Online Appendix

This online appendix contains supplementary information for the paper. Specifically, Appendix A details an extended model with multiple replenishment opportunities, Appendix B provides supplementary information for the numerical study including the heuristics examined in the study and the results for a three-product setting, Appendix C contains proofs of Equation (10) and Example 1 in the paper, as well as statements and proofs of technical lemmas used in the proofs of the propositions, and Appendix D presents the proofs of the propositions in the paper and in Appendices A and B.

A. Multi-Season Problems

In the base model, we consider a single selling season with N periods: the demand is realized at the end of each period, inventory replenishment occurs at the beginning of the season, and left-over inventory is salvaged at the end of the season. In this section we extend the single-season model to multiple seasons, finite or infinite. The seasons are indexed by $t = 1, 2, ...$ Same as in the base model, each season consists of N periods, and at the end of each period product demand is realized. Let the demand be an independent and identically distributed random sequence, $\{\mathbf{D_t}, t = 1, 2, ...\}$, where $\mathbf{D_t} = (D_{int}, i = 1, ..., k, n = 1, ..., N)$ denotes the demand matrix in season t and D_{int} is the demand for product i in period n of season t. Within each season demand is independent across different periods (i.e., D_{int} is independent of D_{jmt} for any i, j, and $m \neq n$). Within each period, however, demand for different products can be dependent (i.e., D_{int} can be dependent of D_{jnt} for $i \neq j$). As in the base model, assume that demand of each product in each period of any season has a finite and positive mean: $0 < E(D_{int}) < \infty$ for all i, n, and t. The selling prices, component costs, and holding costs are stationary over time. For simplification, we assume that the starting inventory at the beginning of the first season (before replenishment) is zero. This assumption could be relaxed for the infinite-horizon problem, as we shall explain in §A.2.

A.1. Finite Horizon

Consider a finite horizon of T selling seasons, where the first season is referred to as season 1, and the season after season t as season $t+1$. Inventory replenishment is allowed at the beginning of each season, but inventory salvage occurs only at the end of season T. That is, inventory salvage takes place only at the end of the multiple-season horizon, and is disallowed in season $t = 1, ..., T-1$. As in the base model, inventory is salvaged at cost.

In the following we will first establish a connection between the expected total profit in the multi-season horizon and that in a single selling season under a feasible policy. The connection will then be applied to analyze the decisions in the multi-season horizon.

Consider a general feasible policy $\varpi = \{(X_t^\varpi, Y_t^\varpi), t = 1, 2, 3, ...\}$, where (X_t^ϖ, Y_t^ϖ) specifies the two control actions in season t: the replenishment decision (order-up-to level) $X_t^\varpi = (x_{0t}^\varpi, x_{1t}^\varpi, ..., x_{kt}^\varpi)$ at the beginning of season t, and the allocation decision $Y_t^\varpi = (y_{int}^\varpi)$ with y_{int} as the sales quantity of product $i = 1, ..., k$ in period $n = 1, ..., N$ of season t (after demand is realized in the period). Furthermore, let X_t^0 denote the starting inventory at the beginning of season t before replenishment. Note that X_t^0 is also the left-over inventory at the end of season $t-1$ after the order fulfillment in season $t-1$ is completed.

In particular, for a policy to be feasible, it must satisfy, for each season t, each period n and each product i,

$$X_t^\varpi \geq X_t^0, \quad 0 \leq y_{int}^\varpi \leq d_{int}, \quad \sum_{i=1}^{k}\sum_{m=1}^{n} y_{imt}^\varpi \leq x_{0t}^\varpi, \quad \sum_{m=1}^{n} y_{imt}^\varpi \leq x_{it}^\varpi,$$

where d_{int} is the realized demand of product i in period n of season t.

Following the single-season analysis, define

$$\begin{aligned}
&R(X_t^\varpi, Y_t^\varpi)\\
&:= \Bigg[\sum_{i=1}^{k}\sum_{n=1}^{N}\Bigg(p_i y_{int}^\varpi - h_i(x_{it}^\varpi - \sum_{m=1}^{n} y_{imt}^\varpi)\Bigg) - h_0 \sum_{n=1}^{N}\Bigg(x_{0t}^\varpi - \sum_{i=1}^{k}\sum_{m=1}^{n} y_{imt}^\varpi\Bigg) + \sum_{i=1}^{k} c_i(x_{it}^\varpi - \sum_{n=1}^{N} y_{int}^\varpi)\\
&\qquad + c_0(x_{0t}^\varpi - \sum_{i=1}^{k}\sum_{n=1}^{N} y_{int}^\varpi)\Bigg]\\
&= \Bigg[-N\sum_{j=0}^{k}(h_j x_{jt}^\varpi) + \sum_{j=0}^{k}(c_j x_{jt}^\varpi) + \sum_{n=1}^{N}\sum_{i=1}^{k}((p_i + (N-n+1)h_i + (N-n+1)h_0 - c_i - c_0)y_{int}^\varpi)\Bigg]
\end{aligned}$$

as the profit obtained in a single season (excluding component procurement cost) where the starting inventory (i.e., before replenishment) is zero, the inventory after replenishment is X_t^ϖ, the sales quantities are Y_t^ϖ, and any leftover inventory is salvaged at cost.

Denote by $\Pi_T^\varpi(X_1^0)$ the expected total profit over the T-season horizon obtained by policy ϖ with starting inventory X_1^0 at the beginning of the horizon (i.e., before replenishment). We have:

$$\begin{aligned}
\Pi_T^\varpi(X_1^0) &= E_\varpi\left[\sum_{t=1}^{T-1}[R(X_t^\varpi, Y_t^\varpi) - \sum_{j=0}^{k} c_j x_{j,t+1}^0 - \sum_{j=0}^{k} c_j(x_{jt}^\varpi - x_{jt}^0)] + R(X_T^\varpi, Y_T^\varpi) - \sum_{j=0}^{k} c_j(x_{jT}^\varpi - x_{jT}^0)\right]\\
&= E_\varpi\left[\sum_{t=1}^{T}\left(R(X_t^\varpi, Y_t^\varpi) - \sum_{j=0}^{k} c_j x_{jt}^\varpi\right) + \sum_{j=0}^{k} c_j x_{j1}^0\right] \qquad \text{(EC.1)}
\end{aligned}$$

where the first equality is due to the following two cases:

- For the last season, i.e., season T, given inventory X_T^0 before replenishment and inventory X_T^ϖ after replenishment, with inventory salvage at the end of the season (as this is the last season), the profit obtained by policy ϖ in season T is $R(X_T^\varpi, Y_T^\varpi) - \sum_{j=0}^{k} c_j(x_{jT}^\varpi - x_{jT}^0)$, where $\sum_{j=0}^{k} c_j(x_{jT}^\varpi - x_{jT}^0)$ is the inventory replenishment cost.

- For season $t \in \{1,...,T-1\}$, with inventory X_t^0 before replenishment and X_t^ϖ after replenishment, the profit obtained by policy ϖ in season t is $R(X_t^\varpi, Y_t^\varpi) - \sum_{j=0}^k c_j x_{j,t+1}^0 - \sum_{j=0}^k c_j(x_{jt}^\varpi - x_{jt}^0)$, where $\sum_{j=0}^k c_j(x_{jt}^\varpi - x_{jt}^0)$ is the inventory replenishment cost. Note that the function $R(\cdot,\cdot)$ assumes leftover inventory salvaged at cost at the end of a season, while inventory salvage does not occur in season t. Thus, the salvage value $\sum_{j=0}^k c_j x_{j,t+1}^0$ is deducted from the profit, where recall that $x_{j,t+1}^0$ is the starting inventory in the next season and also the left-over inventory in the current season t.

(EC.1) establishes an important relationship between multi-season profit and single-season profit. It allows us to convert the control problem for multiple seasons into one for a single season. Below we exemplify the transformation using the optimal control policy and the heuristic policy q.

Specifically, denote by X_t^* the optimal inventory at the beginning of season t, and Y_t^* the optimal allocation in season t. Recall from the single-season analysis that, given inventory X right after replenishment, $R_1(X)$ is the expected total profit of the season under optimal fulfillment policy, excluding the inventory replenishment cost and including the season-end salvage values. Assuming that the starting inventory is zero, the expected profit of the season (net of the inventory replenishment cost) is $\pi(X) = R_1(X) - \sum_{j=0}^k c_j x_j$ and the optimal inventory $X^* = \arg\max_X \pi(X)$. Thus, $R_1(X) \equiv E_{Y_1^*}[R(X, Y_1^*)]$ and $\pi(X) \equiv E_{Y_1^*}[R(X, Y_1^*)] - \sum_{j=0}^k c_j x_j$. Note that by the definition of X^*, for any feasible (X^ϖ, Y^ϖ),

$$E_{Y^\varpi}[R(X^\varpi, Y^\varpi)] - \sum_{j=0}^k c_j x_j^\varpi \leq \pi(X^*) \tag{EC.2}$$

(EC.1) implies that, given the starting inventory $X_1^0 \leq X^*$, the optimal policy in each season is a base-stock policy with an order-up-to inventory level X^*, combined with the optimal allocation policy Y_1^*. Thus, provided that $X_1^0 \leq X^*$, $X_t^* = X^*$ and the expected total profit over the T-season horizon obtained by the optimal replenishment and fulfillment policies, denoted by $\Pi_T(X_1^0)$, is

$$\Pi_T(X_1^0) = \sum_{t=1}^T \pi(X^*) + \sum_{j=0}^k (c_j x_{j1}^0) = T\pi(X^*) + \sum_{j=0}^k (c_j x_{j1}^0). \tag{EC.3}$$

Similarly, suppose that the fulfillment policy q with an inventory order-up-to level X^q is implemented in every season of the selling horizon. That is, in each season order fulfillment follows the static nested allocation policy and inventory replenishment follows a base-stock policy with order-up-to inventory level X^q. In this case, (EC.1) implies that, provided that $X_1^0 \leq X^q$, the expected total profit over the T-season horizon obtained by such a policy, denoted by $\Pi_T^q(X_1^0)$, is

$$\Pi_T^q(X_1^0) = \sum_{t=1}^T \pi^q(X^q) + \sum_{j=0}^k (c_j x_{j1}^0) = T\pi^q(X^q) + \sum_{j=0}^k (c_j x_{j1}^0), \tag{EC.4}$$

where recall that $\pi^q(X^q)$ is the expected single-season profit given initial inventory X^q (after replenishment and before fulfillment) and the static nested allocation policy and, more specifically, $\pi^q(X^q) \equiv E_{Y_1^q}[R(X^q, Y_1^q)] - \sum_{j=0}^k c_j x_j^q$, where Y_1^q consists of the sales quantities y_{in} as in (9).

Analogous to the single-season analysis, we define asymptotic optimality of a policy in the multi-season problem as follows. We define a stochastic problem with scale $m = 1, 2, \ldots$ as a stochastic problem in which the demand for product i in period n is $D_{int}^{(m)}$ and the starting inventory at the beginning of the horizon (before replenishment) is $X_1^{0(m)}$. The superscript (m) labels the scale-m problem and is omitted for the scale-1 problem. Assume that for each problem m, $E(D_{int}^{(m)}) = mE(D_{int})$ and that the normalized demand $m^{-1}D_{int}^{(m)}$ converges in distribution to the expected demand in the scale-1 problem, $E(D_{int})$. That is, $m^{-1}D_{int}^{(m)} \xrightarrow{\mathscr{D}} E(D_{int})$.

Recall the notations for the scale-m problem: for a single-season problem with zero starting inventory, denote by $X^{*(m)}$ the optimal inventory level for replenishment and by $\pi^{(m)}(X^{*(m)})$ the single-season profit obtained by the optimal replenishment and fulfillment policies. Similarly, denote by $X^{q(m)}$ the heuristic inventory order-up-to level and by $\pi^{q(m)}(X^{q(m)})$ the single-season profit obtained by adopting the order-up-to level $X^{q(m)}$ and following the heuristic fulfillment policy q.

Next we will prove that, in the multi-season problem, provided that $X_1^{0(m)} \le \min(X^{q(m)}, X^{*(m)})$ (recall that we assume $X_1^{0(m)} \equiv 0$), the expected profit obtained by applying the order-up-to level $X^{q(m)}$ and fulfillment policy q in each season, denoted by $\Pi_T^{q(m)}(X_1^{0(m)})$, approaches the expected profit obtained by applying the optimal order-up-to level $X^{*(m)}$ and the optimal fulfillment policy in each season, denoted by $\pi_T^{(m)}(X_1^{0(m)})$, in the high-demand regime. That is,

$$\lim_{m\to\infty} \frac{\Pi_T^{q(m)}(X_1^{0(m)})}{\Pi_T^{(m)}(X_1^{0(m)})} = 1.$$

By (EC.3) and (EC.4), it is equivalent to prove

$$\lim_{m\to\infty} \frac{T\pi^{q(m)}(X^{q(m)}) + \sum_{j=0}^k (c_j x_{j1}^{0(m)})}{T\pi(X^{*(m)}) + \sum_{j=0}^k (c_j x_{j1}^{0(m)})} = 1.$$

To this end, first note that because $\pi^{q(m)}(X^{q(m)}) \le \pi(X^{*(m)})$,

$$\frac{\pi^{q(m)}(X^{q(m)})}{\pi(X^{*(m)})} = \frac{T\pi^{q(m)}(X^{q(m)})}{T\pi(X^{*(m)})} \le \frac{T\pi^{q(m)}(X^{q(m)}) + \sum_{j=0}^k (c_j x_{j1}^{0(m)})}{T\pi(X^{*(m)}) + \sum_{j=0}^k (c_j x_{j1}^{0(m)})} \le 1. \tag{EC.5}$$

Furthermore, recall $\lim_{m\to\infty} \frac{\pi^{q(m)}(X^{q(m)})}{\pi^{(m)}(X^{*(m)})} = 1$ (proved in the single-season analysis). Thus, by (EC.5), $\lim_{m\to\infty} \frac{\Pi_T^{q(m)}(X_1^{0(m)})}{\Pi_T^{(m)}(X_1^{0(m)})} = 1$.

Proposition A.1 *For the finite-horizon problem with zero inventory at the beginning of the horizon, a heuristic in which an order-up-to level $X^{q(m)}$ and demand fulfillment policy q are adopted in each season is asymptotically optimal.*

A.2. Infinite Horizon

Now we consider the problem of infinite horizon with long-run average expected profit. For any feasible policy ϖ, denote the long-run average expected profit by

$$\bar{\Pi}^{\varpi}(X_1^0) = \liminf_{T\to\infty} \frac{\Pi_T^{\varpi}(X_1^0)}{T}.$$

Proposition A.2 *Assume $X_1^0 = 0$. The optimal policy that maximizes the long-run average profit is as follows: in season t, order up to X^* and follow the optimal fulfillment rule. Moreover, the optimal long-run average profit, denoted by $\bar{\Pi}^*$, equals to $\pi(X^*)$.*

Similarly, we define policy q: in season t, order up to X^q and follow the static nested policy. Denote the long-run average profit achieved by policy q by $\bar{\Pi}^q(X_1^0)$. Same as before, we define the scale-m problem and add superscript (m) in the notations for the scale-m problem.

Proposition A.3 *Assume $X_1^{0(m)} = 0$. Policy q is asymptotically optimal for the infinite-horizon problem. Specifically, $\lim_{m\to\infty} \frac{\bar{\Pi}^{q(m)}(X_1^{0(m)})}{\bar{\Pi}^{*(m)}} = 1$.*

We can easily show that the condition $X_1^0 = 0$ in Propositions A.2 and A.3 is not necessary. Note that for infinite horizon problem with long-run average profit, if it is optimal to follow a stationary policy π in all seasons, then it is also optimal to follow another feasible policy π' (which may not be optimal) for a finite number of T seasons, and switch to policy π after that. Thus, for any given positive X_1^0, the firm may always follow some feasible policy to exhaust all on-hand inventories first, and then follow the optimal policy π after that.

B. Supplementary Information for Numerical Study

B.1. Definitions of Fulfillment Heuristics

B.1.1. Fixed Priority Heuristic by Zhang (1997) Zhang (1997) considers backlogging of unsatisfied demand and proposes an allocation scheme based on fixed product type-based priority. Specifically, according to pp. 313 of Zhang (1997) the scheme follows the three rules below:

- First-come-first-served: Demands arriving in an earlier period have priority in receiving component stocks over demands arriving in a later period, regardless of product types.
- Product type-based priority: Of the demands that arrive in the same period, product 1 has priority over product 2, which has priority over product 3, and so on.
- Stock commitment: Once component stock units are issued to a particular demand based on the first two rules, these units remain committed to the allocated product and become unavailable to other demands.

Here, "the numbering of product types is such that the most important product is labeled type 1; such a ranking may be determined by exogenous factors such as the revenue margins or marketing orientations of each product." In our lost-sales model of the generalized W systems, under this fixed product type-based priority, in period $n=1,...,N$, given demand realization d_{in} for product $i=1,...,k$, the realized demand is fulfilled in an increasing order of the product index, and the sales quantities satisfy

$$\begin{aligned} y_{1n} &= \min(d_{1n}, x_1, x_0), \\ y_{in} &= \min\left(d_{in}, x_i, \left(x_0 - \sum_{j=1}^{i-1} y_{jn}\right)^+\right), \qquad i=2,...,k. \end{aligned}$$

B.1.2. Fair-Shares Heuristic by Agrawal and Cohen (2001) Agrawal and Cohen (2001) consider a fair-shares heuristic, whereby component allocation in a period is proportional to products' realized demand in the period. In the context of our generalized W systems, the fair-shares heuristic stipulates that the sales quantity of product $i=1,...,k$ in period $n=1,...,N$ is $d_{in} \wedge x_i \wedge (\frac{d_{in}x_0}{\sum_{i=1}^k d_{in}})$. That is, x_0 units of common components are shared among all the realized demands across products. Hence, even in a balanced system (i.e., $x_0 = \sum_{i=1}^k x_i$), when $x_0 < \sum_{i=1}^k d_{in}$ and $d_{1n} \le x_1$, the sales quantity of the most-profitable product 1 is $d_{1n} \wedge x_1 \wedge (\frac{d_{1n}x_0}{\sum_{i=1}^k d_{in}}) < d_{1n}$, implying that some demand of the most-profitable product 1 is rejected under the fair-shares heuristic.

B.1.3. Order-Based Heuristic by Akçay and Xu (2004) By Algorithm 1 on pp. 105 of Akçay and Xu (2004), an order-based heuristic applied to our model of the generalized W systems is as follows: in period n and for a given $\eta^{ax} \in (0,1]$:

- Step 1, initialize the set of unfilled demand $\mathfrak{E} = \{i | d_{in} > 0\}$; initialize the fulfillment $y_{in} = 0$ for all $i = \{1, ..., k\}$; initialize STOP = 0;
- Step 2, if $x_i \wedge x_0 = 0$ for all $i \in \mathfrak{E}$, set STOP =1; Otherwise, for product

$$j = \arg\max_{i \in \mathfrak{E}} [(p_i - c_i - c_0 + (N - n + 1)h_i + (N - n + 1)h_0) \min(x_i, x_0)],$$

fill $\Delta y_{jn} = d_{jn} \wedge \max(\lfloor \eta^{ax}(x_j \wedge x_0) \rfloor, 1)$ units. Then, update y_{jn} by replacing it with $y_{jn} + \Delta y_{jn}$, update the on-hand inventory of component j, i.e., x_j, by replacing it with $x_j - \Delta y_{jn}$, update that of common component, x_0, by replacing it with $x_0 - \Delta y_{jn}$, update d_{jn} by replacing it with $d_{jn} - \Delta y_{jn}$; replace $\mathfrak{E}$ by dropping j if $\eta^{ax} = 1$ or the updated d_{jn} is 0.
- Step 3, repeat Step 2 until the set $\mathfrak{E}$ is empty or STOP = 1.

We assume $\eta^{ax} = 0.5$ in our numerical studies.

B.1.4. Heuristics by Reiman and Wang (2015) Reiman and Wang (2015) consider a setting with continuous-review, positive replenishment lead time, and demand backlogging. They use a two-stage stochastic program (SP) to develop inventory replenishment and allocation policies. Specifically, the **inventory replenishment** policy is a base-stock policy in which the base-stock levels of components are determined by the first-stage optimal solution of the SP. The **inventory allocation** policy is based on target backlog levels derived from the second-stage SP recourse linear program.

The replenishment policy proposed by Reiman and Wang (2015) is readily applicable to our model of the generalized W systems. Nevertheless, since our model assumes lost sales of unmet demand, their allocation policy cannot be directly applied. In the following we first review the policies in Reiman and Wang (2015), and then develop an allocation heuristics for the generalized W systems according to their idea.

- Inventory Replenishment

On pp. 721 of Reiman and Wang (2015), a family of base-stock policies is proposed that uses weighted average of two solutions, $\mathbf{y}^*$ and $\mathbf{y}^o$, as base-stock levels, where the two solutions are derived from two different stochastic programs. That is, as in (18) of Reiman and Wang (2015), the proposed base-stock levels are:

$$\mathbf{y}^\gamma = \gamma \mathbf{y}^* + (1-\gamma)\mathbf{y}^o, \quad 0 \le \gamma \le 1$$

The authors further recommend using $\gamma = 0$ based on their numerical results. Thus, we shall use $\mathbf{y}^o$ in our study. Specially, with the notations in our model, the base-stock levels $\mathbf{X}^o = (x_1^o, x_2^o, ..., x_k^o)$ are as below:

$$\mathbf{X}^o = \arg\min_{\mathbf{X} \ge 0} \left[\mathbf{h} \cdot \mathbf{X} + \mathbf{b} \cdot \mathrm{E}(\overrightarrow{\mathcal{D}}) - \mathrm{E}_{\overrightarrow{\mathcal{D}}}[\phi_+(\mathbf{X}; \overrightarrow{\mathcal{D}})] \right],$$
$$\phi_+(\mathbf{X}, \overrightarrow{\mathcal{D}}) = \max_{\mathbf{z} \ge 0} \{ \mathbf{c} \cdot \mathbf{z} | \mathbf{z} \le \overrightarrow{\mathcal{D}}, \ A\mathbf{z} \le \mathbf{X} \}$$

where the bill of materials is given by matrix A of which element a_{ji} represents the amount of component j needed to assemble one unit of product i (in our context A is a $(k+1)\times k$ matrix), $\mathbf{h}$ is the vector of holding costs of components, $\mathbf{b}$ is the vector of backlog costs of components, $\mathbf{c}$ is the vector with element $c_i = b_i + \sum_{j=1}^{k+1} a_{ji}h_j$, and $\overrightarrow{\mathcal{D}}$ is the demand.

Applying this heuristic to our model of the generalized W systems, at the beginning of each season, a base-stock policy is applied with base-stock levels $\mathbf{X}^{SP} = (X_1^{SP}, X_2^{SP}, ..., X_k^{SP})$ as below:

$$\mathbf{X}^{SP} = \arg\max_{\mathbf{X}} \mathrm{E}_{\overrightarrow{\mathcal{D}}}[-N\sum_{j=0}^{k}(h_j x_j) + \phi(\mathbf{X}, \overrightarrow{\mathcal{D}})]$$

where

$$\phi(\mathbf{X}, \overrightarrow{\mathcal{D}}) = \max_{y_1,\ldots,y_k} \left[\sum_{i=1}^{k}(p_i + h_i + h_0 - c_i - c_0)y_i\right]$$
$$\text{s.t.} \quad 0 \le y_i \le x_i \wedge \mathcal{D}_i, \quad i = 1,\ldots,k, \ \sum_{i=1}^{k} y_i \le x_0.$$

with $\overrightarrow{\mathcal{D}} = (\mathcal{D}_1, \mathcal{D}_2, ..., \mathcal{D}_k)$ and $\mathcal{D}_i$ has the same distribution as $\sum_n D_{in}$. This policy is based on the hypothetical scenario in which components are allocated in the last period after observing demand in all the periods and, thus, the holding cost saving from fulfilling a unit of product-i demand is $h_i + h_0$, i.e., for one period.

• Inventory Allocation

In Reiman and Wang (2015), at the end of each period, a backlog target is computed based on the current inventory level and demand arrival in the current period. The firm then fills demand such that the actual backlog for each product, if higher than the target, gets close to the target as much as possible. Specifically, as in (19) of Reiman and Wang (2015), the backlog targets that minimize the total cost are

$$\mathbf{B}^*(t) = \arg\min\{\mathbf{c}\cdot\mathbf{B}|\mathbf{B} \ge 0, \ A\mathbf{B} \ge Q(t)\},$$

where $Q(t)$ is the shortage of on-hand inventory for clearing existing backlogs at time t. The allocation principle is as follows (pp. 722 of Reiman and Wang (2015)):

No product should have its backlog level strictly above its target if all required components have sufficient inventories, i.e., an allocation policy must yield

$$[B_i(t) - B_i^*(t)]^+ \wedge \left[\min_{j:a_{ji}>0}(I_j(t) - a_{ji} + 1)^+\right] = 0, \quad 1 \le i \le m.$$

In addition, any product whose backlog level is below or at the target is not served, i.e.,

$$z_i(t) \le [B_i^-(t) - B_i^*(t)]^+, \quad 1 \le i \le m.$$

where $B_i(t)$ and $I_j(t)$ are the backlog level of product i and the inventory level of component j at time t after demand fulfillment and backlogging, $B_i^-(t)$ is the backlog level of product i at time t after demand arrival and component receipt but before component allocation, and $z_i(t)$ is product-i demand served at time t.

In particular, for a two-product W system, the authors note that the allocation principle leads to "clear as many backlogs as possible while giving the priority of using component 0 to product 1" and, as in (23) of Reiman and Wang (2015), the backlog targets are

$$\begin{aligned}\mathbf{B}^*(t) &= \arg\min\left\{c_1B_1 + c_2B_2 | B_i \geq 0,\ B_i \geq Q_i(t),\ B_1 + B_2 \geq Q_0(t),\ i = 1, 2\right\},\\ &= \left(Q_1^+(t), Q_2^+(t) \wedge (Q_0^+(t) - Q_1^+(t))^+\right).\end{aligned}$$

As noted earlier, the heuristic of backlog targets cannot be directly applied to our model due to our lost-sales assumption. Thus, we develop a heuristic based on lost-sale targets, which are computed based on the current inventory level and demand arrival in the current period. The firm fills demand such that the actual lost sales for each product, if higher than the target, gets close to the target as much as possible. Specifically, let d_{in} be the realized demand of product i in period n, and x_j be the remaining inventory of component j at the beginning of the period. The component shortage $Q_j, j = 0, ..., k$ is defined as:

$$Q_0 = \sum_i d_{in} - x_0, \quad Q_j = d_{jn} - x_j,\ j = 1, ..., k$$

The lost-sales target is defined as:

$$\mathbf{L}_n^* = (L_{1n}^*, L_{2n}^*, ..., L_{kn}^*) = \arg\min_{L_{1n},...,L_{kn}} \sum_{i=1}^{k} \left[(p_i + (N-n+1)h_i + (N-n+1)h_0 - c_i - c_0)L_{in}\right]$$

$$\text{subject to} \quad \sum_{i=1}^{k} L_{in} \geq Q_0,\ L_{jn} \geq Q_j,\ j = 1, ..., k;\ \text{all } L_{jn}\text{'s are nonnegative.}$$

For $i = 1, ..., k$, the fulfilled quantity of product i, denoted by y_{in}^{SP}, satisfies:

$$y_{in}^{SP} \leq (d_{in} - L_{in}^*)^+ \tag{EC.6}$$

and

$$\min\left((d_{in} - L_{in}^*)^+ - y_{in}^{SP}, x_0 - \sum_{i=1}^{k} y_{in}^{SP}, x_i - y_{in}^{SP}\right) = 0 \tag{EC.7}$$

There may exist more than one solution $\{y_{in}^{SP}, i = 1, ..., k\}$ that satisfy both (EC.6) and (EC.7). As the authors noted in their backlogging setting (e.g., Reiman and Wang 2015), the backlogs can be cleared in any sequence that the firm "finds appropriate, e.g., giving higher priority to products with higher $c_i s$", where c_i represents backlogging cost in their papers. Hence, in the heuristic that we develop based on their paper, we shall follow the sequence of our product priorities, i.e., in an increasing order of the product index. That it, y_{1n}^{SP} is first determined and followed by y_{2n}^{SP}, ..., and y_{kn}^{SP}. Thus, y_{in}^{SP} satisfies:

$$y_{in}^{SP} \le (d_{in} - L_{in}^*)^+, \; i = 1, ..., k \tag{EC.8}$$

$$\min\left((d_{1n} - L_{1n}^*)^+ - y_{1n}^{SP}, x_0 - y_{1n}^{SP}, x_1 - y_{1n}^{SP}\right) = 0, \tag{EC.9}$$

$$\min\left((d_{in} - L_{in}^*)^+ - y_{in}^{SP}, x_0 - \sum_{j=1}^{i} y_{jn}^{SP}, x_i - y_{in}^{SP}\right) = 0, \; i = 2, ..., k \tag{EC.10}$$

In the following we derive the optimal lost-sales targets and the corresponding fulfillment quantities. Note that we have assumed $p_i \ge c_i + c_0$, which implies $p_i + (N-n+1)h_i + (N-n+1)h_0 - c_i - c_0 \ge 0$. The optimal lost-sales targets are as below:

Lemma B.1

$$\begin{aligned}
L_{in}^* =& (d_{in} - x_i)^+, \; i = 1, ..., k-1, \\
L_{kn}^* =& (d_{kn} - x_k)^+ + \left[\left(\sum_i d_{in} - x_0\right) - \sum_i (d_{in} - x_i)^+\right]^+
\end{aligned}$$

Lemma B.2 *Given the optimal lost-sales targets in Lemma B.1 and the fulfillment following an increasing order of the product index, the sales quantities satisfying both (EC.6) and (EC.7) are*

$$\begin{aligned}
y_{1n}^{SP} &= \min(d_{1n}, x_0, x_1) \\
y_{in}^{SP} &= \min\left(d_{in}, \left(x_0 - \sum_{j=1}^{i-1} y_{jn}^{SP}\right)^+, x_i\right), \quad i = 2, ..., k.
\end{aligned}$$

This implies a nested policy without component reservation for future demand.

