E-Companion to:
Dynamic Relocations in Car-Sharing Networks

Appendix A: Proofs and Preliminary Results

A.1. Notation

We introduce the following notation to simplify the exposition of the proofs. We use $\mathbf{e}$ to denote the vector of all ones, and $\mathbf{e}_i$ to denote the i^{th} unit vector, whose dimensions will be typically clear by context. We use E^i to denote a matrix of all zeros except for its i^{th} column which is $\mathbf{e}$.

A.2. Equivalence of Fluid Approximations

In this section, we describe the original LP fluid approximation model from Braverman et al. (2019) for the dynamic car relocation problem described in Section 3.2, and we show it is equivalent to Problem P in Section 3. Problem P is more amenable for our analysis, since it removes redundant constraints and constraints that will never be tight at optimality if the expected relocation travel times and costs satisfy the triangle inequality. Indeed, since our analysis is based on studying the subset of tight constraints at optimality, then removing redundant constraints and constraints that will never be tight at optimality is useful.

We introduce some notation to match the exposition in Braverman et al. (2019). Let a_i be the average availability at location i, i.e., the long-run fraction of time that there is at least one empty car at location i. Then, the LP fluid approximation model from Braverman et al. (2019) is

$$\max_{e_{ij},f_{ij},a_i} \sum_i \sum_j a_i \lambda_i \alpha_{ij} p_{ij} - \sum_{i,j\neq i} c_{ij}\mu^e_{ij} e_{ij} \tag{EC.1a}$$

$$\text{s.t. } a_i\lambda_i\alpha_{ij} = \mu^f_{ij} f_{ij} \qquad 1\le i,j\le n \tag{EC.1b}$$

$$\mu^e_{ij} e_{ij} \le \sum_{k=1}^n \mu^f_{ki} f_{ki} \qquad 1\le i,j\le n, j\neq i \tag{EC.1c}$$

$$\sum_{k=1,k\neq i}^n \mu^e_{ki} e_{ki} \le \lambda_i a_i \le \sum_{k=1,k\neq i}^n \mu^e_{ki} e_{ki} + \sum_{k=1}^n \mu^f_{ki} f_{ki} \qquad 1\le i\le n \tag{EC.1d}$$

$$\lambda_i a_i + \sum_{j=1,j\neq i}^n \mu^e_{ij} e_{ij} = \sum_{k=1,k\neq i}^n \mu^e_{ki} e_{ki} + \sum_{k=1}^n \mu^f_{ki} f_{ki} \qquad 1\le i\le n \tag{EC.1e}$$

$$\sum_{i=1}^n \sum_{j=1}^n (e_{ij} + f_{ij}) = 1 \tag{EC.1f}$$

$$0 \le a_i \le 1 \qquad 1\le i\le n \tag{EC.1g}$$

$$f_{ij} \ge 0 \qquad 1\le i,j\le n \tag{EC.1h}$$

$$e_{ij} \ge 0 \qquad 1\le i,j\le n \tag{EC.1i}$$

To avoid repetition, we omit explaining the LP fluid model (EC.1) in detail. Indeed, in Lemma EC.1 below we show it is equivalent to Problem P, and we explain this version of the problem in detail in Section 3.3.

Lemma EC.1. *If the expected relocation-times and the relocation costs satisfy the triangle inequality, then Problem* (EC.1) *is equivalent to Problem* P*.*

Proof. Adding up constraint (EC.1b) over all j we can replace $a_i\lambda_i$ by $\sum_k \mu^f_{ik} f_{ik}$, thus Problem (EC.1) can be rewritten as follows:

$$\max_{e_{ij},f_{ij}} \sum_i \bar{p}_i \sum_j \mu^f_{ij} f_{ij} - \sum_{i,j\neq i} c_{ij}\mu^e_{ij} e_{ij} \tag{EC.2a}$$

$$\text{s.t.} \sum_k \mu^f_{ik} f_{ik}\alpha_{ij} = \mu^f_{ij} f_{ij} \qquad 1 \le i,j \le n \tag{EC.2b}$$

$$\mu^e_{ij} e_{ij} \le \sum_{k=1}^n \mu^f_{ki} f_{ki} \qquad 1 \le i, j \neq i \le n, \tag{EC.2c}$$

$$\sum_{k=1,k\neq i}^n \mu^e_{ki} e_{ki} \le \sum_j \mu^f_{ij} f_{ij} \le \sum_{k=1,k\neq i}^n \mu^e_{ki} e_{ki} + \sum_{k=1}^n \mu^f_{ki} f_{ki} \qquad 1 \le i \le n \tag{EC.2d}$$

$$\sum_j \mu^f_{ij} f_{ij} + \sum_{k=1,k\neq i}^n e_{ik}\mu_{ik} = \sum_{k=1,k\neq i}^n \mu^e_{ki} e_{ki} + \sum_{k=1}^n \mu^f_{ki} f_{ki} \qquad 1 \le i \le n \tag{EC.2e}$$

$$\sum_i e_{ii} + \sum_i \sum_{j\neq i} e_{ij} + \sum_i \sum_j f_{ij} = 1 \tag{EC.2f}$$

$$0 \le \sum_j \mu^f_{ij} f_{ij} \le \lambda_i \qquad i \le i \le n \tag{EC.2g}$$

$$f_{ij} \ge 0 \qquad 1 \le i,j \le n \tag{EC.2h}$$

$$e_{ij} \ge 0 \qquad 1 \le i,j \le n \tag{EC.2i}$$

Since $e_{ii} \ge 0$, and these variables only appear in constraint (EC.2f), then without loss of generality we can remove them from the problem changing the equality in constraint (EC.2f) to an inequality, making it equivalent to constraint (1e).

It remains to show that constraints (EC.2c) and (EC.2d) can be dropped from problem (EC.2).

We now show that constraint (EC.2d) is redundant. First, the fact that the second inequality in constraint (EC.2d) is redundant follows directly from constraint (EC.2e) and $e_{ik} \ge 0$. Second, the first inequality in constraint (EC.2d) is implied by constraints (EC.2c) and (EC.2e).

Finally, we show that if the expected relocation-times and relocation costs satisfy the triangle inequality then constraint (EC.2c) can be dropped from problem (EC.2). Specifically, we prove that if there exists an optimal solution to Problem P, then there always is an optimal solution to

Problem P that will satisfy constraint (EC.2c)[9]. In fact, let $(f^*_{ij}, e^*_{ij})_{i,j\in\mathcal{N}}$ be an optimal solution to Problem P. If $(f^*_{ij}, e^*_{ij})_{i,j\in\mathcal{N}}$ satisfies constraint (EC.2c) we are done. Therefore, assume that $\mu^e_{kl} e^*_{kl} > \sum_{i=1}^n \mu^f_{ik} f^*_{ik}$ for some nodes k and l. From the equilibrium constraint (1c) it then follows that $\sum_j \mu^f_{kj} f^*_{kj} < \sum_{l\neq k} \mu^e_{lk} e^*_{lk}$. Namely, the solution $(f^*_{ij}, e^*_{ij})_{i,j\in\mathcal{N}}$ simultaneously has a larger relocation rate out of node k than the rate at which matched cars arrive into node k, and a larger relocation rate into node k than the rate at which matched cars leave node k. It follows that the same matchings f^*_{ij} can be attained in equilibrium by relocations $\tilde{e}_{ij} \le e^*_{ij}$ for each $i, j \neq i \in \mathcal{N}$, with $\tilde{e}_{kl} < e^*_{kl}$, such that $\sum_{j\neq i} \mu^e_{ji}\tilde{e}_{ji} - \sum_{j\neq i} \mu^e_{ij}\tilde{e}_{ij} = \sum_{j\neq k} \mu^e_{ji} e^*_{ji} - \sum_{j\neq i} \mu^e_{ij} e^*_{ij}$ for each node i, i.e., the net relocation rate into each node i stays the same as with the relocations e^*_{ij}, and $\mu^e_{kl}\tilde{e}_{kl} \le \sum_{i=1}^n \mu^f_{ik} f^*_{ik}$. Since by assumption the expected relocation-times and relocation costs satisfy the triangle inequality, then $(f^*_{ij}, \tilde{e}_{ij})_{i,j\in\mathcal{N}}$ is feasible in Problem P and it attains an objective value that is no smaller than the one attained by the optimal solution $(f^*_{ij}, e^*_{ij})_{i,j\in\mathcal{N}}$, completing the proof. ☐

A.3. Proof of Lemma 1

Proof. Let $(f^*_{ij}, e^*_{ij})_{i,j\in\mathcal{N}}$ be an optimal solution to Problem P and assume, in order to arrive to a contradiction, that $\sum_j \mu^f_{ij} f^*_{ij} < \lambda_i$ for all i, and $\sum_i \sum_{j\neq i} e^*_{ij} + \sum_i \sum_j f^*_{ij} < 1$.

Then, it follows that there exists $\epsilon > 0$ small enough such that the solution $(\tilde{f}_{ij}, \tilde{e}_{ij})_{i,j\in\mathcal{N}}$, with $\tilde{f}_{ij} = f^*_{ij} + \epsilon\pi_i \frac{\alpha_{ij}}{\mu^f_{ij}} \ge 0$ for all i, and $\tilde{e}_{ij} = e^*_{ij} \ge 0$ for all i, j, is feasible for Problem P and it attains an objective value strictly larger than the optimal solution $(f^*_{ij}, e^*_{ij})_{i,j\in\mathcal{N}}$, a contradiction.

We first verify that the solution $(\tilde{f}_{ij}, \tilde{e}_{ij})_{i,j\in\mathcal{N}}$ is feasible for Problem P.

Indeed, constraint (1b) is satisfied since

$$\mu^f_{ij}\tilde{f}_{ij} = \mu^f_{ij} f^*_{ij} + \epsilon\pi_i\alpha_{ij} = \alpha_{ij}\left(\sum_k \mu^f_{ik} f^*_{ik} + \epsilon\pi_i\right) = \alpha_{ij}\sum_k \mu^f_{ik}\tilde{f}_{ik}.$$

Constraint (1c) is satisfied since

$$\begin{aligned}\sum_j \mu^f_{ij}\tilde{f}_{ij} + \sum_{j\neq i}\mu^e_{ij}\tilde{e}_{ij} &= \sum_j \mu^f_{ij} f^*_{ij} + \epsilon\pi_i + \sum_{j\neq i}\mu^e_{ij} e^*_{ij} = \sum_j \mu^f_{ji} f^*_{ji} + \epsilon\pi_i + \sum_{j\neq i}\mu^e_{ji} e^*_{ji} \\ &= \sum_j \mu^f_{ji}\tilde{f}_{ji} + \sum_{j\neq i}\mu^e_{ji}\tilde{e}_{ji} + \epsilon\left(\pi_i - \sum_j \pi_j\alpha_{ji}\right) = \sum_j \mu^f_{ji}\tilde{f}_{ji} + \sum_{j\neq i}\mu^e_{ji}\tilde{e}_{ji},\end{aligned}$$

where the first and third equalities follow from the definition of $(\tilde{f}_{ij}, \tilde{e}_{ij})_{i,j\in\mathcal{N}}$, the second equality follows since $(f^*_{ij}, e^*_{ij})_{i,j\in\mathcal{N}}$ satisfies constraint (1c), and the last equality follows since $\pi = \pi A$.

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[9] A stronger result, where any optimal solution to Problem P is optimal in problem (EC.2) holds if we assume relocations are costly. However, Braverman et al. (2019) assume cost-less relocations and for consistency we show a weaker result.

