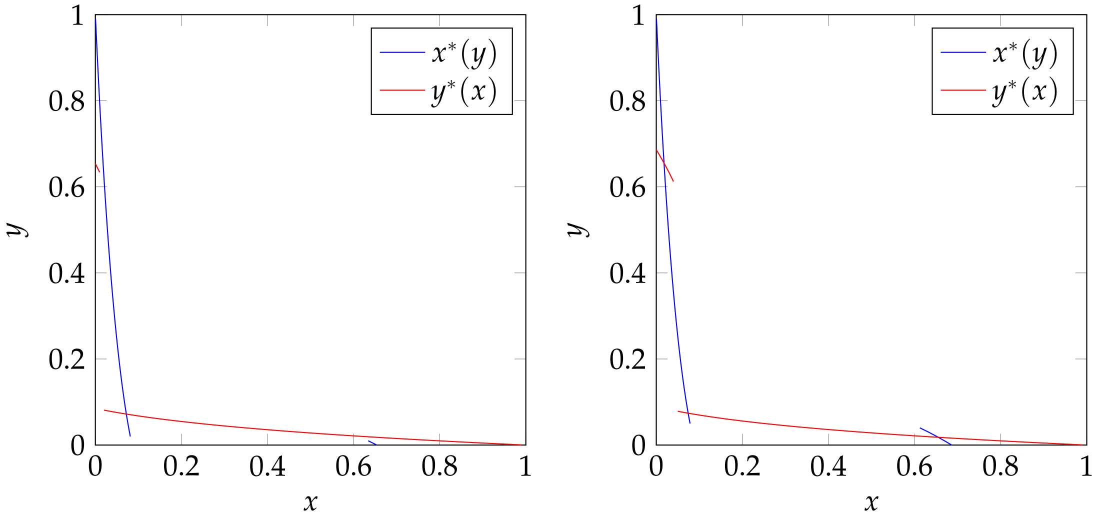
Observe that although the network itself can be asymmetric ($\gamma_{12} \neq \gamma_{21}$), the individual values of γ_{12} and γ_{21} do not play a role in the first-order conditions as long as their sum $\gamma_{12} + \gamma_{21}$ remains the same.

EC.3.8. Additional Figures for the Numerical Experiment

Following the setup in Section 4.4, we find a discontinuous best response need not lead to type-II equilibria, as demonstrated by Figure EC.4. Such complex nature of the equilibrium structure makes our theoretical results particularly valuable. We have also tested convex/concave network effects, and the Nash equilibria follow the same structure. See Section EC.3.9 in the appendix for more details.

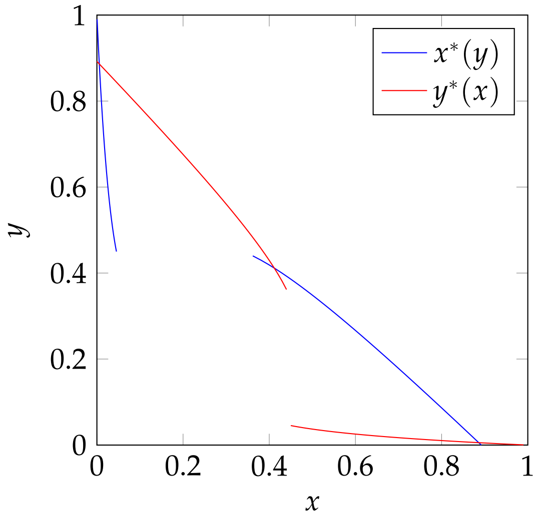
EC.3.9. Nonlinear Network Effects

In this paper, the network effects appear in a linear form according to (1) and (2). Although linear network effects are common in the literature due to the analytical tractability (Candogan et al. 2012, Du et al. 2016, Chen et al. 2018), it is also argued



(a) Type I with $\alpha = -2$, $\gamma = 8.5$.

(b) Type III with $\alpha = -2$, $\gamma = 8.9$.