B.2. Endogenous Inventory for Each Fulfillment Heuristic

B.2.1. Fixed Priority Heuristic by Zhang (1997) For exposition, given the initial inventory X, we define $\pi^z(X|\overrightarrow{\mathcal{D}})$ as the profit achieved under realized demand $\overrightarrow{\mathcal{D}} = (D_{11}, ..., D_{kn})$ by adopting the fixed priority heuristic. That is,

$$\pi^z(X|\overrightarrow{\mathcal{D}}) = \widetilde{V}_1^z(X|\overrightarrow{\mathcal{D}}) - \sum_{i=0}^{k} c_i x_i$$

where, for $n \in \{1, ..., N\}$,

$$\widetilde{V}_n^z(X|\overrightarrow{\mathcal{D}}) = \widetilde{J}_n^z(X, D_{1n}, ..., D_{kn}),$$

with

$$\begin{aligned}&\widetilde{J}_n^z(X, D_{1n}, ..., D_{kn})\\&= \sum_{i=1}^{k}[p_i y_{in} - h_i(x_i - y_{in})] - h_0(x_0 - \sum_{i=1}^{k} y_{in}) + \widetilde{V}_{n+1}^z(X - \sum_{i=1}^{k} y_{in} a_i | \overrightarrow{\mathcal{D}})\end{aligned}$$

where $(y_{1n}, ..., y_{kn})$ is obtained through the following, sequentially from y_{1n} to y_{kn}:

$$\begin{aligned}y_{1n} &= \min(x_1, D_{1n}, x_0)\\y_{in} &= \min(x_i, D_{in}, (x_0 - \sum_{j=1}^{i-1} y_{jn})^+), \text{ for } i = 2, ..., k\end{aligned}$$

with $x^+ = \max(x, 0)$. The boundary condition for the problem is: at the end of last period, $\widetilde{V}_{N+1}^z(X|\overrightarrow{\mathcal{D}}) = \sum_{i=0}^{k} c_i x_i$.

With definition of $\pi^z(X|\overrightarrow{\mathcal{D}})$, the endogenous inventory, X^z, is obtained according to the following algorithm:

- Step 0: let $M = 1,000$, $S = 50$ and $S' = 1,000$; Initialize $l \leftarrow 0$;
- Step 1: $l \leftarrow l + 1$; generate a vector $(\overrightarrow{\mathcal{D}}_l^S(1), ..., \overrightarrow{\mathcal{D}}_l^S(S))$, which has S elements, i.e., $\overrightarrow{\mathcal{D}}_l^S(j)$ with $j = 1, ..., S$, and each element $\overrightarrow{\mathcal{D}}_l^S(j)$ is a random sample of $(D_{11}, ..., D_{kN})$. Then, given $\overrightarrow{\mathcal{D}}_l^S(j)$ for $j = 1, ..., S$, we find an initial inventory X_l^z to maximize the average of $\pi^z(X|\overrightarrow{\mathcal{D}})$, i.e.,

$$X_l^z = \arg\max_X \frac{\sum_{j=1}^{S} \pi^z(X|\overrightarrow{\mathcal{D}}_l^S(j))}{S};$$

In searching for the maximizer X, for product i, the upper bound of x_i is the highest season-wide demand for product i in all the samples $\overrightarrow{\mathcal{D}}_l^S(j)$, $j = 1, ..., S$, i.e., $\max\{\sum_{n=1}^{N} D_{in}, s.t., (D_{11}, ..., D_{kN}) \in \{\overrightarrow{\mathcal{D}}_l^S(1), ..., \overrightarrow{\mathcal{D}}_l^S(S)\}\}$, and the upper bound of x_0 is the sum of the upper bound of x_i over all i. If $l \leq M$ go to Step 1, otherwise go to Step 2;

- Step 2: Generate a vector $\overset{\Rightarrow}{\mathcal{D}}^{S'} = (\overrightarrow{\mathcal{D}}^{S'}(1), ..., \overrightarrow{\mathcal{D}}^{S'}(S'))$, which has S' elements, i.e., $\overrightarrow{\mathcal{D}}^{S'}(j)$ with $j = 1, ..., S'$, and each element $\overrightarrow{\mathcal{D}}^{S'}(j)$ is a sample of $(D_{11}, ..., D_{4N})$; Initialize $l \leftarrow 0$;
- Step 3: Set $l \leftarrow l + 1$ and calculate

$$\ddot{\pi}^z(X_l^z|\overset{\Rightarrow}{\mathcal{D}}^{S'}) = \frac{\sum_{j=1}^{S'} \pi^z(X_l^z|\overrightarrow{\mathcal{D}}^{S'}(j))}{S'};$$

If $l \leq M$ go to Step 3; otherwise go to Step 4;

- Step 4: Choose

$$X^z = \arg\max_{X \in \{X_l^z,\ l=1...,M\}} \ddot{\pi}^z(X|\overset{\Rightarrow}{\mathcal{D}}^{S'}).$$

B.2.2. Fair-Shares Heuristic by Agrawal and Cohen (2001) For exposition, we define $\pi^{ac}(X|\overrightarrow{\mathcal{D}})$ as the profit achieved under the initial inventory X and realized demand $\overrightarrow{\mathcal{D}} = (D_{11},...,D_{kn})$ by adopting the fair-shares heuristic by Agrawal and Cohen (2001) in the subsection above. That is,

$$\pi^{ac}(X|\overrightarrow{\mathcal{D}}) = \widetilde{V}_1^{ac}(X|\overrightarrow{\mathcal{D}}) - \sum_{i=0}^{k} c_i x_i$$

where, for $n \in \{1,...,N\}$,

$$\widetilde{V}_n^{ac}(X|\overrightarrow{\mathcal{D}}) = \widetilde{J}_n^{ac}(X, D_{1n},...,D_{kn}),$$

with

$$\begin{aligned}&\widetilde{J}_n^{ac}(X, D_{1n},...,D_{kn})\\ =&\sum_{i=1}^{k}[p_i y_{in} - h_i(x_i - y_{in})] - h_0(x_0 - \sum_{i=1}^{k} y_{in}) + \widetilde{V}_{n+1}^{ac}(X - \sum_{i=1}^{k} y_{in} a_i|\overrightarrow{\mathcal{D}})\end{aligned}$$

where

$$y_{in} = \left\lfloor \min\left(x_i, D_{in}, \frac{D_{in}}{\sum_{j=1}^{k} D_{jn}} x_0\right)\right\rfloor, \text{ for } i = 1,...,k$$

The boundary condition for the problem is: at the end of last period, $\widetilde{V}_{N+1}^{ac}(X|\overrightarrow{\mathcal{D}}) = \sum_{i=0}^{k} c_i x_i$.

With definition of $\pi^{ac}(X|\overrightarrow{\mathcal{D}})$, the endogenous inventory for problem AC, X^{ac}, is obtained according to the following algorithm:

- Step 0: let $M = 1,000$, $S = 50$ and $S' = 1,000$; Initialize $l \leftarrow 0$;
- Step 1: $l \leftarrow l+1$; generate a vector $(\overrightarrow{\mathcal{D}}_l^S(1),...,\overrightarrow{\mathcal{D}}_l^S(S))$, which has S elements, i.e., $\overrightarrow{\mathcal{D}}_l^S(j)$ with $j = 1,...,S$, and each element $\overrightarrow{\mathcal{D}}_l^S(j)$ is a random sample of $(D_{11},...,D_{kN})$. Then, given $\overrightarrow{\mathcal{D}}_l^S(j)$ for $j = 1,...,S$, we find an initial inventory X_l^{ac} to maximize the average of $\pi^{ac}(X|\overrightarrow{\mathcal{D}})$, i.e.,

$$X_l^{ac} = \arg\max_X \frac{\sum_{j=1}^{S} \pi^{ac}(X|\overrightarrow{\mathcal{D}}_l^S(j))}{S};$$

If $l \le M$ go to Step 1, otherwise go to Step 2;

- Step 2: Generate a vector $\overrightarrow{\overrightarrow{\mathcal{D}}}^{S'} = (\overrightarrow{\mathcal{D}}^{S'}(1),...,\overrightarrow{\mathcal{D}}^{S'}(S'))$, which has S' elements, i.e., $\overrightarrow{\mathcal{D}}^{S'}(j)$ with $j = 1,...,S'$, and each element $\overrightarrow{\mathcal{D}}^{S'}(j)$ is a sample of $(D_{11},...,D_{kN})$; Initialize $l \leftarrow 0$;
- Step 3: Set $l \leftarrow l+1$ and calculate

$$\ddot{\pi}^{ac}(X_l^{ac}|\overrightarrow{\overrightarrow{\mathcal{D}}}^{S'}) = \frac{\sum_{j=1}^{S'} \pi^{ac}(X_l^{ac}|\overrightarrow{\mathcal{D}}^{S'}(j))}{S'};$$

If $l \le M$ go to Step 3; otherwise go to Step 4;

- Step 4: Choose

$$X^{ac} = \arg\max_{X \in \{X_l^{ac},\ l=1...,M\}} \ddot{\pi}^{ac}(X|\overrightarrow{\overrightarrow{\mathcal{D}}}^{S'}).$$

B.2.3. Order-Based Heuristic by Akçay and Xu (2004) For exposition, given the initial inventory X, we define $\pi^{ax}(X|\overrightarrow{\mathcal{D}})$ as the profit achieved under realized demand $\overrightarrow{\mathcal{D}} = (D_{11}, ..., D_{kn})$ according to the order-Based Heuristic by Akçay and Xu (2004) in the subsection above. That is,

$$\pi^{ax}(X|\overrightarrow{\mathcal{D}}) = \widetilde{V}_1^{ax}(X|\overrightarrow{\mathcal{D}}) - \sum_{i=0}^{k} c_i x_i$$

where, for $n \in \{1, ..., N\}$,

$$\widetilde{V}_n^{ax}(X|\overrightarrow{\mathcal{D}}) = \widetilde{J}_n^{ax}(X, D_{1n}, ..., D_{kn}),$$

with

$$\begin{aligned}&\widetilde{J}_n^{ax}(X, D_{1n}, ..., D_{kn})\\ &= \sum_{i=1}^{4}[p_i y_{in} - h_i(x_i - y_{in})] - h_0(x_0 - \sum_{i=1}^{k} y_{in}) + \widetilde{V}_{n+1}^{ax}(X - \sum_{i=1}^{k} y_{in} a_i|\overrightarrow{\mathcal{D}})\end{aligned}$$

where $(y_{1n}, ..., y_{kn})$ is obtained through the following algorithm:in period n and for a given $\eta^{ax} \in (0,1]$:

- Step 1, initialize the set of unfilled demand $\mathfrak{E} = \{i|d_{in} > 0\}$; initialize the fulfillment $y_{in} = 0$ for all $i = \{1, ..., k\}$; initialize STOP = 0;
- Step 2, if $x_i \wedge x_0 = 0$ for all $i \in \mathfrak{E}$, set STOP =1; Otherwise, for product

$$j = \arg\max_{i \in \mathfrak{E}}[(p_i - c_i - c_0 + (N - n + 1)h_i + (N - n + 1)h_0) \min(x_i, x_0)],$$

fill $\Delta y_{jn} = d_{jn} \wedge \max(\lfloor \eta^{ax}(x_j \wedge x_0) \rfloor, 1)$ units. Then, update y_{jn} by replacing it with $y_{jn} + \Delta y_{jn}$, update the on-hand inventory of component j, i.e., x_j, by replacing it with $x_j - \Delta y_{jn}$, update that of common component, x_0, by replacing it with $x_0 - \Delta y_{jn}$, update d_{jn} by replacing it with $d_{jn} - \Delta y_{jn}$; replace $\mathfrak{E}$ by dropping j if $\eta^{ax} = 1$ or the updated d_{jn} is 0.
- Step 3, repeat Step 2 until the set $\mathfrak{E}$ is empty or STOP = 1.

We assume $\eta^{ax} = 0.5$ in our numerical studies. The boundary condition for the problem is: at the end of last period, $\widetilde{V}_{N+1}^{ax}(X|\overrightarrow{\mathcal{D}}) = \sum_{i=0}^{k} c_i x_i$.

With definition of $\pi^{ax}(X|\overrightarrow{\mathcal{D}})$, the endogenous inventory for problem AX, X^{ax}, is obtained according to the following algorithm:

- Step 0: let $M = 1,000$, $S = 50$ and $S' = 1,000$; Initialize $l \leftarrow 0$;
- Step 1: $l \leftarrow l + 1$; generate a vector $(\overrightarrow{\mathcal{D}}_l^S(1), ..., \overrightarrow{\mathcal{D}}_l^S(S))$, which has S elements, i.e., $\overrightarrow{\mathcal{D}}_l^S(j)$ with $j = 1, ..., S$, and each element $\overrightarrow{\mathcal{D}}_l^S(j)$ is a random sample of $(D_{11}, ..., D_{kN})$. Then, given $\overrightarrow{\mathcal{D}}_l^S(j)$ for $j = 1, ..., S$, we find an initial inventory X_l^{ax} to maximize the average of $\pi^{ax}(X|\overrightarrow{\mathcal{D}})$, i.e.,

$$X_l^{ax} = \arg\max_X \frac{\sum_{j=1}^{S} \pi^{ax}(X|\overrightarrow{\mathcal{D}}_l^S(j))}{S};$$

If $l \leq M$ go to Step 1, otherwise go to Step 2;

- Step 2: Generate a vector $\vec{\vec{\mathcal{D}}}^{S'} = (\vec{\mathcal{D}}^{S'}(1), ..., \vec{\mathcal{D}}^{S'}(S'))$, which has S' elements, i.e., $\vec{\mathcal{D}}^{S'}(j)$ with $j = 1, ..., S'$, and each element $\vec{\mathcal{D}}^{S'}(j)$ is a sample of $(D_{11}, ..., D_{kN})$; Initialize $l \leftarrow 0$;
- Step 3: Set $l \leftarrow l + 1$ and calculate

$$\ddot{\pi}^{ax}(X_l^{ax}|\vec{\vec{\mathcal{D}}}^{S'}) = \frac{\sum_{j=1}^{S'} \pi^{ax}(X_l^{ax}|\vec{\mathcal{D}}^{S'}(j))}{S'};$$

If $l \leq M$ go to Step 3; otherwise go to Step 4;

- Step 4: Choose

$$X^{ax} = \arg \max_{X \in \{X_l^{ax},\ l=1...,M\}} \ddot{\pi}^{ax}(X|\vec{\vec{\mathcal{D}}}^{S'}).$$

B.2.4. Heuristics by Reiman and Wang (2015) Recall that for the heuristic by Reiman and Wang (2015) in the subsection above,

$$\phi(\mathbf{X}, \vec{\mathcal{D}}) = \max_{y_1,\ldots,y_k} \left[\sum_{i=1}^{k} (p_i + h_i + h_0 - c_i - c_0) y_i \right]$$
$$\text{s.t.} \quad 0 \leq y_i \leq x_i \wedge \mathcal{D}_i, \quad i = 1, \ldots, k, \ \sum_{i=1}^{k} y_i \leq x_0.$$

with $\vec{\mathcal{D}} = (\mathcal{D}_1, ..., \mathcal{D}_k)$, and $\mathcal{D}_i$ is a realization of the total demand for product $i = 1, ..., k$ over the selling season. For exposition, we define $\pi^{rw}(X|\vec{\mathcal{D}}) = -N \sum_{j=0}^{k} (h_j x_j) + \phi(\mathbf{X}, \vec{\mathcal{D}})$. Then, the initial inventory heuristic X^{rw} is obtained according to the following SAA algorithm:

- Step 0: let $M = 1,000$, $S = 50$ and $S' = 1,000$; Initialize $l \leftarrow 0$;
- Step 1: $l \leftarrow l + 1$; generate a vector $(\vec{\mathcal{D}}_l^S(1), ..., \vec{\mathcal{D}}_l^S(S))$, which has S elements, i.e., $\vec{\mathcal{D}}_l^S(j)$ with $j = 1, ..., S$, and each element $\vec{\mathcal{D}}_l^S(j)$ is a random sample of $(D_{11}, ..., D_{kN})$. Then, given $\vec{\mathcal{D}}_l^S(j)$ for $j = 1, ..., S$, we find an initial inventory X_l^{rw} to maximize the average of $\pi^{rw}(X|\vec{\mathcal{D}})$, i.e.,

$$X_l^{rw} = \arg\max_X \frac{\sum_{j=1}^{S} \pi^{rw}(X|\vec{\mathcal{D}}_l^S(j))}{S};$$

If $l \leq M$ go to Step 1, otherwise go to Step 2;

- Step 2: Generate a vector $\vec{\vec{\mathcal{D}}}^{S'} = (\vec{\mathcal{D}}^{S'}(1), ..., \vec{\mathcal{D}}^{S'}(S'))$, which has S' elements, i.e., $\vec{\mathcal{D}}^{S'}(j)$ with $j = 1, ..., S'$, and each element $\vec{\mathcal{D}}^{S'}(j)$ is a sample of $(D_{11}, ..., D_{kN})$; Initialize $l \leftarrow 0$;
- Step 3: Set $l \leftarrow l + 1$ and calculate

$$\ddot{\pi}^{rw}(X_l^{rw}|\vec{\vec{\mathcal{D}}}^{S'}) = \frac{\sum_{j=1}^{S'} \pi^{rw}(X_l^{rw}|\vec{\mathcal{D}}^{S'}(j))}{S'};$$

If $l \leq M$ go to Step 3; otherwise go to Step 4;

- Step 4: Choose

$$X^{rw} = \arg \max_{X \in \{X_l^{rw},\ l=1...,M\}} \ddot{\pi}^{rw}(X|\vec{\vec{\mathcal{D}}}^{S'}).$$

B.3. Additional Numerical Results for Three-Product Systems

Impact of Parameters on Heuristic Performance

For three-product generalized W systems, we vary the problem parameters as follows:

- Problem scale: $m \in \{1, 10, 100\}$
- Common-component procurement cost: $c_0 \in \{300, 350, 400, 450, 500, 550\}$
- Common-component holding cost: $h_0 \in \{5, 10, 15, 20, 25, 30, 35, 40, 45, 50\}$
- Demand proportion: $\rho \in \{0.2, 0.4, 0.6, 0.8\}$

resulting in a total of 720 (=$3 \times 6 \times 10 \times 4$) problem instances.

By Tables EC.1 and EC.2, the results regarding impact of parameters on heuristics performance is similar to those under $k = 4$.

Table EC.1 Average profit gaps (in %) under heuristic q and different c_0: $k = 3$

	$c_0 = 300$	$c_0 = 350$	$c_0 = 400$	$c_0 = 450$	$c_0 = 500$	$c_0 = 550$
$m = 1$	21.15	22.07	23.09	24.19	25.57	26.69
$m = 10$	6.22	6.52	6.68	6.63	6.57	6.10
$m = 100$	1.95	2.06	2.10	2.06	2.03	1.87

Table EC.2 Average profit gaps (in %) under heuristic q and different h_0: $k = 3$

	$h_0 = 5$	$h_0 = 10$	$h_0 = 15$	$h_0 = 20$	$h_0 = 25$	$h_0 = 30$	$h_0 = 35$	$h_0 = 40$	$h_0 = 45$	$h_0 = 50$
$m = 1$	18.09	19.38	20.55	22.30	23.54	25.70	25.19	26.86	29.22	27.10
$m = 10$	5.33	5.65	5.96	6.50	6.63	7.26	6.52	6.85	7.52	6.32
$m = 100$	1.66	1.76	1.87	2.05	2.07	2.28	2.03	2.12	2.34	1.93

C. Proofs of Equation (10), Example 1, and Technical Lemmas

C.1. Proof of Equation (10)

To prove the formula of the remaining quotas, for exposition, we define $\tilde{\zeta}_i = \sum_{j=i}^k v_j$ representing total quotas available for product i. According to the definition of the static nested policy (i.e., policy q), the vector $(\tilde{\zeta}_1, \ldots, \tilde{\zeta}_k)$ satisfies $\tilde{\zeta}_j = \zeta_j^k$, where $(\zeta_1^1, \ldots, \zeta_k^1)$ is defined as follows:

$$\zeta_1^1 = \sum_{j=1}^k u_j - y_{1n},$$
$$\zeta_j^1 = \sum_{j'=j}^k u_{j'} \wedge \zeta_{j-1}^1, \quad j = 2, \ldots, k,$$

(which implies that $\zeta_1^1 \geq \zeta_2^1 \geq \ldots \geq \zeta_k^1$), and $(\zeta_1^i, \ldots, \zeta_k^i)$, $i = 2, \ldots, k$ is defined recursively (with the superscript i referring to the i-th iteration):

$$\begin{aligned} \zeta_j^i &= \zeta_j^{i-1} - y_{in}, \quad j = 1, \ldots, i \\ \zeta_j^i &= \zeta_j^{i-1} \wedge \zeta_{j-1}^i, \quad j = i+1, \ldots, k, \end{aligned} \tag{EC.11}$$

under which $\zeta_1^i \geq \zeta_2^i \geq \ldots \geq \zeta_k^i$ and

$$\tilde{\zeta}_i = \zeta_i^k = \zeta_i^{k-1} - y_{kn} = \ldots = \zeta_i^i - \sum_{j=i+1}^k y_{jn} = \zeta_i^{i-1} - \sum_{j=i}^k y_{jn}. \tag{EC.12}$$

Recall that our objective is to prove

$$\tilde{\zeta}_1 = \sum_{j=1}^k u_j - \sum_{j=1}^k y_{jn},$$
$$\tilde{\zeta}_i = (\sum_{j=i}^k u_j - \sum_{j=i}^k y_{jn}) \wedge \tilde{\zeta}_{i-1}, \quad i = 2, \ldots, k.$$

To this end, first note that the remaining quota for product 1 is

$$\tilde{\zeta}_1 = \zeta_1^k = \zeta_1^{k-1} - y_{kn} = \zeta_1^1 - \sum_{j=2}^k y_{jn} = \sum_{j=1}^k u_j - \sum_{j=1}^k y_{jn}.$$

Then, for $i \geq 2$, we show that, for $j > i$, $\zeta_{j-1}^i \wedge \zeta_j^{i-1} = \zeta_{j-1}^i \wedge \sum_{j'=j}^k u_{j'}$ so that (EC.11) is equivalent to

$$\begin{aligned} \zeta_j^i &= \zeta_j^{i-1} - y_{in}, \quad j = 1, \ldots, i \\ \zeta_j^i &= \zeta_{j-1}^i \wedge \sum_{j'=j}^k u_{j'}, \quad j = i+1, \ldots, k: \end{aligned} \tag{EC.13}$$

If $\zeta_j^{i-1} = \sum_{j'=j}^k u_{j'}$, $\zeta_{j-1}^i \wedge \zeta_j^{i-1} = \zeta_{j-1}^i \wedge \sum_{j'=j}^k u_{j'}$; while if $\zeta_j^{i-1} < \sum_{j'=j}^k u_{j'}$, the individual quota of product j has been used in the $(i-1)$-th iteration or before, that is, the individual quota of product $j \in \{i+1, \ldots, k\}$ has been used in filling the demand of products $1, 2, \ldots, i-1$. As the individual quota for product $j-1$ has a higher priority of being used to fill the demand of products $1, 2, \ldots, i-1$ than the individual quota for product j, the fact that some individual quota for product j has been used implies that all the individual quotas for product $j-1$ have been used up. Thus, after filling the demand of products $1, 2, \ldots, i-1$, the available quotas of product $j-1$ are the same as those of product j because now both have the same source: the (remaining) individual quotas for product $j, j+1, \ldots, k$, so $\zeta_{j-1}^{i-1} = \zeta_j^{i-1}$. This together with $\zeta_{j-1}^i \le \zeta_{j-1}^{i-1}$ (by definition of ζ_{j-1}^i) and the condition $\zeta_j^{i-1} < \sum_{j'=j}^k u_{j'}$ leads to $\zeta_{j-1}^i \le \zeta_{j-1}^{i-1} = \zeta_j^{i-1} < \sum_{j'=j}^k u_{j'}$. Therefore, $\zeta_{j-1}^i \wedge \zeta_j^{i-1} = \zeta_{j-1}^i = \zeta_{j-1}^i \wedge \sum_{j'=j}^k u_{j'}$.

As a consequence,

$$\begin{aligned}\tilde{\zeta}_i &= \zeta_i^k = \zeta_i^{k-1} - y_{kn} = \ldots = \zeta_i^{i-1} - \sum_{j=i}^k y_{jn} = (\zeta_{i-1}^{i-1}) \wedge (\sum_{j=i}^k u_j) - \sum_{j=i}^k y_{jn} \\ &= (\zeta_{i-1}^{i-1} - \sum_{j=i}^k y_{jn}) \wedge (\sum_{j=i}^k u_j - \sum_{j=i}^k y_{jn}) = \tilde{\zeta}_{i-1} \wedge (\sum_{j=i}^k u_j - \sum_{j=i}^k y_{jn}),\end{aligned}$$

where the fifth equality comes from (EC.13) and the last equality comes from (EC.12). Q.E.D.

C.2. Analysis of Example 1

Analysis of Example 1 with $x_2 = 2$:

Recall that in period 1, the sales quantities of products are as follows:

$$y_{11} = d_{11} = 1, \quad 0 \le y_{21} \le d_{21} = 1, \quad \text{and} \quad 0 \le y_{31} \le d_{31} = 1.$$

Given a component profile $(x_0, x_1, x_2, x_3) = (3 - y_{21} - y_{31}, 2, 2 - y_{21}, 1 - y_{31})$ in the last period, the optimality of the nested allocation policy leads to

$$y_{12}^* = x_0 \wedge x_1 \wedge d_{12}, \quad y_{22}^* = (x_0 - y_{12}^*) \wedge x_2 \wedge d_{22}, \quad \text{and} \quad y_{32}^* = (x_0 - y_{12}^* - y_{22}^*) \wedge x_3 \wedge d_{32}.$$

We present the optimal strategy and the generated profit under each demand realization in Table EC.3, where $\mathbf{d_2} = (d_{12}, d_{22}, d_{32})$ and the column "Prob" is the probability of having d_{i2} units of demand for product $i \in \{1, 2, 3\}$.

For example, in the case of $d_{12} = 0$ and $d_{22} = 1$ with a probability 0.2 (the first two rows of Table EC.3), the sales quantity of product 1 is trivially $y_{12}^* = 0$ (since $d_{12} = 0$). The sales quantity of product 2 is $y_{22}^* = x_0 \wedge x_2 \wedge d_{22} = (3 - y_{21} - y_{31}) \wedge (2 - y_{21}) \wedge 1 = 1$ because $0 \le y_{21} \le 1$ and $0 \le y_{31} \le 1$.