Constraint (1d) is satisfied since for all nodes i, and for $\epsilon > 0$ small enough,

$$\sum_j \mu^f_{ij} \tilde{f}_{ij} = \sum_j \mu^f_{ij} f^*_{ij} + \epsilon \pi_i \leq \lambda_i.$$

Constraint (1e) is satisfied since, for $\epsilon > 0$ small enough,

$$\sum_i \sum_{j \neq i} \tilde{e}_{ij} + \sum_i \sum_j \tilde{f}_{ij} = \sum_i \sum_{j \neq i} e^*_{ij} + \sum_i \sum_j f^*_{ij} + \epsilon \sum_i \pi_i t^f_i \leq 1.$$

Hence, $(\tilde{f}_{ij}, \tilde{e}_{ij})_{i,j \in \mathcal{N}}$ is feasible for Problem P. We now verify that it attains an objective value strictly larger than the optimal solution. Indeed, for any $\epsilon > 0$

$$\begin{aligned} \sum_i \bar{p}_i \sum_j \mu^f_{ij} \tilde{f}_{ij} - \sum_i \sum_{j \neq i} c_{ij} \mu^e_{ij} \tilde{e}_{ij} &= \sum_i \bar{p}_i \sum_j \mu^f_{ij} f^*_{ij} + \epsilon \sum_i \bar{p}_i \pi_i - \sum_i \sum_{j \neq i} c_{ij} \mu^e_{ij} e^*_{ij} \\ &> \sum_i \bar{p}_i \sum_j \mu^f_{ij} f^*_{ij} - \sum_i \sum_{j \neq i} c_{ij} \mu^e_{ij} e^*_{ij}, \end{aligned}$$

a contradiction with the optimality of $(f^*_{ij}, e^*_{ij})_{i,j \in \mathcal{N}}$. ☐

A.4. Proof of Proposition 1

Proof. We solve the linear system of equations from the statement of the proposition in closed form. We start by rewriting it in a vector form more convenient for analysis. Indeed, to simplify the exposition, define $z_i = \sum_j \mu^f_{ij} f_{ji}$ to be the total matching rate at node i. From (1b), i.e., $\mu^f_{ij} f_{ij} = \alpha_{ij} \sum_k \mu^f_{ik} f_{ik}$, it follows that

$$\sum_j \mu^f_{ji} f_{ji} = \sum_j \alpha_{ji} \sum_k \mu^f_{jk} f_{jk} = \sum_j \alpha_{ji} z_j.$$

Then, by defining $\mathbf{z}$ and $\boldsymbol{\beta}$ to be row vectors we can rewrite the linear system of equations more compactly as,

$$\begin{aligned} \mathbf{z} &= \mathbf{z} A + \boldsymbol{\beta}, \\ \mathbf{z} \mathbf{e}_l &= \lambda_l. \end{aligned}$$

Since A is the transition matrix of an ergodic Markov Chain, $\mathbf{z} = \mathbf{z} A + \boldsymbol{\beta}$ has only $n-1$ linearly independent equations. Therefore, without loss of generality we drop the l^{th} equation and combine it with $\mathbf{z} \mathbf{e}_l = \lambda_l$ to form a system of n linearly independent equations.

Let $\boldsymbol{\beta}^l$ be the row vector obtained when replacing the l^{th} component of the vector $\boldsymbol{\beta}$ by λ_l. Then,

$$\mathbf{z} A^l = \boldsymbol{\beta}^l.$$

Since A^l is invertible we can solve for $\mathbf{z}$ in closed form as follows,

$$\mathbf{z} = \boldsymbol{\beta}^l (A^l)^{-1}. \tag{EC.3}$$

We now show that the components of the l^{th} row of the matrix $(A^l)^{-1}$ are $\bar{z}^l_k = \frac{\pi_k}{\pi_l}$. Since $\boldsymbol{\pi}(I - A) = \mathbf{0}^T$, then $\frac{\boldsymbol{\pi}}{\pi_l} A^l = \mathbf{e}_l^T$. Therefore,

$$\bar{\boldsymbol{z}}^{\boldsymbol{l}} = \frac{\boldsymbol{\pi}}{\pi_l} = \mathbf{e}_l^T (A^l)^{-1} = (A^l)^{-1}_{l\cdot}. \tag{EC.4}$$

Substituting (EC.4) in (EC.3) and expanding we obtain,

$$z_k = \sum_{i \neq l} \beta_i b^l_{ik} + \bar{z}^l_k \lambda_l = \sum_i \left(\sum_{j \neq i} \mu^e_{ji} e_{ji} - \sum_{j \neq i} \mu^e_{ij} e_{ij} \right) b^l_{ik} + \bar{z}^l_k \lambda_l = \sum_i \sum_{j \neq i} (b^l_{jk} - b^l_{ik}) \mu^e_{ij} e_{ij} + \bar{z}^l_k \lambda_l, \tag{EC.5}$$

where the first equality follows by definition of A^l and $\boldsymbol{\beta}^l$. The second equality follows from the definition of β_i, and recalling that by definition $b^l_{lj} = 0$ for all j. The third equality follows by interchanging the sums in the first term.

Thus, we conclude

$$f^l_{km} = \frac{\alpha_{km} \sum_i \mu^f_{ki} f^l_{ki}}{\mu^f_{km}} = \frac{\alpha_{km}}{\mu^f_{km}} z_k = \frac{\alpha_{km}}{\mu^f_{km}} \left(\bar{z}^l_k \lambda_l + \sum_i \sum_{j \neq i} (b^l_{jk} - b^l_{ik}) \mu^e_{ij} e_{ij} \right),$$

where the first equality follows from (1b), the second equality follows from the definition of z_k, and the last equality follows from (EC.5), completing the proof. □

A.5. Proof of Corollary 1:

Proof. Follows directly from Proposition 1 and the reformulation (5) of Problem P when constraint (1d) is tight for node l. □

A.6. Proof of Proposition 2

Proof. Analogous to the proof of Proposition 1, we solve the linear system of equations from the statement of the proposition in closed form.

We start by rewriting it in a vector form more convenient for analysis. To simplify the exposition, define $z_i = \sum_j \mu^f_{ij} f_{ji}$ to be the total matching rate at node i. From constraint (1b), i.e., $\mu^f_{ij} f_{ij} = \alpha_{ij} \sum_k \mu^f_{ik} f_{ik}$, it follows that

$$\sum_j \mu^f_{ji} f_{ji} = \sum_j \alpha_{ji} \sum_k \mu^f_{jk} f_{jk} = \sum_j \alpha_{ji} z_j,$$

and

$$\sum_i \sum_j f_{ij} = \sum_i \sum_j \frac{\alpha_{ij} \sum_k \mu^f_{ik} f_{ik}}{\mu^f_{ij}} = \sum_i \sum_k \mu^f_{ik} f_{ik} \sum_j \frac{\alpha_{ij}}{\mu^f_{ij}} = \sum_i z_i t^f_i = \mathbf{z} \mathbf{t}^f,$$

where the last equality is by definition of z_i and t^f_i. Then, by defining $\mathbf{z}$ and $\boldsymbol{\beta}$ to be row vectors we can rewrite the linear system of equations more compactly as follows,

$$\mathbf{z} = \mathbf{z} A + \boldsymbol{\beta},$$

$$\mathbf{z} \mathbf{t}^f = 1 - \sum_i \sum_{j \neq i} e_{ij}.$$

Since A is the transition matrix of an ergodic Markov chain, $\mathbf{z} = \mathbf{z}A + \boldsymbol{\beta}$ has only $n-1$ linearly independent equations. Therefore, without loss of generality we drop the last equation and combine it with $\mathbf{z}\mathbf{t}^f = 1 - \sum_i \sum_{j \neq i} e_{ij}$ to form the following system of n linearly independent equations,

$$\mathbf{z}A^0 = \mathbf{z}\Big[(I-A)_{-n}\Big|\mathbf{t}^f\Big] = \left[\boldsymbol{\beta}_{(-n)}\Big|1 - \sum_i \sum_{j \neq i} e_{ij}\right].$$

Namely, A^0 is invertible and we can solve for $\mathbf{z}$ in closed form as follows,

$$\mathbf{z} = \left[\boldsymbol{\beta}_{(-n)}\Big|1 - \sum_i \sum_{j \neq i} e_{ij}\right](A^0)^{-1}. \tag{EC.6}$$

We now show that the components of the last row of $(A^0)^{-1}$ are $\bar{z}^0_k = \frac{\pi_k}{\sum_i \pi_i t^f_i}$. Since $\boldsymbol{\pi}(I-A) = \mathbf{0}^T$, then $\boldsymbol{\pi}\Big[(I-A)_{-n}\Big|\mathbf{t}^f\Big] = \Big[0, \ldots, 0, \sum_i \pi_i t^f_i\Big]$. Therefore,

$$\boldsymbol{\pi} = \left[0, \ldots, 0, \sum_i \pi_i t^f_i\right]\Big[(I-A)_{-n}\Big|\mathbf{t}^f\Big]^{-1} = \sum_i \pi_i t^f_i (A^0)^{-1}_{n\cdot}$$

Hence, we conclude

$$(A^0)^{-1}_{n\cdot} = \frac{\boldsymbol{\pi}}{\sum_i \pi_i t^f_i} = \bar{\mathbf{z}}^{\mathbf{0}}. \tag{EC.7}$$

Substituting (EC.7) in (EC.6) and expanding we obtain,

$$\begin{aligned} z_k &= \sum_{i \neq n} \beta_i d_{ik} + \bar{z}^0_k \left(1 - \sum_i \sum_{j \neq i} e_{ij}\right) \\ &= \sum_i \left(\sum_{j \neq i} \mu^e_{ji} e_{ji} - \sum_{j \neq i} \mu^e_{ij} e_{ij}\right) d_{ik} + \bar{z}^0_k \left(1 - \sum_i \sum_{j \neq i} e_{ij}\right) \\ &= \sum_i \sum_{j \neq i} (d_{jk} - d_{ik}) \mu^e_{ij} e_{ij} + \bar{z}^0_k \left(1 - \sum_i \sum_{j \neq i} e_{ij}\right), \end{aligned} \tag{EC.8}$$

where the second equality follows from the definition of β_i and recalling that by definition $d_{nk} = 0$ for all nodes k. The third equality follows by interchanging the sums in the first term.

Thus, we conclude

$$f^0_{km} = \frac{\alpha_{km} \sum_i \mu^f_{ki} f^0_{ki}}{\mu_{km}} = \frac{\alpha_{km}}{\mu_{km}} z_k = \frac{\alpha_{km}}{\mu_{km}} \left(\bar{z}^0_k \left(1 - \sum_i \sum_{j \neq i} e_{ij}\right) + \sum_i \sum_{j \neq i} (d_{jk} - d_{ik}) \mu^e_{ij} e_{ij}\right),$$

where the first equality follows from (1b), the second equality follows from the definition of z_k, and the last equality follows from (EC.8), completing the proof. □

A.7. Proof of Corollary 2:

Proof. Follows directly from Proposition 2 and the reformulation (11) of Problem P when constraint (1e) is tight. □

A.8. Proof of Theorem 1

Proof. We first show the one-to-one mapping stated in the Theorem.

On the one hand, let (E,F) be a feasible solution to problem P that satisfies at least one of the constraints (1d) or (1e) with equality, and OBJ be its objective value. From Propositions 1 and 2 and by construction of the Problems P^l and P^0, E is feasible under at least one of the Problems P^l, $l \in \mathcal{N}$, and P^0, and it has the same objective value OBJ.

On the other hand, let E^r be a feasible solution to some Problem P^0 or P^l, $l \in \mathcal{N}$, and OBJ^r be its objective value. Moreover, let $F^r = F(E^r)$ be the induced matches according to (2) if E^r is feasible in Problem P^l, for some $l \in \mathcal{N}$, or (8) if E^r is feasible in Problem P^0. Propositions EC.1 and EC.2 in Appendix B show that F^r is well defined. Indeed, even if E^r belongs to the shared facet between Problems P^l, for some $l \in \mathcal{N}$, and P^0, or to the shared facet between Problems P^l and P^m for some $l, m \in \{1, \ldots, n\}$, Propositions EC.1 and EC.2 show that then the respective formulas for $F(E^r)$ coincide. From Propositions 1 and 2 and by construction of P^l and P^0, (E^r, F^r) is feasible in problem P and it has the same objective value OBJ^r.

Finally, by Lemma 1, any optimal solution to Problem P must satisfy at least one of the constraints (1d) or (1e) with equality. Therefore the last statement in the Theorem follows directly from the one-to-one mapping above, completing the proof. □

Appendix B: Additional Results and Proofs for the Partition of the e_{ij}-Space

Recall the linear programs P given in (1), P^l for $l \in \mathcal{N}$ given in (5), and P^0 given in (11). In this section we provide the details that confirm that problems P^l, $l \in \mathcal{N}$, and P^0 span the space of all relocations that induce at least one of the constraints (1d) or (1e) to be tight.

We start with Proposition EC.1, which formalizes that the feasible set of P^0 shares a facet with the feasible set of each P^l, $l \in \mathcal{N}$. In particular, Proposition EC.1 shows that the closed-form expression for the matches from node k to m when the demand rate at node l is fully served, f^l_{km}, derived in (2) in Proposition 1, and when the supply is fully utilized, f^0_{km}, derived in (8) in Proposition 2, are consistent on the shared facet between P^0 and P^l.

