Online Supplement

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S-1 Supplement material for Section 2.2

In this section, we show the properties of the risk aversion implied by the regime-dependent utility function. Under the regime-dependent utility function

$$U(W) = p\frac{W^{1-\gamma_1}}{1-\gamma_1} + (1-p)\frac{W^{1-\gamma_2}}{1-\gamma_2}, \tag{S-1}$$

the relative risk aversion coefficient can be calculated as

$$\gamma(W,p) = -\frac{WU''(W)}{U'(W)} = f(W,p)\gamma_1 + (1-f(W,p))\gamma_2,$$

where

$$f(W,p) = \frac{pW^{\gamma_2}}{(1-p)W^{\gamma_1} + pW^{\gamma_2}}.$$

Figure S-1 plots the relative risk aversion coefficients when the wealth ranges from 0.5 to 1.5. When $W = 1$, the weight function is $f(1,p) = p$, and the risk aversion is a linear function of bull regime probability,

$$\gamma(1,p) = p\gamma_1 + (1-p)\gamma_2.$$

When $W > 1$, we have $f(W,p) \geq p$, and $\gamma(W,p)$ is a concave function of bull regime probability p, which means (i) when the bull regime probability increases slightly, relative risk aversion decreases dramatically and the investor will invest dramatically more in the risky asset; (ii) when the bull regime probability decreases slightly, relative risk aversion decreases slightly and the investor will not dramatically reduce the weight of the risky asset. For $W < 1$, relative risk aversion is a concave function of bull regime probability and the opposite effects happen.

If W is much larger than 1, $\gamma(W,p)$ would be close to γ_1 for all $p \in (0,1]$. Meanwhile, as W becomes close to zero, $\gamma(W,p)$ would converge to γ_2 for all $p \in [0,1)$. In other words, when the initial wealth is far way from 1, even though the investor has a regime-dependent utility function, her risk aversion will be insensitive to the variation of the market regime probability p when $p \in (0,1)$. In the empirical analysis, when investigating the dynamic portfolio allocation under regime-dependent risk preference, we set the initial wealth close to 1.

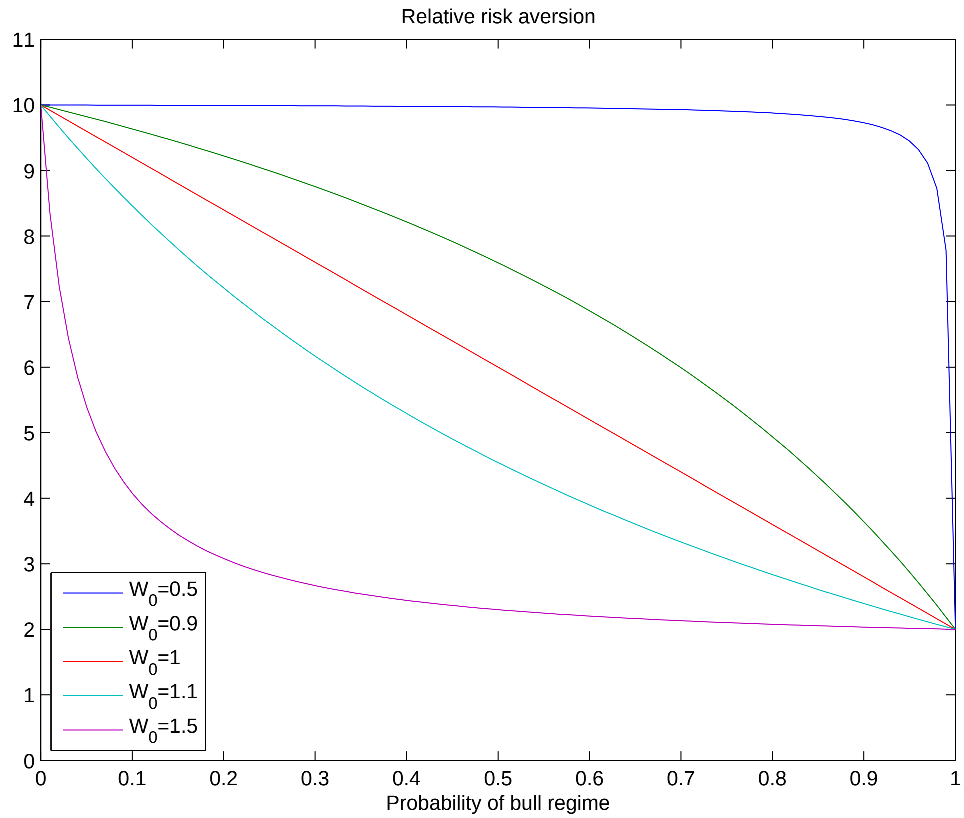

Figure S-1 Dependence of Relative Risk Aversion on Initial Wealth and Probability of Bull Regime

This figure shows how the relatively risk aversion of regime-dependent utility function is dependent on the investor's wealth and bull regime probability. The regime-dependent utility function is given by $U(W) = p\frac{W^{1-\gamma_1}}{1-\gamma_1} + (1-p)\frac{W^{1-\gamma_2}}{1-\gamma_2}$, where W is the wealth, and p is the bull regime probability. In the figure, we set $\gamma_1 = 2$ and $\gamma_2 = 10$, which respectively represent the risk aversion in the bull regime and bear regime.

S-2 Supplement material for Section 2.3

In this section, we first detail the procedure to solve the dynamic asset allocation problem with regime-dependent risk aversion. Then we conduct a numerical experiment to examine the algorithm performance.