Table EC.3 Optimal Component Allocation and Profit in Period 2

Prob	$\mathbf{d_2}$	$y_{21}+y_{31}\le 1$				$y_{21}+y_{31}\ge 1$			
		y_{12}^*	y_{22}^*	y_{32}^*	$J_2(3-y_{21}-y_{31},2,2-y_{21},1-y_{31},d_{12},d_{22},d_{32})$	y_{12}^*	y_{22}^*	y_{32}^*	$J_2(3-y_{21}-y_{31},2,2-y_{21},1-y_{31},d_{12},d_{22},d_{32})$
0.05	$(0,1,1)$	0	1	$1-y_{31}$	$p_2+p_3(1-y_{31})+(c_0-h_0)(1-y_{21})$	0	1	$1-y_{31}$	$p_2+p_3(1-y_{31})+(c_0-h_0)(1-y_{21})$
0.15	$(0,1,0)$	0	1	0	$p_2+(c_0-h_0)(2-y_{21}-y_{31})$	0	1	0	$p_2+(c_0-h_0)(2-y_{21}-y_{31})$
0.05	$(0,2,1)$	0	$2-y_{21}$	$1-y_{31}$	$p_2(2-y_{21})+p_3(1-y_{31})$	0	$2-y_{21}$	$1-y_{31}$	$p_2(2-y_{21})+p_3(1-y_{31})$
0.15	$(0,2,0)$	0	$2-y_{21}$	0	$p_2(2-y_{21})+(c_0-h_0)(1-y_{31})$	0	$2-y_{21}$	0	$p_2(2-y_{21})+(c_0-h_0)(1-y_{31})$
0.025	$(1,1,1)$	1	1	$1-y_{21}-y_{31}$	$p_1+p_2+p_3(1-y_{21}-y_{31})$	1	$2-y_{21}-y_{31}$	0	$p_1+p_2(2-y_{21}-y_{31})$
0.075	$(1,1,0)$	1	1	0	$p_1+p_2+(c_0-h_0)(1-y_{21}-y_{31})$	1	$2-y_{21}-y_{31}$	0	$p_1+p_2(2-y_{21}-y_{31})$
0.025	$(1,2,1)$	1	$2-y_{21}-y_{31}$	0	$p_1+p_2(2-y_{21}-y_{31})$	1	$2-y_{21}-y_{31}$	0	$p_1+p_2(2-y_{21}-y_{31})$
0.075	$(1,2,0)$	1	$2-y_{21}-y_{31}$	0	$p_1+p_2(2-y_{21}-y_{31})$	1	$2-y_{21}-y_{31}$	0	$p_1+p_2(2-y_{21}-y_{31})$
0.05	$(2,1,1)$	2	$1-y_{21}-y_{31}$	0	$2p_1+p_2(1-y_{21}-y_{31})$	$3-y_{21}-y_{31}$	0	0	$p_1(3-y_{21}-y_{31})$
0.15	$(2,1,0)$	2	$1-y_{21}-y_{31}$	0	$2p_1+p_2(1-y_{21}-y_{31})$	$3-y_{21}-y_{31}$	0	0	$p_1(3-y_{21}-y_{31})$
0.05	$(2,2,1)$	2	$1-y_{21}-y_{31}$	0	$2p_1+p_2(1-y_{21}-y_{31})$	$3-y_{21}-y_{31}$	0	0	$p_1(3-y_{21}-y_{31})$
0.15	$(2,2,0)$	2	$1-y_{21}-y_{31}$	0	$2p_1+p_2(1-y_{21}-y_{31})$	$3-y_{21}-y_{31}$	0	0	$p_1(3-y_{21}-y_{31})$

Then, the remaining number of common components is $3-y_{21}-y_{31}-y_{22}^*=2-y_{21}-y_{31}$. If the realized demand for product 3 is $d_{32}=1$, then

$$y_{32}^*=(2-y_{21}-y_{31})\wedge x_3\wedge d_{32}=(2-y_{21}-y_{31})\wedge(1-y_{31})\wedge 1=1-y_{31}$$

because $0\le y_{21}\le 1$. So, in this case the profit earned in period 2, together with the salvage value of the leftover component, is $p_2+p_3(1-y_{31})-h_0(1-y_{21})+c_0(1-y_{21})$, where $h_0(1-y_{21})$ is the holding cost for the leftover $1-y_{21}$ units of common component, and $c_0(1-y_{21})$ is the salvage value of these $1-y_{21}$ units of common component. If, however, $d_{32}=0$, then in this case the profit earned in period 2, together with the salvage value of the leftover component, is $p_2+(c_0-h_0)(2-y_{21}-y_{31})$, where $h_0(2-y_{21}-y_{31})$ is the holding cost for the leftover $2-y_{21}-y_{31}$ units of common component, and $c_0(2-y_{21}-y_{31})$ is the salvage value of these $2-y_{21}-y_{31}$ units of common component.

Similar analysis can be performed for other cases. Then, we obtain

$$R_2(X-a_1-\sum_{i=2}^{3}y_{i1}a_i)$$
$$=\begin{cases} p_1+(1.3-0.7y_{21}-0.5y_{31})p_2+(0.125-0.025y_{21}-0.125y_{31})p_3 \\ \qquad +(c_0-h_0)(0.575-0.275y_{21}-0.375y_{31}), & \text{if} \quad y_{21}+y_{31}\le 1 \\ (1.4-0.4y_{21}-0.4y_{31})p_1+(1-0.4y_{21}-0.2y_{31})p_2+0.1(1-y_{31})p_3 \\ \qquad +(c_0-h_0)(0.5-0.2y_{21}-0.3y_{31}), & \text{if} \quad y_{21}+y_{31}\ge 1. \end{cases}$$

Back to period 1, selling 1 unit of product 1, y_{21} units of product 2 and y_{31} units of product 3 yield an immediate profit $p_1+y_{21}p_2+y_{31}p_3-h_0(3-y_{21}-y_{31})$. As a result, if $0\le y_{21}+y_{31}\le 1$, the total profit $J_1(X,d_{11},d_{21},d_{31})$ is

$$\max_{y_{21},\ y_{31}}\ [2p_1+(1.3+0.3y_{21}-0.5y_{31})p_2+(0.125-0.025y_{21}+0.875y_{31})p_3$$
$$+c_0(0.575-0.275y_{21}-0.375y_{31})-h_0(3.575-1.275y_{21}-1.375y_{31})]\,.$$

Note that the first derivative of the expression in the bracket with respect to y_{31} is $-0.5p_2+0.875p_3-0.375c_0+1.375h_0$. Under the condition

$$p_3<p_2<\frac{9}{8}p_3-\frac{1}{8}c_0+\frac{1}{8}h_0 \text{ (or } p_3-c_0<p_2-c_0<\tfrac{9}{8}(p_3-c_0)+\tfrac{1}{8}h_0),$$

$$
\begin{aligned}
-0.5p_2+0.875p_3-0.375c_0+1.375h_0 &\geq -0.5(p_2-c_0)+0.875(p_3-c_0)+0.0625h_0 \\
&> 0,
\end{aligned}
$$

thus it is optimal to set $y_{31}=1-y_{21}$, and correspondingly,

$$
\begin{aligned}
&J_1(X,d_{11},d_{21},d_{31}) \\
=&\max_{y_{21}}\left[2p_1+(0.8+0.8y_{21})p_2+(1-0.9y_{21})p_3+c_0(0.2+0.1y_{21})-h_0(2.2+0.1y_{21})\right].
\end{aligned}
$$

Because the first derivative of $J_1(X,d_{11},d_{21},d_{31})$ with respect to y_{21} is $0.8p_2-0.9p_3+0.1c_0-0.1h_0<0$ under the condition $p_3<p_2<\frac{9}{8}p_3-\frac{1}{8}c_0+\frac{1}{8}h_0$, it is optimal to set y_{21} as small as possible, i.e., $y_{21}=0$, which results in $y_{31}=1$.

Meanwhile, if $y_{21}+y_{31}\geq 1$, the total profit $J_1(X,d_{11},d_{21},d_{31})$ is

$$
\begin{aligned}
\max_{y_{21},\ y_{31}}\ &\left[(2.4-0.4y_{21}-0.4y_{31})p_1+(1+0.6y_{21}-0.2y_{31})p_2+(0.1+0.9y_{31})p_3\right. \\
&\left.+c_0(0.5-0.2y_{21}-0.3y_{31})-h_0(3.5-1.2y_{21}-1.3y_{31})\right].
\end{aligned}
$$

Note that the first derivative of the expression in the bracket with respect to y_{21} is $-0.4p_1+0.6p_2-0.2c_0+1.2h_0$. Under the condition $p_1>1.5p_2-0.5c_0+3h_0$ (or $p_1-c_0>1.5(p_2-c_0)+3h_0$), $-0.4(p_1-c_0)+0.6(p_2-c_0)+1.2h_0<0$, thus it is optimal to set $y_{21}=1-y_{31}$, and correspondingly,

$$
\begin{aligned}
&J_1(X,d_{11},d_{21},d_{31}) \\
=&\max_{y_{31}}[2p_1+(1.6-0.8y_{31})p_2+(0.1+0.9y_{31})p_3+c_0(0.3-0.1y_{31})-h_0(2.3-0.1y_{31})].
\end{aligned}
$$

Under the condition $p_3<p_2<\frac{9}{8}p_3-\frac{1}{8}c_0+\frac{1}{8}h_0$, the first derivative of $J_1(X,d_{11},d_{21},d_{31})$ with respect to y_{31} is $-0.8p_2+0.9p_3-0.1c_0+0.1h_0>0$ it is optimal to set y_{31} as large as possible, i.e., $y_{31}=1$, which results in $y_{21}=0$.

Summarizing the two cases above, the overall optimal strategy is $y_{21}^*=0$ and $y_{31}^*=1$ with a total profit $2p_1+0.8p_2+p_3+0.2c_0-2.2h_0$.

Analysis of Example 1 with $x_2=5$:

Given a component profile $X_2=(3-y_{21}-y_{31},2,\infty,1-y_{31})$ in the last period, the generated profit under each demand realization is shown in Table EC.4, where the bolded strategies are different from their counterparts in Table EC.3. That is, they represent changes in the optimal strategy and profits due to an unlimited supply of component 2. From Table EC.4, we obtain

$$
\begin{aligned}
&R_2(X_2) \\
=&\begin{cases}
p_1+(1.3-0.5y_{21}-0.5y_{31})p_2+(0.125-0.075y_{21}-0.125y_{31})p_3 & \\
\qquad +(c_0-h_0)(0.575-0.425y_{21}-0.375y_{31}), & \text{if} \quad 0\leq y_{21}+y_{31}\leq 1 \\
(1.4-0.4y_{21}-0.4y_{31})p_1+(1.2-0.4y_{21}-0.4y_{31})p_2+0.05(1-y_{31})p_3 & \\
\qquad +(c_0-h_0)(0.35-0.2y_{21}-0.15y_{31}), & \text{if} \quad y_{21}+y_{31}\geq 1.
\end{cases}
\end{aligned}
$$

Table EC.4 Revised Example: Optimal Component Allocation and Profit in Period 2 (Assume Unlimited x_2)

Prob	$\mathbf{d_2}$	$y_{21}+y_{31}\le 1$				$y_{21}+y_{31}\ge 1$			
		y_{12}^*	y_{22}^*	y_{32}^*	$J_2(3-y_{21}-y_{31},2,2-y_{21},1-y_{31},d_{12},d_{22},d_{32})$	y_{12}^*	y_{22}^*	y_{32}^*	$J_2(3-y_{21}-y_{31},2,2-y_{21},1-y_{31},d_{12},d_{22},d_{32})$
0.05	$(0,1,1)$	0	1	$1-y_{31}$	$p_2+p_3(1-y_{31})+(c_0-h_0)(1-y_{21})$	0	1	$1-y_{31}$	$p_2+p_3(1-y_{31})+(c_0-h_0)(1-y_{21})$
0.15	$(0,1,0)$	0	1	0	$p_2+(c_0-h_0)(2-y_{21}-y_{31})$	0	1	0	$p_2+(c_0-h_0)(2-y_{21}-y_{31})$
0.05	$(0,2,1)$	0	$\mathbf{2}$	$\mathbf{1-y_{21}-y_{31}}$	$\mathbf{2p_2+p_3(1-y_{21}-y_{31})}$	0	$\mathbf{3-y_{21}-y_{31}}$	$\mathbf{0}$	$\mathbf{p_2(3-y_{21}-y_{31})}$
0.15	$(0,2,0)$	0	$\mathbf{2}$	0	$\mathbf{2p_2+(c_0-h_0)(1-y_{21}-y_{31})}$	0	$\mathbf{3-y_{21}-y_{31}}$	0	$\mathbf{p_2(3-y_{21}-y_{31})}$
0.025	$(1,1,1)$	1	1	$1-y_{21}-y_{31}$	$p_1+p_2+p_3(1-y_{21}-y_{31})$	1	$2-y_{21}-y_{31}$	0	$p_1+p_2(2-y_{21}-y_{31})$
0.075	$(1,1,0)$	1	1	0	$p_1+p_2+(c_0-h_0)(1-y_{21}-y_{31})$	1	$2-y_{21}-y_{31}$	0	$p_1+p_2(2-y_{21}-y_{31})$
0.025	$(1,2,1)$	1	$2-y_{21}-y_{31}$	0	$p_1+p_2(2-y_{21}-y_{31})$	1	$2-y_{21}-y_{31}$	0	$p_1+p_2(2-y_{21}-y_{31})$
0.075	$(1,2,0)$	1	$2-y_{21}-y_{31}$	0	$p_1+p_2(2-y_{21}-y_{31})$	1	$2-y_{21}-y_{31}$	0	$p_1+p_2(2-y_{21}-y_{31})$
0.05	$(2,1,1)$	2	$1-y_{21}-y_{31}$	0	$2p_1+p_2(1-y_{21}-y_{31})$	$3-y_{21}-y_{31}$	0	0	$p_1(3-y_{21}-y_{31})$
0.15	$(2,1,0)$	2	$1-y_{21}-y_{31}$	0	$2p_1+p_2(1-y_{21}-y_{31})$	$3-y_{21}-y_{31}$	0	0	$p_1(3-y_{21}-y_{31})$
0.05	$(2,2,1)$	2	$1-y_{21}-y_{31}$	0	$2p_1+p_2(1-y_{21}-y_{31})$	$3-y_{21}-y_{31}$	0	0	$p_1(3-y_{21}-y_{31})$
0.15	$(2,2,0)$	2	$1-y_{21}-y_{31}$	0	$2p_1+p_2(1-y_{21}-y_{31})$	$3-y_{21}-y_{31}$	0	0	$p_1(3-y_{21}-y_{31})$

Back to period 1, selling 1 unit of product 1, y_{21} units of product 2 and y_{31} units of product 3 yield an immediate profit $p_1+y_{21}p_2+y_{31}p_3-h_0(3-y_{21}-y_{31})$. As a result, if $0\le y_{21}+y_{31}\le 1$, the total profit is

$$\max_{y_{21},\ y_{31}}\ [2p_1+(1.3+0.5y_{21}-0.5y_{31})p_2+(0.125-0.075y_{21}+0.875y_{31})p_3$$
$$+c_0(0.575-0.425y_{21}-0.375y_{31})-h_0(3.575-1.425y_{21}-1.375y_{31})]\,.$$

Note that the first derivative of the expression in the bracket with respect to y_{21} is $0.5p_2-0.075p_3-0.425c_0+1.425h_0>0$, and it is optimal to set $y_{21}=1-y_{31}$, and correspondingly,

$$\begin{aligned}&J_1(X,d_{11},d_{21},d_{31})\\ =&\max_{y_{31}}[2p_1+(1.8-y_{31})p_2+(0.05+0.95y_{31})p_3+c_0(0.15+0.05y_{31})-h_0(2.15+0.05y_{31})],\end{aligned}$$

and it is optimal to set y_{31} as small as possible, i.e., $y_{31}=0$, because the first derivative of the expression with respect to y_{31} is

$$\begin{aligned}-p_2+0.95p_3+0.05c_0-0.05h_0&\le -p_2+0.95p_3+0.05c_0\\ &=-(p_2-c_0)+0.95(p_3-c_0)\\ &\le -(p_2-c_0)+(p_3-c_0)\\ &<0.\end{aligned}$$

Meanwhile, if $y_{21}+y_{31}\ge 1$, the total profit $J_1(X,d_{11},d_{21},d_{31})$ is

$$\max_{y_{21},\ y_{31}}\ [(2.4-0.4y_{21}-0.4y_{31})p_1+(1.2+0.6y_{21}-0.4y_{31})p_2+(0.05+0.95y_{31})p_3$$
$$+c_0(0.35-0.2y_{21}-0.15y_{31})-h_0(3.35-1.2y_{21}-1.15y_{31})]\,.$$

Note that the first derivative of the expression in the bracket with respect to y_{21} is $0.6p_2 - 0.4p_1 - 0.2c_0 + 1.2h_0 < 0$ under the condition $p_1 > 1.5p_2 - 0.5c_0 + 3h_0$, thus it is optimal to set $y_{21} = 1 - y_{31}$, and correspondingly,

$$\begin{aligned}&J_1(X, d_{11}, d_{21}, d_{31})\\ =& \max_{y_{31}} \left[2p_1 + (1.8 - y_{31})p_2 + (0.05 + 0.95y_{31})p_3 + c_0(0.15 + 0.05y_{31}) - h_0(2.15 + 0.05y_{31})\right],\end{aligned}$$

and it is optimal to set y_{31} as small as possible under $p_3 < p_2 < \frac{9}{8}p_3 - \frac{1}{8}c_0 + \frac{1}{8}h_0$, i.e., $y_{31} = 0$, because the first derivative of the expression with respect to y_{31} is

$$\begin{aligned}-p_2 + 0.95p_3 + 0.05c_0 - 0.05h_0 &\leq -p_2 + 0.95p_3 + 0.05c_0\\ &= -(p_2 - c_0) + 0.95(p_3 - c_0)\\ &\leq -(p_2 - c_0) + (p_3 - c_0) < 0.\end{aligned}$$

Summarizing both cases above, we have $y^*_{21} = 1$ and $y^*_{31} = 0$ with a total profit $2p_1 + 1.8p_2 + 0.05p_3 + 0.15c_0 - 2.15h_0$.

Analysis of Example 1 (continued):

At the beginning of period 1, $x_0 = 4$, $x_1 = 1$, $x_2 = 2$ and $x_3 = 1$ result in a balanced system, so all the demand in period 1 should be filled, i.e., $y^*_{11} = y^*_{21} = y^*_{31} = 1$. Then, the remaining component inventory at the end of period 1 is $(1, 0, 1, 0)$, which will incurs a holding cost h_0 at the end of period 1, and yield a profit p_2 in period 2. As a consequence, $R_1(4, 1, 2, 1) = p_1 + 2p_2 + p_3 - h_0$.

Given $x_0 = 4$, $x_1 = 1$, $x_2 = 2$ and $x_3 = 1.1$ at the beginning of period 1, after filling product 1 with $y^*_{11} = 1$, there remains no inventory for component 1, thus the system is reduced to a two-product system, in which the demand for product 2 should be filled as much as possible. Therefore, $y^*_{21} = 1$. Then, the remaining component inventory is $x_0 = 2$, $x_2 = 1$ and $x_3 = 1.1$, which implies that at most $x_2 = 1$ unit of product 2 can be sold, and one unit of product 3 should be sold, i,e., $y^*_{31} = 1$. Hence, the remaining component inventory at the end of period 1 is $(1, 0, 1, 0)$, which will incurs a holding cost h_0 at the end of period 1, and yield a profit p_2 in period 2. So, $R_1(4, 1, 2, 1.1) = p_1 + 2p_2 + p_3 - h_0$.

Example 1 verified that $R_1(4, 3, 2, 1) = 2p_1 + 0.8p_2 + p_3 + 0.2c_0 - 2.2h_0$. It remains to show $R_1(4, 3, 2, 1.1) = 2p_1 + 0.8p_2 + 1.005p_3 + 0.195c_0 - 2.195h_0$. After filling the demand for product 1, i.e., $y^*_{11} = 1$, the optimal strategy and the generated profit under each demand realization in the last period is shown in Table EC.5. For example, in the case of $d_{12} = 0$ and $d_{22} = 1$ with a probability 0.2 (the first two rows of Table EC.5), the sales quantity of product 1 is trivially $y^*_{12} = 0$ (since $d_{12} = 0$). The sales quantity of product 2 is $y^*_{22} = x_0 \wedge x_2 \wedge d_{22} = (3 - y_{21} - y_{31}) \wedge (2 - y_{21}) \wedge 1 = 1$ because $0 \leq y_{21}, y_{31} \leq 1$. Then, the remaining number of common components is $3 - y_{21} - y_{31} - y^*_{22} =$

$2-y_{21}-y_{31}$. If the realized demand for product 3 is $d_{32}=1$, $y_{32}^*=(2-y_{21}-y_{31})\wedge x_3\wedge d_{32}=(2-y_{21}-y_{31})\wedge(1.1-y_{31})\wedge 1=(1.1-y_{31})\wedge 1$ under the condition $0\le y_{21}+y_{31}\le 1$. So, in this case, when $d_{32}=1$, the profit earned in period 2, together with the salvage value of $1-y_{21}-y_{31}$ units of common component , is $p_2+p_3+(c_0-h_0)(1-y_{21}-y_{31})$ if $0\le y_{31}\le 0.1$, while the profit earned in period 2, together with the salvage value of $0.9-y_{21}$ units of common component, is $p_2+p_3(1.1-y_{31})+(c_0-h_0)(0.9-y_{21})$ if $0.1\le y_{31}\le 1$. However, if $d_{32}=0$, the profit earned in period 2, together with the salvage value of $2-y_{21}-y_{31}$ units of common component, is $p_2+(c_0-h_0)(2-y_{21}-y_{31})$.

Table EC.5 Optimal Component Allocation and Profit in Period 2 with $(3-y_{21}-y_{31},2,2-y_{21},1.1-y_{31})$

Prob	$\mathbf{d}_2$	$y_{21}+y_{31}\le 1$							
		$y_{31}\le 0.1$				$y_{31}\ge 0.1$			
		y_{12}^*	y_{22}^*	y_{32}^*	$J_2(3-y_{21}-y_{31},2,2-y_{21},1-y_{31},d_{12},d_{22},d_{32})$	y_{12}^*	y_{22}^*	y_{32}^*	$J_2(3-y_{21}-y_{31},2,2-y_{21},1-y_{31},d_{12},d_{22},d_{32})$
0.05	$(0,1,1)$	0	1	1	$p_2+p_3+(c_0-h_0)(1-y_{21}-y_{31})$	0	1	$1.1-y_{31}$	$p_2+p_3(1.1-y_{31})+(c_0-h_0)(0.9-y_{21})$
0.15	$(0,1,0)$	0	1	0	$p_2+(c_0-h_0)(2-y_{21}-y_{31})$	0	1	0	$p_2+(c_0-h_0)(2-y_{21}-y_{31})$
0.05	$(0,2,1)$	0	$2-y_{21}$	$1-y_{31}$	$p_2(2-y_{21})+p_3(1-y_{31})$	0	$2-y_{21}$	$1-y_{31}$	$p_2(2-y_{21})+p_3(1-y_{31})$
0.15	$(0,2,0)$	0	$2-y_{21}$	0	$p_2(2-y_{21})+(c_0-h_0)(1-y_{31})$	0	$2-y_{21}$	0	$p_2(2-y_{21})+(c_0-h_0)(1-y_{31})$
0.025	$(1,1,1)$	1	1	$1-y_{21}-y_{31}$	$p_1+p_2+p_3(1-y_{21}-y_{31})$	1	1	$1-y_{21}-y_{31}$	$p_1+p_2+p_3(1-y_{21}-y_{31})$
0.075	$(1,1,0)$	1	1	0	$p_1+p_2+(c_0-h_0)(1-y_{21}-y_{31})$	1	1	0	$p_1+p_2+(c_0-h_0)(1-y_{21}-y_{31})$
0.025	$(1,2,1)$	1	$2-y_{21}-y_{31}$	0	$p_1+p_2(2-y_{21}-y_{31})$	1	$2-y_{21}-y_{31}$	0	$p_1+p_2(2-y_{21}-y_{31})$
0.075	$(1,2,0)$	1	$2-y_{21}-y_{31}$	0	$p_1+p_2(2-y_{21}-y_{31})$	1	$2-y_{21}-y_{31}$	0	$p_1+p_2(2-y_{21}-y_{31})$
0.05	$(2,1,1)$	2	$1-y_{21}-y_{31}$	0	$2p_1+p_2(1-y_{21}-y_{31})$	2	$1-y_{21}-y_{31}$	0	$2p_1+p_2(1-y_{21}-y_{31})$
0.15	$(2,1,0)$	2	$1-y_{21}-y_{31}$	0	$2p_1+p_2(1-y_{21}-y_{31})$	2	$1-y_{21}-y_{31}$	0	$2p_1+p_2(1-y_{21}-y_{31})$
0.05	$(2,2,1)$	2	$1-y_{21}-y_{31}$	0	$2p_1+p_2(1-y_{21}-y_{31})$	2	$1-y_{21}-y_{31}$	0	$2p_1+p_2(1-y_{21}-y_{31})$
0.15	$(2,2,0)$	2	$1-y_{21}-y_{31}$	0	$2p_1+p_2(1-y_{21}-y_{31})$	2	$1-y_{21}-y_{31}$	0	$2p_1+p_2(1-y_{21}-y_{31})$
		$y_{21}+y_{31}\ge 1$							
		$y_{21}\le 0.9$				$y_{21}\ge 0.9$			
0.05	$(0,1,1)$	0	1	$1.1-y_{31}$	$p_2+p_3(1.1-y_{31})+(c_0-h_0)(0.9-y_{21})$	0	1	$2-y_{21}-y_{31}$	$p_2+p_3(2-y_{21}-y_{31})$
0.15	$(0,1,0)$	0	1	0	$p_2+(c_0-h_0)(2-y_{21}-y_{31})$	0	1	0	$p_2+(c_0-h_0)(2-y_{21}-y_{31})$
0.05	$(0,2,1)$	0	$2-y_{21}$	$1-y_{31}$	$p_2(2-y_{21})+p_3(1-y_{31})$	0	$2-y_{21}$	$1-y_{31}$	$p_2(2-y_{21})+p_3(1-y_{31})$
0.15	$(0,2,0)$	0	$2-y_{21}$	0	$p_2(2-y_{21})+(c_0-h_0)(1-y_{31})$	0	$2-y_{21}$	0	$p_2(2-y_{21})+(c_0-h_0)(1-y_{31})$
0.025	$(1,1,1)$	1	$2-y_{21}-y_{31}$	0	$p_1+p_2(2-y_{21}-y_{31})$	1	$2-y_{21}-y_{31}$	0	$p_1+p_2(2-y_{21}-y_{31})$
0.075	$(1,1,0)$	1	$2-y_{21}-y_{31}$	0	$p_1+p_2(2-y_{21}-y_{31})$	1	$2-y_{21}-y_{31}$	0	$p_1+p_2(2-y_{21}-y_{31})$
0.025	$(1,2,1)$	1	$2-y_{21}-y_{31}$	0	$p_1+p_2(2-y_{21}-y_{31})$	1	$2-y_{21}-y_{31}$	0	$p_1+p_2(2-y_{21}-y_{31})$
0.075	$(1,2,0)$	1	$2-y_{21}-y_{31}$	0	$p_1+p_2(2-y_{21}-y_{31})$	1	$2-y_{21}-y_{31}$	0	$p_1+p_2(2-y_{21}-y_{31})$
0.05	$(2,1,1)$	$3-y_{21}-y_{31}$	0	0	$p_1(3-y_{21}-y_{31})$	$3-y_{21}-y_{31}$	0	0	$p_1(3-y_{21}-y_{31})$
0.15	$(2,1,0)$	$3-y_{21}-y_{31}$	0	0	$p_1(3-y_{21}-y_{31})$	$3-y_{21}-y_{31}$	0	0	$p_1(3-y_{21}-y_{31})$
0.05	$(2,2,1)$	$3-y_{21}-y_{31}$	0	0	$p_1(3-y_{21}-y_{31})$	$3-y_{21}-y_{31}$	0	0	$p_1(3-y_{21}-y_{31})$
0.15	$(2,2,0)$	$3-y_{21}-y_{31}$	0	0	$p_1(3-y_{21}-y_{31})$	$3-y_{21}-y_{31}$	0	0	$p_1(3-y_{21}-y_{31})$

Then, in the case of $y_{21}+y_{31}\le 1$ we obtain

$$
\begin{aligned}
&R_2(3-y_{21}-y_{31},2,2-y_{21},1.1-y_{31})\\
=&\begin{cases}
p_1+(1.3-0.7y_{21}-0.5y_{31})p_2+(0.125-0.075y_{21}-0.025y_{31})p_3\\
\qquad +(c_0-h_0)(0.575-0.275y_{21}-0.425y_{31}), & \text{if} \quad 0\le y_{31}\le 0.1\\
p_1+(1.3-0.7y_{21}-0.5y_{31})p_2+(0.13-0.025y_{21}-0.125y_{31})p_3\\
\qquad +(c_0-h_0)(0.57-0.275y_{21}-0.375y_{31}), & \text{if} \quad 0.1\le y_{31}\le 1.
\end{cases}
\end{aligned}
$$

Therefore, the total profit under $y_{21}+y_{31}\le 1$ is

$$
\max_{y_{21},y_{31}}\left[p_1+y_{21}p_2+y_{31}p_3-h_0(3-y_{21}-y_{31})+R_2(3-y_{21}-y_{31},2,2-y_{21},1.1-y_{31})\right].
$$

It is easy to check that the first derivative with respect to y_{31} is $0.975p_3 - 0.5p_2 - 0.425c_0 + 1.425h_0$ if $0 \le y_{31} \le 0.1$, while $0.875p_3 - 0.5p_2 - 0.375c_0 + 1.375h_0$ if $0.1 \le y_{31} \le 1$. Both are positive under $p_2 < 9p_3/8 - \frac{1}{8}c_0 + \frac{1}{8}h_0$. So, in this case it is optimal to set $y_{21}^* = 0$ and $y_{31}^* = 1$, which results in a total profit $2p_1 + 0.8p_2 + 1.005p_3 + 0.195c_0 - 2.195h_0$.