(c) Type III with $\alpha = -2$, $\gamma = 15$.

Figure EC.4 Illustration of complex patterns of Nash equilibria.

that network effects can be either convex or concave, depending on specific products. In this section, we explore other forms of network effects and compute the Nash equilibria numerically. The objective is to test the robustness of the theoretical results to the assumption of linear network effects.

We first consider the following convex network effects[18]:

$$x_i = \frac{\exp\left\{\alpha^X - \beta_i p_i^X + \sum_{j=1}^n g_{ij} x_j^2\right\}}{1 + \exp\left\{\alpha^X - \beta_i p_i^X + \sum_{j=1}^n g_{ij} x_j^2\right\} + \exp\left\{\alpha^Y - \beta_i p_i^Y + \sum_{j=1}^n g_{ij} y_j^2\right\}},$$

$$y_i = \frac{\exp\left\{\alpha^Y - \beta_i p_i^Y + \sum_{j=1}^n g_{ij} y_j^2\right\}}{1 + \exp\left\{\alpha^X - \beta_i p_i^X + \sum_{j=1}^n g_{ij} x_j^2\right\} + \exp\left\{\alpha^Y - \beta_i p_i^Y + \sum_{j=1}^n g_{ij} y_j^2\right\}}.$$

To simplify the comparative statics, we assume Assumption 1 to 3 and conduct a numerical experiment similar to Section 4.4. Recall that the parameter α represents the quality of both products and γ represents the strength of the network effects. The types of Nash equilibria are shown in Table EC.1.

Similarly, we use the following model for concave network effects:

$$x_i = \frac{\exp\left\{\alpha^X - \beta_i p_i^X + \sum_{j=1}^n g_{ij} \sqrt{x_j}\right\}}{1 + \exp\left\{\alpha^X - \beta_i p_i^X + \sum_{j=1}^n g_{ij} \sqrt{x_j}\right\} + \exp\left\{\alpha^Y - \beta_i p_i^Y + \sum_{j=1}^n g_{ij} \sqrt{y_j}\right\}},$$

$$y_i = \frac{\exp\left\{\alpha^Y - \beta_i p_i^Y + \sum_{j=1}^n g_{ij} \sqrt{y_j}\right\}}{1 + \exp\left\{\alpha^X - \beta_i p_i^X + \sum_{j=1}^n g_{ij} \sqrt{x_j}\right\} + \exp\left\{\alpha^Y - \beta_i p_i^Y + \sum_{j=1}^n g_{ij} \sqrt{y_j}\right\}}.$$

The types of Nash equilibria are shown in Table EC.2.

The two tables demonstrate the predictive power of the theoretical results in Section 4. Although they are derived for linear network effects, the patterns persist for convex/concave network effects. In general, when the network effects are weak, there is a single symmetric Nash equilibrium; when the network effects are strong, there are three Nash equilibria, two of which are asymmetric; when the quality is low and the network effects are neither too strong nor too weak, there are only two asymmetric Nash equilibria.

	γ						
α	0	5	10	20	30	40	50
6	I	I	I	III	III	III	III
4	I	I	I	III	III	III	III
2	I	I	III	III	III	III	III
0	I	I	III	III	III	III	III
-2	I	I	II	II	III	III	III
-4	I	I	I	II	II	III	III
-6	I	I	I	II	II	II	III

Table EC.1 The equilibrium types for different qualities and network strength when the network effects are convex. The type I, II, and III correspond to one, two and three Nash equilibria, which are defined in Section 4.4.

	γ						
α	0	10	20	30	40	50	150
6	I	I	I	I	I	I	III
4	I	I	I	I	I	I	III
2	I	I	I	I	I	I	III
0	I	I	III	III	III	III	III
-2	I	I	II	III	III	III	III
-4	I	I	II	II	III	III	III
-6	I	I	I	II	II	III	III

Table EC.2 The equilibrium types for different qualities and network strength when the network effects are concave. The type I, II, and III correspond to one, two and three Nash equilibria, which are defined in Section 4.4.