S-2.1 Algorithm to solve the dynamic asset allocation problem

As discussed in Guidolin and Timmermann (2007) and Barberis (2000), a dynamic asset allocation problem can be solved using the standard technique of discretizing the state space and backward induction. We follow this standard technique and discretize the space of the state variables. Our asset allocation problem has four state variables: time t, the wealth accumulate with risk free rate A_t, the posterior probability of bull regime $\pi_{1,t|t} = \Pr(S_t = 1|\mathcal{F}_t)$, and the VIX index V_t. We discretize the time variable to match the portfolio rebalancing frequency. Suppose that we are at time t and the investment horizon is T months. We divide the horizon into T one-month intervals, $[t,\ t+1]$, $[t+1,\ t+2]$,..., $[T-1,\ T]$. We rebalance the portfolio at the beginning of each month, and the control variables are $\omega_t, \omega_{t+1}, ..., \omega_{T-1}$, i.e., asset allocations to the stock index at times $t, t+1, t+2, ..., T-1$, respectively. Next, we discretize the state variable A_t by splitting it into $K_A = 11$ grids in the range between 0.5 and 1.5. We discretize the state variable $\pi_{1,t|t}$ by splitting it into $K_\pi = 11$ equally spaced grid points in the range of 0 and 1. Let $V_{min,t}$ and $V_{max,t}$ denote respectively the minimum and maximum of the historical VIX values. The state variable V_t is split into $K_v = 11$ [S-1] equally spaced grids within the range of $V_{min,t}$ and $V_{max,t}$.

We solve the problem starting from time $T-1$ and moving backward to time 0. For each time t, we take the following steps.

Step 1: For each grid of accumulated wealth $A_t^{g_A}$, each grid of posterior probability $\pi_{1,t|t}^{g_\pi}$, and each grid of the VIX index $V_t^{g_v}$, where g_A, g_π and g_v stand for grids, the optimal risky asset weight

[S-1] In Section 4.1, we also try discretizing each state variable into $K = 21$ discrete grids, and find the optimal weights are quite similar to those using $K = 11$. Considering that using $K = 21$ will consume much more time to solve the optimal allocation problem than using $K = 11$, we set $K = 11$ for each state variable in the empirical study.

ω_t^* can be obtained by solving the following optimization problem:

$$\omega_t^*\left(A_t^{g_A}, \pi_{1,t|t}^{g_\pi}, V_t^{g_v}, t\right) = \arg\max_{\omega_t \in [0,1]} E_t\left[J\left(A_{t+1}, \pi_{1,t+1|t+1}, V_{t+1}, t+1\right)\right].$$

We approximate the conditional expectation $E_t[.]$ using Monte Carlo simulation with the following steps.

Step (1): Given that current bull regime probability is $\pi_{1,t|t}^{g_\pi}$, and current VIX index is $V_t^{g_v}$, the forecasting probability of a bull market regime in the next period $t+1$ can be expressed as

$$\pi_{1,t+1|t}^{g_\pi,g_v} = p_{11}\left(V_t^{g_v}; \hat{\Theta}\right)\pi_{1,t|t}^{g_\pi} + \left(1 - p_{22}\left(V_t^{g_v}; \hat{\Theta}\right)\right)\left(1 - \pi_{1,t|t}^{g_\pi}\right),$$

where the transition probabilities $p_{11}(.)$ and $p_{22}(.)$ are given by Equations (5) and (6) in the main text respectively.

Step (2): Knowing $\pi_{1,t+1|t}^{g_\pi,g_v}$ and from Equations (1)-(3) in the main text, which specify the data generating process in each regime, we generate N ($N = 50,000$) samples of the risky returns $r_{s,t+1}^{(i)}$, VIX difference $r_{v,t+1}^{(i)}$, and the difference in the log T-bill rate $r_{f,t+1}^{(i)}$, where $i = 1, \cdots, N$.

Step (3): For the ith sample, we obtain the posterior probability $\pi_{1,t+1|t+1}^{(i)}$ corresponding to $r_{s,t+1}^{(i)}$, $r_{f,t+1}^{(i)}$, and $r_{v,t+1}^{(i)}$ on the basis of the Hamilton filter. The VIX index $V_{t+1}^{(i)}$ can be obtained as $V_{t+1}^{(i)} = V_t^{g_v} + r_{v,t+1}^{(i)}$, and $\tilde{R}_{t+1}^{(i)}(\omega_t)$ can be expressed as

$$\tilde{R}_{t+1}^{(i)}(\omega_t) = \exp\left[(T-t-1)\, r_{f,t+1}^{(i)}\right]\left[\omega_t\left(\exp\left(r_{s,t+1}^{(i)}\right) - 1\right) + 1\right].$$

Step (4a): If $t = T-1$, from the terminal condition in Equation (24), the objective function can be approximated by

$$\frac{1}{N}\sum_{i=1}^{N}\left(\pi_{1,t+1|t+1}^{(i)}\frac{\left[A_t^{g_A}\tilde{R}_{t+1}^{(i)}(\omega_t)\right]^{1-\gamma_1}}{1-\gamma_1} + \left(1 - \pi_{1,t+1|t+1}^{(i)}\right)\frac{\left[A_t^{g_A}\tilde{R}_{t+1}^{(i)}(\omega_t)\right]^{1-\gamma_2}}{1-\gamma_2}\right). \tag{S-2}$$

Step (4b): If $t < T-1$, we approximate $Q_j\left(A_t^{g_A}\tilde{R}_{t+1}^{(i)}(\omega_t), \pi_{1,t+1|t+1}^{(i)}, V_{t+1}^{(i)}, t+1\right)$, where $j = 1, 2$, using linear interpolation on the grids of the state variables. Thus from Equation (25) the objective function can be calculated as

$$\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{2}\frac{\left[A_t^{g_A}\tilde{R}_{t+1}^{(i)}\left(\omega_t\right)\right]^{1-\gamma_j}}{1-\gamma_j}Q_j\left(A_t^{g_A}\tilde{R}_{t+1}^{(i)}\left(\omega_t\right),\pi_{1,t+1|t+1}^{(i)},V_{t+1}^{(i)},t+1\right). \tag{S-3}$$