Proposition EC.1. *Constraint* (5c) *in Problem* P^l *and constraint* (11b) *in Problem* P^0 *are a shared facet between* P^l *and* P^0*. Specifically,* $\sum_j \mu^f_{lj} f^0_{lj} \le \lambda_l$ *if and only if* $\sum_i \sum_j f^l_{ij} \ge 1 - \sum_i \sum_{j \ne i} e_{ij}$*. Moreover, if constraint* (11b) *is tight (or equivalently constraint* (5c) *is tight) then* $f^l_{km} = f^0_{km}$ *for each node* k *and* m*.*

Proof. We first show the first statement in the proposition. Recall that constraint (11b) in Problem P^0 is equivalent to

$$\sum_j \mu^f_{lj} f^0_{lj} = \sum_i \sum_{j \ne i} (d_{jl} - d_{il}) \mu^e_{ij} e_{ij} + \bar{z}^0_l \left(1 - \sum_i \sum_{j \ne i} e_{ij}\right) \le \lambda_l, \tag{EC.9}$$

where the first equality is by definition, cf. (8).

On the other hand, constraint (5c) in Problem P^l is equivalent to

$$\begin{aligned}
&\sum_i \sum_j f^l_{ij} \le 1 - \sum_i \sum_{j\neq i} e_{ij} \\
\iff\quad & \sum_k \bar{z}^l_k \lambda_l t^f_k + \sum_i \sum_{j\neq i} \sum_k (b^l_{jk} - b^l_{ik}) t^f_k \mu^e_{ij} e_{ij} \le 1 - \sum_i \sum_{j\neq i} e_{ij} \\
\iff\quad & \frac{\lambda_l}{\bar{z}^0_l} - \sum_i \sum_{j\neq i} \frac{(d_{jl} - d_{il})\mu^e_{ij} e_{ij}}{\bar{z}^0_l} \le 1 - \sum_i \sum_{j\neq i} e_{ij} \\
\iff\quad & \sum_j \mu^f_{lj} f^0_{lj} = \sum_i \sum_{j\neq i} (d_{jl} - d_{il})\mu^e_{ij} e_{ij} + \bar{z}^0_l \left(1 - \sum_i \sum_{j\neq i} e_{ij}\right) \ge \lambda_l,
\end{aligned}$$

where the first equivalence is by definition, cf. (2). The second equivalence follows since by definition $\sum_k \bar{z}^l_k t^f_k = \sum_k \frac{\pi_k}{\pi_l} t^f_k = 1/\bar{z}^0_l$, cf. equations (3) and (9), and from (EC.15) in Lemma EC.3 below, which shows that $(d_{jl} - d_{il}) = -\bar{z}^0_l \sum_k (b^l_{jk} - b^l_{ik}) t^f_k$. The third equivalence follows from (EC.9), completing the proof of the first statement in the proposition.

We now show the second statement in the proposition. Assume $\sum_j \mu^f_{lj} f^0_{lj} = \lambda_l$, i.e., assume (EC.9) with equality, then

$$\begin{aligned}
\sum_j \mu^f_{kj} f^0_{kj} &= \sum_i \sum_{j\neq i} (d_{jk} - d_{ik})\mu^e_{ij} e_{ij} + \bar{z}^0_k \left(1 - \sum_i \sum_{j\neq i} e_{ij}\right) \\
&= \sum_i \sum_{j\neq i} (d_{jk} - d_{ik})\mu^e_{ij} e_{ij} + \bar{z}^0_k \left(\frac{\lambda_l}{\bar{z}^0_l} - \sum_i \sum_{j\neq i} \frac{(d_{jl} - d_{il})\mu^e_{ij} e_{ij}}{\bar{z}^0_l}\right) \\
&= \sum_i \sum_{j\neq i} (d_{jk} - d_{ik})\mu^e_{ij} e_{ij} + \bar{z}^0_k \left(\frac{\lambda_l}{\bar{z}^0_l} + \sum_i \sum_{j\neq i} \sum_k (b^l_{jk} - b^l_{ik}) t^f_k \mu^e_{ij} e_{ij}\right) \\
&= \sum_i \sum_{j\neq i} (d_{jk} - d_{ik})\mu^e_{ij} e_{ij} + \pi_k \frac{\lambda_l}{\pi_l} + \sum_i \sum_{j\neq i} (b^l_{jk} - b^l_{ik})\mu^e_{ij} e_{ij} - \sum_i \sum_{j\neq i} (d_{jk} - d_{ik})\mu^e_{ij} e_{ij} \\
&= \pi_k \bar{z}^l_k + \sum_i \sum_{j\neq i} (b^l_{jk} - b^l_{ik})\mu^e_{ij} e_{ij} \\
&= \sum_j \mu^f_{kj} f^l_{kj},
\end{aligned}$$

where the first and last two equalities are by definition, cf. Propositions 1 and 2. The second equality follows from (EC.9) with equality. The third equality follows from (EC.15) in Lemma EC.3 below. The fourth equality follows from the fact that by definition $\bar{z}^0_i$ is proportional to π_i for each node i, cf. (9), and from (EC.14) in Lemma EC.3 below, which shows that $b^l_{jk} = d_{jk} - d_{lk} + \sum_i b^l_{ji} t^f_i \bar{z}^0_k$.

To conclude the proof, note that $f^l_{km} = f^0_{km}$ for each node k and m then follows directly from constraint (1b) and the chain of equalities above. □

Similarly, Proposition EC.2 formalizes that the feasible sets of each pair P^l and P^m, for $l, m \in \{1, \dots, n\}$ share a facet. In particular, Proposition EC.2 shows that the closed-form expressions for the matches from node i to j when the demand rate at node l is fully served, f_{ij}^l, derived in (2) in Proposition 1, is consistent when the demand rate at node $m \neq l$ is also fully served, f_{ij}^m, i.e., on the shared facet between P^l and P^m.

Proposition EC.2. *Constraint* (5b) *for node* m *in Problem* P^l *when the demand rate for node* l *is fully served, and constraint* (5b) *for node* l *in problem* P^m *when the demand rate for node* m *is fully served are a shared facet between* P^l *and* P^m. *Specifically,* $\sum_j \mu_{mj}^f f_{mj}^l \leq \lambda_m$ *if and only if* $\sum_j \mu_{lj}^f f_{lj}^m \geq \lambda_l$. *Moreover, if* $\sum_j \mu_{mj}^f f_{mj}^l = \lambda_m$ *(or equivalently* $\sum_j \mu_{lj}^f f_{lj}^m = \lambda_l$*) then* $f_{ij}^l = f_{ij}^m$ *for each node* i *and* j.

Proof. We begin by showing the first statement in the proposition. Constraint (11b) for node m in Problem P^0 when the demand rate for node l is fully served is equivalent to

$$\sum_j \mu_{mj}^f f_{mj}^l = \lambda_l \bar{z}_m^l + \sum_i \sum_{j \neq i} (b_{jm}^l - b_{im}^l) \mu_{ij}^e e_{ij} = \lambda_l \frac{\pi_m}{\pi_l} + \sum_i \sum_{j \neq i} (b_{jm}^l - b_{im}^l) \mu_{ij}^e e_{ij} \leq \lambda_m, \tag{EC.10}$$

where the equalities are by definition, cf. Proposition 1.

Analogously, constraint (11b) for node l in Problem P^0 when the demand rate for node m is fully served is equivalent to

$$\begin{aligned}
& \sum_j \mu_{lj}^f f_{lj}^m = \lambda_m \frac{\pi_l}{\pi_m} + \sum_i \sum_{j \neq i} (b_{jl}^m - b_{il}^m) \mu_{ij}^e e_{ij} \leq \lambda_l \\
\iff \quad & \lambda_m \frac{\pi_l}{\pi_m} - \frac{\pi_l}{\pi_m} \sum_i \sum_{j \neq i} (b_{jm}^l - b_{im}^l) \mu_{ij}^e e_{ij} \leq \lambda_l \\
\iff \quad & \lambda_m \leq \lambda_l \frac{\pi_m}{\pi_l} + \sum_i \sum_{j \neq i} (b_{jm}^l - b_{im}^l) \mu_{ij}^e e_{ij} \\
\iff \quad & \sum_j \mu_{mj}^f f_{mj}^l \geq \lambda_m,
\end{aligned}$$

where the first equivalence follows from (EC.17) in Lemma EC.3 below, which shows that $(b_{jl}^m - b_{il}^m) = -\pi_l/\pi_m (b_{jm}^l - b_{im}^l)$. The third equivalence follows from (EC.10), proving the first statement in the proposition.

We now show the second statement in the proposition. Assume $\sum_j \mu_{mj}^f f_{mj}^l = \lambda_m$, i.e., assume (EC.10) with equality. Then,

$$\begin{aligned}
\sum_j \mu_{kj}^f f_{kj}^l &= \lambda_l \bar{z}_k^l + \sum_i \sum_{j \neq i} (b_{jk}^l - b_{ik}^l) \mu_{ij}^e e_{ij} \\
&= \pi_k \left(\frac{\lambda_l}{\pi_l} - \frac{\sum_i \sum_{j \neq i} (b_{jm}^k - b_{im}^k) \mu_{ij}^e e_{ij}}{\pi_m} - \frac{\sum_i \sum_{j \neq i} (b_{jl}^m - b_{il}^m) \mu_{ij}^e e_{ij}}{\pi_l} \right) \\
&= \pi_k \left(\frac{\lambda_l}{\pi_l} - \frac{\sum_i \sum_{j \neq i} (b_{jm}^k - b_{im}^k) \mu_{ij}^e e_{ij}}{\pi_m} + \frac{\sum_i \sum_{j \neq i} (b_{jm}^l - b_{im}^l) \mu_{ij}^e e_{ij}}{\pi_m} \right) \\
&= \pi_k \left(\frac{\lambda_l}{\pi_l} - \frac{\sum_i \sum_{j \neq i} (b_{jm}^k - b_{im}^k) \mu_{ij}^e e_{ij}}{\pi_m} + \frac{\lambda_m}{\pi_m} - \frac{\lambda_l}{\pi_l} \right)
\end{aligned}$$

$$= \lambda_m \bar{z}_k^m + \sum_i \sum_{j \neq i} (b_{jk}^m - b_{ik}^m) \mu_{ij}^e e_{ij}$$
$$= \sum_j \mu_{kj}^f f_{kj}^m$$

where the first and last equalities are by definition, cf. Proposition 1. The second equality follows by definition of $\bar{z}_k^l$ in (3) and (EC.16) in Lemma EC.3, which shows $(b_{jk}^l - b_{ik}^l)/\pi_k + (b_{jm}^k - b_{im}^k)/\pi_m + (b_{jl}^m - b_{il}^m)/\pi_l = 0$. The third equality follows from (EC.17) in Lemma EC.3. Finally, the fourth equality follows from (EC.10) with equality.

To conclude the proof, note that $f_{ij}^l = f_{ij}^m$ for each node i and j then follows directly from constraint (1b) and the chain of equalities above. □

For completeness, we now provide the statements and proofs of the auxiliary lemmas that were used in the analysis in this section.

First, Lemma EC.2 shows an identity connecting the expanded fundamental matrix when node l is absorbing, B^l, with the original transition matrix, A. In fact, (EC.2) can be interpreted as the equilibrium condition for each row i of the matrix B^l, namely $B_{i\cdot}^l A = B_{i\cdot}^l - \mathbf{e}_i^T + \mathbf{e}_l^T$.

Lemma EC.2. *Let A be the transition matrix of an ergodic Markov chain, and B^l be the expanded fundamental matrix when the Markov chain is modified to make node l absorbing.*

Recall that I is the identity matrix, and E_l is such that $E_{\cdot l} = \mathbf{e}$ and $E_{\cdot i} = \mathbf{0}$ for all $i \neq l$, then

$$B^l A = B^l - I + E_l.$$

Proof. To simplify the exposition of the proof, assume without loss of generality that $l = n$. Then,

$$B^n A = \left[\begin{array}{c|c} (I - Q^n)^{-1} & \mathbf{0} \\ \hline \mathbf{0}^T & 0 \end{array}\right] \left[\begin{array}{c|c} Q^n & (A_{\cdot n})_{-n} \\ \hline (A_{n\cdot})_{-n} & \alpha_{nn} \end{array}\right] = \left[\begin{array}{c|c} (I - Q^n)^{-1} Q^n & (I - Q)^{-1} (A_{\cdot n})_{-n} \\ \hline \mathbf{0}^T & 0 \end{array}\right]. \tag{EC.11}$$

Since $(I - Q^n)^{-1}(I - Q^n) = I$, then

$$(I - Q^n)^{-1} Q^n = (I - Q^n)^{-1} - I. \tag{EC.12}$$

Moreover, we also have

$$(I - Q)^{-1} (A_{\cdot n})_{-n} = \mathbf{e}, \tag{EC.13}$$

because the i^{th} component of the vector $(I - Q)^{-1}(A_{\cdot n})_{-n}$ is the probability of being absorbed in the absorbing state n when starting from the transient state i, see for example Resnick (1992). Since n is the only absorbing state, then (EC.13) trivially holds.