On the other hand, in the case of $y_{21} + y_{31} \ge 1$ we obtain

$$\begin{aligned}
&R_2(3 - y_{21} - y_{31}, 2, 2 - y_{21}, 1.1 - y_{31}) \\
&= \begin{cases} (1.4 - 0.4y_{21} - 0.4y_{31})p_1 + (1 - 0.4y_{21} - 0.2y_{21})p_2 + (0.105 - 0.1y_{31})p_3 \\ \qquad + (c_0 - h_0)(0.495 - 0.2y_{21} - 0.3y_{31}), & \text{if} \quad 0 \le y_{21} \le 0.9 \\ (1.4 - 0.4y_{21} - 0.4y_{31})p_1 + (1 - 0.4y_{21} - 0.2y_{21})p_2 + (0.15 - 0.05y_{21} - 0.1y_{31})p_3 \\ \qquad + (c_0 - h_0)(0.45 - 0.15y_{21} - 0.3y_{31}), & \text{if} \quad 0.9 \le y_{21} \le 1. \end{cases}
\end{aligned}$$

Therefore, the total profit under $y_{21} + y_{31} \ge 1$ is

$$\max_{y_{21}, y_{31}} \left[p_1 + y_{21}p_2 + y_{31}p_3 - h_0(3 - y_{21} - y_{31}) + R_2(3 - y_{21} - y_{31}, 2, 2 - y_{21}, 1.1 - y_{31})\right].$$

It is easy to check that the first derivative with respect to y_{21} is $0.6p_2 - 0.4p_1 - 0.2c_0 + 1.2h_0$ if $0 \le y_{21} \le 0.9$, while $0.6p_2 - 0.4p_1 - 0.05p_3 - 0.15c_0 + 1.15h_0$ if $0.9 \le y_{21} \le 1$. Both are negative under $p_1 > 1.5p_2 - 0.5c_0 + 3h_0$. So, in this case it is optimal to set $y_{21}^* = 0$ and $y_{31}^* = 1$, which results in a total profit $2p_1 + 0.8p_2 + 1.005p_3 + 0.195c_0 - 2.195h_0$.

Summarizing both cases above, we obtain $y_{21}^* = 0$ and $y_{31}^* = 1$ under component profile $(4, 3, 2, 1.1)$, which results in $R_1(4, 3, 2, 1.1) = 2p_1 + 0.8p_2 + 1.005p_3 + 0.195c_0 - 2.195h_0$.

C.3. Technical Lemmas about the Joint Concavity

LEMMA EC.1. *If $f(X)$ is jointly concave in $X = (x_0, x_1, \ldots, x_k) \in \Xi \subseteq \mathcal{R}^{k+1}$, where $\Xi = \{X | x_j \ge 0,\ j = 0, \ldots, k\}$, then $\tilde{f}(x_0, x_1, \ldots, x_k, y_1, \ldots, y_k) := f(x_0 - \sum_{j=1}^k y_j, x_1 - y_1, \ldots, x_j - y_j, \ldots, x_k - y_k)$ is jointly concave in $(x_0, x_1, \ldots, x_k, y_1, \ldots, y_k)$ for $X \in \Xi$ and $Y \in A(X) \subseteq \mathcal{R}^k$, where*

$$A(X) = \{(y_1, \ldots, y_k) | \sum_{j=1}^k y_j \le x_0,\ 0 \le y_j \le x_j,\ j = 1, \ldots, k\}.$$

Proof: Note that both Ξ and $A(X)$ are convex sets, and $(x_0 - \sum_{j=1}^k y_j, x_1 - y_1, \ldots, x_j - y_j, \ldots, x_k - y_k) \in \Xi$ for $Y \in A(X)$. It suffices to prove that given $\lambda \in [0, 1]$, $\lambda \tilde{f}(X, Y) + (1 - \lambda)\tilde{f}(X', Y') \le \tilde{f}(\lambda X + (1 - \lambda)X', \lambda Y + (1 - \lambda)Y')$ for $X, X' \in \Xi$, $Y \in A(X)$ and $Y' \in A(X')$, where

$$\begin{aligned}
&\tilde{f}(X, Y) = f(x_0 - \sum_{j=1}^k y_j, x_1 - y_1, \ldots, x_j - y_j, \ldots, x_k - y_k), \\
&\tilde{f}(X', Y') = f(x_0' - \sum_{j=1}^k y_j', x_1' - y_1', \ldots, x_j' - y_j', \ldots, x_k' - y_k'), \text{ and} \\
&\tilde{f}(\lambda X + (1 - \lambda)X', \lambda Y + (1 - \lambda)Y') \\
&= f\left(\lambda(x_0 - \sum_{j=1}^k y_j) + (1 - \lambda)(x_0' - \sum_{j=1}^k y_j'), \lambda(x_1 - y_1) + (1 - \lambda)(x_1' - y_1'), \ldots, \lambda(x_k - y_k) + (1 - \lambda)(x_k' - y_k')\right).
\end{aligned}$$

Let $\tilde{X} = (x_0 - \sum_{j=1}^k y_j, x_1 - y_1, ..., x_k - y_k)$ and $\tilde{X}' = (x_0' - \sum_{j=1}^k y_j', x_1' - y_1', ..., x_k' - y_k')$. The joint concavity of $f(\cdot)$ implies $\lambda f(\tilde{X}) + (1-\lambda) f(\tilde{X}') \le f(\lambda \tilde{X} + (1-\lambda)\tilde{X}')$. That is:

$$\lambda \tilde{f}(X,Y) + (1-\lambda)\tilde{f}(X^{'},Y^{'}) \le \tilde{f}(\lambda X + (1-\lambda)X^{'}, \lambda Y + (1-\lambda)Y^{'}).$$

Q.E.D.

Lemma EC.2 **(Simchi-Levi et al. (2004), Proposition 2.1.15(b))**. *Given a convex set* $\Xi \subseteq \mathcal{R}^m$*, for any* $Z = (z_1, \ldots, z_m) \in \Xi$*, there exists an associated convex set* $A(Z) \in \mathcal{R}^k$ *which is closed and bounded, and* $\mathcal{C} := \{(Z,Y) | Z \in \Xi, Y = (y_1, \ldots, y_k) \in A(Z)\}$ *is a convex set. If* $f(Z,Y)$ *is jointly concave in* (Z,Y) *and the function* $\tilde{f}(Z) := \max_{Y \in A(Z)} f(Z,Y)$ *is well defined, then* $\tilde{f}(Z)$ *is jointly concave in* Z *over* Ξ.

Lemma EC.3. *For a generalized W system with component profile* $X = (x_0, x_1, \ldots, x_k)$*,* $R_n(X)$ *in (1) is jointly concave in* X*. Moreover,* $J_n(X, d_{1n}, \ldots, d_{kn})$ *in (1) is jointly concave in* X *for given* $(d_{1n}, \ldots, d_{kn})$.

Proof: Recall the definition of $R_n(X)$ in (1): $R_{N+1}(X) = \sum_{j=0}^k c_j x_j$, and for $n \ge 1$, $R_n(X) = E_{D_{1n},\ldots,D_{kn}} J_n(X, D_{1n}, \ldots, D_{kn})$, where $J_n(X, d_{1n}, \ldots, d_{kn})$ is

$$\max_{y_{1n},\ldots,y_{kn}} \left[-\sum_{j=0}^k (h_j x_j) + \sum_{i=1}^k ((p_i + h_i + h_0) y_{in}) + R_{n+1}(x_0 - \sum_{i=1}^k y_{in}, x_1 - y_{1n}, \ldots, x_k - y_{kn})\right]$$

$$\text{s.t.} \quad 0 \le y_{in} \le x_i \wedge d_{in}, \;\; i = 1, \ldots, k, \;\; \sum_{i=1}^k y_{in} \le x_0.$$

We prove the lemma by induction. First note that as a linear function, $R_{N+1}(X) = \sum_{j=0}^k (c_j x_j)$ is jointly concave in $(x_0, x_1, \ldots, x_k)$. Assume $R_{n+1}(x_0, x_1, \ldots, x_k)$ is jointly concave in $(x_0, x_1, \ldots, x_k)$.

To show the concavity, note that $\Xi = \{x_j \ge 0, j = 0, 1, \ldots, k\}$ is a convex set. Also, fixing d_{jn}, $j = 1, \ldots, k$, for any $X = (x_0, x_1, \ldots, x_k)$, it is easy to check that $A(X) = \{(y_{1n}, \ldots, y_{kn}) | 0 \le y_{in} \le x_i, y_{in} \le d_{in}, \sum_{i=1}^k y_{in} \le x_0, i = 1, \ldots, k\}$ is a convex set, because all the constraints are linear. Furthermore, for any given X, $A(X)$ is a closed and bounded set in $\mathcal{R}^k$.

Now we prove that $\mathcal{C} = \{(X,Y) | X \in \Xi, Y \in A(X)\}$ is a convex set. Note that for $(X,Y), (X^{'}, Y^{'}) \in \mathcal{C}$ and $0 \le \lambda \le 1$, we have

$$\begin{aligned}
&\lambda x_j + (1-\lambda) x_j^{'} \ge 0, \quad j = 0, 1, \ldots, k \\
&0 \le \lambda y_{in} + (1-\lambda) y_{in}^{'} \le \lambda x_i + (1-\lambda) x_i^{'}, i = 1, \ldots, k \\
&\lambda y_{in} + (1-\lambda) y_{in}^{'} \le d_{in}, \quad i = 1, \ldots, k, \\
&\lambda \sum_{i=1}^k y_{in} + (1-\lambda) \sum_{i=1}^k y_{in}^{'} \le \lambda x_0 + (1-\lambda) x_0^{'},
\end{aligned}$$

since all the constraints are linear. Thus, $(\lambda X+(1-\lambda)X', \lambda Y+(1-\lambda)Y')\in\mathcal{C}$ and $\mathcal{C}$ is convex.

By the induction hypothesis and Lemma EC.1, $R_{n+1}(x_0-\sum_{i=1}^k y_{in}, x_1-y_{1n},\ldots,x_k-y_{kn})$ is jointly concave in $(x_0,x_1,\ldots,x_k,y_{1n},\ldots,y_{kn})$. Meanwhile, $-\sum_{j=0}^k(h_jx_j)+\sum_{i=1}^k((p_i+h_i+h_0)y_{in})$ is clearly jointly concave in $(x_0,x_1,\ldots,x_k,y_{1n},\ldots,y_{kn})$. Thus,

$$-\sum_{j=0}^{k}(h_jx_j)+\sum_{i=1}^{k}((p_i+h_i+h_0)y_{in})+R_{n+1}(x_0-\sum_{i=1}^{k}y_{in},x_1-y_{1n},\ldots,x_k-y_{kn})$$

is jointly concave in $(x_0,x_1,\ldots,x_k,y_{1n},\ldots,y_{kn})$. By Lemma EC.2, this result, together with that $A(X)$ is closed and bounded convex set and $\mathcal{C}$ is a convex set, implies the joint concavity of $J_n(X,d_{1n},\ldots,d_{kn})$ in $X=(x_0,x_1,\ldots,x_k)$ for given $(d_{1n},\ldots,d_{kn})$. As taking expectation preserves concavity, taking expectation of $J_n(X,d_{1n},\ldots,d_{kn})$ over $(d_{1n},\ldots,d_{kn})$, we have the joint concavity of $R_n(X)$ in $X=(x_0,x_1,\ldots,x_k)$. This completes the proof. Q.E.D.

C.4. Technical Lemmas for Proposition 2

In this subsection we first show the optimality of the nested allocation policy for a W system in Lemma EC.4. To characterize the critical function $\bar{y}(X)$ in Proposition 2, we then recall some important concepts and results on anti-multimodularity from Li and Yu (2014). These are presented in Lemmas EC.5 through EC.7 (and the definitions prior to the lemmas), and are used to prove Proposition EC.1, which is then used to prove the properties of the critical function $\bar{y}(X)$ in Lemma EC.8.

C.4.1. Optimality of Nested Policy in W system

Lemma EC.4. *Consider a W system, i.e., $k=2$.*
(i) $y^*_{1n}=x_0\wedge x_1\wedge d_{1n}$. Thus, for a W system, a nested policy is always optimal.
(ii) The optimal profit-to-go after observing d_{in} units of demand for product $i\in\{1,2\}$, as defined in (2), can be rewritten as

$$J_n(X,d_{1n},d_{2n})=(p_1+h_1+h_0)y^*_{1n}-\sum_{j=0}^{2}(h_jx_j)+G_n(X-y^*_{1n}a_1,d_{2n}), \tag{EC.14}$$

where $y^*_{1n}=x_1\wedge x_0\wedge d_{1n}$ and

$$G_n(X,d_{2n})=\max_{y_{2n}}[(p_2+h_2+h_0)y_{2n}+R_{n+1}(X-y_{2n}a_2)] \tag{EC.15}$$
$$\text{subject to}\quad 0\le y_{2n}\le x_2\wedge d_{2n},\quad y_{2n}\le x_0.$$

Proof: (i) The result is due to the facts that product 1 is of the highest "effective" margin among all the products (i.e., $p_i+h_i+h_0-c_0-c_i$ is the highest when $i=1$) and $p_1>c_0+c_1$. A formal proof is provided below.

It suffices to prove that $R_n(X+\epsilon a_1)-R_n(X)\le \epsilon(p_1+h_1+h_0)$ for any arbitrarily small $\epsilon\ge 0$. Specifically, we compare two systems: system 1 has a component profile X, while system 2 has a component profile $X+\epsilon a_1$.

Next, we show $R_n(X+\epsilon a_1)-R_n(X)\le \epsilon(p_1+h_1+h_0)$ by induction. The boundary case is when $n=N+1$:

$$R_{N+1}(X+\epsilon a_1)-R_{N+1}(X)=\epsilon(c_1+c_0)\le \epsilon(p_1+h_1+h_0)$$

for any X. For $n\le N$, given the induction hypothesis $R_{n+1}(X+\epsilon a_1)-R_{n+1}(X)\le \epsilon(p_1+h_1+h_0)$ for any X, we shall show that

$$J_n(X+\epsilon,d_{1n},d_{2n})-J_n(X,d_{1n},d_{2n})\le \epsilon(p_1+h_1+h_0).$$

With this result, $R_n(X+\epsilon a_1)-R_n(X)\le \epsilon(p_1+h_1+h_0)$ then follows by taking expectation over (d_{1n},d_{2n}).

For expositional convenience, we only consider $x_1\le x_0$ in the following proof (it is easy to verify that the result holds when $x_1>x_0$ since $(x_1-x_0)^+$ units of component 1 generate no additional revenue). Denote by y_{1n}^* and y_{2n}^*, respectively, the optimal sales quantity of product 1 and product 2 in system 2 for period n. Clearly, $0\le y_{1n}^*\le d_{1n}$, $0\le y_{2n}^*\le d_{2n}$, $y_{1n}^*\le x_1+\epsilon\le x_0+\epsilon$ and $y_{2n}^*\le x_2\wedge(x_0+\epsilon-y_{1n}^*)$. It suffices to consider *(a)* $0\le y_{1n}^*\le x_1$ and *(b)* $x_1<y_{1n}^*\le x_1+\epsilon$, respectively. For exposition, within this proof we define $\tilde{p}_i:=p_i+h_i+h_0$, $i=1,2$.

In the following, we first investigate two sub-cases for **scenario (a)** $0\le y_{1n}^*\le x_1$.

- (a.1): $0\le y_{1n}^*\le x_1$ and $0\le y_{2n}^*\le x_2\wedge(x_0-y_{1n}^*)$. Selling y_{1n}^* units of product-1 and y_{2n}^* units of product-2 is feasible in system 1, because $0\le y_{1n}^*\le x_0\wedge x_1=x_1$ and $0\le y_{2n}^*\le x_2\wedge(x_0-y_{1n}^*)$. Hence, the difference between the profit-to-go in system 2 and system 1 is

$$\begin{aligned}
&J_n(X+\epsilon a_1,d_{1n},d_{2n})-J_n(X,d_{1n},d_{2n})\\
\le\ & -h_0(x_0+\epsilon)-h_1(x_1+\epsilon)-h_2x_2+\sum_{i=1}^{2}(\tilde{p}_iy_{ij}^*)+R_{n+1}(X+(\epsilon-y_{1n}^*)a_1-y_{2n}^*a_2)\\
&-[-\sum_{j=0}^{2}(h_jx_j)+\sum_{i=1}^{2}(\tilde{p}_iy_{ij}^*)+R_{n+1}(X-y_{1n}^*a_1-y_{2n}^*a_2)]\\
=\ & -\epsilon(h_0+h_1)+R_{n+1}(X+(\epsilon-y_{1n}^*)a_1-y_{2n}^*a_2)-R_{n+1}(X-y_{1n}^*a_1-y_{2n}^*a_2)\\
\le\ & \epsilon\tilde{p}_i=\epsilon(p_1+h_1+h_0),
\end{aligned}$$

where the first inequality is by the feasibility of (y_{1n}^*,y_{2n}^*) in system 1 and the second inequality is from the induction hypothesis and $h_0+h_1\ge 0$.

- (a.2): $0\le y_{1n}^*\le x_1$ and $0\le x_2\wedge(x_0-y_{1n}^*)<y_{2n}^*\le x_2\wedge(x_0+\epsilon-y_{1n}^*)$. Note that, if $x_2\le x_0-y_{1n}^*$, this case is invalid. Thus, we focus on $x_2>x_0-y_{1n}^*$ below. Under this condition, the sub-case implies $x_0 - y^*_{1n} < y^*_{2n} \le x_2 \wedge d_{2n}$. Hence, selling y^*_{1n} units of product 1 and $x_0 - y^*_{1n}$ units of product 2 is feasible in system 1. Thus,

$$\begin{aligned}
&J_n(X+\epsilon a_1, d_{1n}, d_{2n}) - J_n(X, d_{1n}, d_{2n})\\
\le\ &-h_0(x_0+\epsilon) - h_1(x_1+\epsilon) - h_2x_2 + \sum_{i=1}^{2}(\tilde{p}_i y^*_{ij}) + R_{n+1}(X + (\epsilon - y^*_{1n})a_1 - y^*_{2n}a_2)\\
&-[-\sum_{j=0}^{2}(h_jx_j) + \tilde{p}_1 y^*_{1n} + \tilde{p}_2(x_0 - y^*_{1n}) + R_{n+1}(X - y^*_{1n}a_1 - (x_0 - y^*_{1n})a_2)]\\
=\ &-\epsilon(h_0+h_1) + R_{n+1}(X + (\epsilon - y^*_{1n})a_1 - y^*_{2n}a_2) + \tilde{p}_2(y^*_{1n} + y^*_{2n} - x_0) - R_{n+1}(X - y^*_{1n}a_1 - (x_0 - y^*_{1n})a_2)\\
\le\ &R_{n+1}(X + (\epsilon - y^*_{1n})a_1 - y^*_{2n}a_2) + \tilde{p}_2(y^*_{1n} + y^*_{2n} - x_0) - R_{n+1}(X - y^*_{1n}a_1 - (x_0 - y^*_{1n})a_2)\\
\le\ &R_{n+1}(X + (\epsilon - y^*_{1n})a_1 - y^*_{2n}a_2) + \tilde{p}_1(y^*_{1n} + y^*_{2n} - x_0) - R_{n+1}(0, x_1 + y^*_{2n} - x_0, x_2 - y^*_{2n})\\
\le\ &(p_1+h_1+h_0)(x_0+\epsilon - y^*_{1n} - y^*_{2n}) + \tilde{p}_1(y^*_{1n} + y^*_{2n} - x_0)\\
=\ &\epsilon(p_1+h_1+h_0),
\end{aligned}$$

Next, we examine **scenario (b)** $x_1 < y^*_{1n} \le x_1+\epsilon$ and consider the following two subcases.

- (b.1): $x_1 < y^*_{1n} \le x_1+\epsilon$ and $0 \le y^*_{2n} \le x_2 \wedge (x_0 - x_1)$. In system 1, a feasible policy is to sell x_1 units of product 1 and y^*_{2n} units of product 2 because $0 \le x_1 < y^*_{1n} \le d_{1n}$, $x_1 \le x_0$ and $0 \le y^*_{2n} \le x_2 \wedge (x_0 - x_1)$. Clearly, the induction hypothesis implies

$$\begin{aligned}
&R_{n+1}(X + (\epsilon - y^*_{1n})a_1 - y^*_{2n}a_2) - R_{n+1}(X - x_1a_1 - y^*_{2n}a_2)\\
\le\ &(\epsilon + x_1 - y^*_{1n})(p_1+h_1+h_0).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&J_n(X+\epsilon a_1, d_{1n}, d_{2n}) - J_n(X, d_{1n}, d_{2n})\\
\le\ &-h_0(x_0+\epsilon) - h_1(x_1+\epsilon) - h_2x_2 + \tilde{p}_1 y^*_{1n} + \tilde{p}_2 y^*_{2n} + R_{n+1}(X + (\epsilon - y^*_{1n})a_1 - y^*_{2n}a_2)\\
&-[-\sum_{j=0}^{2}(h_jx_j) + \tilde{p}_1 x_1 + \tilde{p}_2 y^*_{2n} + R_{n+1}(X - x_1a_1 - y^*_{2n}a_2)]\\
=\ &-\epsilon(h_0+h_1) + (p_1+h_1+h_0)(y^*_{1n} - x_1) + R_{n+1}(X + (\epsilon - y^*_{1n})a_1 - y^*_{2n}a_2) - R_{n+1}(X - x_1a_1 - y^*_{2n}a_2)\\
\le\ &\epsilon p_1 \le \epsilon(p_1+h_1+h_0).
\end{aligned}$$

- (b.2): $x_1 < y^*_{1n} \le x_1+\epsilon$ and $0 \le x_2 \wedge (x_0 - x_1) < y^*_{2n} \le x_2 \wedge (x_0 + \epsilon - y^*_{1n})$. If $x_2 \le x_0 - x_1$, this case is invalid. Thus, we focus on $x_2 > x_0 - x_1$ below. Note that under this condition, $x_1 < y^*_{1n} \le x_1 + \epsilon$ and $x_0 - x_1 < y^*_{2n} \le x_2 \wedge (x_0 + \epsilon - y^*_{1n})$. Then, it suffices to check the following two subcases.

(b.2.1): Suppose $y^*_{2n} \le x_0$. In system 1, a feasible policy is to sell $x_0 - y^*_{2n}$ units of product 1 and y^*_{2n} units of product 2, because $y^*_{2n} \le x_0$ (from the condition of Subcase b.2.1), $y^*_{2n} \le x_2$ (by the optimality of y^*_{2n} in system 2) and $x_0 - y^*_{2n} < x_1 < d_{1n}$ (the first inequality is due to $x_0 - x_1 < y^*_{2n}$

from the conditions of Case b.2, and the second inequality follows from $x_1 < y^*_{1n} \le d_{1n}$ with $x_1 < y^*_{1n}$ from the conditions of Case b.2). Clearly, we have

$$\begin{aligned}
&R_{n+1}(X + (\epsilon - y^*_{1n})a_1 - y^*_{2n}a_2) - R_{n+1}(X - (x_0 - y^*_{2n})a_1 - y^*_{2n}a_2) \\
\le\ &(\epsilon + x_0 - y^*_{1n} - y^*_{2n})(p_1 + h_1 + h_0)
\end{aligned}$$

according to the induction hypothesis. Therefore,

$$\begin{aligned}
&J_n(X + \epsilon a_1, d_{1n}, d_{2n}) - J_n(X, d_{1n}, d_{2n}) \\
\le\ &-h_0(x_0 + \epsilon) - h_1(x_1 + \epsilon) - h_2x_2 + \tilde{p}_1 y^*_{1n} + \tilde{p}_2 y^*_{2n} + R_{n+1}(X + (\epsilon - y^*_{1n})a_1 - y^*_{2n}a_2) \\
&-[-\sum_{j=0}^{2}(h_jx_j) + \tilde{p}_1(x_0 - y^*_{2n}) + \tilde{p}_2 y^*_{2n} + R_{n+1}(X - (x_0 - y^*_{2n})a_1 - y^*_{2n}a_2)] \\
=\ &-\epsilon(h_0 + h_1) + \tilde{p}_1(y^*_{1n} + y^*_{2n} - x_0) + R_{n+1}(X + (\epsilon - y^*_{1n})a_1 - y^*_{2n}a_2) - R_{n+1}(X - (x_0 - y^*_{2n})a_1 - y^*_{2n}a_2) \\
\le\ &\epsilon(p_1 + h_1 + h_0).
\end{aligned}$$

(b.2.2): Suppose $x_0 < y^*_{2n} \le x_2 \wedge (x_0 + \epsilon - y^*_{1n})$. In system 1, a feasible policy is to sell $y^*_{1n} \wedge x_1 = x_1$ units of product 1 and $x_0 - x_1$ units of product 2, because $x_1 \le x_0$, $x_1 < y^*_{1n} \le d_{1n}$ (from the conditions of Case b.2), $x_0 - x_1 \le x_0 < y^*_{2n} \le d_{2n}$ (the second inequality is due to the condition of Subcase b.2.2) and $x_0 - x_1 < x_2$. This will deplete all the common components in system 1, and thus in system 1 the firm makes no sales in the remaining season and only incurs the salvage value and holding costs. As a consequence,

$$R_{n+1}(0, 0, x_2 - (x_0 - x_1)) = -h_2(N - n)(x_2 - (x_0 - x_1)) + c_2(x_2 - (x_0 - x_1))$$

Similarly,

$$\begin{aligned}
&R_{n+1}(0, -x_0 + x_1 + y^*_{2n}, x_2 - y^*_{2n}) \\
=\ &-h_1(N - n)(-x_0 + x_1 + y^*_{2n}) - h_2(N - n)(x_2 - y^*_{2n}) + c_1(-x_0 + x_1 + y^*_{2n}) + c_2(x_2 - y^*_{2n}).
\end{aligned}$$

Hence,

$$\begin{aligned}
&R_{n+1}(0, -x_0 + x_1 + y^*_{2n}, x_2 - y^*_{2n}) - R_{n+1}(0, 0, x_2 - (x_0 - x_1)) \\
=&- h_1(N - n)(-x_0 + x_1 + y^*_{2n}) - h_2(N - n)(x_2 - y^*_{2n}) + c_1(-x_0 + x_1 + y^*_{2n}) \\
&+ c_2(x_2 - y^*_{2n}) + h_2(N - n)(x_2 - (x_0 - x_1)) - c_2(x_2 - (x_0 - x_1)) \\
=&- (h_1 - h_2)(N - n)(-x_0 + x_1 + y^*_{2n}) + (c_1 - c_2)(-x_0 + x_1 + y^*_{2n})
\end{aligned}$$

Thus,

$$\begin{aligned}
& R_{n+1}(x_0+\epsilon-y_{1n}^*-y_{2n}^*, x_1+\epsilon-y_{1n}^*, x_2-y_{2n}^*)-R_{n+1}(0,0,x_2-(x_0-x_1)) \\
= & R_{n+1}(x_0+\epsilon-y_{1n}^*-y_{2n}^*, x_1+\epsilon-y_{1n}^*, x_2-y_{2n}^*)-R_{n+1}(0,-x_0+x_1+y_{2n}^*, x_2-y_{2n}^*) \\
& -(h_1-h_2)(N-n)(-x_0+x_1+y_{2n}^*)+(c_1-c_2)(-x_0+x_1+y_{2n}^*) \\
\leq & R_{n+1}(x_0+\epsilon-y_{1n}^*-y_{2n}^*, x_1+\epsilon-y_{1n}^*, x_2-y_{2n}^*)-R_{n+1}(0,-x_0+x_1+y_{2n}^*, x_2-y_{2n}^*) \\
& +(c_1-c_2)(-x_0+x_1+y_{2n}^*) \\
\leq & (p_1+h_1+h_0)(x_0+\epsilon-y_{1n}^*-y_{2n}^*)+(c_1-c_2)(-x_0+x_1+y_{2n}^*),
\end{aligned} \tag{EC.16}$$

where the first inequality follows from $h_1 \geq h_2$ and $-x_0+x_1+y_{2n}^* \geq 0$, and the second inequality holds because of the induction hypothesis.