Step 2: After obtaining the optimal risky asset weight ω_t^* by maximizing the value function of (S-2) or (S-3), for each grid of state variables, $Q_j\left(A_t^{g_A},\pi_{1,t|t}^{g_\pi},V_t^{g_V},t\right)$ can be calculated using Equation (26) as

$$Q_j\left(A_t^{g_A},\pi_{1,t|t}^{g_\pi},V_t^{g_V},t\right)=\frac{1}{N}\sum_{i=1}^{N}\left(\tilde{R}_{t+1}^{(i)}\left(\omega_t^*\right)\right)^{1-\gamma_j}Q_j\left(A_t^{g_A}\tilde{R}_{t+1}^{(i)}\left(\omega_t^*\right),\pi_{1,t+1|t+1}^{(i)},V_{t+1}^{(i)},t+1\right)$$

S-2.2 Algorithm performance

In this section, we conduct a numerical experiment to examine the algorithm performance. To demonstrate the benefit of factoring out the highly-nonlinear part, i.e. $A_t^{1-\gamma_j}$ from the value function $J\left(A_t,\pi_{1,t|t},V_t,t\right)$ in each regime, we solve the dynamic portfolio problem using two algorithms. The first algorithm solves the optimization problem

$$J\left(A_t,\pi_{1,t|t},V_t,t\right)=\max_{\omega_t\in[0,1]}E_t\left[J\left(A_t\tilde{R}_{t+1}\left(\omega_t\right),\pi_{1,t+1|t+1},V_{t+1},t+1\right)\right], \tag{S-4}$$

with the terminal condition given by

$$J\left(A_T,\pi_{1,T|T},V_T,T\right)=\pi_{1,T|T}\frac{A_T^{1-\gamma_1}}{1-\gamma_1}+\left(1-\pi_{1,T|T}\right)\frac{A_T^{1-\gamma_2}}{1-\gamma_2}. \tag{S-5}$$

In this algorithm, when calculating the values of $J\left(A_t,\pi_{1,t|t},V_t,t\right)$ on the grids of state variables, we use the cubic-spline interpolation method. [S-2] In the second algorithm, we decompose the value function $J\left(A_t,\pi_{1,t|t},V_t,t\right)$ into two parts in each regime: one is the nonlinear power function $\frac{A_t^{1-\gamma_j}}{1-\gamma_j}$, and the other is $Q_j\left(A_t,\pi_{1,t|t},V_t,t\right)$, which is a function of portfolio returns. To obtain the value of $J\left(A_t,\pi_{1,t|t},V_t,t\right)$ on the grid of state variables, the second algorithm firstly uses the linear interpolation method to calculate $Q_j\left(A_t,\pi_{1,t|t},V_t,t\right)$, and then multiplies it by the power function

[S-2] In the first algorithm, we also try the linear interpolation to calculate the value function on the grids. However, the performance are quite poor and are available upon request.

$\frac{A_t^{1-\gamma_j}}{1-\gamma_j}$.

In the numerical experiment, we set the investment horizon 12 months, the initial wealth to be 1 and the initial VIX to be the median of its historical values. The initial bull regime probability $\pi_0 = \Pr(S_0 = 1)$ can be 0, 0.5 or 1. Figure S-2 plots the optimal initial weights when the grid size of A decreases from 0.16 to 0.02, or the approximation accuracy increases. We mark the weights from the first algorithm with asterisks, and use the solid line to represent our proposed algorithm.