Substituting (EC.12) and (EC.13) into (EC.11) we conclude that

$$B^n A = \left[\begin{array}{c|c} (I - Q)^{-1} - I & \mathbf{e} \\ \hline \mathbf{0}^T & 0 \end{array}\right] = B^n - I + E_n.$$

Since $l = n$ was chosen arbitrarily, we conclude that $B^l A = B^l - I + E_l$, completing the proof. □

Lemma EC.3 below shows several useful identities between the linear coefficients used in Proposition 1, b^l_{ij}, and the linear coefficients used in Proposition 2, d_{ij}.

Lemma EC.3. *Recall that* $A^0 = \Big[(I-A)_{-n}\big|\mathbf{t}^f\Big]$*;* $d_{ni} = d_{nj} = 0$ *for all* i*,* j*; the coefficients* d_{ij}*, for* $i \neq n$*, are the* $(i,j)^{th}$ *elements of* $(A^0)^{-1}$*; and* $\bar{z}^0_k = \frac{\pi_k}{\sum_i \pi_i t^f_i}$ *are the* $(n,k)^{th}$ *elements of* $(A^0)^{-1}$*.*

While b^l_{ij} *are the* $(i,j)^{th}$ *elements of* B^l*, the expanded fundamental matrix when node* l *is absorbing. Then, for each* $j,k,l \in \mathcal{N}$*,*

$$b^l_{jk} = d_{jk} - d_{lk} + \sum_i b^l_{ji} t^f_i \bar{z}^0_k. \tag{EC.14}$$

Therefore, for each $i,j,l \in \mathcal{N}$*,*

$$(d_{jl} - d_{il}) = -\bar{z}^0_l \sum_k (b^l_{jk} - b^l_{ik}) t^f_k. \tag{EC.15}$$

Moreover, for each $i,j,k,l,m \in \mathcal{N}$*,*

$$\frac{(b^k_{jl} - b^k_{il})}{\pi_l} + \frac{(b^l_{jm} - b^l_{im})}{\pi_m} + \frac{(b^m_{jk} - b^m_{ik})}{\pi_k} = 0. \tag{EC.16}$$

Hence, for each $i,j,l,m \in \mathcal{N}$*,*

$$(b^m_{jl} - b^m_{il}) = -\frac{\pi_l}{\pi_m}(b^l_{jm} - b^l_{im}) \tag{EC.17}$$

Proof. From Lemma EC.2 we have $B^l(I-A) = I - E_l$. Then, we have the following system of n linearly independent equations,

$$B^l A^0 = B^l \Big[(I-A)_{-n}\big|\mathbf{t}^f\Big] = \Big[(I-E_l)_{-n}\big|B^l\mathbf{t}^f\Big],$$

and since A is the transition matrix of an ergodic Markov chain, then A^0 is invertible and we can solve for B^l as follows,

$$B^l = \Big[(I-E_l)_{-n}\big|B^l\mathbf{t}^f\Big](A^0)^{-1} \quad \Longleftrightarrow \quad b^l_{jk} = d_{jk} - d_{lk} + \sum_i b^l_{ji} t^f_i \bar{z}^0_k.$$

This completes the proof of (EC.14), which implies

$$b^k_{jl} - b^k_{il} = d_{jl} - d_{il} + \sum_m (b^k_{jm} - b^k_{im}) t^f_m \bar{z}^0_l, \tag{EC.18}$$

for any $i,j,k,l \in \mathcal{N}$. Letting $l = k$ in (EC.18), and recalling $b^l_{il} = 0$ for all i completes the proof of (EC.15). From equations (EC.15) and (EC.18) we have that, for each $i,j,k,l,m \in \mathcal{N}$,:

$$\begin{aligned}
\frac{(b^k_{jl} - b^k_{il})}{\bar{z}^0_l} &= -\sum_n (b^l_{jn} - b^l_{in}) t_n + \sum_n (b^k_{jn} - b^k_{in}) t_n \\
\frac{(b^l_{jm} - b^l_{im})}{\bar{z}^0_m} &= -\sum_n (b^m_{jn} - b^m_{in}) t_n + \sum_n (b^l_{jn} - b^l_{in}) t_n \\
\frac{(b^m_{jk} - b^m_{ik})}{\bar{z}^0_k} &= -\sum_n (b^k_{jn} - b^k_{in}) t_n + \sum_n (b^m_{jn} - b^m_{in}) t_n
\end{aligned}$$

Adding up these three equations, and recalling that $\bar{z}_k^0$ is proportional to π_k for each k, results in (EC.16). Letting $k=m$ in (EC.16) and recalling $b_{il}^l=0$ for all i gives (EC.17), completing the proof of the lemma. $\square$

Appendix C: Projected Gradients and their Properties

In this section, we define the projected gradients used in our method for dynamic relocations in car-sharing networks described in Section 5.1. We additionally prove they have interesting properties, which simplify the implementation of our method.

We first introduce some notation to have a consistent exposition with Section 5.1 and the Appendix in Section 8. For any node $l\in\mathcal{N}$, let $r_{ij}^{lm}=(b_{jm}^l-b_{im}^l)\mu_{ij}^e$ denote the coefficients of the demand constraint (5b) in Problem P^l for node $m\neq l$ and $r_{ij}^{l0}=1+\sum_k t_k^f(b_{jk}^l-b_{ik}^l)\mu_{ij}^e$ denote the coefficients of the supply constraint (5c) in Problem P^l. Recall, from Corollary 1, that $g_{ij}^l=\left(\sum_k \bar{p}_k(b_{jk}^l-b_{ik}^l)-c_{ij}\right)\mu_{ij}^e$ denotes the coefficients of the gradient in Problem P^l, $l\in\mathcal{N}$.

Similarly, let $r_{ij}^{0k}=(d_{jk}-d_{ik})\,\mu_{ij}^e-\bar{z}_k^0$ denote the coefficients of the demand constraint (11b) for node k in Problem P^0. Recall, from Corollary 2, that $g_{ij}^0=\left(\sum_k \bar{p}_k(d_{jk}-d_{ik}-\bar{z}_k^0/\mu_{ij}^e)-c_{ij}\right)\mu_{ij}^e$ denotes the coefficients of the gradient in Problem P^0.

Moreover, consider any subset $\mathcal{S}$ of the indices for P^0 and P^l, $l\in\mathcal{N}$, with at least two members, i.e., $\mathcal{S}\subseteq\mathcal{N}\cup\{0\}$, $s=|\mathcal{S}|-1\geq 1$. Then, for any index $l\in\mathcal{S}$ let $W_l(\mathcal{S}\setminus\{l\})$ denote the subspace defined by the demand (and supply) constraints indexed by $\mathcal{S}\setminus\{l\}$ in Problem P^l being tight. Specifically, let $\mathcal{S}\setminus\{l\}=\{k_1,\ldots,k_s\}$ and define

$$W_l(\mathcal{S}\setminus\{l\})=\mathrm{Nul}\begin{pmatrix} — \mathbf{r}_{\mathbf{i\cdot}}^{\mathbf{lk_1}} — \\ \vdots \\ — \mathbf{r}_{\mathbf{i\cdot}}^{\mathbf{lk_s}} — \end{pmatrix}=\mathrm{Span}\left\{\mathbf{r}_{\mathbf{i\cdot}}^{\mathbf{lk_1}},\ldots,\mathbf{r}_{\mathbf{i\cdot}}^{\mathbf{lk_s}}\right\}^{\perp}. \tag{EC.19}$$

Namely, $W_l(\mathcal{S}\setminus\{l\})$ is the orthogonal subspace to the subspace spanned by the vectors $\{\mathbf{r}_{\mathbf{i\cdot}}^{\mathbf{lk_1}},\ldots,\mathbf{r}_{\mathbf{i\cdot}}^{\mathbf{lk_s}}\}$ (Strang 1993).

Interestingly, we first show in Lemma EC.4 below that the subspace $W_l(\mathcal{S}\setminus\{l\})$ only depends on the subset of indexes $\mathcal{S}$ and it is independent of the index $l\in\mathcal{S}$. Namely, for any indexes $l,m\in\mathcal{S}$, $l\neq m$, the subspace defined by the demand (and supply) constraints indexed by $\mathcal{S}\setminus\{l\}$ in Problem P^l being tight is equal to the subspace defined by the demand (and supply) constraints indexed by $\mathcal{S}\setminus\{m\}$ in problem P^m being tight. This symmetry is an important feature of the reformulation of the fluid model for car relocations, which helps to simplify our analysis.

Lemma EC.4. *For any subset of indices $\mathcal{S}\subseteq\mathcal{N}\cup\{0\}$, $s=|\mathcal{S}|-1\geq 1$, and indices $l,m\in\mathcal{S}$, $l\neq m$, $W_l(\mathcal{S}\setminus\{l\})=W_m(\mathcal{S}\setminus\{m\})$.*

Proof. Let $\mathcal{S}\setminus\{l\}=\{k_1,\dots,k_s\}$. Assume, without loss of generality, that $m=k_1$, then

$$\begin{aligned} W_l(\mathcal{S}\setminus\{l\}) &= \text{Span}\left\{\mathbf{r_{i\cdot}^{lm}},\mathbf{r_{i\cdot}^{lk_2}},\dots,\mathbf{r_{i\cdot}^{lk_s}}\right\}^{\perp} \\ &= \text{Span}\left\{\mathbf{r_{i\cdot}^{lm}},\mathbf{r_{i\cdot}^{mk_2}},\dots,\mathbf{r_{i\cdot}^{mk_s}}\right\}^{\perp} \\ &= \text{Span}\left\{\mathbf{r_{i\cdot}^{ml}},\mathbf{r_{i\cdot}^{mk_2}},\dots,\mathbf{r_{i\cdot}^{mk_s}}\right\}^{\perp} \\ &= W_m(\mathcal{S}\setminus\{m\}), \end{aligned}$$

where the first and last equalities are by definition, c.f. (EC.19). The second equality has two cases. If $l,m\in\mathcal{N}$ then the second equality follows from (EC.16) and (EC.17), which imply $\mathbf{r_{i\cdot}^{lk_i}}=(\pi_k/\pi_m)\mathbf{r_{i\cdot}^{lm}}+\mathbf{r_{i\cdot}^{mk_i}}$, hence $\text{Span}\{\mathbf{r_{i\cdot}^{lk_i}}\}=\text{Span}\{\mathbf{r_{i\cdot}^{lm}},\mathbf{r_{i\cdot}^{mk_i}}\}$, for each $i\in\{2,\dots,s\}$. Alternatively, if without loss of generality $l=0$, then the second equality follows from (EC.15) and (EC.18), which imply $\mathbf{r_{i\cdot}^{0k_i}}=(\pi_k/\pi_m)\mathbf{r_{i\cdot}^{0m}}+\mathbf{r_{i\cdot}^{mk_i}}$, hence $\text{Span}\{\mathbf{r_{i\cdot}^{0k_i}}\}=\text{Span}\{\mathbf{r_{i\cdot}^{0m}},\mathbf{r_{i\cdot}^{mk_i}}\}$, for each $i\in\{2,\dots,s\}$. The third equality also has two cases. If $l,m\in\mathcal{N}$ then the third equality follows from (EC.17), which implies $\mathbf{r_{i\cdot}^{lm}}=-(\pi_m/\pi_l)\mathbf{r_{i\cdot}^{ml}}$, hence $\text{Span}\{\mathbf{r_{i\cdot}^{lm}}\}=\text{Span}\{\mathbf{r_{i\cdot}^{ml}}\}$. Alternatively, if without loss of generality $l=0$, then the third equality follows from (EC.15), which implies $\mathbf{r_{i\cdot}^{0m}}=-\bar{z}_m^0\mathbf{r_{i\cdot}^{m0}}$, hence $\text{Span}\{\mathbf{r_{i\cdot}^{0m}}\}=\text{Span}\{\mathbf{r_{i\cdot}^{m0}}\}$, completing the proof. ☐

We are now ready to define the projected gradients we use in our method for dynamic relocations, which apply standard machinery in orthogonal projections, see Strang (1993).

