Technical proofs for companion

EC.1. Proof of Lemma 1

Proof. Suppose actions follow a PIL policy $\mathrm{P}_t^U(\mathbf{x}_t)$ for given $U \geq 0$. Suppose the state $\mathbf{x}_0$ of period 0 satisfies $\mathbb{E}[J_{\tau-1}|\mathbf{x}_0] \leq U$. Denote by $q_\tau \geq 0$ the non-negative order placed in period 0 that attains $\mathbb{E}[I_\tau|\mathbf{x}_0] = \mathbb{E}[J_{\tau-1} + q_\tau|\mathbf{x}_0] = U$.

Conditional on D_0 being d_0, the projected inventory level as seen in period 1 before the order $q_{\tau+1}$ is placed is given by $\mathbb{E}[J_\tau \mid D_0 = d_0]$. It is sufficient to show that

$$\forall d_0 \geq 0 : \mathbb{E}[J_\tau \mid \mathbf{x}_0, D_0 = d_0] \leq U, \tag{EC.1}$$

since this implies that for period 1 it is always possible to place a non-negative order $q_{\tau+1}$ to attain the projected inventory level U. (Periods $2, 3, \ldots$ then follow by induction.) Since J_τ is decreasing in D_0 for all $D_1, \ldots, D_\tau$ we find $\mathbb{E}[J_\tau \mid D_0 = d_0] \leq \mathbb{E}[J_\tau \mid D_0 = 0]$, and hence to show (EC.1) it suffices to demonstrate that $\mathbb{E}[J_\tau \mid D_0 = 0] \leq \mathbb{E}[I_\tau] = U$, which will be the subject of the remainder of this proof.

Define the random variable $G(y)$ such that $\mathbb{P}(G(y) \leq g) = \mathbb{P}(I_\tau \leq g \mid \mathbf{x}_t, J_0 = y)$, or equivalently $G(y) = ((\ldots((y + q_1 - D_1)^+ + q_2 - D_2)^+ \ldots)^+ + q_{\tau-1} - D_{\tau-1})^+ + q_\tau$. Observe that we have $\mathbb{E}[I_\tau|\mathbf{x}_0] = \mathbb{E}[G((I_0 - D)^+)]$ and $\mathbb{E}[J_\tau \mid D_0 = 0] = \mathbb{E}[(G(I_0) - D)^+]$. Note that $0 \leq dG(y)/dy \leq 1$ and $G(y) \geq 0$ so that

$$(G(I_0) - D)^+ \leq G((I_0 - D)^+). \tag{EC.2}$$

It follows from (EC.2) that $\mathbb{E}[J_\tau \mid D_0 = 0] = \mathbb{E}[(G(I_0) - D)^+] \leq \mathbb{E}[G((I_0 - D)^+)] = \mathbb{E}[I_\tau]$. □

EC.2. Proof of Lemma 2

Proof. First note that the inventory at the end of a period under C^r satisfies the Lindley recursion $J_{t+1} = (J_t + r - D_t)^+$ so that $\lim_{t\to\infty} \mathbb{E}[J_t]$ can be determined with the Pollaczek Khinchine equation for the mean waiting time in an $M/D/1$ queue. Therefore we can express g^r in closed form as

$$g^r = p(\mu - r) + h\frac{r^2}{2(\mu - r)}. \tag{EC.3}$$

We can now directly verify that inserting (7) and (EC.3) into the right hand side of (6) again equals (7) by using integration by parts and tedious but otherwise straightforward algebra:

$$\begin{aligned}
&\mathbb{E}_D\left[h(x-D)^+ + p(D-x)^+ + \mathcal{H}^r((x-D)^+ + r)\right] - g^r \\
&= p\mathbb{E}\left[(D-x)^+\right] + h\mathbb{E}\left[(x-D)^+\right] + \int_0^x \frac{\mathcal{H}^r(x-y+r)}{\mu} e^{-y/\mu}\,\mathrm{d}y + \mathcal{H}^r(r)e^{-x/\mu} - g^r \\
&= p\mu e^{-x/\mu} + h(x-\mu+\mu e^{-x/\mu}) + e^{-x/\mu}x\left(\frac{h(3r^2+3rx+x^2)}{6\mu(\mu-r)} + \frac{pr(2r+x)}{2\mu(\mu-r)} - \frac{p/(2r+x)}{2(\mu-r)}\right) \quad \text{(EC.4)} \\
&\quad + e^{-x/\mu}\left(\frac{hr^2}{2(\mu-r)} - pr^2\right) - p(\mu-r) - h\frac{r^2}{2(\mu-r)} \\
&= \frac{h}{2(\mu-r)}x^2 - px = \mathcal{H}^r(x). \quad \text{(EC.5)}
\end{aligned}$$