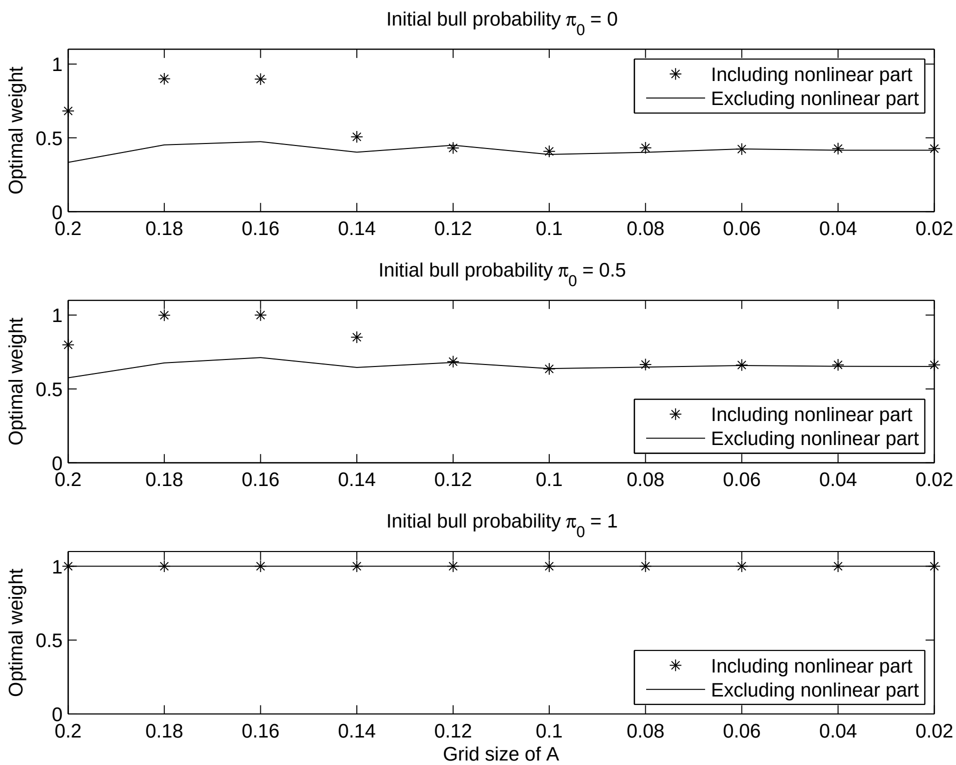

Figure S-2 Convergence of Dynamic Programming Algorithms
This figure shows the optimal initial portfolio weight in the risky asset when the grid size of A decrease from 0.16 to 0.02, or the approximation of the value function becomes more accurate. While the asterisks points are from the algorithm that the value function includes the $A^{1-\gamma_j}$ component, the solid line is obtained from our proposed algorithm, which factors out $A^{1-\gamma_j}$ from the value function in each regime. The initial bull regime probability is 0 in the top panel, 0.5 in the middle panel, and 1 in the bottom panel.

As we can see, both algorithms can converge when the grid size of A decreases to 0.1. However, compared with the first algorithm, the second algorithm can converge more quickly. Meanwhile the second algorithm uses the simple linear interpolation instead of the nonlinear cubic-spline interpolation, so it is more efficient. Therefore in the empirical study we use the second algorithm to solve the dynamic portfolio allocation problem.

S-3 Supplement material for Section 4.1

In this section, we examine how initial wealth affects the optimal initial risky asset weight based on the estimated regime switching model in Section 3.2.

We consider the initial wealth to be 0.9, 1 and 1.1. Figure S-3 plots the risky asset weights where the investment horizon T is one month. Consistent with our expectation, the investor with low initial wealth will behave more conservatively than the investor with high initial wealth, allocating less fraction of her wealth to the risky asset. This observation is true for different initial market conditions and different VIX values. Figure S-4 further presents the risky asset weights when investment horizon is 12 months. Compared with the investor with $W_0 = 1(W_0 = 0.9)$, the investor with $W_0 = 1.1(W_0 = 1)$ becomes more aggressive and increase her risky asset with a larger amount as the investment horizon extends.

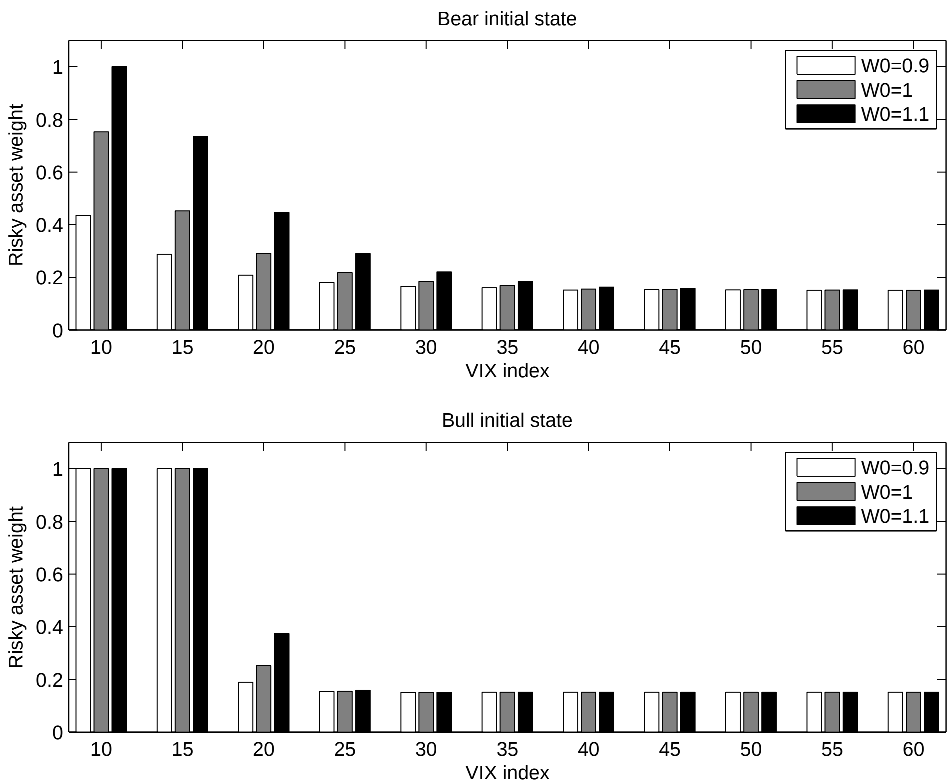

Figure S-3 Optimal Initial Portfolio Weight in the Risky Asset with One-month Investment Horizon

The figure plots the optimal initial portfolio weight invested in the S&P 500 index as a function of the VIX index for an investor with regime-dependent risk aversion ($\gamma = 2$ in bull regime and 10 in bear regime) and initial wealth of 0.1, 1 and 1.1. The initial market regime is bear in the top panel and bull in the bottom panel.