Definition EC.1. *Let* $(\mathbf{g_{i\cdot}^{l}})_{W_l(\mathcal{S}\setminus\{l\})}$ *denote the gradient of relocations out of node* i *in Problem* P^l, $l\in\mathcal{S}\subseteq\mathcal{N}\cup\{0\}$, *projected on* $W_l(\mathcal{S}\setminus\{l\})$, *the subspace defined by the demand (and supply) constraints indexed by* $\mathcal{S}\setminus\{l\}$ *in Problem* P^l *being tight. Moreover, let* $J_l(\mathcal{S}\setminus\{l\})$ *denote a matrix such that its columns form a basis for the subspace* $W_l(\mathcal{S}\setminus\{l\})$. *Then,*

$$(\mathbf{g_{i\cdot}^{l}})_{W_l(\mathcal{S}\setminus\{l\})}=J_l(\mathcal{S}\setminus\{l\})\left(J_l(\mathcal{S}\setminus\{l\})^TJ_l(\mathcal{S}\setminus\{l\})\right)^{-1}J_l(\mathcal{S}\setminus\{l\})^T\mathbf{g_{i\cdot}^{l}}. \tag{EC.20}$$

In order to provide further intuition, we note that when considering only two nodes (analogously one node and the index 0 for P^0 and the associated supply constraint), i.e., if $\mathcal{S}=\{l,m\}\subseteq\mathcal{N}\cup\{0\}$, then (EC.20) simplifies to,

$$(\mathbf{g_{i\cdot}^{l}})_{W(l,\{l,m\})}=\mathbf{g_{i\cdot}^{l}}-\left(\frac{\mathbf{g_{i\cdot}^{l}}\cdot\mathbf{r_{i\cdot}^{lm}}}{\mathbf{r_{i\cdot}^{lm}}\cdot\mathbf{r_{i\cdot}^{lm}}}\right)\mathbf{r_{i\cdot}^{lm}}.$$

Namely, $(\mathbf{g_{i\cdot}^{l}})_{W(l,\{l,m\})}$ is the projection of $\mathbf{g_{i\cdot}^{l}}$ onto the hyperplane defined by the normal $\mathbf{r_{i\cdot}^{lm}}$.

Proposition EC.3 below shows the main result in this section. Namely, that the projection of $\mathbf{g_{i\cdot}^{l}}$ –the gradient for relocations out of node i in Problem P^l– is equivalent to the projection of $\mathbf{g_{i\cdot}^{m}}$ –the gradient for relocations out of node i in problem P^m– onto $W_l(\mathcal{S}\setminus\{l\})=W_m(\mathcal{S}\setminus\{m\})$ –the subspace defined by the demand (and supply) constraints indexed by $\mathcal{S}\setminus\{l\}$ in Problem P^l being tight (or equivalently from Lemma EC.4, the constraints indexed by $\mathcal{S}\setminus\{m\}$ in problem P^m being tight)–. In order to prove it, we introduce the following notation.

Recall that $\bar{p}_i=\sum_j \alpha_{ij}p_{ij}$ is the expected revenue per match at location i. Analogous to the definition A^0 in Proposition 2 we now define $A^p=\left[(I-A)_{-n}\middle|\bar{\mathbf{p}}\right]$, $h_{ni}=h_{nj}=0$ for all i, j, the coefficients h_{ij}, for $i\neq n$, are the $(i,j)^{\text{th}}$ elements of $(A^p)^{-1}$, and $\bar{z}^p_k=\frac{\pi_k}{\sum_i \pi_i\bar{p}_i}$ are the $(n,k)^{\text{th}}$ elements of $(A^p)^{-1}$. Importantly, note that by substituting the coefficients d_{ij} by h_{ij}, $\bar{z}^0_i$ by $\bar{z}^p_i$, and t^f_i by $\bar{p}_i$, all the results in Lemma EC.3 apply (following exactly the same proof). The proof of Proposition EC.3 below relies heavily on these results.

Proposition EC.3. *For any subset of indices $\mathcal{S}\subseteq\mathcal{N}\cup\{0\}$, $s=|\mathcal{S}|-1\geq 1$, and indices $l,m\in\mathcal{S}$, $l\neq m$, $(\mathbf{g}^{\mathbf{l}}_{\mathbf{i}\cdot})_{W_l(\mathcal{S}\setminus\{l\})}=(\mathbf{g}^{\mathbf{m}}_{\mathbf{i}\cdot})_{W_m(\mathcal{S}\setminus\{m\})}$.*

Proof. It is sufficient to show that for any nodes $l,m\in\mathcal{N}$, $l\neq m$, it must be the case that

$$\mathbf{g}^{\mathbf{l}}_{\mathbf{i}\cdot}-\mathbf{g}^{\mathbf{m}}_{\mathbf{i}\cdot}=\frac{\mathbf{r}^{lm}_{i\cdot}}{\bar{z}^p_m}, \tag{EC.21}$$

$$\mathbf{g}^{\mathbf{l}}_{\mathbf{i}\cdot}-\mathbf{g}^{\mathbf{0}}_{\mathbf{i}\cdot}=\mathbf{r}^{\mathbf{l0}}_{\mathbf{i}\cdot}. \tag{EC.22}$$

Indeed, (EC.21) and (EC.22) imply that for any $l,m\in\mathcal{S}\subseteq\mathcal{N}\cup\{0\}$, $l\neq m$, $\mathbf{g}^{\mathbf{l}}_{\mathbf{i}\cdot}-\mathbf{g}^{\mathbf{m}}_{\mathbf{i}\cdot}\in\operatorname{Span}\{\mathbf{r}^{\mathbf{lm}}_{\mathbf{i}\cdot}\}$. Hence, letting $\mathcal{S}\setminus\{l\}=\{k_1,\ldots,k_s\}$ and noting that $m=k_i$ for some $i\in\{1\ldots,s\}$, it follows that $\mathbf{g}^{\mathbf{l}}_{\mathbf{i}\cdot}-\mathbf{g}^{\mathbf{m}}_{\mathbf{i}\cdot}\in\operatorname{Span}\{\mathbf{r}^{\mathbf{lk_1}}_{\mathbf{i}\cdot},\ldots,\mathbf{r}^{\mathbf{lk_s}}_{\mathbf{i}\cdot}\}$. Since by its definition in (EC.19) $W_l(\mathcal{S}\setminus\{l\})=\operatorname{Span}\{\mathbf{r}^{\mathbf{lk_1}}_{\mathbf{i}\cdot},\ldots,\mathbf{r}^{\mathbf{lk_s}}_{\mathbf{i}\cdot}\}^{\perp}$ then we conclude

$$\left(\mathbf{g}^{\mathbf{l}}_{\mathbf{i}\cdot}-\mathbf{g}^{\mathbf{m}}_{\mathbf{i}\cdot}\right)_{W_l(\mathcal{S}\setminus\{l\})}=0\iff\left(\mathbf{g}^{\mathbf{l}}_{\mathbf{i}\cdot}\right)_{W_l(\mathcal{S}\setminus\{l\})}=\left(\mathbf{g}^{\mathbf{m}}_{\mathbf{i}\cdot}\right)_{W_l(\mathcal{S}\setminus\{l\})}\iff\left(\mathbf{g}^{\mathbf{l}}_{\mathbf{i}\cdot}\right)_{W_l(\mathcal{S}\setminus\{l\})}=\left(\mathbf{g}^{\mathbf{m}}_{\mathbf{i}\cdot}\right)_{W_m(\mathcal{S}\setminus\{m\})},$$

where the first equivalence follows from the linearity of orthogonal projections, and the second equivalence follows from Lemma EC.4.

We now prove (EC.21). Note that

$$\begin{aligned}\frac{g^l_{ij}-g^m_{ij}}{\mu^e_{ij}}&=\sum_k\bar{p}_k(b^l_{jk}-b^l_{ik})-\sum_k\bar{p}_k(b^m_{jk}-b^m_{ik})\\&=\frac{b^l_{jm}-b^l_{im}}{\bar{z}^p_m}-\frac{h_{jm}-h_{im}}{\bar{z}^p_m}-\frac{b^m_{jl}-b^m_{il}}{\bar{z}^p_l}+\frac{h_{jl}-h_{il}}{\bar{z}^p_l}\\&=\frac{b^l_{jm}-b^l_{im}}{\bar{z}^p_m}\\&=\frac{r^{lm}_{ij}}{\bar{z}^p_m\mu^e_{ij}},\end{aligned}$$

where the first and last equality are by definition. The second equality follows from (EC.18) – substituting d_{ij} by h_{ij}, $\bar{z}^0_i$ by $\bar{z}^p_i$, and t^f_i by $\bar{p}_i$. The third equality follows from (EC.15) and (EC.18) –substituting d_{ij} by h_{ij}, $\bar{z}^0_i$ by $\bar{z}^p_i$, and t^f_i by $\bar{p}_i$– which combined show $\frac{h_{jl}-h_{il}}{\bar{z}^p_l}-\frac{h_{jm}-h_{im}}{\bar{z}^p_m}-\frac{b^m_{jl}-b^m_{il}}{\bar{z}^p_l}=0$, completing the proof of (EC.21).

We now prove (EC.22). Note that

$$\begin{aligned}\frac{g_{ij}^{l}-g_{ij}^{0}}{\mu_{ij}^{e}} &= \sum_{k}\bar{p}_k(b_{jk}^{l}-b_{ik}^{l})-\sum_{k}\bar{p}_k\left(d_{jk}-d_{ik}-\frac{\bar{z}_k^0}{\mu_{ij}^{e}}\right)\\ &= \sum_{k}\bar{p}_k\left(\sum_{m}t_m^{f}\bar{z}_k^{0}(b_{jm}^{l}-b_{im}^{l})\right)+\frac{1}{\mu_{ij}^{e}}\\ &= \sum_{m}t_m^{f}(b_{jm}^{l}-b_{im}^{l})+\frac{1}{\mu_{ij}^{e}}\\ &= \frac{r_{ij}^{l0}}{\mu_{ij}^{e}},\end{aligned}$$

where the first and last equality are by definition. The second equality follows from (EC.18) and $\sum_k \bar{p}_k\bar{z}_k^0 = 1$. The third equality follows from $\sum_k \bar{p}_k\bar{z}_k^0 = 1$, completing the proof. □

Appendix D: Asymptotic Optimality of the DG Policy in Networks with Two Nodes

In this section, we prove the asymptotic optimality of the DG policy in networks with $n = 2$ nodes. In networks with two nodes, the state-dependent routing induced by the DG policy is continuous, which allows building on Mandelbaum and Pats (1998) to extend the approach from Braverman et al. (2019) to state-dependent stochastic networks. Since we follow Braverman et al. (2019), we focus on maximizing availability and assume costless relocations.

Analyzing the asymptotic optimality of the DG policy for networks with $n \geq 3$ nodes would require generalizing the approach from Braverman et al. (2019) to large networks with multiple discontinuous state-dependent routing and is beyond the scope of this paper. Motivated by service systems with customer returns and service speedup, Chan et al. (2014) use Filippov analysis (Filippov 1988) to analyze the multiple possible cases of the fluid limit of a queueing network with two stations and one state-dependent discontinuity in both network routing and the service rate of one of the queues. In contrast, a car-sharing network with $n = 3$ nodes is modeled by 18 queues (15 infinite server queues for customer trips and relocations, and 3 single server queues for idle cars at each node), with up to 9 network routing discontinuities, making the approach from Chan et al. (2014) intractable for large car-sharing networks.