Clearly $\mathcal{H}^r(x) = \frac{h}{2(\mu-r)}x^2 - px$ also satisfies $\mathcal{H}^r(0) = 0$. □

EC.3. Proof of Lemma 4

Proof. We prove the following statement by induction on t_2, starting at $t_2 = t_1$:

$$\mathcal{H}^r(I_{t_1}) = \mathbb{E}_{D_{t_1},\dots,D_{t_2}}\left[c[t_1,t_2](\mathrm{C}^r) + \mathcal{H}^r(I_{t_2+1})|I_{t_1}\right] - (t_2+1-t_1)g^r. \quad \text{(EC.6)}$$

The base case $t_2 = t_1$ holds by (6), the definition of c_t, and $I_{t_1+1} = (I_{t_1} - D_{t_1}) + r$. Now for the inductive step, assume that the statement holds for some $t_2 \geq t_1$; we will show it holds also for $t_2 + 1$. Use (6) to conclude that

$$\begin{aligned}
\mathbb{E}_{D_{t_1},\dots,D_{t_2}}\left[\mathcal{H}^r(I_{t_2+1}) \mid I_{t_1}\right] &= \mathbb{E}_{D_{t_1},\dots,D_{t_2}}\left[\mathbb{E}_{D_{t_2+1}}[c_{t_2+1} + \mathcal{H}^r((I_{t_2+1} - D_{t_2+1})^+ + r) - g^r|I_{t_2+1}]\Big|I_{t_1}\right] \\
&= \mathbb{E}_{D_{t_1},\dots,D_{t_2}}\left[\mathbb{E}_{D_{t_2+1}}\left[c_{t_2+1} + \mathcal{H}^r(I_{t_2+2}) \mid I_{t_2+1}\right]\Big|I_{t_1}\right] - g^r \\
&= \mathbb{E}_{D_{t_1},\dots,D_{t_2},D_{t_2+1}}\left[c_{t_2+1} + \mathcal{H}^r(I_{t_2+2}) \mid I_{t_1}\right] - g^r. \quad \text{(EC.7)}
\end{aligned}$$

Now substitution of (EC.7) back into the induction hypothesis (EC.6) yields:

$$\begin{aligned}
\mathcal{H}^r(I_{t_1}) &= \mathbb{E}_{D_{t_1},\dots,D_{t_2}}\left[c[t_1,t_2](\mathrm{C}^r) + \mathcal{H}^r(I_{t_2+1}) \mid I_{t_1}\right] - (t_2+1-t_1)g^r \\
&= \mathbb{E}_{D_{t_1},\dots,D_{t_2+1}}\left[c[t_1,t_2+1](\mathrm{C}^r) + \mathcal{H}^r(I_{t_2+2}) \mid I_{t_1}\right] - (t_2+2-t_1)g^r. \quad \text{(EC.8)}
\end{aligned}$$

By induction, this shows that (EC.6) holds for all $t_2 \geq t_1$. □

EC.4. Proof of Lemma 5

Proof. We have

$$\mathbb{E}[(X+Y)^+] = \int_{x+y\geq 0} (x+y)dF(x,y) = \int_{x=-\infty}^{\infty}\int_{y=-x}^{\infty} (x+y)dF(x,y). \tag{EC.9}$$

For $x+y\geq 0$ we can write

$$x+y = \int_{z=-y}^{x} dz. \tag{EC.10}$$

Substitution of (EC.10) into (EC.9) yields

$$\begin{aligned}\mathbb{E}[(X+Y)^+] &= \int_{x=-\infty}^{\infty}\int_{y=-x}^{\infty}\int_{z=-y}^{x} dz dF(x,y) = \int_{-y\leq z\leq x} dF(x,y)dz\\ &= \int_{z=-\infty}^{\infty}\int_{x=z}^{\infty}\int_{y=-z}^{\infty} dF(x,y)dz = \int_{z=-\infty}^{\infty} \mathbb{P}(X\geq z, Y\geq -z)dz. \quad \square\end{aligned}$$