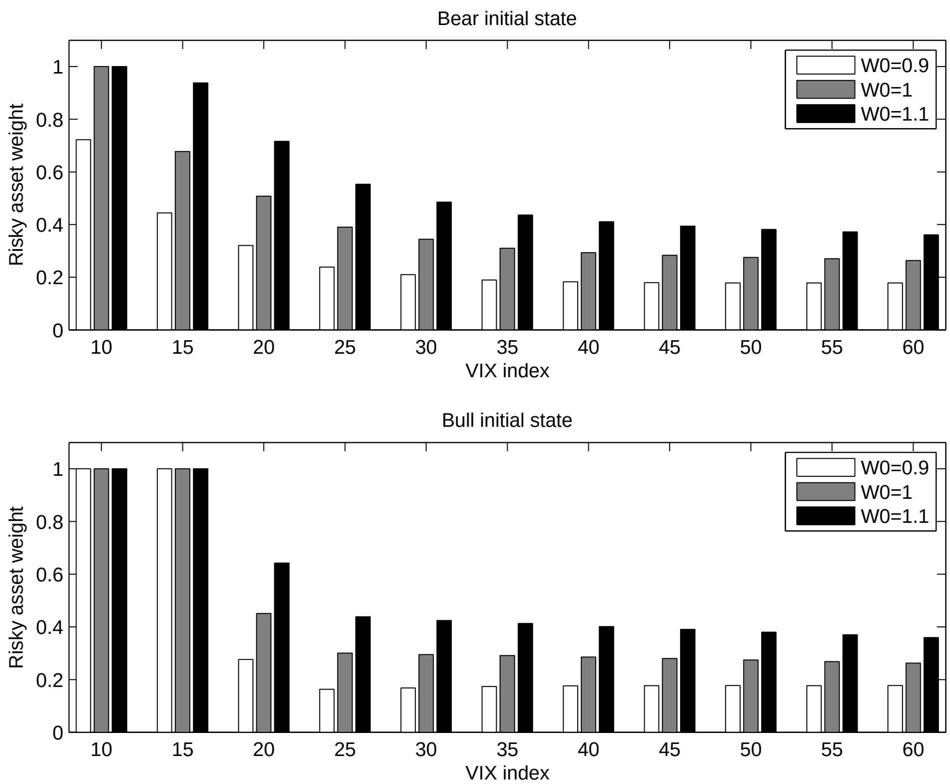

Figure S-4 Optimal Initial Portfolio Weight in the Risky Asset with Twelve-month Investment Horizon

The figure plots the optimal initial portfolio weight invested in the S&P 500 index as a function of the VIX index for an investor with regime-dependent risk aversion ($\gamma = 2$ in bull regime and 10 in bear regime) and initial wealth of 0.1, 1 and 1.1. The initial market regime is bear in the top panel and bull in the bottom panel.

S-4 Supplement material for Section 4.3

In this section, we conduct more robust tests to examine the investment performance of the strategy with regime-dependent risk aversion.

S-4.1 Strategies with regime-dependent and constant risk aversions

To examine whether strategy with regime-dependent risk aversion can consistently generate higher performance than strategy with constant risk aversion, in Table S-1 we set the constant risk aversion to be 4, 6 or 8. The out-of-sample results show that the strategy with regime-dependent risk aversion can obtain significant higher Sharpe, Sortino and reward-to-VaR ratios than the strategies with constant risk aversions. Meanwhile in most cases the strategy with regime-dependent risk aversion can yield significant higher utility-based performance measures than the strategies with constant risk aversions.

Table S-1 Performance of Optimal Portfolio Strategies for Investors with Constant and Regime-Dependent Risk Aversions

This table reports out-of-sample performance of optimal portfolio strategies for investors with constant ($\gamma = 4, 6$ or 8) and regime-dependent risk aversion (γ_S: $\gamma_1 = 2$ in bull regime and γ_2=10 in bear regime) between January 2002 and December 2015. The strategy with regime-dependent preference is used as a benchmark for comparison. The minimum, maximum, annualized mean return, standard deviation (SD), Sharpe ratio (SR), Sortino ratio (ST) and reward-to-VaR ratio at 99% confidence level (RV) are reported, and the returns are expressed in percentage points. Δ_2 and Δ_{10} represent the annualized certainty equivalent rates of excess returns (CEER) expressed in percentage points for risk aversion coefficient of 2 and 10, respectively. The one-sided p-values for the null hypothesis that the benchmark strategy has a smaller performance measure than the alternative strategy are reported in parentheses. A small p-value indicates the null can be rejected. The asterisks *, **, and *** denote 10%, 5%, and 1% statistical significance level, respectively.

	Min	Max	Mean	SD	SR	ST	RV	Δ_2	Δ_{10}
T = 1 month									
γ_S	-5.85	4.83	3.69	6.22	0.59	0.94	0.08	3.29	1.73
$\gamma = 4$	-8.46***	6.31	3.82	8.77	0.44*	0.62*	0.05	3.04	-0.23*
	(0.000)	(1.000)	(0.552)	(1.000)	(0.086)	(0.062)	(0.108)	(0.413)	(0.060)
$\gamma = 6$	-5.85***	4.20	3.38	7.06	0.48**	0.70**	0.05*	2.87	0.82**
	(0.000)	(0.135)	(0.263)	(1.000)	(0.036)	(0.024)	(0.080)	(0.193)	(0.037)
$\gamma = 8$	-5.60	3.95***	3.13**	6.23	0.50***	0.76***	0.06**	2.74**	1.16***
	(0.714)	(0.003)	(0.014)	(0.550)	(0.007)	(0.005)	(0.025)	(0.012)	(0.007)
T= 6 months									
γ_S	-10.96	13.98	4.39	7.80	0.56	1.16	0.23	3.80	1.48
$\gamma = 4$	-23.37***	14.24	4.16	10.26	0.41*	0.61**	0.10***	3.06	-2.99**
	(0.000)	(0.895)	(0.414)	(0.999)	(0.079)	(0.010)	(0.007)	(0.263)	(0.044)
$\gamma = 6$	-15.99***	13.68**	3.65**	7.99	0.46**	0.76***	0.14***	3.00**	0.10**
	(0.001)	(0.021)	(0.017)	(0.649)	(0.041)	(0.002)	(0.004)	(0.015)	(0.011)
$\gamma = 8$	-12.13*	12.42***	3.30***	6.81***	0.48**	0.86***	0.17***	2.83***	0.91**
	(0.051)	(0.004)	(0.002)	(0.000)	(0.027)	(0.004)	(0.005)	(0.003)	(0.017)
T= 12 months									
γ_S	-13.12	29.69	5.61	9.15	0.61	1.63	0.54	4.85	2.00
$\gamma = 4$	-25.32***	28.09**	5.51	10.84	0.51	0.89**	0.24**	4.34	-2.03*
	(0.008)	(0.022)	(0.461)	(0.917)	(0.204)	(0.017)	(0.023)	(0.325)	(0.076)
$\gamma = 6$	-17.21**	26.78***	4.75***	8.69	0.55	1.12**	0.32**	4.03***	0.88***
	(0.012)	(0.005)	(0.001)	(0.232)	(0.150)	(0.013)	(0.024)	(0.002)	(0.008)
$\gamma = 8$	-12.97	23.75***	4.17***	7.45***	0.56*	1.28***	0.38***	3.64***	1.53*
	(0.887)	(0.001)	(0.001)	(0.000)	(0.095)	(0.007)	(0.009)	(0.002)	(0.070)