Hence,

$$\begin{aligned}
& J_n(X+\epsilon a_1, d_{1n}, d_{2n})-J_n(X, d_{1n}, d_{2n}) \\
\leq & -h_0(x_0+\epsilon)-h_1(x_1+\epsilon)-h_2x_2+\tilde{p}_1y_{1n}^*+\tilde{p}_2y_{2n}^*+R_{n+1}(X+(\epsilon-y_{1n}^*)a_1-y_{2n}^*a_2) \\
& -[-\sum_{j=0}^{2}(h_jx_j)+\tilde{p}_1x_1+\tilde{p}_2(x_0-x_1)+R_{n+1}(0,0,x_2-(x_0-x_1))] \\
= & -\epsilon(h_0+h_1)+(p_1+h_1+h_0)(y_{1n}^*-x_1)+(p_2+h_2+h_0)(x_1+y_{2n}^*-x_0) \\
& +R_{n+1}(x_0+\epsilon-y_{1n}^*-y_{2n}^*, x_1+\epsilon-y_{1n}^*, x_2-y_{2n}^*)-R_{n+1}(0,0,x_2-(x_0-x_1)) \\
\leq & -\epsilon(h_0+h_1)+(p_1+h_1+h_0)(y_{1n}^*-x_1)+(p_2+h_2+h_0)(x_1+y_{2n}^*-x_0) \\
& +(p_1+h_1+h_0)(x_0+\epsilon-y_{1n}^*-y_{2n}^*)+(c_1-c_2)(x_1+y_{2n}^*-x_0) \\
\leq & (p_1+h_1+h_0)(x_0+\epsilon-x_1-y_{2n}^*)+(p_2+h_2+h_0+c_1-c_2)(x_1+y_{2n}^*-x_0) \\
\leq & \epsilon(p_1+h_1+h_0),
\end{aligned}$$

where the first inequality is due to the feasibility of selling x_1 units of product 1 and x_0-x_1 units of product 2 in system 1, the second inequality is by (EC.16), the third inequality is due to $-\epsilon(h_0+h_1) \leq 0$, and the fourth inequality is implied by $p_1+h_1+h_0-c_1 \geq p_2+h_2+h_0-c_2$ and $-x_0+x_1+y_{2n}^* \geq 0$.

Combining all the cases (a) and (b) above, we obtain

$$J_n(X+\epsilon a_1, d_{1n}, d_{2n})-J_n(X, d_{1n}, d_{2n}) \leq \epsilon(p_1+h_1+h_0).$$

Thus, $R_n(X+\epsilon a_1)-R_n(X) \leq \epsilon(p_1+h_1+h_0)$ implying

$$\frac{dR_n(X+ya_1)}{dy} = \lim_{\epsilon \to 0}\frac{1}{\epsilon}(R_n(X+\epsilon a_1)-R_n(X)) \leq p_1+h_1+h_0.$$

As a result, $p_1 + h_1 + h_0 + \frac{dR_{n+1}(X - y_{1n}a_1)}{dy_{1n}} \geq 0$, i.e., the first derivative of the expected-profit function with respect to y_{1n} is always nonnegative, which implies that the demand for product 1 is always fulfilled if both component 0 and component 1 are available. In other words, it is always optimal to set $y^*_{1n} = x_0 \wedge x_1 \wedge d_{1n}$.

(ii) As implied in part (i), we can reformulate the decisions at the end of period n as a two-stage dynamic programming problem: the first stage is to fill as much of the product-1 demand as possible, while the second stage is to process (that is, either accept or reject) each demand for product 2. These two stages of the dynamic programming problem are represented by (EC.14) and (EC.15). Q.E.D.

C.4.2. Results on Anti-multimodularity

Definitions (Li and Yu (2014)). A set $\Xi \subseteq \mathbb{R}^n$ is called a polyhedron if there exist $\alpha_i \in \mathbb{R}^n$ and $\beta_i \in \mathbb{R}$, such that $\Xi = \{\mathbf{v} = (v_1, \ldots, v_n) \in \mathbb{R}^n | \alpha_i \cdot \mathbf{v} \geq \beta_i, i = 1, 2, \ldots, m\}$ for some $m \in \mathbb{N}$. Anti-multimodular (multimodular) functions are defined on a special polyhedral form:

(P1) Each n-dimensional vector α_i has the form $\pm(0, \ldots, 0, 1, \ldots, 1, 0, \ldots, 0)$; that is, the nonzero components of α_i are either consecutive 1's or consecutive -1's.

Let $\Xi \subseteq \mathbb{R}^n$ and $W \subseteq \mathbb{R}$ be polyhedrons satisfying (P1). A function $g : \Xi \to \mathbb{R}$ is anti-multimodular (multimodular) if $\psi(\mathbf{x}, y) = g(x_1 - y, x_2 - x_1, \ldots, x_n - x_{n+1})$ is supermodular (submodular) on $\mathbf{S} = \{(\mathbf{x}, y) \subseteq \mathbb{R}^n \times \mathbb{R} | y \in W, (x_1 - y, x_2 - x_1, \ldots, x_n - x_{n+1}) \in \Xi\}$.

The lemma below shows that anti-multimodularity implies decreasing difference and concavity.

Lemma EC.5 **(Li and Yu (2014))**. *Suppose that $g(\boldsymbol{v})$ is anti-multimodular.*

(i) If $g(\boldsymbol{v})$ is continuous, then it is jointly concave.

(ii) If $g(\boldsymbol{v})$ is twice differentiable, then for all i,

$$\frac{\partial^2 g}{\partial v_i^2} \leq \frac{\partial^2 g}{\partial v_i \partial v_{i+1}} \leq \frac{\partial^2 g}{\partial v_i \partial v_{i+2}} \leq \cdots \leq \frac{\partial^2 g}{\partial v_i \partial v_n} \leq 0,$$
$$\frac{\partial^2 g}{\partial v_i^2} \leq \frac{\partial^2 g}{\partial v_i \partial v_{i-1}} \leq \frac{\partial^2 g}{\partial v_i \partial v_{i-2}} \leq \cdots \leq \frac{\partial^2 g}{\partial v_i \partial v_1} \leq 0.$$

The following are basic operations preserving anti-multimodularity.

Lemma EC.6 **(Li and Yu (2014))**. *(i) If $g(\boldsymbol{v})$ is anti-multimodular and $k > 0$, then $kg(\boldsymbol{v})$ is anti-multimodular.*

(ii) If $g(\boldsymbol{v})$ is anti-multimodular, then $g(-\boldsymbol{v})$ is anti-multimodular.

(iii) If $f(\boldsymbol{v})$ and $g(\boldsymbol{v})$ is anti-multimodular, then $f(\boldsymbol{v}) + g(\boldsymbol{v})$ is anti-multimodular.

(iv) If $g(\boldsymbol{v}, d)$ is anti-multimodular in $\boldsymbol{v}$ for any given d and D is a random variable, then $\boldsymbol{E}g(\boldsymbol{v}, D)$ is anti-multimodular in $\boldsymbol{v}$.

(v) If $f : \mathbb{R} \to \mathbb{R}$ is concave, then $g(\boldsymbol{v}) = f(v_1 + v_2 + \cdots + v_n)$ is anti-multimodular.

(vi) If $g(\boldsymbol{v})$ is anti-multimodular, then $\tilde{g}(v_n, v_{n+1}, \ldots, v_1)$ is anti-multimodular.

(vii) If $g(\boldsymbol{v})$ is anti-multimodular, then $g(v_1, v_2, \ldots, v_{i-1}, w_1 + \cdots + w_m, v_{i+1}, \ldots, v_n)$ is also anti-multimodular in $(v_1, v_2, \ldots, v_{i-1}, w_1, w_2, \ldots, w_m, v_{i+1}, \ldots, v_n)$.

Note that the order of arguments in Lemma EC.6(vi) and (vii) matters. Additionally, we have the following properties.

Lemma EC.7 **(Li and Yu (2014))**. *(i) If $\mathcal{C} = \{(\boldsymbol{v}, \boldsymbol{w}) | \boldsymbol{v} \in \boldsymbol{S} \subseteq \mathbb{R}^n, \boldsymbol{w} \in A(\boldsymbol{v}) \subseteq \mathbb{R}^m\}$ is a polyhedron satisfying (P1), $g(\boldsymbol{v}, \boldsymbol{w})$ is anti-multimodular in $(\boldsymbol{v}, \boldsymbol{w})$ on $\mathcal{C}$, then $f(\boldsymbol{v}) = max\{g(\boldsymbol{v}, \boldsymbol{w}) | \boldsymbol{w} \in A(\boldsymbol{v})\}$ is anti-multimodular in $\boldsymbol{v}$ on $\boldsymbol{S}$.*

(ii) Suppose that $g(\boldsymbol{v}, \zeta)$ is anti-multimodular on $\mathcal{C}$, where $\mathcal{C} \subseteq \mathbb{R}^n \times \mathbb{R}$ is a polyhedron satisfying (P1). Let $\zeta^*(\boldsymbol{v})$ be the largest value of ζ that maximizes $g(\boldsymbol{v}, \zeta)$. Then, $\zeta^*(\boldsymbol{v})$ is nonincreasing in $\boldsymbol{v}$, and if $\zeta^*(\boldsymbol{v})$ is differentiable in $\boldsymbol{v}$, then

$$-1 \le \frac{\partial \zeta^*}{\partial v_n} \le \frac{\partial \zeta^*}{\partial v_{n+1}} \le \cdots \le \frac{\partial \zeta^*}{\partial v_1} \le 0.$$

Based on the results in Li and Yu (2014), next we will show the anti-multimodularity of $R_n(X)$, $J_n(X, d_{1n}, d_{2n})$, and $G_n(X, d_{2n})$ for the W system and as defined in Lemma EC.4. To this end, we will first apply a change of state variables, from $X = (x_0, x_1, x_2)$ to $\tilde{X} := (x_1, \beta, x_2)$, where $\beta = x_0 - x_1 - x_2$. Accordingly, we rewrite and redefine the value functions also with respect to $\tilde{X}$. That is, let $\tilde{R}_n(\tilde{X}) := R_n(X)$, $\tilde{J}_n(\tilde{X}, d_{1n}, d_{2n}) := J_n(X, d_{1n}, d_{2n})$, and $\tilde{G}_n(\tilde{X}, d_{2n}) := G_n(X, d_{2n})$. The following proposition characterizes the value functions.

Proposition EC.1. *(i) $\tilde{R}_n(\tilde{X})$ is anti-multimodular in $\tilde{X} = (x_1, \beta, x_2)$. Moreover, for any given d_{1n} and d_{2n}, both $\tilde{G}_n(\tilde{X}, d_{2n})$ and $\tilde{J}_n(\tilde{X}, d_{1n}, d_{2n})$ are anti-multimodular in $\tilde{X} = (x_1, \beta, x_2)$.*
(ii) Given $n \ge 1$, for $\tilde{R}_n(\tilde{X})$,

$$\frac{\partial^2 \tilde{R}_n}{\partial x_1^2} \le \frac{\partial^2 \tilde{R}_n}{\partial x_1 \partial \beta} \le \frac{\partial^2 \tilde{R}_n}{\partial x_1 \partial x_2} \le 0, \ \frac{\partial^2 \tilde{R}_n}{\partial \beta^2} \le \frac{\partial^2 \tilde{R}_n}{\partial \beta \partial x_1} \le 0,$$
$$\frac{\partial^2 \tilde{R}_n}{\partial \beta^2} \le \frac{\partial^2 \tilde{R}_n}{\partial \beta \partial x_2} \le 0, \ \frac{\partial^2 \tilde{R}_n}{\partial x_2^2} \le \frac{\partial^2 \tilde{R}_n}{\partial x_2 \partial \beta} \le \frac{\partial^2 \tilde{R}_n}{\partial x_2 \partial x_1} \le 0.$$

(iii) For the W system, in period n and given component inventory X at the beginning of the period, there exists a critical function $\bar{y}_n(X)$, with $(x_0 - x_1)^+ \le \bar{y}_n(X) \le x_0 \wedge x_2$, such that it is optimal to first fill as much of the demand for product 1 as possible, i.e., $y_{1n}^* = x_0 \wedge x_1 \wedge d_{1n}$, and then to fill at most $\bar{y}_n(X - y_{1n}^* a_1)$ units of product 2 demand, i.e., $y_{2n}^* = d_{2n} \wedge \bar{y}_n(X - y_{1n}^* a_1)$.

Proof: (i) Since $\tilde{R}_{N+1}(\tilde{X}) = (c_1 + c_0)x_1 + (c_2 + c_0)x_2 + c_0\beta$ is a linear function, $\tilde{R}_{N+1}(\tilde{X})$ is anti-multimodular in (x_1, β, x_2). Next, we prove the anti-multimodularity of $\tilde{R}_n(\tilde{X})$ for any integer

$n \in [1, N]$ by induction, that is, we will show that, if $\tilde{R}_{n+1}(\tilde{X})$ is anti-multimodular in (x_1, β, x_2), so is $\tilde{R}_n(\tilde{X})$.

As shown in Lemma EC.4(ii), the decisions on processing the realized demand in period n can be made in two sequential stages: in the first stage fulfill the product-1 demand as much as allowed by component inventory, and then process the product-2 demand in the second stage. That is,

$$\tilde{R}_n(x_1, \beta, x_2) = E_{D_{1n}, D_{2n}}[\tilde{J}_n(x_1, \beta, x_2, D_{1n}, D_{2n})],$$

where

$$\begin{aligned} \tilde{J}_n(x_1, \beta, x_2, d_{1n}, d_{2n}) = \ & \max_{y_{1n}} \left[-\sum_{j=1}^{2}((h_0 + h_j)x_j) - h_0\beta + (p_1 + h_1 + h_0)y_{1n} + \tilde{G}_n(x_1 - y_{1n}, \beta, x_2, d_{2n}) \right] \\ \text{subject to} \quad & y_{1n} \le x_1,\ y_{1n} \le x_1 + \beta + x_2,\ 0 \le y_{1n} \le d_{1n}. \end{aligned}$$

and

$$\begin{aligned} \tilde{G}_n(x_1, \beta, x_2, d_{2n}) = \ & \max_{y_{2n}} \left[(p_2 + h_2 + h_0)y_{2n} + \tilde{R}_{n+1}(x_1, \beta, x_2 - y_{2n}) \right] \\ \text{subject to} \quad & y_{2n} \le x_2,\ y_{2n} \le x_1 + \beta + x_2,\ 0 \le y_{2n} \le d_{2n}. \end{aligned}$$

To apply the results on anti-multimodularity, we define $\tilde{y}_{1n} = -y_{1n}$ and $\tilde{y}_{2n} = -y_{2n}$, and rewrite the expressions of $\tilde{J}_n$ and $\tilde{G}_n$ as follows:

$$\begin{aligned} \tilde{J}_n(x_1, \beta, x_2, d_{1n}, d_{2n}) = \ & \max_{\tilde{y}_{1n}} \left[-\sum_{j=1}^{2}((h_0 + h_j)x_j) - h_0\beta - (p_1 + h_1 + h_0)\tilde{y}_{1n} + \tilde{G}_n(x_1 + \tilde{y}_{1n}, \beta, x_2, d_{2n}) \right] \\ \text{subject to} \quad & x_1 + \tilde{y}_{1n} \ge 0,\ x_1 + \beta + x_2 + \tilde{y}_{1n} \ge 0,\ -\tilde{y}_{1n} \ge 0,\ \tilde{y}_{1n} + d_{1n} \ge 0. \end{aligned}$$

and

$$\begin{aligned} \tilde{G}_n(x_1, \beta, x_2, d_{2n}) = \ & \max_{\tilde{y}_{2n}} \left[-(p_2 + h_2 + h_0)\tilde{y}_{2n} + \tilde{R}_{n+1}(x_1, \beta, x_2 + \tilde{y}_{2n}) \right] \\ \text{subject to} \quad & x_2 + \tilde{y}_{2n} \ge 0,\ x_1 + \beta + x_2 + \tilde{y}_{2n} \ge 0,\ -\tilde{y}_{2n} \ge 0,\ \tilde{y}_{2n} + d_{2n} \ge 0. \end{aligned}$$

Our proof follows two steps: (i-a) we show that given that $\tilde{R}_{n+1}(x_1, \beta, x_2)$ is anti-multimodular in (x_1, β, x_2), $\tilde{G}_n(x_1, \beta, x_2, d_{2n})$ is anti-multimodular in (x_1, β, x_2) for given d_{2n}; and (i-b) we then prove that given that $\tilde{G}_n(x_1, \beta, x_2, d_{2n})$ is anti-multimodular in (x_1, β, x_2) for given d_{2n}, $\tilde{J}_n(x_1, \beta, x_2, d_{1n}, d_{2n})$ is anti-multimodular in (x_1, β, x_2) for given (d_{1n}, d_{2n}).

(i-a) According to the induction hypothesis, $\tilde{R}_{n+1}(x_1, \beta, x_2)$ is anti-multimodular in (x_1, β, x_2). Thus, by Lemma EC.6(vii), $\tilde{R}_{n+1}(x_1, \beta, x_2 + \tilde{y}_{2n})$ is anti-multimodular in $(x_1, \beta, x_2, \tilde{y}_{2n})$. Also note that $-(p_2 + h_2 + h_0)\tilde{y}_{2n}$ is anti-multimodular in $(x_1, \beta, x_2, \tilde{y}_{2n})$. Hence,

$$-(p_2 + h_2 + h_0)\tilde{y}_{2n} + \tilde{R}_{n+1}(x_1, \beta, x_2 + \tilde{y}_{2n})$$

is anti-multimodular in $(x_1, \beta, x_2, \tilde{y}_{2n})$. It is also clear to see that the feasible set of $\tilde{y}_{2n}$ is a polyhedron satisfying (P1). By Lemma EC.7(i), $\tilde{G}_n(x_1, \beta, x_2, d_{2n})$ is anti-multimodular in (x_1, β, x_2).
(i-b) The anti-multimodularity of $\tilde{G}_n(x_1, \beta, x_2, d_{2n})$ in (x_1, β, x_2), together with Lemma EC.6(vii), implies that $\tilde{G}_n(\tilde{y}_{1n} + x_1, \beta, x_2, d_{2n})$ is anti-multimodular in $(\tilde{y}_{1n}, x_1, \beta, x_2)$. Also note that $-(p_1 + h_1 + h_0)\tilde{y}_{1n}$ is anti-multimodular in $(\tilde{y}_{1n}, x_1, \beta, x_2)$. Hence,

$$-\sum_{j=1}^{2}((h_0 + h_j)x_j) - h_0\beta - (p_1 + h_1 + h_0)\tilde{y}_{1n} + \tilde{G}_n(\tilde{y}_{1n} + x_1, \beta, x_2)$$

is anti-multimodular in $(\tilde{y}_{1n}, x_1, \beta, x_2)$. It is also clear to see that the feasible set of $\tilde{y}_{1n}$ is a polyhedron satisfying (P1). By Lemma EC.7(i), $\tilde{J}_n(x_1, \beta, x_2, d_{1n}, d_{2n})$ is anti-multimodular in (x_1, β, x_2) for given (d_{1n}, d_{2n}). Further applying Lemma EC.6(iv), as

$$\tilde{R}_n(x_1, \beta, x_2) = E_{D_{1n}, D_{2n}} \tilde{J}_n(x_1, \beta, x_2, D_{1n}, D_{2n}),$$

we conclude that $\tilde{R}_n(x_1, \beta, x_2)$ is anti-multimodular in (x_1, β, x_2).
(ii) It follows from part (i) and Lemma EC.5.
(iii) As shown in Lemma EC.4, the nested fulfillment policy is optimal. Further, as in the proof of part (i) above, the fulfillment of the demand for product 2 is characterized by

$$\begin{aligned} \tilde{G}_n(x_1, \beta, x_2, d_{2n}) = \; & \max_{\tilde{y}_{2n}} \Big[-(p_2 + h_2 + h_0)\tilde{y}_{2n} + \tilde{R}_{n+1}(x_1, \beta, x_2 + \tilde{y}_{2n}) \Big] \\ \text{subject to} \quad & x_2 + \tilde{y}_{2n} \geq 0,\; x_1 + \beta + x_2 + \tilde{y}_{2n} \geq 0,\; -\tilde{y}_{2n} \geq 0,\; \tilde{y}_{2n} + d_{2n} \geq 0. \end{aligned}$$

Now we derive the optimal solution $\tilde{y}^*_{2n}$. Recall from part (i-a) that $\tilde{R}_{n+1}(x_1, \beta, x_2 + \tilde{y}_{2n})$ is anti-multimodular in $(x_1, \beta, x_2, \tilde{y}_{2n})$. Thus, by Lemma EC.5(i), $\tilde{R}_{n+1}(x_1, \beta, x_2 + \tilde{y}_{2n})$ is concave in $\tilde{y}_{2n}$. So is $-(p_2 + h_2 + h_0)\tilde{y}_{2n} + \tilde{R}_{n+1}(x_1, \beta, x_2 + \tilde{y}_{2n})$. Hence, the optimal solution is $\tilde{y}^*_{2n} = d_{2n} \wedge \tilde{y}_n(x_1, \beta, x_2)$, where

$$\tilde{y}_n(x_1, \beta, x_2) = \operatorname*{arg\,max}_{\tilde{y} \in [-((x_1+\beta+x_2)\wedge x_2), 0]} [-(p_2 + h_2 + h_0)\tilde{y} + \tilde{R}_{n+1}(x_1, \beta, x_2 + \tilde{y})].$$

Specifically, if

$$-(p_2 + h_2 + h_0) + \frac{\partial \tilde{R}_{n+1}(x_1, \beta, x_2 + \tilde{y})}{\partial \tilde{y}}\Big|_{\tilde{y} = -((x_1+\beta+x_2)\wedge x_2)} < 0,$$

then $\tilde{y}_n(x_1, \beta, x_2) = -((x_1 + \beta + x_2) \wedge x_2)$; if

$$-(p_2 + h_2 + h_0) + \frac{\partial \tilde{R}_{n+1}(x_1, \beta, x_2 + \tilde{y})}{\partial \tilde{y}}\Big|_{\tilde{y}=0} > 0,$$

then $\tilde{y}_n(x_1, \beta, x_2) = 0$; otherwise, $\tilde{y}_n(x_1, \beta, x_2)$ is the solution in $[-((x_1 + \beta + x_2) \wedge x_2), 0]$ to

$$-(p_2 + h_2 + h_0) + \frac{\partial \tilde{R}_{n+1}(x_1, \beta, x_2 + \tilde{y})}{\partial \tilde{y}} = 0.$$

If multiple solutions exist in $[-((x_1 + \beta + x_2) \wedge x_2), 0]$, define $\tilde{y}_n(x_1, \beta, x_2)$ to be the largest solution.

Per definition, $\tilde{y}_{2n} = -y_{2n}$. Thus, the optimal selling quantity of product 2 is $y^*_{2n} = d_{2n} \wedge \bar{y}_n(X)$, where $\bar{y}_n(X) := -\tilde{y}_n(x_1, \beta, x_2)$ and $x_0 = x_1 + \beta + x_2$. That is,

$$\bar{y}_n(X) = \underset{y \in [0, x_0 \wedge x_2]}{\arg\max} \, [(p_2 + h_2 + h_0)y + R_{n+1}(X - y a_2)]. \tag{EC.17}$$

More specifically, if $(p_2 + h_2 + h_0) + \frac{\partial R_{n+1}(X - y a_2)}{\partial y}|_{y=(x_0 \wedge x_2)} > 0$, then $\bar{y}_n(X) = x_0 \wedge x_2$; if $(p_2 + h_2 + h_0) + \frac{\partial R_{n+1}(X - y a_2)}{\partial y}|_{y=0} < 0$, then $\bar{y}_n(X) = 0$; otherwise, $\bar{y}_n(X)$ is the solution in $[0, x_0 \wedge x_2]$ to $(p_2 + h_2 + h_0) + \frac{\partial R_{n+1}(X - y a_2)}{\partial y} = 0$. If multiple solutions exist in $[0, x_0 \wedge x_2]$, define $\bar{y}_n(X)$ to be the largest solution. Q.E.D.

C.4.3. Characterization of the critical threshold $\bar{y}_n(X)$

Lemma EC.8. *$\bar{y}_n(X)$ decreases in x_1, and increases in x_0 and x_2. Additionally, $\bar{y}_n(X)$ decreases when either product's demand in period $n+1$ (i.e., the next period) stochastically increases.*

Proof. We first prove the monotonicity of $\bar{y}_n(X)$ in x_0, x_1, and x_2. As in the proof of Proposition EC.1(iii), $\tilde{y}_n(x_1, \beta, x_2)$ is the largest $\tilde{y}_{2n} \in [-((x_1 + \beta + x_2) \wedge x_2), 0]$ that maximizes $-(p_2 + h_2 + h_0)\tilde{y}_{2n} + \tilde{R}_{n+1}(x_1, \beta, x_2 + \tilde{y}_{2n})$. Also recall that $-(p_2 + h_2 + h_0)\tilde{y}_{2n} + \tilde{R}_{n+1}(x_1, \beta, x_2 + \tilde{y}_{2n})$ is anti-multimodular in $(x_1, \beta, x_2, \tilde{y}_{2n})$ and that the feasible set of $\tilde{y}_{2n}$ is a polyhedron satisfying (P1). Thus, by Lemma EC.7(ii), for $\tilde{y}_n(x_1, \beta, x_2)$,

$$-1 \le \frac{\partial \tilde{y}_n(x_1, \beta, x_2)}{\partial x_2} \le \frac{\partial \tilde{y}_n(x_1, \beta, x_2)}{\partial \beta} \le \frac{\partial \tilde{y}_n(x_1, \beta, x_2)}{\partial x_1} \le 0.$$

Since $\beta = x_0 - x_1 - x_2$ and $\bar{y}_n(X) = -\tilde{y}_n(x_1, \beta, x_2)$,

$$-1 \le \frac{\partial \bar{y}_n(x_0, x_1, x_2)}{\partial x_1} \le 0, \qquad 0 \le \frac{\partial \bar{y}_n(x_0, x_1, x_2)}{\partial x_0} \le 1 \qquad \text{and} \qquad 0 \le \frac{\partial \bar{y}_n(x_0, x_1, x_2)}{\partial x_2} \le 1,$$

which follow from

$$\begin{aligned} \frac{\partial \bar{y}_n(X)}{\partial x_0} &= -\frac{\partial \tilde{y}_n(x_1, \beta, x_2)}{\partial \beta} \\ \frac{\partial \bar{y}_n(X)}{\partial x_j} &= -[\frac{\partial \tilde{y}_n(x_1, \beta, x_2)}{\partial x_j} - \frac{\partial \tilde{y}_n(x_1, \beta, x_2)}{\partial \beta}], \qquad j = 1, 2. \end{aligned}$$

In the following, we prove that $\bar{y}_{n-1}(X)$ decreases when either product's demand in period n (i.e., the next period) stochastically increases. Consider two markets A and B, which are identical except for the demand distribution in period n. Demand in period n for product $i \in \{1, 2\}$ in market $\eta \in \{A, B\}$ is D^η_{in} and follows distribution F^η_{in}. By Müller and Stoyan (2002), if $F^A_{in}(x) \le F^B_{in}(x)$ for all x, then the demand for product i in market A, D^A_{in}, stochastically dominates D^B_{in} in the first order, denoted by $D^B_{in} \le_{st} D^A_{in}$. We will show that if $D^B_{1n} \le_{st} D^A_{1n}$ or $D^B_{2n} \le_{st} D^A_{2n}$, $\bar{y}_{n-1}$ is lower in market A than its counterpart in market B.

To this end, for market $\eta = A, B$, denote by $R_n^{\eta}(X)$ the optimal profit-to-go in the remaining n periods, by $G_n^{\eta}(X, d_{2n})$ the optimal profit-to-go from the second stage of period n to the end of the season, and by $J_n^{\eta}(X, d_{1n}, d_{2n})$ the optimal profit-to-go from the first stage of period n to the end of the season. To show that $\bar{y}_{n-1}$ is lower in market A than its counterpart in market B, by definition of $\bar{y}_n(X)$ in (EC.17), it suffices to show $\frac{\partial R_n^A(X-ya_2)}{\partial y} \le \frac{\partial R_n^B(X-ya_2)}{\partial y}$.