EC.5. Proof of Lemma 9

Proof. We first derive an identity for $q[1,T+\tau](P^U)|D_0,\ldots,D_{T-1}$. From (1-3), we find $J_t - L_t = I_t - D_t = J_{t-1} + q_t - D_t$, which implies $J_t - J_{t-1} = L_t + q_t - D_t$. Summing this identity for $t = 1,\ldots,T+\tau-1$ yields: $\sum_{t=1}^{T+\tau-1} J_t - J_{t-1} = J_{T+\tau-1} - J_0 = L[1,T+\tau-1] + q[1,T+\tau-1] - D[1,T+\tau-1]$. Since $J_0 = L_0 + I_0 - D_0$ by (1-2), we obtain

$$J_{T+\tau-1} = I_0 + L[0,T+\tau-1] + q[1,T+\tau-1] - D[0,T+\tau-1] \tag{EC.11}$$

Next, note that $\mathbb{E}(J_{T+\tau-1}(\mathrm{P}^U)|D_0,\ldots,D_{T-1}) = \mathbb{E}(J_{T+\tau-1}(\mathrm{P}^U)|\mathbf{x}_T)$, since $\mathbf{x}_T$ is known given $D_t,\ldots,D_{T-1}$. Thus, Lemma 1 and (4) imply $\mathbb{E}(J_{T+\tau-1}(\mathrm{P}^U)|D_0,\ldots,D_{T-1}) = U - q_{T+\tau}(\mathrm{P}^U)|D_0,\ldots,D_{T-1}$. Taking the conditional expectation on both sides of (EC.11) and plugging in this latter identity yields $U - q_{T+\tau}(\mathrm{P}^U)|D_0,\ldots,D_{T-1} = I_0 + \mathbb{E}(L[0,T+\tau-1] + q[1,T+\tau-1] - D[0,T+\tau-1]|D_0,\ldots,D_{T-1})$. Using $\mathbb{E}(D[0,T+\tau-1]|D_0,\ldots,D_{T-1}) = D[0,T-1] + \tau\mu$ and rearranging terms yields:

$$\begin{aligned}&q[1,T+\tau](P^U)|D_0,\ldots,D_{T-1}\\ &= U - I_0 - D[0,T-1] - \tau\mu - \mathbb{E}[L[1,T+\tau-1](P^U)|D_0,\ldots,D_{T-1}]\end{aligned} \tag{EC.12}$$

We are now ready to prove Properties 1 and 2 by induction. As induction hypothesis, let $T \in \mathbb{N}$ and suppose that $\forall t < T$ and for all demand sequences $D_0, D_1, \dots$ the random variables $q[1, t+\tau](\mathrm{P}^U)|D_0, \dots, D_{t-1}$ are non-decreasing and concave, while the random variables $L[1, t+\tau](\mathrm{P}^U)|D_0, \dots, D_{t+\tau}$ are non-increasing and convex. We will prove that these random variables are also non-decreasing and concave (resp. non-increasing and convex) for $t = T$.

Since $L[1, T+\tau-1](P^U)|D_0, \dots, D_{T+\tau-1}$ is non-increasing and convex in U by induction hypothesis, and since these properties are preserved in expectation, we find that $\mathbb{E}[(L[1, T+\tau-1](P^U)|D_0, \dots, D_{T+\tau-1})|D_0, \dots, D_{T-1}] = \mathbb{E}[L[1, T+\tau-1](P^U)|D_0, \dots, D_{T-1}]$ is non-increasing and convex in U for every realization $D_0, \dots, D_{T-1}$. By (EC.12) and since U is non-decreasing and concave in U, this implies that $q[1, T+\tau](P^U)|D_0, \dots, D_{T-1}$ is non-decreasing and concave in U. Next, specializing (17) to P^U, we find:

$$L[0, T+\tau](P^U) = \max_{k \in \{0, \dots, T+\tau\}} \left(D[0,k] - I_0 - q[1,k](P^U)\right)^+, \tag{EC.13}$$

Since $L[0, T+\tau](P^U)$ is the maximum of a number of functions, that by induction hypothesis and by the result we just proved are decreasing and convex in U, $L[0, T+\tau](P^U)$ must also be decreasing and convex in U. This proves the inductive step.