S-4.2 Strategy performance when changing the initial wealth

In Table S-2, we examine how initial wealth affects the performance of strategy with regime-dependent risk aversion. As regime-dependent utility function implies a DRRA risk aversion, the investor with a higher initial wealth would invest more fraction of her wealth in the risky asset than the investor with a lower initial wealth. Consistent with our expectation, compared with the benchmark strategy with initial wealth $W_0 = 1$, strategy with $W_0 = 0.9$ yields lower risk and lower average return, while strategy with $W_0 = 1.1$ yields higher risk and higher average return.

Table S-2 Performance of Optimal Portfolio Strategies for Investors with Different Initial Wealth
This table reports out-of-sample performance of optimal portfolio strategies for investors with regime-dependent risk aversion (γ_S: $\gamma_1 = 2$ in bull regime and γ_2=10 in bear regime) and initial wealth of 0.9, 1 and 1.1 between January 2002 and December 2015. The strategy with initial wealth of 1 is used as a benchmark for comparison. The minimum, maximum, annualized mean return, standard deviation (SD), Sharpe ratio (SR), Sortino ratio (ST) and reward-to-VaR ratio at 99% confidence level (RV) are reported, and the returns are expressed in percentage points. Δ_2 and Δ_{10} represent the annualized certainty equivalent rates of excess returns (CEER) expressed in percentage points for risk aversion coefficient of 2 and 10, respectively. The one-sided p-values for the null hypothesis that the benchmark strategy has a smaller performance measure than the alternative strategy are reported in parentheses. A small p-value indicates the null can be rejected. The asterisks *, **, and *** denote 10%, 5%, and 1% statistical significance level, respectively.

	Min	Max	Mean	SD	SR	ST	RV	Δ_2	Δ_{10}
T = 1 month									
$W_0 = 0.9$	-5.18	3.69***	2.78***	5.47***	0.51**	0.77**	0.07***	2.47***	1.25*
	(0.917)	(0.000)	(0.001)	(0.000)	(0.014)	(0.012)	(0.003)	(0.009)	(0.050)
$W_0 = 1$	-5.85	4.83	3.69	6.22	0.59	0.94	0.08	3.29	1.73
$W_0 = 1.1$	-5.85**	5.26	4.37	6.65	0.66	1.06	0.09	3.92	2.15
	(0.031)	(1.000)	(0.998)	(1.000)	(0.969)	(0.955)	(0.995)	(0.987)	(0.950)
T = 6 months									
$W_0 = 0.9$	-8.95	11.41***	2.75***	5.84***	0.47*	0.93*	0.22**	2.40*	1.08
	(0.990)	(0.000)	(0.007)	(0.000)	(0.062)	(0.090)	(0.019)	(0.067)	(0.280)
$W_0 = 1$	-10.96	13.98	4.39	7.80	0.56	1.16	0.23	3.80	1.48
$W_0 = 1.1$	-14.51***	20.25	5.80	9.82	0.59	1.15	0.21	4.87	0.97
	(0.003)	(1.000)	(0.982)	(1.000)	(0.715)	(0.498)	(0.938)	(0.758)	(0.321)
T= 12 months									
$W_0 = 0.9$	-9.99	25.62***	4.15***	7.59***	0.55	1.58	0.52**	3.60*	1.68
	(0.984)	(0.000)	(0.007)	(0.001)	(0.154)	(0.395)	(0.020)	(0.081)	(0.329)
$W_0 = 1$	-13.12	29.69	5.61	9.15	0.61	1.63	0.54	4.85	2.00
$W_0 = 1.1$	-16.77***	32.87	6.89	10.65	0.65	1.49	0.47	5.86	1.35
	(0.009)	(0.997)	(0.985)	(0.998)	(0.743)	(0.249)	(0.935)	(0.735)	(0.298)

In terms of composite performance measures, strategy with $W_0 = 1.1$ ($W_0 = 1$) can produce significant higher Sharpe ratio, Sortino ratio and reward-to-VaR ratio than strategy with $W_0 = 1$ ($W_0 = 0.9$) when $T = 1$. However, as the investment horizon extends, strategy with higher initial wealth might perform worse. For instance, when the investment horizon is 6 or 12 months, strategy with $W_0 = 1.1$ yields lower Sortino ratios than strategy with $W_0 = 1$.

For an investor with regime-dependent risk aversion, a high wealth leads to a low risk aversion and a high weight in the risky asset. Especially when making multi-period investment, high initial wealth makes the investor invest more than single-period investment. As we document in Section S-1 of the online supplement, the investor with $W > 1$ is relatively slow to decrease her risk exposure even if bear market regime is more likely. Therefore, while the investor with $W = 1.1$ can earn relatively higher average returns, she is exposed to large downside risks and gets lower Sortino ratios.