To show $\frac{\partial R_n^A(X-ya_2)}{\partial y} \le \frac{\partial R_n^B(X-ya_2)}{\partial y}$, it suffices to show that for small $\epsilon > 0$,

$$J_n(X+\epsilon a_2, d_{1n}, d_{2n}) - J_n(X, d_{1n}, d_{2n}) \text{ increases in } d_{in},\ i = 1, 2. \tag{EC.18}$$

To see the sufficiency, note that $D_{in}^A \ge_{st} D_{in}^B$ implies $E[g(D_{in}^A)] \ge E[g(D_{in}^B)]$ for any increasing function $g(x)$. Thus, (EC.18) implies that

$$E[J_n^A(X+\epsilon a_2, d_{1n}, d_{2n})] - E[J_n^A(X, d_{1n}, d_{2n})] \ge E[J_n^B(X+\epsilon a_2, d_{1n}, d_{2n})] - E[J_n^B(X, d_{1n}, d_{2n})].$$

That is equivalent to

$$R_n^A(X+\epsilon a_2) - R_n^A(X) \ge R_n^B(X+\epsilon a_2) - R_n^B(X),$$

which further implies

$$\frac{\partial R_n^A(X-ya_2)}{\partial y} \le \frac{\partial R_n^B(X-ya_2)}{\partial y}.$$

To prove (EC.18), we first prove it for $i = 2$ and it is equivalent to proving that

$$\Delta_1(\delta) := J_n(X+\epsilon a_2, d_{1n}, d_{2n}+\delta) - J_n(X, d_{1n}, d_{2n}+\delta) - (J_n(X+\epsilon a_2, d_{1n}, d_{2n}) - J_n(X, d_{1n}, d_{2n})) \ge 0$$

for any $n \ge 1$, $\epsilon > 0$ and $\delta > 0$. By the proof of Proposition 2 (i),

$$J_n(X, d_{1n}, d_{2n}) = -\sum_{j=0}^{2}(h_j x_j) + (p_1 + h_1 + h_0)y_{1n}^*(X) + G_n(X - y_{1n}^*(X)a_1, d_{2n}),$$

where $y_{1n}^*(X) = x_0 \wedge x_1 \wedge d_{1n}$. Thus, $y_{1n}^*(X) \le y_{1n}^*(X+\epsilon a_2)$. Consider the following two cases:

- **Case (i)** $y_{1n}^*(X) = y_{1n}^*(X+\epsilon a_2)$: In this case,

$$\begin{aligned}\Delta_1(\delta) := {} & G_n(X+\epsilon a_2 - y_{1n}^*(X)a_1, d_{2n}+\delta) - G_n(X - y_{1n}^*(X)a_1, d_{2n}+\delta) \\ & -(G_n(X+\epsilon a_2 - y_{1n}^*(X)a_1, d_{2n}) - G_n(X - y_{1n}^*(X)a_1, d_{2n})).\end{aligned}$$

The monotonicities of $\bar{y}_n$ in x_0 and x_2 imply that $\bar{y}_n(X - y_{1n}^*(X)a_1) \le \bar{y}_n(X+\epsilon a_2 - y_{1n}^*(X)a_1)$. As a result, there are three sub-cases:

Case (i.1) $d_{2n} > \bar{y}_n(X+\epsilon a_2 - y_{1n}^*(X)a_1) \ge \bar{y}_n(X - y_{1n}^*(X)a_1)$,

Case (i.2) $\bar{y}_n(X+\epsilon a_2 - y_{1n}^*(X)a_1) \ge d_{2n} > \bar{y}_n(X - y_{1n}^*(X)a_1)$,

Case (i.3) $\bar{y}_n(X+\epsilon a_2 - y_{1n}^*(X)a_1) \ge \bar{y}_n(X - y_{1n}^*(X)a_1) \ge d_{2n}$.

Below we show that $\Delta_1(\delta) \ge 0$ under each case by using the definition of $\bar{y}_n$:

—**Case (i.1)** $d_{2n} > \bar{y}_n(X+\epsilon a_2 - y^*_{1n}(X)a_1) \ge \bar{y}_n(X - y^*_{1n}(X)a_1)$: With d_{2n} units of demand for product 2, the optimal allocation under component profile $X - y^*_{1n}(X)a_1$ is

$$y^*_{2n}(X - y^*_{1n}(X)a_1) = d_{2n} \wedge \bar{y}_n(X - y^*_{1n}(X)a_1) = \bar{y}_n(X - y^*_{1n}(X)a_1),$$

and the optimal allocation under component profile $X+\epsilon a_2 - y^*_{1n}(X)a_1$ is

$$y^*_{2n}(X+\epsilon a_2 - y^*_{1n}(X)a_1) = d_{2n} \wedge \bar{y}_n(X+\epsilon a_2 - y^*_{1n}(X)a_1) = \bar{y}_n(X+\epsilon a_2 - y^*_{1n}(X)a_1).$$

Moreover, the condition $d_{2n} > \bar{y}_n(X+\epsilon a_2 - y^*_{1n}(X)a_1) \ge \bar{y}_n(X - y^*_{1n}(X)a_1)$ implies

$$d_{2n} + \delta \ge d_{2n} > \bar{y}_n(X+\epsilon a_2 - y^*_{1n}(X)a_1) \ge \bar{y}_n(X - y^*_{1n}(X)a_1),$$

so with $d_{2n}+\delta$ units of demand for product 2, the optimal allocation under component profile $X - y^*_{1n}(X)a_1$ is

$$y^*_{2n}(X - y^*_{1n}(X)a_1) = (d_{2n}+\delta) \wedge \bar{y}_n(X - y^*_{1n}(X)a_1) = \bar{y}_n(X - y^*_{1n}(X)a_1),$$

and the optimal allocation under component profile $X+\epsilon a_2 - y^*_{1n}(X)a_1$ is

$$y^*_{2n}(X+\epsilon a_2 - y^*_{1n}(X)a_1) = (d_{2n}+\delta) \wedge \bar{y}_n(X+\epsilon a_2 - y^*_{1n}(X)a_1) = \bar{y}_n(X+\epsilon a_2 - y^*_{1n}(X)a_1).$$

Thus, (EC.15) results in

$$\begin{aligned}
&G_n(X+\epsilon a_2 - y^*_{1n}(X)a_1, d_{2n}+\delta) = G_n(X+\epsilon a_2 - y^*_{1n}(X)a_1, d_{2n}) \\
=\ &(p_2+h_2+h_0)\bar{y}_n(X+\epsilon a_2 - y^*_{1n}(X)a_1) + R_{n+1}(X+\epsilon a_2 - y^*_{1n}(X)a_1 - \bar{y}_n(X+\epsilon a_2 - y^*_{1n}(X)a_1)a_2), \\
&G_n(X - y^*_{1n}(X)a_1, d_{2n}+\delta) = G_n(X - y^*_{1n}(X)a_1, d_{2n}) \\
=\ &(p_2+h_2+h_0)\bar{y}_n(X - y^*_{1n}(X)a_1) + R_{n+1}(X - y^*_{1n}(X)a_1 - \bar{y}_n(X - y^*_{1n}(X)a_1)a_2).
\end{aligned}$$

As such, $\Delta_1(\delta) = 0$.

—**Case (i.2)** $\bar{y}_n(X+\epsilon a_2 - y^*_{1n}(X)a_1) \ge d_{2n} > \bar{y}_n(X - y^*_{1n}(X)a_1)$: We consider two subcases

$$d_{2n} + \delta > \bar{y}_n(X+\epsilon a_2 - y^*_{1n}(X)a_1) \ge d_{2n} > \bar{y}_n(X - y^*_{1n}(X)a_1)$$

and

$$\bar{y}_n(X+\epsilon a_2 - y^*_{1n}(X)a_1) \ge d_{2n} + \delta \ge d_{2n} > \bar{y}_n(X - y^*_{1n}(X)a_1).$$

$*$ **Subcase (i.2.a)** $\bar{y}_n(X+\epsilon a_2-y^*_{1n}(X)a_1)\geq d_{2n}+\delta\geq d_{2n}>\bar{y}_n(X-y^*_{1n}(X)a_1)$: With d_{2n} units of demand for product 2, the optimal allocation under component profile $X-y^*_{1n}(X)a_1$ is

$$y^*_{2n}(X-y^*_{1n}(X)a_1)=d_{2n}\wedge\bar{y}_n(X-y^*_{1n}(X)a_1)=\bar{y}_n(X-y^*_{1n}(X)a_1),$$

and the optimal allocation under component profile $X+\epsilon a_2-y^*_{1n}(X)a_1$ is

$$y^*_{2n}(X+\epsilon a_2-y^*_{1n}(X)a_1)=d_{2n}\wedge\bar{y}_n(X+\epsilon a_2-y^*_{1n}(X)a_1)=d_{2n}.$$

Moreover, with $d_{2n}+\delta$ units of demand for product 2, the optimal allocation under component profile $X-y^*_{1n}(X)a_1$ is

$$y^*_{2n}(X-y^*_{1n}(X)a_1)=(d_{2n}+\delta)\wedge\bar{y}_n(X-y^*_{1n}(X)a_1)=\bar{y}_n(X-y^*_{1n}(X)a_1),$$

and the optimal allocation under component profile $X+\epsilon a_2-y^*_{1n}(X)a_1$ is

$$y^*_{2n}(X+\epsilon a_2-y^*_{1n}(X)a_1)=(d_{2n}+\delta)\wedge\bar{y}_n(X+\epsilon a_2-y^*_{1n}(X)a_1)=d_{2n}+\delta.$$

So, (EC.15) results in

$$\begin{aligned}&G_n(X-y^*_{1n}(X)a_1,d_{2n}+\delta)=G_n(X-y^*_{1n}(X)a_1,d_{2n})\\ =\;&(p_2+h_2+h_0)\bar{y}_n(X-y^*_{1n}(X)a_1)+R_{n+1}(X-y^*_{1n}(X)a_1-\bar{y}_n(X-y^*_{1n}(X)a_1)a_2).\end{aligned}$$

As such,

$$\begin{aligned}\Delta_1(\delta)&=G_n(X+\epsilon a_2-y^*_{1n}(X)a_1,d_{2n}+\delta)-G_n(X+\epsilon a_2-y^*_{1n}(X)a_1,d_{2n})\\ &=[(p_2+h_2+h_0)(d_{2n}+\delta)+R_{n+1}(X+\epsilon a_2-y^*_{1n}(X)a_1-(d_{2n}+\delta)a_2)]\\ &\quad-[(p_2+h_2+h_0)d_{2n}+R_{n+1}(X+\epsilon a_2-y^*_{1n}(X)a_1-d_{2n}a_2)]\geq 0,\end{aligned}$$

where the inequality is due to the concavity of

$$(p_2+h_2+h_0)y+R_{n+1}(X+\epsilon a_2-y^*_{1n}(X)a_1-ya_2)$$

in y and the definition of $\bar{y}_n(X+\epsilon a_2-y^*_{1n}(X)a_1)$, both of which jointly imply that

$$(p_2+h_2+h_0)y+R_{n+1}(X+\epsilon a_2-y^*_{1n}(X)a_1-ya_2)$$

increases in $y\in[0,\bar{y}_n(X+\epsilon a_2-y^*_{1n}(X)a_1)]$ and consequently, $\Delta_1(\delta)\geq 0$ as $\bar{y}_n(X+\epsilon a_2-y^*_{1n}(X)a_1)\geq d_{2n}+\delta\geq d_{2n}$.

$*$ **Subcase (i.2.b)** $d_{2n}+\delta > \bar{y}_n(X+\epsilon a_2 - y^*_{1n}(X)a_1) \geq d_{2n} > \bar{y}_n(X - y^*_{1n}(X)a_1)$: With d_{2n} units of demand for product 2, the optimal allocation under component profile $X - y^*_{1n}(X)a_1$ is

$$y^*_{2n}(X - y^*_{1n}(X)a_1) = d_{2n} \wedge \bar{y}_n(X - y^*_{1n}(X)a_1) = \bar{y}_n(X - y^*_{1n}(X)a_1),$$

and the optimal allocation under component profile $X+\epsilon a_2 - y^*_{1n}(X)a_1$ is

$$y^*_{2n}(X+\epsilon a_2 - y^*_{1n}(X)a_1) = d_{2n} \wedge \bar{y}_n(X+\epsilon a_2 - y^*_{1n}(X)a_1) = d_{2n}.$$

Moreover, with $d_{2n}+\delta$ units of demand for product 2, the optimal allocation under component profile $X - y^*_{1n}(X)a_1$ is

$$y^*_{2n}(X - y^*_{1n}(X)a_1) = (d_{2n}+\delta) \wedge \bar{y}_n(X - y^*_{1n}(X)a_1) = \bar{y}_n(X - y^*_{1n}(X)a_1),$$

and the optimal allocation under component profile $X+\epsilon a_2 - y^*_{1n}(X)a_1$ is

$$y^*_{2n}(X+\epsilon a_2 - y^*_{1n}(X)a_1) = (d_{2n}+\delta) \wedge \bar{y}_n(X+\epsilon a_2 - y^*_{1n}(X)a_1) = \bar{y}_n(X+\epsilon a_2 - y^*_{1n}(X)a_1).$$

So, (EC.15) results in

$$\begin{aligned} &G_n(X - y^*_{1n}(X)a_1, d_{2n}+\delta) = G_n(X - y^*_{1n}(X)a_1, d_{2n}) \\ &= (p_2+h_2+h_0)\bar{y}_n(X - y^*_{1n}(X)a_1) + R_{n+1}(X - y^*_{1n}(X)a_1 - \bar{y}_n(X - y^*_{1n}(X)a_1)a_2). \end{aligned}$$

As such,

$$\begin{aligned} &\Delta_1(\delta) = G_n(X+\epsilon a_2 - y^*_{1n}(X)a_1, d_{2n}+\delta) - G_n(X+\epsilon a_2 - y^*_{1n}(X)a_1, d_{2n}) \\ &= [(p_2+h_2+h_0)\bar{y}_n(X+\epsilon a_2 - y^*_{1n}(X)a_1) + R_{n+1}(X+\epsilon a_2 - y^*_{1n}(X)a_1 - \bar{y}_n(X+\epsilon a_2 - y^*_{1n}(X)a_1)a_2)] \\ &- [(p_2+h_2+h_0)d_{2n} + R_{n+1}(X+\epsilon a_2 - y^*_{1n}(X)a_1 - d_{2n}a_2)] \geq 0 \end{aligned}$$

because, similar to subcase (i.2.a), $(p_2+h_2+h_0)y + R_{n+1}(X+\epsilon a_2 - y^*_{1n}(X)a_1 - ya_2)$ increases in $y \in [0, \bar{y}_n(X+\epsilon a_2 - y^*_{1n}(X)a_1)]$ and consequently, $\Delta_1(\delta) \geq 0$ as $\bar{y}_n(X+\epsilon a_2 - y^*_{1n}(X)a_1) \geq d_{2n}$.

— **Case (i.3)** $\bar{y}_n(X+\epsilon a_2 - y^*_{1n}(X)a_1) \geq \bar{y}_n(X - y^*_{1n}(X)a_1) \geq d_{2n}$: With d_{2n} units of demand for product 2, the optimal allocation under component profile $X - y^*_{1n}(X)a_1$ is

$$y^*_{2n}(X - y^*_{1n}(X)a_1) = d_{2n} \wedge \bar{y}_n(X - y^*_{1n}(X)a_1) = d_{2n},$$

and the optimal allocation under component profile $X+\epsilon a_2 - y^*_{1n}(X)a_1$ is

$$y^*_{2n}(X+\epsilon a_2 - y^*_{1n}(X)a_1) = d_{2n} \wedge \bar{y}_n(X+\epsilon a_2 - y^*_{1n}(X)a_1) = d_{2n}.$$

Thus, (EC.15) results in

$$\begin{aligned}
&G_n(X+\epsilon a_2-y^*_{1n}(X)a_1,d_{2n})-G_n(X-y^*_{1n}(X)a_1,d_{2n}))\\
=&(p_2+h_2+h_0)d_{2n}+R_{n+1}(X+\epsilon a_2-y^*_{1n}(X)a_1-d_{2n}a_2)\\
&-[(p_2+h_2+h_0)d_{2n}+R_{n+1}(X-y^*_{1n}(X)a_1-d_{2n}a_2)]\\
=&R_{n+1}(X+\epsilon a_2-y^*_{1n}(X)a_1-d_{2n}a_2)-R_{n+1}(X-y^*_{1n}(X)a_1-d_{2n}a_2).
\end{aligned}$$

In the meanwhile, given $d_{2n}+\delta$ units of product-2 demand, (EC.15) results in

$$G_n(X-y^*_{1n}(X)a_1,d_{2n}+\delta)=(p_2+h_2+h_0)y^\delta_{2n}+R_{n+1}(X-y^*_{1n}(X)a_1-y^\delta_{2n}a_2),$$

where $y^\delta_{2n}:=(d_{2n}+\delta)\wedge\bar{y}_n(X-y^*_{1n}(X)a_1)$. This, together with the fact that with $d_{2n}+\delta$ units of demand for product 2, the optimal allocation under component profile $X-y^*_{1n}(X)a_1$ is feasible under component profile $X+\epsilon a_2-y^*_{1n}(X)a_1$, implies

$$G_n(X+\epsilon a_2-y^*_{1n}(X)a_1,d_{2n}+\delta)\geq(p_2+h_2+h_0)y^\delta_{2n}+R_{n+1}(X+\epsilon a_2-y^*_{1n}(X)a_1-y^\delta_{2n}a_2),$$

so

$$\begin{aligned}
\Delta_1(\delta)\geq\ &R_{n+1}(X-y^*_{1n}(X)a_1-(y^\delta_{2n}-\epsilon)a_2)-R_{n+1}(X-y^*_{1n}(X)a_1-y^\delta_{2n}a_2)\\
&-[R_{n+1}(X-y^*_{1n}(X)a_1-(d_{2n}-\epsilon)a_2)-R_{n+1}(X-y^*_{1n}(X)a_1-d_{2n}a_2)]\geq 0,
\end{aligned}$$

because $R_{n+1}(X+\epsilon a_2-y^*_{1n}(X)a_1-ya_2)$ is concave in y, and

$$y^\delta_{2n}=(d_{2n}+\delta)\wedge\bar{y}_n(X-y^*_{1n}(X)a_1)\geq d_{2n}.$$

- **Case (ii)** $y^*_{1n}(X)<y^*_{1n}(X+\epsilon a_2)$: This case is valid only if under the component profile X the availability of common components limits the fulfillment of the demand for product 1. That is, no common component is left after filling the demand for product 1 under the component profile X, i.e., $y^*_{1n}(X)=x_0$ and for any d_{2n},

$$G_n(X-y^*_{1n}(X)a_1,d_{2n})=[c_1-(N-n+1)h_1](x_1-y^*_{1n}(X))+[c_2-(N-n+1)h_2]x_2,$$

thus

$$\begin{aligned}
\Delta_1(\delta):=\ &(p_1+h_1+h_0)y^*_{1n}(X+\epsilon a_2)+G_n(X+\epsilon a_2-y^*_{1n}(X+\epsilon a_2)a_1,d_{2n}+\delta)-(p_1+h_1+h_0)y^*_{1n}(X)\\
&-[(p_1+h_1+h_0)y^*_{1n}(X+\epsilon a_2)+G_n(X+\epsilon a_2-y^*_{1n}(X+\epsilon a_2)a_1,d_{2n})-(p_1+h_1+h_0)y^*_{1n}(X)]\\
=\ &G_n(X+\epsilon a_2-y^*_{1n}(X+\epsilon a_2)a_1,d_{2n}+\delta)-G_n(X+\epsilon a_2-y^*_{1n}(X+\epsilon a_2)a_1,d_{2n})\geq 0,
\end{aligned}$$

where the inequality holds because $G_n(X,d_{2n})$ increases in d_{2n}, as a higher d_{2n} expands the feasible region of y_{2n} in (EC.15).

Following a similar way, we can prove that $J_n(X+\epsilon a_2,d_{1n},d_{2n})-J_n(X,d_{1n},d_{2n})$ increases in d_{1n}. This completes the proof.

C.5. Technical Lemma for Proposition 7

LEMMA EC.9. *(i) When* $(x_0, x_1, \ldots, x_k, u_1, \ldots, u_k)$ *satisfies* $\sum_{i=1}^k u_i \le x_0$ *and* $0 \le u_i \le x_i, i = 1, ...k$, $V_1^{mts}(X, u_1, \ldots, u_k)$ *is jointly concave in* $(x_0, x_1, \ldots, x_k, u_1, \ldots, u_k)$ *and*

$$0 \le \frac{\partial V_1^{mts}(X, u_1, \ldots, u_k)}{\partial u_i} \le (p_i + Nh_i + Nh_0 - c_i - c_0)\bar{F}_i(u_i), \tag{EC.19}$$

where $\bar{F}_i(u_i) = \mathcal{P}\left(\sum_{n=1}^N D_{in} \ge u_i\right)$ *is the complementary cumulative distribution function of the demand for product* i *over the whole selling season. Furthermore,* u_i^* *as defined in (8) satisfies* $u_i^* \le x_i$ *when* $x_i \le E(\sum_{n=1}^N D_{in})$ *and* $x_0 \le \sum_{i=1}^k x_i$.

(ii) Let $R_b^{mts}(x_0) := R^{mts}(x_0, x_1^{b(m)}, \ldots, x_k^{b(m)}) = V_1^{mts}(x_0, x_1^{b(m)}, \ldots, x_k^{b(m)}, u_1^*, \ldots, u_k^*)$ *and* $\pi_b^{mts}(x_0) := R_b^{mts}(x_0) - c_0 x_0 - \sum_{j=1}^k (c_j x_j^{b(m)})$. *Given* $x_i^b \le E(\sum_{n=1}^N D_{in})$ *for all* $i = 1, \ldots, k$, *both* $R_b^{mts}(x_0)$ *and* $\pi_b^{mts}(x_0)$ *are concave in* x_0 *when* $x_0 \le \sum_{i=1}^k x_i^{b(m)}$.

(iii) Given inventory profile $X = (x_0, x_1, ..., x_k)$ *with* $x_0 < \sum_{j=1}^k x_j$ *and the corresponding sales quota* $(u_1^*, \ldots, u_k^*)$ *in the MTS heuristic:*

(iii-1) assume that there exist three product indexes, $i, i_1, i_2 \in \{1, ..., k\}$, *such that* $u_i^* = x_i > 0$, $0 < u_{i_1}^* < x_{i_1}$, *and* $0 < u_{i_2}^* < x_{i_2}$. *Then, at the point* $(u_1, \ldots, u_k) = (u_1^*, \ldots, u_k^*)$,

$$\frac{\partial V_1^{mts}(X, u_1, \ldots, u_k)}{\partial u_{i_1}} = \frac{\partial V_1^{mts}(X, u_1, \ldots, u_k)}{\partial u_{i_2}} \le \frac{\partial V_1^{mts}(X, u_1, \ldots, u_k)}{\partial u_i}.$$

(iii-2) assume that there exists a product index $i_3 \in \{1, ..., k\}$ *such that* $u_{i_3}^* = 0$. *Then, at the point* $(u_1, \ldots, u_k) = (u_1^*, \ldots, u_k^*)$,

$$\frac{\partial V_1^{mts}(X, u_1, \ldots, u_k)}{\partial u_{i_3}} \le \frac{\partial V_1^{mts}(X, u_1, \ldots, u_k)}{\partial u_{i_4}}$$

for any $u_{i_4}^* > 0$ *and* $i_4 \in \{1, ..., k\}$.

(iv) Given $x_0 < \sum_{i=1}^k x_i$, *the optimal quota* $(u_1^*, \ldots, u_k^*)$ *satisfies* $\sum_{i=1}^k u_i^* = x_0$, *and* $0 \le \frac{du_i^*}{dx_0} \le 1$.

Proof. (i) We first note that under the make-to-stock heuristic defined in subsection 4.2.1, order fulfillment is independent across product. Thus, the multi-product problem can be decomposed into k independent sub-problems, each in a single-product sub-system. Specifically, given $\sum_{i=1}^k u_i \le x_0$ and $0 \le u_i \le x_i, i = 1, ...k$, we rewrite:

$$V_1^{mts}(X, u_1, \ldots, u_k) = -N\sum_{j=0}^k (h_j x_j) + \sum_{j=0}^k (c_j x_j) + \sum_{i=1}^k E_{D_{i1}}\left[\bar{\bar{J}}_1^{mts,i}(u_i, D_{i1})\right],$$

where

$$\bar{\bar{J}}_n^{mts,i}(u_i, d_{in}) = (p_i + (N-n+1)h_i + (N-n+1)h_0 - c_i - c_0)y_{in} + E_{D_{i,n+1}}\left[\bar{\bar{J}}_{n+1}^{mts,i}(u_i - y_{in}, D_{i,n+1})\right]$$

with $y_{in} = x_i \wedge u_i \wedge d_{in} = u_i \wedge d_{in}, i = 1, \ldots, k$ since $0 \le u_i \le x_i$ for all i and $\bar{\bar{J}}_{N+1}^{mts,i}(u_i, \cdot) = 0$. Here we adopt the same approach as stated in the proof of Proposition 7 (i.1.a) to account for the holding costs and salvage values. That is, at the beginning of the season the firm incurs the cost for holding all the initial inventory through the entire season and also receives the total salvage values of all the initial inventory. During the selling season, when a sales takes place, an addition representing savings of holding cost and a deduction of the salvage value are taken into account in the selling margin.

Next we prove by induction the following result:

$$0 \le \frac{dE_{D_{in}}(\bar{\bar{J}}_n^{mts,i}(u_i, D_{in}))}{du_i} \le (p_i + (N-n+1)h_i + (N-n+1)h_0 - c_i - c_0)\mathcal{P}\left(\sum_{j=n}^{N} D_{ij} \ge u_i\right)$$

First note that $\bar{\bar{J}}_{N+1}^{mts,i}(u_i, \cdot) = 0$ implies

$$\frac{dE_{D_{i,N+1}}(\bar{\bar{J}}_{N+1}^{mts,i}(u_i, D_{i,N+1}))}{du_i} = 0.$$

Now, suppose that the result holds for period $n+1$, that is,

$$0 \le \frac{dE_{D_{i,n+1}}(\bar{\bar{J}}_{n+1}^{mts,i}(u_i, D_{i,n+1}))}{du_i} \le (p_i + (N-n)h_i + (N-n)h_0 - c_i - c_0)\mathcal{P}\left(\sum_{j=n+1}^{N} D_{ij} \ge u_i\right).$$

With u_i units of sales quotas at the beginning of period n, consider the following two cases:

- $u_i < d_{in}$: it is optimal to fill u_i units of demand in period n. In such a case $d_{in} - u_i$ units of demand are unfilled and sale quota is zero for the subsequent periods. The realized profit on this sample path is $\bar{\bar{J}}_n^{mts,i}(u_i, d_{in}) = (p_i + (N-n+1)h_i + (N-n+1)h_0 - c_i - c_0)u_i$, with derivative

$$\frac{d\bar{\bar{J}}_n^{mts,i}(u_i, d_{in})}{du_i} = p_i + (N-n+1)h_i + (N-n+1)h_0 - c_i - c_0 \ge 0$$

because of our assumption $p_i \ge c_i + c_0$;

- $d_{in} \le u_i$: it is optimal to fill all d_{in} units of demand in period n. On this sample path, the profit-to-go is

$$\bar{\bar{J}}_n^{mts,i}(u_i, d_{in}) = (p_i + (N-n+1)h_i + (N-n+1)h_0 - c_i - c_0)d_{in} + E_{D_{i,n+1}}(\bar{\bar{J}}_{n+1}^{mts,i}(u_i - d_{in}, D_{i,n+1})).$$

The derivative is

$$\frac{d\bar{\bar{J}}_n^{mts,i}(u_i, d_{in})}{du_i} = \frac{dE_{D_{i,n+1}}(\bar{\bar{J}}_{n+1}^{mts,i}(u_i - d_{in}, D_{i,n+1}))}{du_i} \ge 0$$

because of the induction hypothesis.