D.1. DG Policy for Car-Sharing Networks with Two Nodes

Consider a car-sharing network with two nodes denoted by i and j. Then, the DG policy is fully characterized by the following continuous network routing:

$$Q_{kk}^{\mathrm{DG}}(E(t),F(t)) = \left(\left(\frac{\kappa z_k^{\mathrm{out}}-\mu_{lk}^{e}E_{lk}(t)-E_{kk}(t)}{\sum_{m\in\{i,j\}}\mu_{mk}^{f}F_{mk}(t)}\right)^{+}\wedge 1\right), \quad k,l\in\{i,j\}, k\neq l, \tag{EC.23}$$

$$z_k^{\text{out}} = \begin{cases} \sum_{m\in\{i,j\}} \mu_{km}^f \hat{f}_{km}^* & \text{if } \hat{e}_{kl}^* > 0, \\ \hat{\lambda}_k & \text{if } \hat{e}_{kl}^* = \hat{e}_{lk}^* = 0, \\ (1+s_k)\hat{\lambda}_k & \text{if } \hat{e}_{lk}^* > 0. \end{cases} \quad s_k = \frac{\hat{e}_{lk}^*}{\hat{e}_{lk}^* + \hat{e}_{kl}^*} \quad k,l \in \{i,j\},\ k \neq l \tag{EC.24}$$

$$Q_{kl}^{\text{DG}}(E(t),F(t)) = 1 - Q_{kk}^{\text{DG}}(E(t),F(t)), \quad k,l\in\{i,j\}, k\neq l. \tag{EC.25}$$

as specified in Algorithm 1 in Section 5.1, where we explicitly consider that there are two nodes in the network (i and j).

D.2. The Fluid Model Induced by the DG Policy and its Equilibrium Point

In this section, we specify the fluid model induced by the DG policy. For conciseness, we largely follow the exposition of Section 4 in Braverman et al. (2019), specialized to car-sharing networks with two nodes i and j. We focus on the main points where the analysis differs. Specifically, consider the following extension of the fluid model Braverman et al. (2019) in this special case.

$$f_{kl}(t) = f_{kl}(0) + \lambda_k P_{kl}(t - u_k(t)) - \mu_{kl}^f \int_0^t f_{kl}(s)ds, \qquad k,l\in\{i,j\} \tag{EC.26}$$

$$e_{kl}(t) = e_{kl}(0) - \mu_{kl}^e \int_0^t e_{kl}(s)ds + Q_{kl}^{\text{DG}}(\kappa \boldsymbol{e}(t), \kappa \boldsymbol{f}(t)) \sum_{m\in\{i,j\}} \mu_{mk}^f \int_0^t f_{mk}(s)ds, \quad k,l\in\{i,j\},\ k\neq l, \tag{EC.27}$$

$$e_{kk}(t) = e_{kk}(0) - \lambda_k(t - u_k(t)) + \mu_{lk}^e \int_0^t e_{lk}(s)ds + Q_{kk}^{\text{DG}}(\kappa \boldsymbol{e}(t), \kappa \boldsymbol{f}(t)) \sum_{m\in\{i,j\}} \mu_{mk}^f \int_0^t f_{mk}(s)ds, \quad k,l\in\{i,j\},\ k\neq l, \tag{EC.28}$$

$$u_k(t) \text{ is non-decreasing with } u_k(0) = 0 \text{ and } \int_0^\infty e_{kk}(s)du_k(s) = 0, \qquad k\in\{i,j\}. \tag{EC.29}$$

We focus on car availability maximization. Let $\bar{a}_k = \lim_{t\to\infty} 1 - \dot{u}_k(t)$ denote the car availability at node $k\in\{i,j\}$ in equilibrium.

Proposition EC.4. *Any equilibrium* $(\bar{a}, \bar{e}, \bar{f})$ *of the fluid model* (EC.26)-(EC.29) *must satisfy*

$$\sum_{m\in\{i,j\}} \mu_{mk}^f f_{mk}^* \leq \sum_{m\in\{i,j\}} \mu_{mk}^f \bar{f}_{mk}, \quad k\in\{i,j\}.$$

Hence, any equilibrium attained by the DG policy in the LP fluid model is asymptotically optimal as $\kappa\to\infty$ *for the dynamic car relocation problem in networks with two nodes.*

Proof. The process-level convergence as $\kappa \to \infty$ of the car relocation problem dynamics induced by the DG policy in networks with two nodes to the fluid model (EC.26)-(EC.29) follows from the analysis in Braverman et al. (2019). Therefore, we focus here on its steady-state convergence.

Since the state-dependent network routing induced by the DG policy is continuous in car-sharing networks with two nodes, we can take derivates in (EC.26)-(EC.28) to get,

$$\dot{f}_{kl}(t) = \lambda_k P_{kl}(1 - \dot{u}_k(t)) - \mu^f_{kl} f_{kl}(t), \qquad k,l \in \{i,j\}, \tag{EC.30}$$

$$\begin{aligned}\dot{e}_{kl}(t) = &- \mu^e_{kl} e_{kl}(t) + Q^{\mathrm{DG}}_{kl}(\kappa \boldsymbol{e}(t), \kappa \boldsymbol{f}(t)) \sum_{m\in\{i,j\}} \mu^f_{mk} f_{mk}(t) \\ &+ \dot{Q}^{\mathrm{DG}}_{kl}(\kappa \boldsymbol{e}(t), \kappa \boldsymbol{f}(t)) \sum_{m\in\{i,j\}} \mu^f_{mk} \int_0^t f_{mk}(s) ds, \qquad k,l \in \{i,j\},\ k \neq l,\end{aligned} \tag{EC.31}$$

$$\begin{aligned}\dot{e}_{kk}(t) = &- \lambda_k(1 - \dot{u}_k(t)) + \mu^e_{lk} e_{lk}(t) + Q^{\mathrm{DG}}_{kk}(\kappa \boldsymbol{e}(t), \kappa \boldsymbol{f}(t)) \sum_{m\in\{i,j\}} \mu^f_{mk} f_{mk}(t) \\ &+ \dot{Q}^{\mathrm{DG}}_{kk}(\kappa \boldsymbol{e}(t), \kappa \boldsymbol{f}(t)) \sum_{m\in\{i,j\}} \mu^f_{mk} \int_0^t f_{mk}(s) ds, \qquad k,l \in \{i,j\},\ k \neq l,\end{aligned} \tag{EC.32}$$

while from (EC.29) we get,

$$e_{kk}(t)\dot{u}_k(t) = 0 \qquad k \in \{i,j\}. \tag{EC.33}$$

By imposing $(\dot{e}(t), \dot{f}(t))$ and $\dot{Q}^{\mathrm{DG}}$ to be equal to zero we conclude that any equilibrium $(\bar{a}, \bar{e}, \bar{f})$ of the fluid model (EC.26)-(EC.29), if it exists, must satisfy,

$$\lambda_k P_{kl} \bar{a}_k = \mu^f_{kl} \bar{f}_{kl}, \qquad k,l \in \{i,j\}, \tag{EC.34}$$

$$\mu^e_{kl} \bar{e}_{kl} = Q^{\mathrm{DG}}_{kl}(\kappa \bar{e}, \kappa \bar{f}) \sum_{m\in\{i,j\}} \mu^f_{mk} \bar{f}_{mk}, \qquad k,l \in \{i,j\},\ k \neq l, \tag{EC.35}$$

$$\lambda_k \bar{a}_k = \mu^e_{lk} \bar{e}_{lk} + Q^{\mathrm{DG}}_{kk}(\kappa \bar{e}, \kappa \bar{f}) \sum_{m\in\{i,j\}} \mu^f_{mk} \bar{f}_{mk}, \qquad k \in \{i,j\}, \tag{EC.36}$$

$$\bar{e}_{kk}(1 - \bar{a}_k) = 0, \qquad k \in \{i,j\}. \tag{EC.37}$$

Hence, we conclude

$$Q^{\mathrm{DG}}_{kk}(\kappa \bar{e}, \kappa \bar{f}) = \left(\left(\frac{z^{\mathrm{out}}_k - \mu^e_{lk} \bar{e}_{lk} - \bar{e}_{kk}}{\sum_{m\in\{i,j\}} \mu^f_{mk} \bar{f}_{mk}} \right)^+ \wedge 1 \right) = \frac{\sum_{m\in\{i,j\}} \mu^f_{km} \bar{f}_{km} - \mu^e_{lk} \bar{e}_{lk}}{\sum_{m\in\{i,j\}} \mu^f_{mk} \bar{f}_{mk}}, \quad k,l \in \{i,j\}, k \neq l \tag{EC.38}$$

where the first equality follows from (EC.23), and the second equality follows from (EC.34) and (EC.36). Therefore, we get

$$\sum_{m\in\{i,j\}} \mu^f_{mk} f^*_{mk} \le z^{\text{out}}_k \le \sum_{m\in\{i,j\}} \mu^f_{mk} \bar{f}_{mk} + \bar{e}_{kk}, \quad k\in\{i,j\}, \tag{EC.39}$$

where the first inequality follows from (EC.24) and the second inequality follows from (EC.38).

The result in the proposition then follows from (EC.37) and (EC.39). Indeed, if $\bar{e}_{kk}=0$ in (EC.39) we are done. Alternatively, if $\bar{e}_{kk}>0$ then from (EC.37) we have $\bar{a}_k=1$ or equivalently $\sum_{m\in\{i,j\}} \mu^f_{mk}\bar{f}_{mk}=\lambda_k \ge \sum_{m\in\{i,j\}} \mu^f_{mk} f^*_{mk}$, concluding the proof.

Appendix E: Additional Numerical Results, Plots and Tables

E.1. Steady-State Performance for the Greedy and JLCR-0.5 Policies

Percentile	100	500	1000	1500	2000	2500	3000	3500	4000	4500	5000
95%	14.48%	1.2%	-7.77%	-15.56%	-20.24%	-25.82%	-28.69%	-33.4%	-37.04%	-41.03%	-40.59%
75%	8.73%	-10.76%	-28.35%	-41.97%	-50.51%	-61.68%	-72.53%	-75.38%	-81.66%	-87.69%	-96.85%
50%	3.29%	-30.54%	-60.96%	-85.49%	-105.46%	-124.53%	-138.93%	-151.43%	-158.5%	-174.75%	-178.5%
25%	-5.4%	-65.76%	-108.97%	-140.45%	-165.24%	-199.24%	-226.03%	-250.81%	-262.57%	-282.07%	-293.52%
5%	-30.1%	-144.75%	-226.85%	-295.35%	-334.09%	-392.11%	-426.71%	-442.96%	-493.81%	-521.57%	-505.33%
Average	-1.03%	-45.31%	-82.58%	-110.49%	-131.06%	-155.33%	-174.37%	-187.34%	-199.91%	-214.15%	-220.21%
95 % Conf. Interval	[-4.08%, 2.03%]	[-55.08%, -35.55%]	[-98.15%, -67.00%]	[-129.15%, -91.83%]	[-152.03%, -110.09%]	[-180.53%, -130.13%]	[-202.79%, -145.96%]	[-215.96%, -158.72%]	[-231.05%, -168.77%]	[-247.76%, -180.53%]	[-251.92%, -188.50%]

Table EC.1 Percentiles of availability gap closed by the Greedy policy in steady-state for fleet sizes $\kappa\in\{100,\dots,5000\}$.

Percentile	100	500	1000	1500	2000	2500	3000	3500	4000	4500	5000
95%	-3.38%	-5.89%	-10.16%	-16.66%	-19.87%	-27.05%	-28.98%	-37.64%	-31.64%	-40.18%	-45.11%
75%	-12.55%	-16.43%	-24.3%	-35.55%	-37.89%	-45.88%	-51.07%	-56.49%	-59.3%	-64.78%	-68.05%
50%	-22.18%	-31.15%	-41.74%	-51.19%	-61.1%	-70.54%	-78.77%	-85.1%	-89.78%	-96.52%	-106.39%
25%	-37.6%	-50.75%	-65.82%	-73.11%	-78.88%	-92.85%	-103.13%	-115.33%	-124.23%	-130.01%	-137.01%
5%	-54.07%	-94.21%	-144.24%	-188.32%	-208.29%	-242.02%	-306.33%	-287.72%	-320.28%	-343.22%	-332.2%
Average	-25.69%	-38.24%	-52.8%	-64.7%	-74.07%	-86.87%	-98.09%	-105.02%	-110.95%	-118.55%	-123.03%
95% Conf. Interval	[-29.20%, -22.17%]	[-44.42%, -32.06%]	[-61.85%, -43.74%]	[-75.02%, -54.38%]	[-85.68%, -62.46%]	[-100.40%, -73.34%]	[-113.97%, -82.20%]	[-121.12%, -88.91%]	[-127.49%, -94.40%]	[-135.87%, -101.24%]	[-140.60%, -105.46%]

Table EC.2 Percentiles of availability gap closed by the JLCR-0.5 policy in steady-state for fleet sizes $\kappa\in\{100,\dots,5000\}$.