S-4.3 Strategy performance when changing γ_1 and γ_2

In the analysis of previous sections, the investor's risk aversion depends on the market regimes: $\gamma_1 = 2$ in the bull regime, and $\gamma_2 = 10$ in the bear regime. In this section, we change the values of γ_1 and γ_2, and examine how the changes of γ_1 and γ_2 affect the strategy performance. In Table S-3, we consider the following four cases: fix $\gamma_1 = 2$, and set γ_2 to be 8 or 12; fix $\gamma_2 = 10$, and set $\gamma_1 = 4$.

The results shown in Table S-3 are intuitive that the strategy will produce lower average return and lower volatility if either γ_1 or γ_2 increases. If using Sharpe, Sortino or reward-to-VaR ratio as the performance measure, the strategy with $\gamma_2 - \gamma_1 = 8$ ($\gamma_2 - \gamma_1 = 10$) can generally yield higher performance than the strategy with $\gamma_2 - \gamma_1 = 6$ ($\gamma_2 - \gamma_1 = 8$), and in some cases, the differences are statistically significant. Finally, the utility-based performance results show that increasing γ_1 or γ_2 makes the strategy more attractive to the highly risk-averse investor, while decreasing γ_1 or γ_2 makes the strategy more attractive to the investor with low risk aversion.

Table S-3 Performance of Optimal Portfolio Strategies for Investors with Regime-Dependent Risk Aversions

This table reports out-of-sample performance of optimal portfolio strategies for investors with regime-dependent risk aversion (γ_S: $\gamma_1 = 2$ or 4 in bull regime and $\gamma_2 = 8, 10$ or 12 in bear regime) between January 2002 and December 2015. The strategy with $\gamma_1 = 2$ and $\gamma_2 = 10$ is used as a benchmark for comparison. The minimum, maximum, annualized mean return, standard deviation (SD), Sharpe ratio (SR), Sortino ratio (ST) and reward-to-VaR ratio at 99% confidence level (RV) are reported, and the returns are expressed in percentage points. Δ_2 and Δ_{10} represent the annualized certainty equivalent rates of excess returns (CEER) expressed in percentage points for risk aversion coefficient of 2 and 10, respectively. The one-sided p-values for the null hypothesis that the benchmark strategy has a smaller performance measure than the alternative strategy are reported in parentheses. A small p-value indicates the null can be rejected. The asterisks *, **, and *** denote 10%, 5%, and 1% statistical significance level, respectively.

(γ_1, γ_2)	Min	Max	Mean	SD	SR	ST	RV	Δ_2	Δ_{10}
T= 1 month									
(2,10)	-5.85	4.83	3.69	6.22	0.59	0.94	0.08	3.29	1.73
(2,8)	-5.85***	5.24	3.85	6.65	0.58	0.90	0.08	3.40	1.62
	(0.003)	(1.000)	(0.806)	(1.000)	(0.305)	(0.256)	(0.464)	(0.717)	(0.301)
(4,10)	-5.76	3.89***	3.44***	6.07***	0.57**	0.88**	0.07***	3.07***	1.58**
	(0.909)	(0.002)	(0.003)	(0.000)	(0.011)	(0.010)	(0.006)	(0.004)	(0.018)
(2,12)	-5.71	4.46***	3.55	5.90***	0.60	0.96	0.08	3.19	1.79
	(1.000)	(0.002)	(0.161)	(0.000)	(0.675)	(0.710)	(0.296)	(0.241)	(0.645)
T= 6 months									
(2,10)	-10.96	13.98	4.39	7.80	0.57	1.16	0.23	3.80	1.48
(2,8)	-13.41**	14.39	4.58	8.43	0.54	1.04*	0.19**	3.88	1.02
	(0.012)	(1.000)	(0.796)	(1.000)	(0.238)	(0.071)	(0.035)	(0.648)	(0.152)
(4,10)	-10.51	13.62***	3.93**	7.19***	0.55	1.09	0.22	3.42**	1.41
	(0.999)	(0.004)	(0.021)	(0.000)	(0.182)	(0.141)	(0.210)	(0.042)	(0.367)
(2,12)	-9.26	13.75***	4.21	7.33***	0.58	1.24	0.26	3.68	1.68
	(0.992)	(0.004)	(0.138)	(0.000)	(0.680)	(0.869)	(0.951)	(0.249)	(0.761)
T= 12 months									
(2,10)	-13.12	29.69	5.61	9.15	0.62	1.63	0.54	4.85	2.00
(2,8)	-15.77**	31.23	5.87	9.72	0.61	1.45*	0.46**	5.01	1.53
	(0.015)	(0.992)	(0.877)	(0.982)	(0.404)	(0.054)	(0.030)	(0.739)	(0.206)
(4,10)	-12.04	27.43***	4.95***	8.24***	0.60	1.57	0.53	4.32**	1.95
	(0.985)	(0.000)	(0.008)	(0.000)	(0.269)	(0.196)	(0.350)	(0.020)	(0.414)
(2,12)	-11.28	28.87**	5.36*	8.73**	0.62	1.77	0.61	4.67	2.18
	(0.985)	(0.022)	(0.079)	(0.015)	(0.513)	(0.919)	(0.960)	(0.178)	(0.692)

S-4.4 Strategy performance when using interest rate as regime predictor

The short term interest rate was shown to be informative to market regimes (Yang et al., 2009; Aslanidis and Christiansen, 2012), and Ang and Bekaert (2002) use the interest rate as the predictor for the regime shifts and solve the international asset allocation problem.