Summarizing these two cases, $\frac{dE_{D_{in}}(\bar{\bar{J}}_n^{mts,i}(u_i, D_{in}))}{du_i} \ge 0$. Furthermore, we have

$$\begin{aligned}
&\frac{dE_{D_{in}}(\bar{\bar{J}}_n^{mts,i}(u_i, D_{in}))}{du_i} \\
&= (p_i + (N-n+1)h_i + (N-n+1)h_0 - c_i - c_0)\mathcal{P}(D_{in} > u_i) + \frac{d}{du_i}E_{D_{in}}\left(E_{D_{i,n+1}}\left[\bar{\bar{J}}_{n+1}^{mts,i}(u_i - u_i \wedge D_{in}, D_{i,n+1})\right]\right)
\end{aligned}$$

$$
\begin{aligned}
&= (p_i + (N-n+1)h_i + (N-n+1)h_0 - c_i - c_0)\mathcal{P}\left(D_{in} > u_i\right) + E_{D_{in}}\left(\frac{d}{du_i}E_{D_{i,n+1}}\left[\bar{\bar{J}}^{mts,i}_{n+1}(u_i - D_{in}, D_{i,n+1})\right]\cdot 1_{D_{in}\le u_i}\right)\\
&\le (p_i + (N-n+1)h_i + (N-n+1)h_0 - c_i - c_0)\mathcal{P}\left(D_{in} > u_i\right)\\
&\quad + E_{D_{in}}\left((p_i + (N-n)h_i + (N-n)h_0 - c_i - c_0)\mathcal{P}\left(\sum_{j=n+1}^{N} D_{ij} \ge u_i - D_{in}\right)\cdot 1_{D_{in}\le u_i}\right)\\
&\le (p_i + (N-n+1)h_i + (N-n+1)h_0 - c_i - c_0)\mathcal{P}\left(D_{in} > u_i\right)\\
&\quad + (p_i + (N-n+1)h_i + (N-n+1)h_0 - c_i - c_0)E_{D_{in}}\left(\mathcal{P}\left(\sum_{j=n+1}^{N} D_{ij} \ge u_i - D_{in}\right)\cdot 1_{D_{in}\le u_i}\right)\\
&= (p_i + (N-n+1)h_i + (N-n+1)h_0 - c_i - c_0)\left[\mathcal{P}\left(D_{in} > u_i\right) + E_{D_{in}}\left(\mathcal{P}\left(\sum_{j=n+1}^{N} D_{ij} \ge u_i - D_{in}\right)\cdot 1_{D_{in}\le u_i}\right)\right]\\
&= (p_i + (N-n+1)h_i + (N-n+1)h_0 - c_i - c_0)\mathcal{P}\left(\sum_{j=n}^{N} D_{ij} \ge u_i\right),
\end{aligned}
$$

where the first inequality is by the induction hypothesis, 1_A is the indicator function for event A, and the last equality is from

$$
\begin{aligned}
&\mathcal{P}\left(D_{in} > u_i\right) + E_{D_{in}}\left(\mathcal{P}\left(\sum_{j=n+1}^{N} D_{ij} \ge u_i - D_{in}\right)\cdot 1_{D_{in}\le u_i}\right)\\
&= E_{D_{in}}\left(1_{D_{in}>u_i} + \mathcal{P}\left(\sum_{j=n+1}^{N} D_{ij} \ge u_i - D_{in}\right)\cdot 1_{D_{in}\le u_i}\right)\\
&= \mathcal{P}(\sum_{j=n}^{N} D_{ij} \ge u_i).
\end{aligned}
$$

Therefore,

$$
0 \le \frac{dE_{D_{in}}(\bar{\bar{J}}^{mts,i}_{n}(u_i, D_{in}))}{du_i} \le (p_i + (N-n+1)h_i + (N-n+1)h_0 - c_i - c_0)\mathcal{P}\left(\sum_{j=n}^{N} D_{ij} \ge u_i\right).
$$

Accordingly,

$$
0 \le \frac{dE_{D_{i1}}(\bar{\bar{J}}^{mts,i}_{1}(u_i, D_{i1}))}{du_i} \le (p_i + Nh_i + Nh_0 - c_i - c_0)\bar{F}_i(u_i).
$$

Furthermore, we show that for given demand realization d_{in}, $\bar{\bar{J}}^{mts,i}_{n}(u_i, d_{in})$ is concave in u_i. First note that $\bar{\bar{J}}^{mts,i}_{N+1}(u_i, \cdot) = 0$ is concave in u_i. For $n \le N$, suppose that the result holds for period $n+1$. For period n, recall the following two cases.

- if $u_i \le d_{in}$, $y_{in} = u_i$, and $\frac{d\bar{\bar{J}}^{mts,i}_{n}(u_i,d_{in})}{du_i} = p_i + (N-n+1)h_i + (N-n+1)h_0 - c_i - c_0$;
- otherwise, $y_{in} = d_{in}$, and

$$
\begin{aligned}
&\frac{d\bar{\bar{J}}^{mts,i}_{n}(u_i, d_{in})}{du_i}\\
&= \frac{dE_{D_{i,n+1}}(\bar{\bar{J}}^{mts,i}_{n+1}(u_i - d_{in}, D_{i,n+1}))}{du_i}
\end{aligned}
$$

$$
\begin{aligned}
&\leq (p_i+(N-n)h_i+(N-n)h_0-c_i-c_0)\mathcal{P}\left(\sum_{j=n+1}^{N} D_{ij}\geq u_i-d_{in}\right)\\
&\leq p_i+(N-n)h_i+(N-n)h_0-c_i-c_0\\
&\leq p_i+(N-n+1)h_i+(N-n+1)h_0-c_i-c_0.
\end{aligned}
$$

These two cases, together with the induction hypothesis, imply that $\frac{d\bar{\bar{J}}_n^{mts,i}(u_i,d_{in})}{du_i}$ decreases in u_i. Thus, $\bar{\bar{J}}_n^{mts,i}(u_i,d_{in})$ is concave in u_i (for given d_{in}), so is $E_{D_{in}}(\bar{\bar{J}}_n^{mts,i}(u_i,D_{in}))$. This further implies that $V_1^{mts}(X,u_1,\ldots,u_k)$ is jointly concave in $(x_0,x_1,\ldots,x_k,u_1,\ldots,u_k)$.

In addition, for $x_i\leq E(\sum_{n=1}^N D_{in})$, we will show that $\frac{dE_{D_{i1}}(\bar{\bar{J}}_1^{mts,i}(u_i,D_{i1}))}{du_i}>0$ for $u_i\in[0,x_i)$. That is, we will show that increasing the sales quota from u_i to $u_i+\epsilon_u$ by an arbitrarily small $\epsilon_u>0$ strictly improves the expected profit in sub-system i. Consider the following two cases:

- The realized demand over the season is $\sum_{n=1}^N d_{in}\leq u_i$: the realized profit with the sales quota u_i is the same as that with the sales quota $u_i+\epsilon_u$, because the realized sales quantity is the same;
- The realized demand over the season is $\sum_{n=1}^N d_{in}>u_i$: the realized sales quantity with the sales quota $u_i+\epsilon_u$ is strictly greater than that with the sales quota u_i, so is the realized profit with the sales quota $u_i+\epsilon_u$ than that with the sales quota u_i.

It then suffices to prove $\mathcal{P}\left(\sum_{n=1}^N D_{in}>u_i\right)>0$ for $u_i<x_i$. First consider a trivial case in which $\mathcal{P}(\sum_{n=1}^N D_{in}=E(\sum_{n=1}^N D_{in}))=1$. Since $x_i\leq E(\sum_{n=1}^N D_{in})$, $\mathcal{P}(\sum_{n=1}^N D_{in}\geq x_i>u_i)=1$. Now, consider the case $\mathcal{P}(\sum_{n=1}^N D_{in}=E(\sum_{n=1}^N D_{in}))<1$. To show $\mathcal{P}\left(\sum_{n=1}^N D_{in}>u_i\right)>0$ for $u_i<x_i$, it suffices to show $\mathcal{P}\left(\sum_{n=1}^N D_{in}>x_i\right)>0$, since it implies

$$
\mathcal{P}\left(\sum_{n=1}^N D_{in}>u_i\right)\geq\mathcal{P}\left(\sum_{n=1}^N D_{in}>x_i\right)>0.
$$

To this end, consider the following two sub-cases:

- If $x_i=0$, suppose $\mathcal{P}\left(\sum_{n=1}^N D_{in}>x_i\right)=0$. It implies $\mathcal{P}\left(\sum_{n=1}^N D_{in}=0\right)=1$, which further implies $E(\sum_{n=1}^N D_{in})=0$ and thus contradicts with $E(\sum_{n=1}^N D_{in})>0$;
- If $x_i>0$, suppose $\mathcal{P}\left(\sum_{n=1}^N D_{in}>x_i\right)=0$, which implies $\mathcal{P}\left(\sum_{n=1}^N D_{in}\leq x_i\right)=1$. Since

$$
\mathcal{P}(\sum_{n=1}^N D_{in}=E(\sum_{n=1}^N D_{in}))<1,\quad \mathcal{P}\left(\sum_{n=1}^N D_{in}=x_i\right)<1,\quad\text{and}\quad \mathcal{P}\left(\sum_{n=1}^N D_{in}<x_i\right)>0.
$$

Hence,

$$
E(\sum_{n=1}^N D_{in})=\int_0^{x_i}(\eta)d\mathcal{P}\left(\sum_{n=1}^N D_{in}\leq\eta\right)<\int_0^{x_i}(x_i)d\mathcal{P}\left(\sum_{n=1}^N D_{in}\leq\eta\right)=x_i.
$$

This, however, contradicts with $x_i\leq E(\sum_{n=1}^N D_{in})$.

Lastly, given $x_0 \le \sum_{i=1}^{k} x_i$ and $x_i \le E(\sum_{n=1}^{N} D_{in})$ for all $i = 1, \ldots, k$, we will show by contradiction that $u_i^* \le x_i$ where u_i^* is as defined in (8). Suppose there exists some product index $i_1 \in \{1, \ldots, k\}$ such that $u_{i_1}^* > x_{i_1}$. This, together with the constraints $\sum_{i=1}^{k} u_i^* \le x_0 \le \sum_{i=1}^{k} x_i$, implies that there exists a product index $i_2 \ne i_1$ with $u_{i_2}^* < x_{i_2}$. Then, given $x_{i_2} \le E(\sum_{n=1}^{N} D_{i_2,n})$ and $E(\sum_{n=1}^{N} D_{i_2,n}) > 0$, the proof above implies that at this point $(u_1, \ldots, u_k) = (u_1^*, \ldots, u_k^*)$,

$$\frac{\partial V_1^{mts}(X, u_1, \ldots, u_k)}{\partial u_{i_2}}|_{(u_1,\ldots,u_k)=(u_1^*,\ldots,u_k^*)} > 0.$$

However, for product i_1 with $u_{i_1}^* > x_{i_1}$,

$$\frac{\partial V_1^{mts}(X, u_1, \ldots, u_k)}{\partial u_{i_1}}|_{(u_1,\ldots,u_k)=(u_1^*,\ldots,u_k^*)} = 0$$

. In the MTS heuristic defined in subsection 4.2.1, when $u_{i_1} > x_{i_1}$, further increasing the sales quota for product i_1 does not change the sales quantity of product i_1. Moreover, further increasing the sales quota for product i_1 does not increase the sales quantity of other products since in the MTS heuristic demand fulfillment of one product is decoupled from that of another product.

Thus, as $\frac{\partial V_1^{mts}(X,u_1,\ldots,u_k)}{\partial u_{i_2}} > \frac{\partial V_1^{mts}(X,u_1,\ldots,u_k)}{\partial u_{i_1}} = 0$ at this point $(u_1, ..., u_k) = (u_1^*, \ldots, u_k^*)$, we can decrease $u_{i_1}^*$ and increase $u_{i_2}^*$ simultaneously by a small amount to increase $V_1^{mts}(X, u_1^*, \ldots, u_k^*)$, contradicting the optimality of $(u_1^*, \ldots, u_k^*)$.

(ii) The result in (i) above implies that given $x_0 \le \sum_{i=1}^{k} x_i$ and $x_i \le E(\sum_{n=1}^{N} D_{in})$ for all $i = 1, \ldots, k$, (8) is equivalent to

$$(u_1^*, \ldots, u_k^*) := \arg\max_{u_1,\ldots,u_k} \left\{ V_1^{mts}(x_0, x_1, \ldots, x_k, u_1, \ldots, u_k), s.t., \sum_{i=1}^{k} u_i \le x_0,\ 0 \le u_i \le x_i,\ i = 1, \ldots, k \right\}.$$

As implied by (i) above, $V_1^{mts}(x_0, x_1, \ldots, x_k, u_1, \ldots, u_k)$ is jointly concave in $(x_0, x_1, \ldots, x_k, u_1, \ldots, u_k)$ when $\sum_{i=1}^{k} u_i \le x_0$ and $0 \le u_i \le x_i$ for $i = 1, ..., k$. Thus, together with the fact that both $\{(u_1, \ldots, u_k) | \sum_{i=1}^{k} u_i \le x_0,\ 0 \le u_i \le x_i,\ i = 1, ..., k\}$ and $\{(u_1, \ldots, u_k, x_0, x_1, \ldots, x_k) | x_0 \ge 0,\ x_i \ge 0,\ x_0 \le \sum_{i=1}^{k} x_i,\ x_i \le E(\sum_{n=1}^{N} D_{in}),\ \sum_{i=1}^{k} u_i \le x_0,\ 0 \le u_i \le x_i,\ i = 1, \ldots, k\}$ are convex sets, Lemma EC.2 implies that $R^{mts}(X) = V_1^{mts}(X, u_1^*, \ldots, u_k^*)$ is jointly concave in X for $x_0 \le \sum_{i=1}^{k} x_i$ and $x_i \le E(\sum_{n=1}^{N} D_{in})$. Thus, given $x_i = x_i^{b(m)}$ and $x_i^{b(m)} \le E(\sum_{n=1}^{N} D_{in}^{(m)})$ for $i = 1, \ldots, k$, $R_b^{mts}(x_0) = R^{mts}(x_0, x_1^{b(m)}, \ldots, x_k^{b(m)})$ is concave in x_0 when $x_0 \le \sum_{i=1}^{k} x_i^{b(m)}$. This further implies that $\pi_b^{mts}(x_0) = R_b^{mts}(x_0) - c_0 x_0 - \sum_{j=1}^{k} (c_j x_j^{b(m)})$ is concave in x_0 when $x_0 \le \sum_{i=1}^{k} x_i^{b(m)}$.

(iii) For (iii-1), if $\frac{\partial V_1^{mts}}{\partial u_{i_1}} > \frac{\partial V_1^{mts}}{\partial u_{i_2}}$, we can increase $u_{i_1}^*$ and decrease $u_{i_2}^*$ simultaneously by a small amount to increase $V_1^{mts}(X, u_1^*, u_2^*, ..., u_k^*)$, contradicting the optimality of $(u_1^*, \ldots, u_k^*)$. Similarly, $\frac{\partial V_1^{mts}}{\partial u_{i_1}} < \frac{\partial V_1^{mts}}{\partial u_{i_2}}$ contradicts the optimality of $(u_1^*, \ldots, u_k^*)$. Thus, we have $\frac{\partial V_1^{mts}}{\partial u_{i_1}} = \frac{\partial V_1^{mts}}{\partial u_{i_2}}$.

Similarly, if $\frac{\partial V_1^{mts}}{\partial u_{i_1}} > \frac{\partial V_1^{mts}}{\partial u_i}$, we can increase $u_{i_1}^*$ and decrease u_i^* simultaneously by a small amount to increase $V_1^{mts}(X, u_1^*, u_2^*, ..., u_k^*)$, contradicting the optimality of $(u_1^*, \ldots, u_k^*)$. Thus, we have $\frac{\partial V_1^{mts}}{\partial u_{i_1}} \leq \frac{\partial V_1^{mts}}{\partial u_i}$.

For (iii-2), if $\frac{\partial V_1^{mts}}{\partial u_{i_3}} > \frac{\partial V_1^{mts}}{\partial u_{i_4}}$, we can increase $u_{i_3}^*$ and decrease $u_{i_4}^*$ simultaneously by a small amount to increase $V_1^{mts}(X, u_1^*, u_2^*, ..., u_k^*)$, contradicting the optimality of $(u_1^*, \ldots, u_k^*)$.

(iv) We first prove by contradiction that, given $x_0 < \sum_{i=1}^k x_i$, the optimal quota $(u_1^*, \ldots, u_k^*)$ satisfies $\sum_{i=1}^k u_i^* = x_0$. Suppose $\sum_{i=1}^k u_i^* < x_0$. This, together with $x_0 < \sum_{i=1}^k x_i$, implies that there exists some $i_1 \in \{1, \ldots, k\}$ with $u_{i_1}^* < x_{i_1}$. Furthermore, part (i) implies $\frac{\partial V_1^{mts}(X,u_1^*,u_2^*,\ldots,u_k^*)}{\partial u_{i_1}}|_{u_{i_1}=u_{i_1}^*} \geq 0$. If $\frac{\partial V_1^{mts}(X,u_1^*,u_2^*,\ldots,u_k^*)}{\partial u_{i_1}}|_{u_{i_1}=u_{i_1}^*} > 0$, slightly increasing $u_{i_1}^*$ while keeping all other quotas unchanged would result in a larger $V_1^{mts}(X, u_1^*, u_2^*, \ldots, u_k^*)$, contradicting the optimality of $u_{i_1}^*$; If, however, $\frac{\partial V_1^{mts}(X,u_1^*,u_2^*,\ldots,u_k^*)}{\partial u_{i_1}}|_{u_{i_1}=u_{i_1}^*} = 0$, we can construct a vector of sales quotas which is lexicographical larger than $(u_1^*, \ldots, u_k^*)$ by replacing $u_{i_1}^*$ with $u_{i_1}^* + \epsilon$, for an arbitrarily small $\epsilon \in (0, (x_0 - \sum_{i=1}^k u_i^*) \wedge (x_{i_1} - u_{i_1}^*)]$, and achieve the same profit as $V_1^{mts}(X, u_1^*, u_2^*, \ldots, u_k^*)$. This contradicts the assumption that $(u_1^*, u_2^*, \ldots, u_k^*)$ is the largest maximizer in lexicographical order.

To explicitly account for the dependence of u_i^* on x_0, in the rest of the proof we will use $u_i^*(x_0)$ to denote the optimal product-i quota for a given x_0 units of common component. To show $\frac{du_i^*(x_0)}{dx_0} \geq 0$, first note that since all the sales quotas are non-negative, it holds trivially when $u_i^*(x_0) = 0$. Now, for $u_i^*(x_0) > 0$, suppose $\frac{du_i^*(x_0)}{dx_0} < 0$, then $u_i^*(x_0) > u_i^*(x_0 + \epsilon)$ for arbitrarily small $\epsilon > 0$. In this case, the result $\sum_{i=1}^k u_i^*(x_0 + \epsilon) = x_0 + \epsilon$ implies $u_j^*(x_0) < u_j^*(x_0 + \epsilon)$ for some product $j \neq i$, which further implies $u_j^*(x_0) < x_j$. By parts (iii-1) and (iii-2), at the point $(u_1, ..., u_k) = (u_1^*(x_0), ..., u_k^*(x_0))$,

$$\frac{\partial V_1^{mts}(x_0, x_1, ..., x_k, u_1, ..., u_k)}{\partial u_i} \geq \frac{\partial V_1^{mts}(x_0, x_1, ..., x_k, u_1, ..., u_k)}{\partial u_j}$$

That is,

$$\left.\frac{dE_{D_{i1}}(\bar{\bar{J}}_1^{mts,i}(u_i, D_{i1}))}{du_i}\right|_{u_i=u_i^*(x_0)} \geq \left.\frac{dE_{D_{j1}}(\bar{\bar{J}}_1^{mts,j}(u_j, D_{j1}))}{du_j}\right|_{u_j=u_j^*(x_0)}$$

By concavity of $E_{D_{i1}}(\bar{\bar{J}}_1^{mts,i}(u_i, D_{i1}))$ in u_i, it further implies

$$\frac{dE_{D_{i1}}(\bar{\bar{J}}_1^{mts,i}(u_i, D_{i1}))}{du_i}|_{u_i=u_i^*(x_0+\epsilon)} \geq \frac{dE_{D_{j1}}(\bar{\bar{J}}_1^{mts,j}(u_j, D_{j1}))}{du_j}|_{u_j=u_j^*(x_0+\epsilon)}$$

since

$$\begin{aligned} &\left.\frac{dE_{D_{i1}}(\bar{\bar{J}}_1^{mts,i}(u_i, D_{i1}))}{du_i}\right|_{u_i=u_i^*(x_0+\epsilon)} \geq \left.\frac{dE_{D_{i1}}(\bar{\bar{J}}_1^{mts,i}(u_i, D_{i1}))}{du_i}\right|_{u_i=u_i^*(x_0)} \\ \geq &\left.\frac{dE_{D_{j1}}(\bar{\bar{J}}_1^{mts,j}(u_j, D_{j1}))}{du_j}\right|_{u_j=u_j^*(x_0)} \geq \left.\frac{dE_{D_{j1}}(\bar{\bar{J}}_1^{mts,j}(u_j, D_{j1}))}{du_j}\right|_{u_j=u_j^*(x_0+\epsilon)}. \end{aligned} \tag{EC.20}$$

Consider the following two cases:

- $$\frac{dE_{D_{i1}}(\bar{\bar{J}}_1^{mts,i}(u_i,D_{i1}))}{du_i}|_{u_i=u_i^*(x_0+\epsilon)} > \frac{dE_{D_{j1}}(\bar{\bar{J}}_1^{mts,j}(u_j,D_{j1}))}{du_j}|_{u_j=u_j^*(x_0+\epsilon)}:$$

It implies that at the point $(u_1,...,u_k)=(u_1^*(x_0+\epsilon),...,u_k^*(x_0+\epsilon))$,

$$\frac{\partial V_1^{mts}(x_0+\epsilon,x_1,...,x_k,u_1,...,u_k)}{\partial u_i} \geq \frac{\partial V_1^{mts}(x_0+\epsilon,x_1,...,x_k,u_1,...,u_k)}{\partial u_j}.$$

Thus, we can increase $u_i^*(x_0+\epsilon)$ and decrease $u_j^*(x_0+\epsilon)$ simultaneously to increase $V_1^{mts}(x_0+\epsilon,x_1,...,x_k,u_1^*(x_0+\epsilon),...,u_k^*(x_0+\epsilon))$, contradicting the optimality of $(u_1^*(x_0+\epsilon),\ldots,u_k^*(x_0+\epsilon))$.

- $$\frac{dE_{D_{i1}}(\bar{\bar{J}}_1^{mts,i}(u_i,D_{i1}))}{du_i}|_{u_i=u_i^*(x_0+\epsilon)} = \frac{dE_{D_{j1}}(\bar{\bar{J}}_1^{mts,j}(u_j,D_{j1}))}{du_j}|_{u_j=u_j^*(x_0+\epsilon)}:$$

The inequalities in (EC.20) imply that all of the four marginal profits are equal, i.e.,

$$\begin{aligned}\frac{dE_{D_{i1}}(\bar{\bar{J}}_1^{mts,i}(u_i,D_{i1}))}{du_i}|_{u_i=u_i^*(x_0)} &= \frac{dE_{D_{j1}}(\bar{\bar{J}}_1^{mts,j}(u_j,D_{j1}))}{du_j}|_{u_j=u_j^*(x_0)}\\ &= \frac{dE_{D_{i1}}(\bar{\bar{J}}_1^{mts,i}(u_i,D_{i1}))}{du_i}|_{u_i=u_i^*(x_0+\epsilon)}\\ &= \frac{dE_{D_{j1}}(\bar{\bar{J}}_1^{mts,j}(u_j,D_{j1}))}{du_j}|_{u_j=u_j^*(x_0+\epsilon)}.\end{aligned}$$

In such a case, the concavity of the profit function in each single-product system implies

$$\frac{dE_{D_{i1}}(\bar{\bar{J}}_1^{mts,i}(u_i,D_{i1}))}{du_i} = \frac{dE_{D_{j1}}(\bar{\bar{J}}_1^{mts,j}(u_j,D_{j1}))}{du_j} \quad \text{for} \quad u_i\in[u_i^*(x_0+\epsilon),u_i^*(x_0)]$$

and $u_j\in[u_j^*(x_0),u_j^*(x_0+\epsilon)]$. Based on this result, we can construct a vector of sales quotas contradicting the assumption that $(u_1^*(x_0+\epsilon),...,u_k^*(x_0+\epsilon))$ and $(u_1^*(x_0),...,u_k^*(x_0))$ are the largest maximizer in the lexicographical order for $x_0+\epsilon$ and x_0 units of common components, respectively.

Specifically, if $i<j$, then with $x_0+\epsilon$ units of common components, we can construct a vector of sales quotas which is lexicographically larger than $(u_1^*(x_0+\epsilon),...,u_k^*(x_0+\epsilon))$ by replacing $u_i^*(x_0+\epsilon)$ with $u_i^*(x_0+\epsilon)+\epsilon'$ and $u_j^*(x_0+\epsilon)$ with $u_j^*(x_0+\epsilon)-\epsilon'$, where $0<\epsilon'\leq(u_i^*(x_0)-u_i^*(x_0+\epsilon))\wedge(u_j^*(x_0+\epsilon)-u_j^*(x_0))$. This constructed vector sales quotas leads to the same profit as $(u_1^*(x_0+\epsilon),...,u_k^*(x_0+\epsilon))$ and is thus also an optimal solution to the problem. It thus contradicts with the assumption that $(u_1^*(x_0+\epsilon),...,u_k^*(x_0+\epsilon))$ is the largest maximizer in the lexicographical order.

Similarly, if $i>j$, then with x_0 units of common components, we can construct a vector of sales quotas which is lexicographical larger than $(u_1^*(x_0),...,u_k^*(x_0))$ by replacing $u_i^*(x_0)$ with $u_i^*(x_0)-\epsilon''$ and $u_j^*(x_0)$ with $u_j^*(x_0)+\epsilon''$, with an arbitrarily small $\epsilon''\in(0,(u_i^*(x_0)-u_i^*(x_0+\epsilon))\wedge(u_j^*(x_0+\epsilon)-u_j^*(x_0))]$. This constructed vector sales quotas leads to the same profit as $(u_1^*(x_0),...,u_k^*(x_0))$ and is thus also an optimal solution to the problem. It thus contradicts with the assumption that $(u_1^*(x_0),...,u_k^*(x_0))$ is the largest maximizer in the lexicographical order.

Summarizing the two cases, $\frac{du_i^*(x_0)}{dx_0}\geq 0$. This, together with $\sum_{i=1}^k\frac{du_i^*(x_0)}{dx_0}=1$ (as implied by $\sum_{i=1}^k u_i^*(x_0)=x_0$ above), results in $0\leq\frac{du_i^*(x_0)}{dx_0}\leq 1$. Q.E.D.

D. Proofs of Propositions in the Main Body and in Appendices A and B

D.1. Proof of Proposition 1

We will show that the following two definitions of nested policy, (A.1) and (A.2), are equivalent. In both definitions, assume that the allocation in each period is feasible, i.e., $0 \le y_{in} \le x_i \wedge d_{in}$ and $\sum_{i=1}^{k} y_i \le x_0$, for all i and n.

(A.1) For any $n \ge 1$, $i_1, i_2 \in \{1, \dots, k\}$, and $i_1 < i_2$, $y_{i_2,n} > 0$ implies $y_{i_1,n} = x_{i_1} \wedge d_{i_1,n} \wedge (x_0 - \sum_{j<i_1} y_{jn})$. That is, no lower-margin order is fulfilled unless all the higher-margin orders have been met to the extent allowed by the component inventory.

(A.2) Assume that the allocation in each period, $y_{1n}, y_{2n}, ..., y_{kn}$, is determined in the decreasing order of product margins. For any $n \ge 1$, there exist nonnegative sales quotas $u_1, u_2, ..., u_k$ with $\sum_{i=1}^{k} u_i \le x_0$ such that, for $i = 1, ..., k$, at most $\sum_{l=i}^{k} u_l$ units of the common components can be used to satisfy demand for products i, $i+1$,..., and k. That is, $\sum_{l=i}^{k} y_{ln} \le \sum_{l=i}^{k} u_l$ for $i = 1, ..., k$. In other words, for $i = 1, ..., k$, y_{in} is the maximum nonnegative number satisfying $y_{in} \le x_i \wedge d_{in}$ and $\sum_{l=j}^{i} y_{ln} \le \sum_{l=j}^{k} u_l$ for $j = 1, ..., i$.

In preparation, we first prove that in (A.2),

$$y_{in} \ge 0 \quad \text{and} \quad \sum_{l=i}^{k} y_{ln} \le \sum_{l=i}^{k} u_l \quad \text{for } i = 1, ..., k, \tag{EC.21}$$

is equivalent to

$$y_{in} \ge 0 \quad \text{and} \quad \sum_{l=j}^{i} y_{ln} \le \sum_{l=j}^{k} u_l \quad \text{for } i = 1, ..., k, \text{ and } j = 1, ..., i. \tag{EC.22}$$

(EC.21) ⇐ (EC.22) Let $i = k$ in (EC.22), we have $\sum_{l=j}^{k} y_{ln} \le \sum_{l=j}^{k} u_l$ for $j = 1, ..., k$, i.e., (EC.21).