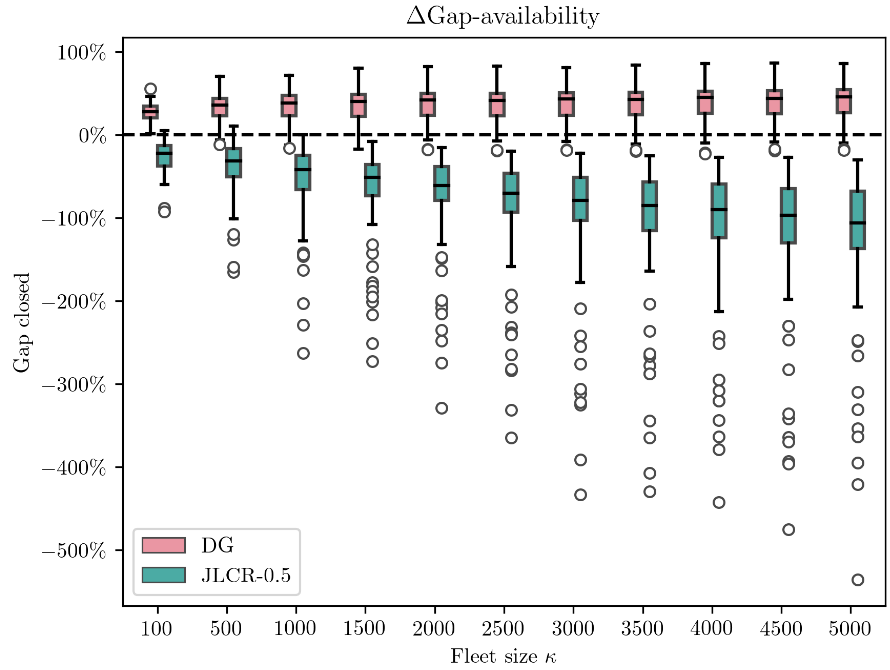

Figure EC.1 Boxplots of availability gap closed by DG and JLCR-0.5 policies in steady-state.

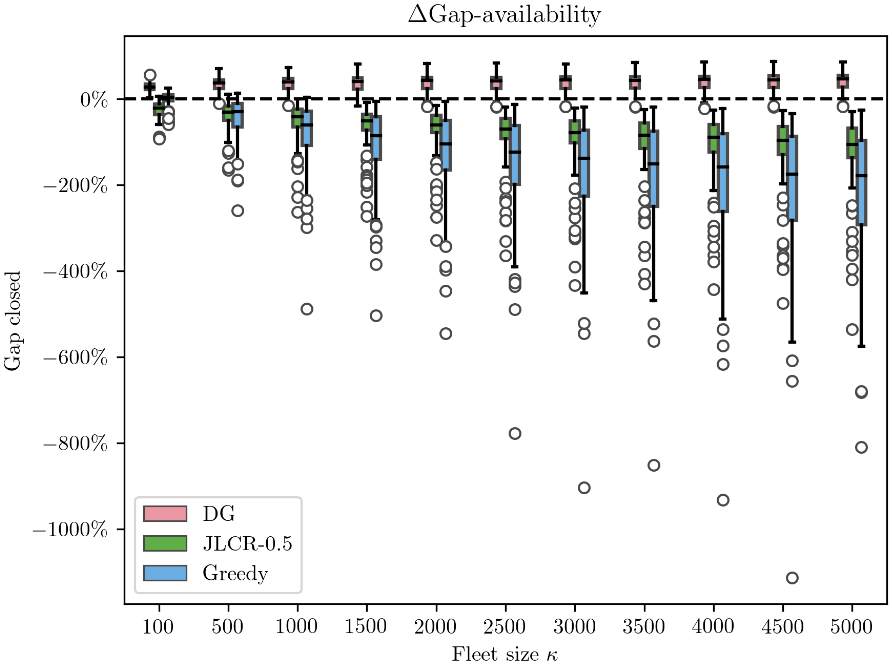

Figure EC.2 Boxplots of availability gap closed by DG, JLCR-0.5, and Greedy policies in steady-state.

E.2. Steady-State Performance of the DG over the Static Policy in Percentage Improvement

In this section, we make the results from Section 6.3, on the steady-state performance of the DG policy with stationary parameters, directly comparable to the results in Sections 6.4 and 6.5, on the short-term performance of the DG policy with time-varying parameters.

Namely, we use a different metric to report the results from Section 6.3. Specifically, in Section 6.3, we reported a lower bound on the static policy's optimality gap closed by the DG policy, i.e.,

$$\Delta \text{Gap} = \frac{\Pi^{\text{DG}} - \Pi^{\text{Static}}}{P^* - \Pi^{\text{Static}}} \times 100\%.$$

In contrast, below, we report the *percentage improvement* of the DG policy over the Static policy, i.e.,

$$\%\text{Imp} = \frac{\Pi^{\text{DG}} - \Pi^{\text{Static}}}{\Pi^{\text{Static}}} \times 100\%.$$

Recall that the latter is the metric we use in Sections 6.4 and 6.5, because there is no known tight upper bound on the performance of the optimal policy in a short-term setup with time-varying parameters.

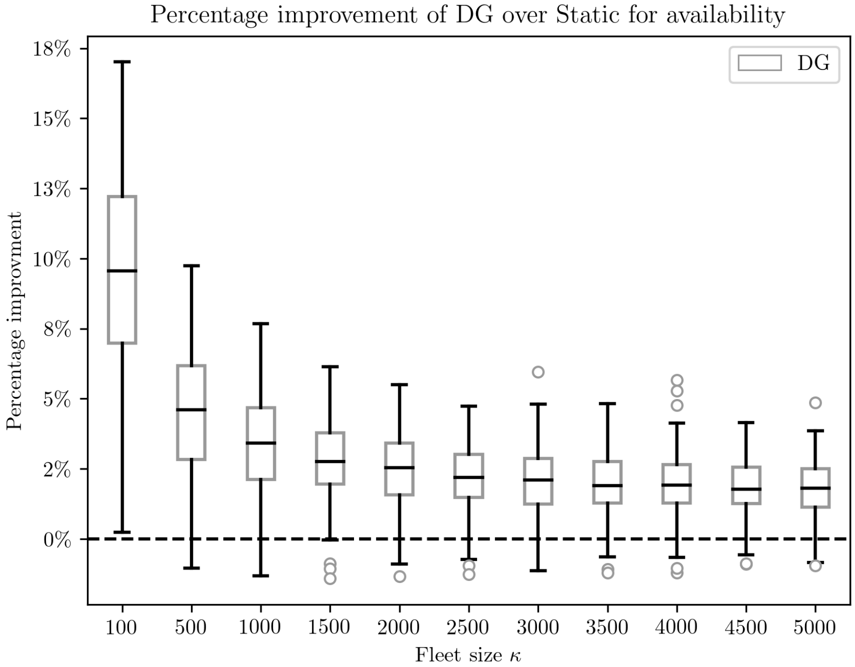

Figure EC.3 Boxplot of availability percentage improvement of DG over Static in steady-state.

Percentile	100	500	1000	1500	2000	2500	3000	3500	4000	4500	5000
95%	14.38%	7.84%	6.08%	4.94%	4.56%	4.13%	3.87%	3.69%	3.89%	3.41%	3.44%
75%	12.21%	6.18%	4.68%	3.79%	3.41%	3.02%	2.87%	2.77%	2.65%	2.56%	2.5%
50%	9.56%	4.61%	3.41%	2.77%	2.54%	2.2%	2.1%	1.9%	1.92%	1.77%	1.82%
25%	6.99%	2.84%	2.12%	1.96%	1.58%	1.49%	1.25%	1.29%	1.28%	1.25%	1.13%
5%	2.48%	0.77%	0.3%	0.15%	0.09%	0.01%	-0.01%	0%	-0.07%	0.03%	-0.01%
Average	9.28%	4.54%	3.31%	2.72%	2.47%	2.17%	2.04%	1.92%	1.9%	1.79%	1.78%
95% Conf. Interval	[8.54%, 10.03%]	[4.08%, 4.99%]	[2.95%, 3.67%]	[2.41%, 3.03%]	[2.19%, 2.75%]	[1.92%, 2.43%]	[1.79%, 2.29%]	[1.69%, 2.15%]	[1.66%, 2.14%]	[1.58%, 2.00%]	[1.57%, 1.99%]

Table EC.3 Percentiles of availability percentage improvement of DG over Static in steady-state for fleet sizes $\kappa \in \{100, \ldots, 5000\}$.

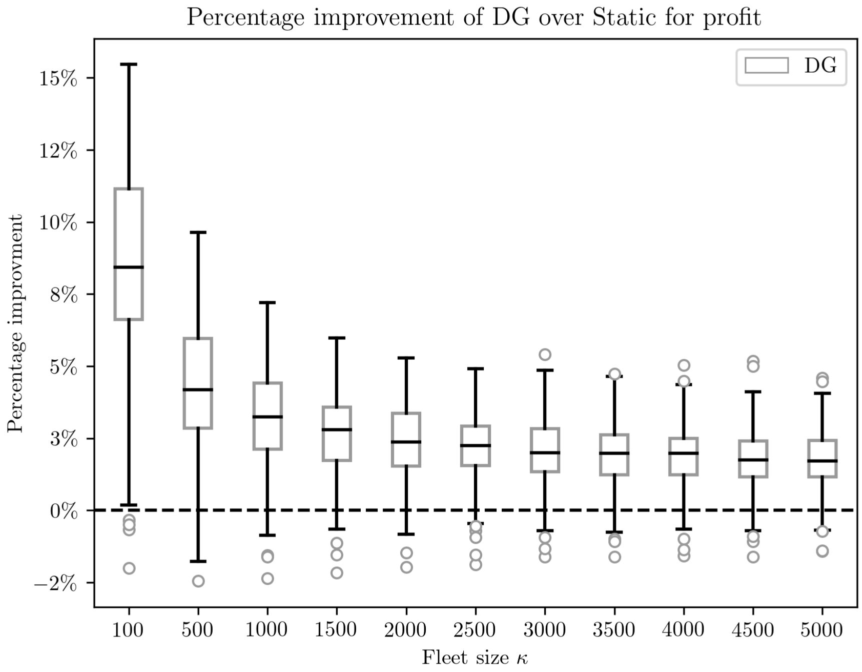

Figure EC.4 Boxplot of profit percentage improvement of DG over Static in steady-state.

Percentile	100	500	1000	1500	2000	2500	3000	3500	4000	4500	5000
95%	13.43%	7.57%	5.78%	4.93%	4.54%	3.99%	3.85%	3.66%	3.66%	3.48%	3.51%
75%	11.15%	5.96%	4.41%	3.58%	3.36%	2.92%	2.84%	2.62%	2.49%	2.41%	2.43%
50%	8.43%	4.17%	3.25%	2.79%	2.36%	2.25%	2%	1.98%	1.97%	1.74%	1.7%
25%	6.61%	2.85%	2.12%	1.73%	1.53%	1.55%	1.34%	1.24%	1.23%	1.15%	1.17%
5%	0.15%	-0.17%	-0.36%	-0.45%	-0.41%	-0.47%	-0.33%	-0.29%	-0.31%	-0.33%	-0.25%
Average	8.11%	4.14%	3.09%	2.61%	2.34%	2.12%	2.01%	1.87%	1.82%	1.75%	1.72%
95% Conf. Interval	[7.30%, 8.92%]	[3.64%, 4.64%]	[2.71%, 3.47%]	[2.28%, 2.94%]	[2.04%, 2.64%]	[1.85%, 2.39%]	[1.74%, 2.27%]	[1.63%, 2.11%]	[1.58%, 2.06%]	[1.51%, 1.98%]	[1.49%, 1.95%]

Table EC.4 Percentiles of profit percentage improvement of DG over static in steady-state for fleet sizes $\kappa \in \{100, \ldots, 5000\}$.