Using the same notations in Section 2.1, denote $r_{s,t}$ as the excess log return of S&P 500 index, $R_{f,t}$ as the log T-bill rate, and $r_{f,t} = R_{f,t} - R_{f,t-1}$. Similar to Ang and Bekaert (2002), we construct the following regime switching model

$$r_{s,t} = \mu_{s,S_t} + \sigma_{s,S_t}\varepsilon_{s,t}, \tag{S-6}$$

$$r_{f,t} = \mu_{f,S_t} + \kappa_{S_t}R_{f,t-1} + \sigma_{f,S_t}\sqrt{R_{f,t-1}}\varepsilon_{r,t}, \tag{S-7}$$

and use the log T-bill rate as the transition variable in the time varying transition probability

$$p_{11}\left(R_{f,t-1}\right) = \frac{\exp\left(\alpha_1 + \beta_1 R_{f,t-1}\right)}{1 + \exp\left(\alpha_1 + \beta_1 R_{f,t-1}\right)}, \tag{S-8}$$

$$p_{22}\left(R_{f,t-1}\right) = \frac{\exp\left(\alpha_2 + \beta_2 R_{f,t-1}\right)}{1 + \exp\left(\alpha_2 + \beta_2 R_{f,t-1}\right)}. \tag{S-9}$$

Table S-4 shows that using interest rate as regime predictor, the strategy with regime-dependent risk aversion can produce better performance than the strategy with constant risk aversion but the superiorities are in general not significant. In the last row of each panel, we report the strategy with regime-dependent risk aversion where VIX is used as the predictor for regime shifts. As we can see, using interest rate as the regime predictor cannot produce better performance than using VIX as the regime predictor, yielding lower Sharpe, Sortino and reward-to-VaR ratios and utility-based performances measures.

Table S-4 Performance of Optimal Portfolio Strategies Using Interest Rate as Regime Predictor
This table reports out-of-sample performance of optimal portfolio strategies using interest rate as regime predictor for investors with constant ($\gamma = 2$ or 10) and regime-dependent risk aversion (γ_S: $\gamma_1 = 2$ in bull regime and γ_2=10 in bear regime) between January 2002 and December 2015. A strategy using VIX as regime predictor is included for comparison. The strategy using interest rate as regime predictor with regime-dependent preference is used as a benchmark for comparison. The minimum, maximum, annualized mean return, standard deviation (SD), Sharpe ratio (SR), Sortino ratio (ST) and reward-to-VaR ratio at 99% confidence level (RV) are reported, and the returns are expressed in percentage points. Δ_2 and Δ_{10} represent the annualized certainty equivalent rates of excess returns (CEER) expressed in percentage points for risk aversion coefficient of 2 and 10, respectively. The one-sided p-values for the null hypothesis that the benchmark strategy has a smaller performance measure than the alternative strategy are reported in parentheses. A small p-value indicates the null can be rejected. The asterisks *, **, and *** denote 10%, 5%, and 1% statistical significance level, respectively.

	Min	Max	Mean	SD	SR	ST	RV	Δ_2	Δ_{10}
T= 1 month									
$\gamma_S(R_f)$	-10.45	9.12	2.95	8.91	0.33	0.50	0.03	2.15	-1.13
$\gamma = 2$	-10.45	9.12	2.66	10.61	0.25	0.35	0.02	1.52	-3.33*
	(0.654)	(0.966)	(0.407)	(1.000)	(0.260)	(0.220)	(0.281)	(0.314)	(0.079)
$\gamma = 10$	-10.45	9.12	2.74	8.60***	0.32	0.48	0.03	2.00	-1.07
	(0.990)	(0.877)	(0.197)	(0.000)	(0.323)	(0.358)	(0.217)	(0.266)	(0.581)
$\gamma_S(VIX)$	-5.85	4.83***	3.69	6.22***	0.59	0.94	0.08	3.29	1.73
	(1.000)	(0.000)	(0.680)	(0.000)	(0.910)	(0.878)	(0.924)	(0.760)	(0.945)
T= 6 months									
$\gamma_S(R_f)$	-28.14	16.80	3.23	9.55	0.34	0.54	0.09	2.31	-2.83
$\gamma = 2$	-31.91	17.43	2.75	12.47	0.22	0.30	0.05	1.05	-10.24*
	(0.269)	(0.746)	(0.395)	(1.000)	(0.222)	(0.186)	(0.207)	(0.270)	(0.094)
$\gamma = 10$	-28.14	16.79***	2.94	8.83***	0.33	0.52	0.08	2.15	-2.59
	(0.123)	(0.000)	(0.197)	(0.000)	(0.485)	(0.459)	(0.323)	(0.309)	(0.798)
$\gamma_S(VIX)$	-10.96	13.98**	4.39	7.80*	0.56	1.16	0.23	3.80	1.48
	(0.878)	(0.010)	(0.765)	(0.077)	(0.894)	(0.879)	(0.875)	(0.822)	(0.919)
T= 12 months									
$\gamma_S(R_f)$	-24.56	33.27	4.79	10.61	0.45	0.94	0.22	3.76	-0.88
$\gamma = 2$	-34.89	33.27	4.61	13.29	0.35	0.55	0.16	2.78	-8.16
	(0.161)	(0.975)	(0.468)	(0.969)	(0.258)	(0.175)	(0.205)	(0.318)	(0.154)
$\gamma = 10$	-24.56	32.83*	4.49	9.83***	0.46	0.99	0.21	3.62	-0.43
	(0.248)	(0.092)	(0.177)	(0.000)	(0.540)	(0.665)	(0.413)	(0.333)	(0.814)
$\gamma_S(VIX)$	-13.12	29.69	5.61	9.15*	0.61	1.63	0.54	4.85	2.00
	(0.965)	(0.237)	(0.686)	(0.069)	(0.879)	(0.885)	(0.883)	(0.759)	(0.930)

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