Now note that our induction hypothesis holds for $T = 1$ since in that case the functions are independent of U, and hence non-increasing and convex (non-decreasing and concave). This completes our proof by induction of Properties 1 and 2. For Property 3, note that it follows directly from Property 2. □

EC.6. Explicit recursions for inventory projection

All probabilities below will be conditional on the state at time 0, $\mathbf{x}_0$. We omit this condition as it makes the derivations more readable. From equations (1)-(3), we obtain the following recursive expressions for any $t \in \{0, \dots, \tau\}$:

$$\mathbb{P}\left(\tilde{J}_t = x\right) = \sum_{k=0}^{\infty} \mathbb{P}\left((\tilde{I}_t - K_t)^+ = x \mid K_t = k\right) \mathbb{P}(K_t = k) = \sum_{k=0}^{x} \mathbb{P}\left(\tilde{I}_t = x - k\right) \theta_k \tag{EC.14}$$

$$\mathbb{P}\left(\tilde{L}_t = x\right) = \sum_{k=0}^{\infty} \mathbb{P}\left((K_t - \tilde{I}_t)^+ = x \mid K_t = k\right) \mathbb{P}(K_t = k) = \sum_{k=x+1}^{\infty} \mathbb{P}\left(\tilde{I}_t = k - x\right) \theta_k \tag{EC.15}$$

$$\mathbb{P}\left(\tilde{I}_t = x\right) = \mathbb{P}\left(\tilde{J}_{t-1} + Q_t = x\right) = \sum_{y=0}^{\infty} \mathbb{P}\left(\tilde{J}_{t-1} = x - y\right) \mathbb{P}(Q_t = y), \tag{EC.16}$$

These recursions can be applied running from $t = 0$ till $t = \tau$ as $\tilde{I}_0$ and Q_t, $t \in \{1, \ldots, \tau - 1\}$ have Poisson distributions with means λI_0 and λq_t conditional on $\mathbf{x}_0 = (I_0, q_1, \ldots, q_{\tau-1})$ when demand has a ME distribution with scale λ:

$$\mathbb{P}\left(\tilde{I}_0 = x\right) = e^{-\lambda I_0} \frac{(\lambda I_0)^x}{x!}, \qquad \mathbb{P}\left(Q_t = x\right) = e^{-\lambda q_t} \frac{(\lambda q_t)^x}{x!}. \tag{EC.17}$$

EC.7. Two moment fits for mixed Erlang distributions

This Section provides the two moment fits that can also be found in Tijms (2003) or van Houtum (2006). Suppose that $D := \sum_{i=1}^{K} E_i$ where $\{E_i\}_{i=1}^{\infty}$ is a sequence of i.i.d. exponential random variables with rate λ and that K is an independent discrete random variable on the integers with mass function $\theta_k = \mathbb{P}(K = k)$. When the mean $\mu = \mathbb{E}[D]$ of demand and the squared coefficient of variation $c_D^2 = \mathbf{Var}[D]/\mu^2$ are known, the following two point distribution for K_t and choice for λ will match these moments when $c_D^2 \le 1$:

$$\theta_k = \begin{cases} p & \text{if } k = k_1 \\ 1 - p & \text{if } k = k_2 \\ 0 & \text{otherwise} \end{cases}, \quad \lambda = \frac{k_1 - p}{\mu},$$

with

$$k_1 := \left\lfloor \frac{1}{c_D^2} \right\rfloor, \quad k_2 = k_1 + 1, \quad p = \frac{k_1 c_D^2 - \sqrt{k_1(1 + c_D^2) - k_1^2 c_D^2}}{1 + c_D^2}.$$

When $c_D^2 > 1$, then an appropriate parameterization of a mixed Erlang distribution is

$$\theta_k = \begin{cases} p & \text{if } k = k_1 \\ 1 - p & \text{if } k = k_2 \\ 0 & \text{otherwise} \end{cases}, \quad \lambda = \frac{p + k_2(1 - p)}{\mu},$$

with

$$k_1 = 1, \quad k_2 = \max\left(3, \left\lceil \frac{4c_D^2 + \sqrt{16(c_D^2 - 1)}}{2} \right\rceil\right), \quad p = \frac{k_1 c_D^2 - \sqrt{k_1(1 + c_D^2) - k_1^2 c_D^2}}{1 + c_D^2}.$$