E.3. Time-varying Results for Lookahead Parameter $T = 0.75$

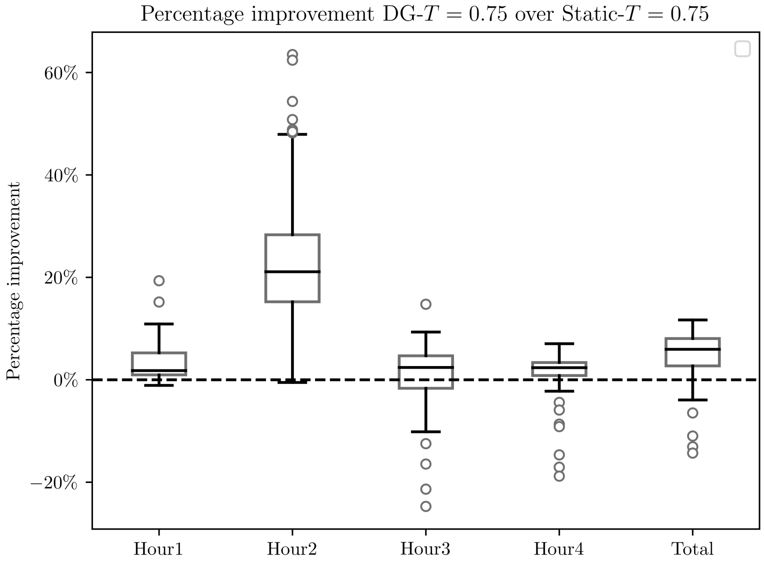

Figure EC.5 Boxplot of percentage improvement of DG-0.75 over static-0.75 for $\kappa = 2000$ in the time-varying setting.

Percentile	Hour1	Hour2	Hour3	Hour4	Total
95%	9.65%	48.47%	8.53%	4.58%	10.95%
75%	5.24%	28.31%	4.69%	3.37%	8.06%
50%	1.81%	21.11%	2.38%	2.34%	5.96%
25%	0.95%	15.21%	-1.69%	0.81%	2.7%
5%	0.14%	5.99%	-9.23%	-6.08%	-3.86%
Average	3.44%	23.11%	1%	1.32%	4.89%
95 % Conf. Interval	[2.73%, 4.15%]	[20.53%, 25.69%]	[-0.23%, 2.22%]	[0.52%, 2.13%]	[3.91%, 5.86%]

Table EC.5 Percentiles of percentage improvement of DG-0.75 over Static-0.75 for $\kappa = 2000$ in the time-varying setting.

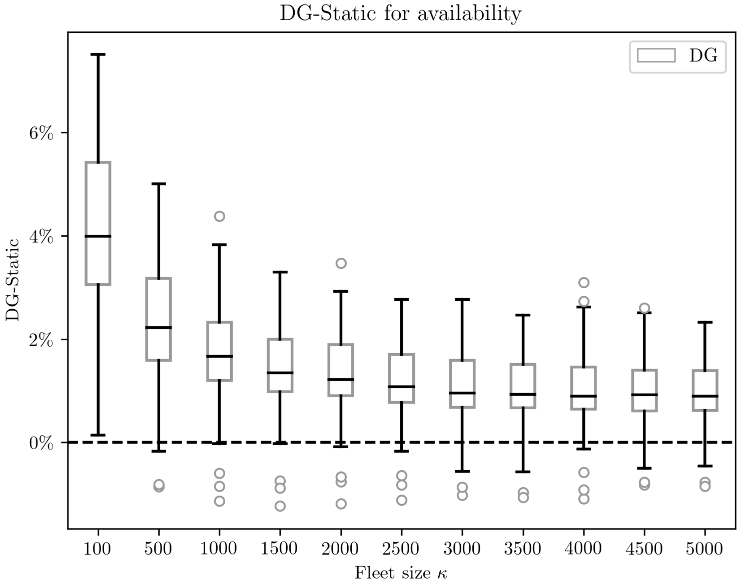

Figure EC.6 Boxplot of availability absolute improvement of DG over Static in steady-state

E.4. Absolute Improvement in Availability by DG Policy Compared to Static

In this section, we use absolute improvement $(\Pi^{\mathrm{DG-T}} - \Pi^{\mathrm{Static-T}}) \times 100\%$ to evaluate the performance of the DG policy. We do so for the availability objective only, since it is scale-free, as opposed to the profit objective.

Specifically, Figure EC.6 and Table EC.6 in this section use absolute improvement to illustrate the same results Figure EC.3 and Table EC.3 in Section E.2 show for percentage improvement.

Similarly, Figure EC.7 and Table EC.7 in this section use absolute improvement to illustrate the same results Figure 2 and Table 4 in Section 6.4 show for percentage improvement.

Finally, Figure EC.8 and Table EC.8 in this section use absolute improvement to illustrate the same results Figure EC.5 and Table EC.5 in Section E.3 show for percentage improvement.

Percentile	100	500	1000	1500	2000	2500	3000	3500	4000	4500	5000
95%	6.76%	4.14%	3.41%	2.87%	2.56%	2.35%	2.22%	2.09%	2.19%	1.92%	2.02%
75%	5.42%	3.18%	2.33%	1.99%	1.89%	1.7%	1.6%	1.52%	1.46%	1.4%	1.39%
50%	3.99%	2.23%	1.67%	1.35%	1.22%	1.08%	0.96%	0.93%	0.9%	0.92%	0.9%
25%	3.05%	1.59%	1.2%	0.99%	0.9%	0.78%	0.68%	0.67%	0.65%	0.61%	0.62%
5%	1.66%	0.63%	0.25%	0.1%	0.05%	0.01%	-0.01%	0%	-0.07%	0.03%	-0.01%
Average	4.14%	2.31%	1.73%	1.43%	1.32%	1.16%	1.1%	1.03%	1.03%	0.97%	0.97%
95% Conf.	[3.81%,	[2.07%,	[1.53%,	[1.26%,	[1.15%,	[1.01%,	[0.95%,	[0.89%,	[0.89%,	[0.85%,	[0.84%,
Interval	4.47%]	2.54%]	1.93%]	1.61%]	1.48%]	1.31%]	1.24%]	1.17%]	1.18%]	1.10%]	1.09%]

Table EC.6 Percentiles of availability DG-Static in steady-state for fleet sizes $\kappa \in \{100, \ldots, 5000\}$.

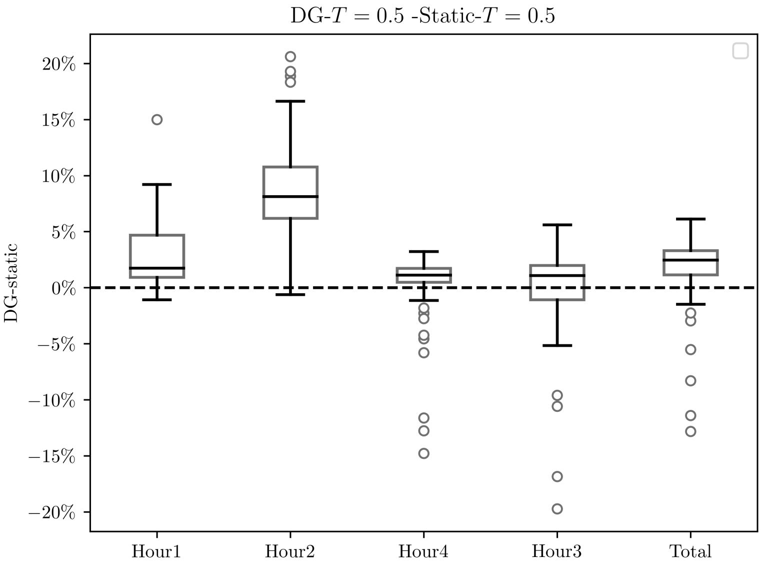

Figure EC.7 Boxplot of availability absolute improvement of DG $T = 0.5$ lookahead policy over static $T = 0.5$ lookahead policy for $\kappa = 2000$ in the time-varying setting.

Percentile	Hour1	Hour2	Hour3	Hour4	Total
95%	7.92%	16.24%	3.77%	2.71%	4.28%
75%	4.69%	10.77%	1.98%	1.74%	3.3%
50%	1.76%	8.14%	1.08%	1.13%	2.48%
25%	0.92%	6.19%	-1.07%	0.48%	1.15%
5%	0.14%	2.86%	-5%	-4.26%	-2.29%
Average	2.92%	8.69%	0.08%	0.51%	1.7%
95% Conf. Interval	[2.37%, 3.47%]	[7.85%, 9.53%]	[-0.67%, 0.83%]	[-0.05%, 1.07%]	[1.13%, 2.27%]

Table EC.7 Percentiles of availability absolute improvement of DG $T = 0.5$ lookahead policy over static $T = 0.5$ lookahead policy for $\kappa = 2000$ in the time-varying setting.

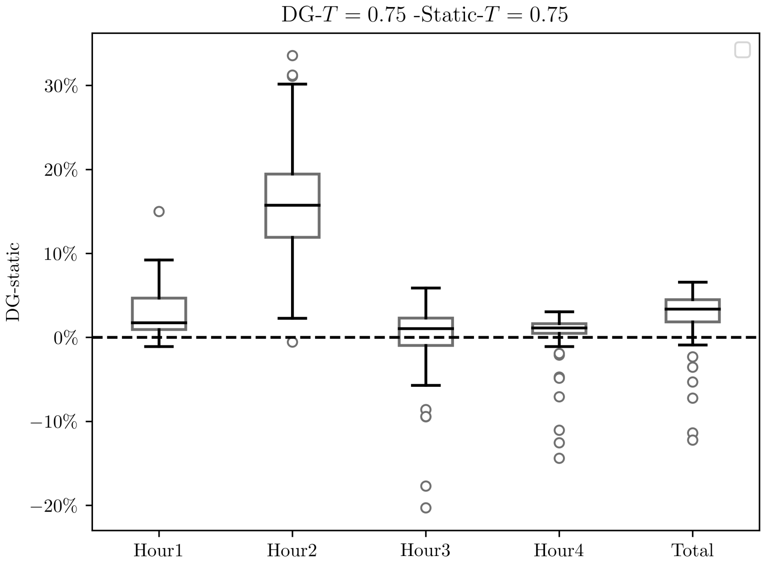

Figure EC.8 Boxplot of availability absolute improvement of DG $T = 0.75$ lookahead policy over static $T = 0.75$ lookahead policy for $\kappa = 2000$ in the time-varying setting.

Percentile	Hour1	Hour2	Hour3	Hour4	Total
95%	7.92%	28.98%	4.05%	2.57%	5.6%
75%	4.69%	19.46%	2.31%	1.65%	4.5%
50%	1.75%	15.75%	1.04%	1.12%	3.39%
25%	0.94%	11.93%	-0.95%	0.48%	1.84%
5%	0.14%	5.61%	-5.85%	-4.73%	-2.38%
Average	2.92%	15.97%	0.06%	0.49%	2.6%
95% Conf. Interval	[2.37%, 3.47%]	[14.62%, 17.31%]	[-0.75%, 0.87%]	[-0.06%, 1.04%]	[1.98%, 3.22%]

Table EC.8 Percentiles of availability absolute improvement of DG $T = 0.75$ lookahead policy over static $T = 0.75$ lookahead policy for $\kappa = 2000$ in the time-varying setting.

Appendix F: Appendix for Section 6

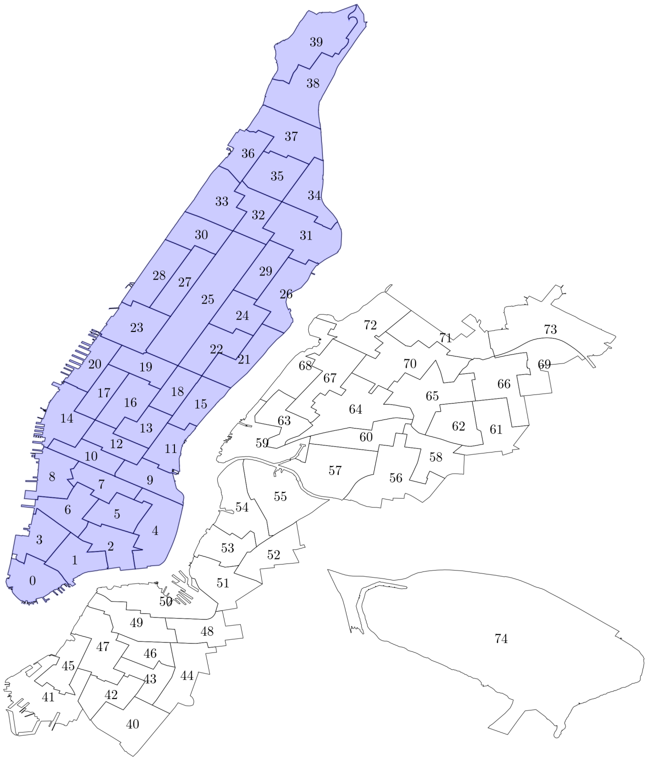

Figure EC.9 Neighborhoods included in our NYC instance are colored.