(EC.21) ⇒ (EC.22) (EC.21) implies $\sum_{l=i}^{k} y_{ln} \le \sum_{l=i}^{k} u_l$ for $i = 1, ..., k$. This implies that, for $j = 1, ..., i$, $\sum_{l=j}^{k} y_{ln} \le \sum_{l=j}^{k} u_l$, which further implies $\sum_{l=j}^{i} y_{ln} \le \sum_{l=j}^{k} y_{ln} \le \sum_{l=j}^{k} u_l$ since $y_{ln} \ge 0$ for $l = i+1, ..., k$. Thus, we have (EC.22).

By (EC.22) and $y_{in} \le x_i \wedge d_{in}$, the sales quantities in (A.2) satisfy the following equations:

$$y_{1n} = x_1 \wedge d_{1n} \wedge \sum_{j=1}^{k} u_j, \tag{EC.23}$$

and for $i = 2, \dots, k$,

$$y_{in} = x_i \wedge d_{in} \wedge \left[(\sum_{j=1}^{k} u_j - \sum_{j=1}^{i-1} y_{jn}) \wedge (\sum_{j=2}^{k} u_j - \sum_{j=2}^{i-1} y_{jn}) \wedge \dots \wedge (\sum_{j=i-1}^{k} u_j - y_{i-1,n}) \wedge \sum_{j=i}^{k} u_j \right] \tag{EC.24}$$

Next we prove the equivalence between (A.1) and (A.2), i.e., (A.1)⇔(A.2):

"$\Leftarrow$": We first show that, by (A.2), for $i_1 < i_2$, $y_{i_2,n} > 0$ implies $y_{i_1,n} = x_{i_1} \wedge d_{i_1,n}$: By (EC.24),

$$y_{i_2,n} = x_{i_2} \wedge d_{i_2,n} \wedge (\sum_{j=1}^{k} u_j - \sum_{j=1}^{i_2-1} y_{jn}) \wedge (\sum_{j=2}^{k} u_j - \sum_{j=2}^{i_2-1} y_{jn}) \wedge \ldots \wedge (\sum_{j=i_2-1}^{k} u_j - y_{i_2-1,n}) \wedge \sum_{j=i_2}^{k} u_j,$$

Hence, $y_{i_2,n} > 0$ implies

$$0 < y_{i_2,n} \leq (\sum_{j=1}^{k} u_j - \sum_{j=1}^{i_2-1} y_{jn}) \wedge (\sum_{j=2}^{k} u_j - \sum_{j=2}^{i_2-1} y_{jn}) \wedge \ldots \wedge (\sum_{j=i_2-1}^{k} u_j - y_{i_2-1,n}) \wedge \sum_{j=i_2}^{k} u_j.$$

Furthermore, as $i_1 < i_2$, $i_1 \leq i_2 - 1$. Thus,

$$\sum_{j=2}^{k} u_j - \sum_{j=2}^{i_2-1} y_{jn} \leq \sum_{j=2}^{k} u_j - \sum_{j=2}^{i_1} y_{jn}, \quad \ldots, \quad \sum_{j=i_1-1}^{k} u_j - \sum_{j=i_1-1}^{i_2-1} y_{jn} \leq \sum_{j=i_1-1}^{k} u_j - \sum_{j=i_1-1}^{i_1} y_{jn},$$
$$\sum_{j=i_1}^{k} u_j - \sum_{j=i_1}^{i_2-1} y_{jn} \leq \sum_{j=i_1}^{k} u_j - y_{i_1,n},$$
$$\sum_{j=i_1+l}^{k} u_j - \sum_{j=i_1+l}^{i_2-1} y_{jn} \leq \sum_{j=i_1+l}^{k} u_j, \quad l = 1, \ldots, i_2 - i_1 - 1$$

Thus,

$$\begin{aligned}
0 < & \Bigg[(\sum_{j=1}^{k} u_j - \sum_{j=1}^{i_2-1} y_{jn}) \wedge (\sum_{j=2}^{k} u_j - \sum_{j=2}^{i_2-1} y_{jn}) \wedge \ldots \wedge (\sum_{j=i_1-1}^{k} u_j - \sum_{j=i_1-1}^{i_2-1} y_{jn}) \wedge (\sum_{j=i_1}^{k} u_j - \sum_{j=i_1}^{i_2-1} y_{jn}) \\
& \wedge (\sum_{j=i_1+1}^{k} u_j - \sum_{j=i_1+1}^{i_2-1} y_{jn}) \wedge \ldots \wedge (\sum_{j=i_2-1}^{k} u_j - y_{i_2-1,n}) \wedge \sum_{j=i_2}^{k} u_j \Bigg] \\
\leq & \Bigg[(\sum_{j=1}^{k} u_j - \sum_{j=1}^{i_1} y_{jn}) \wedge (\sum_{j=2}^{k} u_j - \sum_{j=2}^{i_1} y_{jn}) \wedge \ldots \wedge (\sum_{j=i_1-1}^{k} u_j - \sum_{j=i_1-1}^{i_1} y_{jn}) \wedge (\sum_{j=i_1}^{k} u_j - y_{i_1,n}) \\
& \wedge \sum_{j=i_1+1}^{k} u_j \wedge \ldots \wedge \sum_{j=i_2-1}^{k} u_j \wedge \sum_{j=i_2}^{k} u_j \Bigg]
\end{aligned}$$

Hence, when $y_{i_2,n} > 0$,

$$y_{i_1,n} < (\sum_{j=1}^{k} u_j - \sum_{j=1}^{i_1-1} y_{jn}) \wedge (\sum_{j=2}^{k} u_j - \sum_{j=2}^{i_1-1} y_{jn}) \wedge \ldots \wedge (\sum_{j=i_1-1}^{k} u_j - y_{i_1-1,n}) \wedge (\sum_{j=i_1}^{k} u_j).$$

Thus, according to (EC.24), $y_{i_1,n} = x_{i_1} \wedge d_{i_1,n}$. Moreover, $y_{i_1,n} < \sum_{j=1}^{k} u_j - \sum_{j=1}^{i_1-1} y_{jn} \leq x_0 - \sum_{j<i_1} y_{jn}$ since $\sum_{j=1}^{k} u_j \leq x_0$. Therefore, $y_{i_1,n} = x_{i_1} \wedge d_{i_1,n} \wedge (x_0 - \sum_{j<i_1} y_{jn})$ when $y_{i_2,n} > 0$.

"$\Rightarrow$": For exposition, here we use superscript g to denote the variables related to definition (A.2) (e.g., u_j^g for the individual quota for product j in (A.2), and $y_{jn}^g(u_1^g, \ldots u_k^g)$ for the sales quantity for product j satisfying (A.2)), while no superscript for definition (A.1) (e.g., y_{jn} for the sales quantity for product j satisfying (A.1)). We will show that, for any sales quantity $(y_{1n}, \ldots, y_{kn})$ satisfying

(A.1), we can construct a vector of quotas $(u_1^g, \dots, u_k^g)$ under which (A.2) leads to the same sale quantities as (A.1), i.e., $y_{jn}^g(u_1^g, \dots u_k^g) = y_{jn}$ for all $j = 1, ..., k$.

To this end, first note that, by (A.1), y_{jn}'s must satisfy: either $y_{jn} = 0$ for all $j = 1, \dots, k$, or there exists a critical index $i_0 = \max\{i = 1, \dots, k | y_{in} > 0\}$ such that

$$y_{jn} = \begin{cases} x_j \wedge d_{jn} & \text{if } j = 1, \dots, i_0 - 1 \\ y_{i_0,n} \in (0, x_{i_0} \wedge d_{i_0,n}] & \text{if } j = i_0 \\ 0 & \text{otherwise.} \end{cases} \tag{EC.25}$$

If $y_{jn} = 0$ for all $j = 1, \dots, k$, we define $u_j^g = 0$ for $j \in \{1, \dots, k\}$ and thus $y_{jn}^g(u_1^g, \dots u_k^g) = 0$ for all j. Now, consider the second case with i_0. For this case, define $u_j^g = y_{jn}$ for $j = 1, \dots, k$. As such, $y_{1n}^g(u_1^g, \dots u_k^g) = x_1 \wedge d_{1n} \wedge (\sum_{j=1}^k u_j^g) = x_1 \wedge d_{1n} \wedge (\sum_{j=1}^k y_{jn}) = y_{1n}$, where the last equality holds because in (EC.25):

- if $i_0 = 1$, we have $\sum_{j=2}^k y_{jn} = 0$ and $y_{1n} = \sum_{j=1}^k y_{jn} \le x_1 \wedge d_{1n}$;
- if $i_0 > 1$, we have $\sum_{j=2}^k y_{jn} > 0$ and $y_{1n} = x_1 \wedge d_{1n} < \sum_{j=1}^k y_{jn}$.

Furthermore, by (EC.24), $y_{2n}^g(u_1^g, \dots u_k^g) = x_2 \wedge d_{2n} \wedge (\sum_{j=2}^k u_j^g) \wedge (\sum_{j=1}^k u_j^g - y_{1n}^g) = x_2 \wedge d_{2n} \wedge (\sum_{j=2}^k y_{jn}) = y_{2n}$, where the last equality holds because:

- if $i_0 = 1$, we have $y_{2n}^g(u_1^g, \dots u_k^g) = y_{2n} = \sum_{j=2}^k y_{jn} = 0$;
- if $i_0 = 2$, we have $y_{2n}^g(u_1^g, \dots u_k^g) > \sum_{j=3}^k y_{jn} = 0$ and $y_{2n} = \sum_{j=2}^k y_{jn} \le x_2 \wedge d_{2n}$;
- if $i_0 > 2$, we have $\sum_{j=3}^k y_{jn} > 0$ and $y_{2n} = x_2 \wedge d_{2n} < \sum_{j=2}^k y_{jn}$.

Similarly, it can be proved that $y_{jn}^g(u_1^g, \dots u_k^g) = y_{jn}$ for all $j = 3, ..., k$. Q.E.D.

D.2. Proof of Proposition 2

(i) The optimality of the stated fulfillment policy follows from Lemma EC.4 and Proposition EC.1 (iii).

Next we derive the optimal quotas u_{1n}^* and u_{2n}^* as defined in (6). Recall that $y_{1n}^* = x_0 \wedge x_1 \wedge d_{1n}$ and $y_{2n}^* = \bar{y}_n(X - y_{1n}^* a_1) \wedge d_{2n}$. In the meanwhile, by (6), $y_{1n}^* = x_1 \wedge d_{1n} \wedge (u_{1n}^* + u_{2n}^*)$ and $y_{2n}^* = x_2 \wedge d_{2n} \wedge (u_{1n}^* + u_{2n}^* - y_{1n}^*) \wedge u_{2n}^*$. Thus, given $\bar{y}_n(X - y_{1n}^* a_1) \in [0, (x_0 - y_{1n}^*) \wedge x_2]$, the quotas that are lexicographically largest satisfy $u_{1n}^* + u_{2n}^* = x_0$ and $u_{2n}^* = \bar{y}_n(X - y_{1n}^* a_1) \wedge d_{2n}$, where $y_{1n}^* = x_0 \wedge x_1 \wedge d_{1n}$.

(ii) We will apply Lemma EC.8 to show the properties of u_{2n}^*:

- The impact of $d_{1,n+1}$ or $d_{2,n+1}$ on u_{2n}^*: Note that both d_{2n} and $y_{1n}^* = x_0 \wedge x_1 \wedge d_{1n}$ are independent of $d_{1,n+1}$ and $d_{2,n+1}$. Thus, $X - y_{1n}^* a_1$ is independent of $d_{1,n+1}$ and $d_{2,n+1}$. By Lemma EC.8, $\bar{y}_n(X)$ decreases when $d_{1,n+1}$ or $d_{2,n+1}$ stochastically increases. Thus, the same property also holds for $\bar{y}_n(X - y_{1n}^* a_1)$ and $u_{2n}^* = d_{2n} \wedge \bar{y}_n(X - y_{1n}^* a_1)$.

- The impact of x_0 on u_{2n}^*: We first consider the case of $x_0 \leq x_1$. When $d_{1n} \geq x_0$, $y_{1n}^* = x_0 < x_1 \wedge d_{1n}$, $u_{2n}^* = 0$ because of no common component. When $d_{1n} < x_0$, the condition $x_0 \leq x_1$ implies $y_{1n}^* = d_{1n} < x_0 \leq x_1$. Thus, by Lemma EC.8, $\bar{y}_n(X - y_{1n}^* a_1) = \bar{y}_n(X - d_{1n} a_1)$ increases in x_0 for $d_{1n} < x_0$. Combining the subcases of $d_{1n} < x_0$ and $d_{1n} \geq x_0$, $\bar{y}_n(X - y_{1n}^* a_1)$ increases in x_0 for $x_0 \leq x_1$.

For the case of $x_0 > x_1$, when $d_{1n} \geq x_1$, $y_{1n}^* = x_1 < x_0 \wedge d_{1n}$. Thus, after fulfilling the product-1 order, there is no component 1 left to fulfill any future demand of product 1. That is, starting from period $n+1$, the W system degenerates to a single-product system and thus the demand for product 2 should be satisfied as much as allowed by the inventory. Therefore, $\bar{y}_n(X - y_{1n}^* a_1) = x_2 \wedge (x_0 - y_{1n}^*) = x_2 \wedge (x_0 - x_1)$. When $d_{1n} < x_1$, the condition $x_0 > x_1$ implies $y_{1n}^* = d_{1n} < x_1 \leq x_0$. Thus, by Lemma EC.8, $\bar{y}_n(X - y_{1n}^* a_1) = \bar{y}_n(X - d_{1n} a_1)$ increases in x_0 for $d_{1n} < x_1$. Combining the subcases of $d_{1n} < x_1$ and $d_{1n} \geq x_1$, $\bar{y}_n(X - y_{1n}^* a_1)$ increases in x_0 for $x_0 > x_1$.

Combining the cases of $x_0 \leq x_1$ and $x_0 > x_1$, given the continuity of $\bar{y}_n(\cdot)$, $\bar{y}_n(X - y_{1n}^* a_1)$ increases in x_0. Thus, $u_{2n}^* = d_{2n} \wedge \bar{y}_n(X - y_{1n}^* a_1)$ also increases in x_0.

- The impact of x_1 or x_2 on u_{2n}^*: The proof is similar to that for the impact of x_0. Q.E.D.

D.3. Proof of Proposition 3

Here we adopt the same approach as stated in the proof of Proposition 7 (i.1.a) to account for the holding costs and salvage values. That is, at the beginning of the season the firm incurs the cost for holding all the initial inventory through the entire season and also receives the total salvage values of all the initial inventory. During the selling season, when a sales takes place, an addition representing savings of holding cost and a deduction of the salvage value are taken into account in the selling margin. As the demand for product 1 should always be filled, it suffices to consider the fulfillment decision for $j > 1$. We first prove the result for product $j \in \{2, .., k-2, k-1\}$ and product k, and then show it for any product $j_1 \in \{2, .., k-2\}$ and product $j_2 \in \{j_1+1, \ldots, k-1\}$.

Consider first product $j \in \{2, .., k-2, k-1\}$ and product k. For expositional convenience, consider the fulfillment decision for a unit of demand of product j and a unit of demand of product k. To show that a nested policy is optimal, compare two systems with the same inventory profile in period n: system 1 follows a non-nested policy in the current period (i.e., period n): rejecting the demand for product j and fulfilling that for product k resulting in an immediate profit of $p_k + (N-n+1)h_k + (N-n+1)h_0 - c_k - c_0$. System 1 then follows the optimal fulfillment policy starting from period $n+1$. System 2 follows a nested policy in the current period: rejecting product k's demand and fulfilling product j's demand resulting in an immediate profit of $p_j + (N-n+1)h_j + (N-n+1)h_0 - c_j - c_0$. In the subsequent periods system 2 follows the optimal fulfillment policy. Note that, given that the period-n fulfillment policy in system 1 is feasible, the aforementioned period-n fulfillment policy is also feasible for system 2, since the inventory of component j is ample. Note also that the total expected profit for the subsequent periods (i.e., period $n+1$ through period N) is weakly higher in system 2 than in system 1, because, after the fulfillment in period n, compared with system 1, system 2 has more component k, fewer but still ample component j, and same inventory for all the other components. Since $p_j - c_j \geq p_k - c_k$ and $h_j \geq h_k$ imply $p_j + (N-n+1)h_j + (N-n+1)h_0 - c_j - c_0 \geq p_k + (N-n+1)h_k + (N-n+1)h_0 - c_k - c_0$, compared to system 1, system 2 yields a higher profit in period n and weakly total higher profit for the subsequent periods. Thus, the total profit under system 2 is higher than that under system 1. Following the same way, we can replace a non-nested policy in any period by a nested policy, and improve the expected total profit. As such, a nested policy is optimal.

For product $j_1 \in \{2, .., k-2\}$ and product $j_2 \in \{j_1+1, \ldots, k-1\}$, since the inventory for components j_1 and j_2 is ample, by a similar coupling argument, rejecting product j_1's demand while fulfilling an order for product j_2 is always suboptimal. Hence, we conclude that a nested policy is optimal. Q.E.D.

D.4. Proof of Proposition 4

(i) By Lemma EC.3, $R_n(X)$ and $J_n(X, d_{1n}, \ldots, d_{kn})$ are jointly concave in X. So are $R_1(X)$ and $\pi(X) = R_1(X) - \sum_{j=0}^{k}(c_j x_j)$.

(ii) Bernstein et al. (2011) prove the result for W systems. Below we extend the proof to generalized W systems. We shall focus on the case with two periods. (For more than two periods, a similar proof follows). Assume first that there are L possible demand realizations of the last period (i.e., period-2) demand $(D_{12}, D_{22}, ..., D_{k2})$ and each realization $(d_{12}^l, \ldots, d_{k2}^l)$ occurs with probability g_l, $l = 1, ..., L$. After observing the demand realization $(d_{11}, \ldots, d_{k1})$ in period 1,

$$
\begin{aligned}
& J_1^{(L)}(X, d_{11}, \ldots, d_{k1}) = \\
\max_{y_{11},\ldots,y_{k1}} \quad & -2\sum_{j=0}^{k}(h_j x_j) + \sum_{j=0}^{k}(c_j x_j) + \sum_{i=1}^{k}\left[(p_i + 2(h_i + h_0) - c_i - c_0)y_{i1} + \sum_{l=1}^{L} g_l (p_i + h_i + h_0 - c_i - c_0) y_{i2}^l\right] \\
\text{subject to} \quad & y_{i1} + y_{i2}^l \leq x_i \quad \text{for all } i = 1, ..., k, \ l = 1, ..., L \\
& \sum_{i=1}^{k} y_{i1} + \sum_{i=1}^{k} y_{i2}^l \leq x_0 \quad \text{for all } l = 1, ..., L \\
& 0 \leq y_{i1} \leq d_{i1} \quad \text{for all } i = 1, ..., k \\
& 0 \leq y_{i2}^l \leq d_{i2}^l \quad \text{for all } i = 1, ..., k, \ l = 1, ..., L,
\end{aligned}
$$

Note that $J_1^{(L)}(X, d_{11}, \ldots, d_{k1})$ is supermodular in (x_0, x_i) for given $x_1, ..., x_{i-1}, x_{i+1}, ...x_k$, because we can rewrite it as

$$J_1^{(L)}(X, d_{11}, \ldots, d_{k1}) =$$

$$\begin{aligned}\max_{y_{11},\ldots,y_{k1}} \quad & -2\sum_{j=0}^{k}(h_jx_j)+\sum_{j=0}^{k}(c_jx_j)+\sum_{i=1}^{k}\left[(p_i+2(h_i+h_0)-c_i-c_0)y_{i1}+\sum_{l=1}^{L}g_l(p_i+h_i+h_0-c_i-c_0)y_{i2}^l\right]\\ \text{s.t.} \quad & z_i\le x_i \quad \text{for all } i=1,...,k\\ & \sum_{i=1}^{k} z_i\le x_0 \quad \text{for all } i=1,...,k\\ & y_{i1}+y_{i2}^l-z_i\le 0 \quad \text{for all } i=1,...,k,\ l=1,...,L\\ & 0\le y_{i1}\le d_{i1} \quad \text{for all } i=1,...,k\\ & 0\le y_{i2}^l\le d_{i2}^l \quad \text{for all } i=1,...,k,\ l=1,...,L.\end{aligned}$$

Since $y_{i1}+y_{i2}^l-z_i\le 0$ is a homogeneous-side constraint, by Zipkin (2003), $J_1^{(L)}(X,d_{11},\ldots,d_{k1})$ is supermodular in (x_0,x_i) given $(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_k)$, $i=1,\ldots,k$.

Now, because sales are bounded by demand and demand has a finite mean,

$$J_1(X,d_{11},\ldots,d_{k1})=\lim_{L\to\infty}J_1^{(L)}(X,d_{11},\ldots,d_{k1})$$

is also supermodular in (x_0,x_i). Because expectation preserves supermodularity, $R_1(X)$ is supermodular in (x_0,x_i). So is the optimal profit function $\pi(X)=-\sum_{j=0}^{k}(c_jx_j)+R_1(X)$.

(iii) We prove the claim by induction. Let $e_1=(0,1,0)$ and $e_2=(0,0,1)$. First note that when $n=N+1$, for any $\delta\ge 0$ and $\epsilon\ge 0$,

$$R_{N+1}(X+\delta e_1+\epsilon e_2)-R_{N+1}(X+\epsilon e_2)\le R_{N+1}(X+\delta e_1)-R_{N+1}(X)$$

trivially holds since both sides are linear functions. Now, suppose

$$R_{n+1}(X+\delta e_1+\epsilon e_2)-R_{n+1}(X+\epsilon e_2)\le R_{n+1}(X+\delta e_1)-R_{n+1}(X)$$

holds. We shall show that

$$R_n(X+\delta e_1+\epsilon e_2)-R_n(X+\epsilon e_2)\le R_n(X+\delta e_1)-R_n(X).$$

According to the definition of $R_n(X)$, it suffices to prove, for any $\delta\ge 0$, $\epsilon\ge 0$, $d_{1n}\ge 0$ and $d_{2n}\ge 0$,

$$\begin{aligned}&J_n(X+\delta e_1+\epsilon e_2,d_{1n},d_{2n})-J_n(X+\delta e_1,d_{1n},d_{2n})\\ &\le J_n(X+\epsilon e_2,d_{1n},d_{2n})-J_n(X,d_{1n},d_{2n}).\end{aligned} \tag{EC.26}$$

For given d_{1n} and d_{2n}, denote by $y_{in}^*(X)$ the optimal sales quantity of product $i\in\{1,2\}$ under component profile X. Here, we omit the arguments d_{1n} and d_{2n} in functions, whenever no confusion may arise. Recall equations (EC.14) and (EC.15). Given

$$J_n(X,d_{1n},d_{2n})=-\sum_{j=0}^{2}(h_jx_j)+(p_1+h_1+h_0)y_{1n}^*+G_n(X-y_{1n}^*a_1,d_{2n}),$$

the optimality of the nested allocation policy implies that $y^*_{1n}(X) = x_1 \wedge x_0 \wedge d_{1n}$, which is independent of x_2 or d_{2n}, so $y^*_{1n}(X) = y^*_{1n}(X + \epsilon e_2)$, $y^*_{1n}(X + \delta e_1) = y^*_{1n}(X + \delta e_1 + \epsilon e_2)$, and $y^*_{1n}(X) \leq y^*_{1n}(X + \delta e_1) \leq y^*_{1n}(X) + \delta$. Therefore, (EC.26) holds if

$$\begin{aligned}&G_n(X + \delta e_1 + \epsilon e_2 - y^*_{1n}(X + \delta e_1)a_1, d_{2n}) - G_n(X + \delta e_1 - y^*_{1n}(X + \delta e_1)a_1, d_{2n})\\ \leq\ &G_n(X + \epsilon e_2 - y^*_{1n}(X)a_1, d_{2n}) - G_n(X - y^*_{1n}(X)a_1, d_{2n}).\end{aligned}$$

Consider two cases as below: $y^*_{1n}(X) = y^*_{1n}(X + \delta e_1)$ and $y^*_{1n}(X) < y^*_{1n}(X + \delta e_1) \leq y^*_{1n}(X) + \delta$.

Case (iii.a) $y^*_{1n}(X) = y^*_{1n}(X + \delta e_1)$: In this case, it suffices to prove that

$$G_n(X + \delta e_1 + \epsilon e_2, d_{2n}) - G_n(X + \delta e_1, d_{2n}) \leq G_n(X + \epsilon e_2, d_{2n}) - G_n(X, d_{2n}) \qquad \text{(EC.27)}$$

is true for any $\delta \geq 0$, $\epsilon \geq 0$, and $X \geq 0$. Recall (EC.15):

$$\begin{aligned}&G_n(X, d_{2n}) = \max_{y_{2n}} [(p_2 + h_2 + h_0)y_{2n} + R_{n+1}(X - y_{2n}a_2)]\\ &\text{subject to} \ \ 0 \leq y_{2n} \leq x_2 \wedge d_{2n}, \quad y_{2n} \leq x_0.\end{aligned}$$

Consider the following two subcases, $y^*_{2n}(X + \delta e_1 + \epsilon e_2) < y^*_{2n}(X)$ and $y^*_{2n}(X) \leq y^*_{2n}(X + \delta e_1 + \epsilon e_2)$.

Subcase (iii.a.1) $y^*_{2n}(X + \delta e_1 + \epsilon e_2) < y^*_{2n}(X)$: Since $y^*_{2n}(X)$ is feasible, we have $y^*_{2n}(X + \delta e_1 + \epsilon e_2) < y^*_{2n}(X) \leq x_0$ and $y^*_{2n}(X + \delta e_1 + \epsilon e_2) < y^*_{2n}(X) \leq x_2$, implying that $y^*_{2n}(X + \delta e_1 + \epsilon e_2)$ is a feasible sales quantity under component profile $X + \delta e_1$. This results in

$$\begin{aligned}&G_n\left(X + \delta e_1 + \epsilon e_2, d_{2n}\right) - G_n\left(X + \delta e_1, d_{2n}\right)\\ \leq\ &R_{n+1}\left(X + \delta e_1 + \epsilon e_2 - y^*_{2n}(X + \delta e_1 + \epsilon e_2)a_2\right) - R_{n+1}\left(X + \delta e_1 - y^*_{2n}(X + \delta e_1 + \epsilon e_2)a_2\right).\end{aligned}$$

Meanwhile, as implied by the definition of $y^*_{2n}(X)$, $y^*_{2n}(X) \leq x_0$ and $y^*_{2n}(X) \leq x_2 \leq x_2 + \epsilon$, that is, $y^*_{2n}(X)$ is a feasible sales quantity under component profile $X + \epsilon e_2$. Hence,

$$G_n(X + \epsilon e_2, d_{2n}) - G_n(X, d_{2n}) \geq R_{n+1}\left(X + \epsilon e_2 - y^*_{2n}(X)a_2\right) - R_{n+1}\left(X - y^*_{2n}(X)a_2\right).$$

Then, (EC.27) holds because

$$\begin{aligned}&R_{n+1}\left(X + \delta e_1 + \epsilon e_2 - y^*_{2n}(X + \delta e_1 + \epsilon e_2)a_2\right) - R_{n+1}\left(X + \delta e_1 - y^*_{2n}(X + \delta e_1 + \epsilon e_2)a_2\right)\\ \leq\ &R_{n+1}\left(X + \delta e_1 + \epsilon e_2 - y^*_{2n}(X)a_2\right) - R_{n+1}\left(X + \delta e_1 - y^*_{2n}(X)a_2\right)\\ \leq\ &R_{n+1}\left(X + \epsilon e_2 - y^*_{2n}(X)a_2\right) - R_{n+1}\left(X - y^*_{2n}(X)a_2\right),\end{aligned}$$

where the second inequality is because of the induction hypothesis, and the first inequality is because $R_{n+1}(X+\epsilon e_2+ya_2)-R_{n+1}(X+ya_2)$ decreases in y, which is implied by the proof of Proposition EC.1 (ii):

$$\begin{aligned}
&\frac{\partial^2 R_{n+1}(X+\epsilon e_2+ya_2)}{\partial\epsilon\partial y}\\
=&\frac{\partial^2 R_{n+1}(x_0+y,x_1,x_2+y+\epsilon)}{\partial\epsilon\partial y}\\
=&\frac{\partial^2 \tilde{R}_{n+1}(x_1,x_0+y-x_1-(x_2+y+\epsilon),x_2+y+\epsilon)}{\partial\epsilon\partial y}\\
=&\frac{\partial^2 \tilde{R}_{n+1}(x_1,x_0-x_1-x_2-\epsilon,x_2+y+\epsilon)}{\partial\epsilon\partial y}\\
=&\left.\frac{\partial^2 \tilde{R}_{n+1}(x_1,\beta,z)}{\partial z^2}\right|_{\beta=x_0-x_1-x_2-\epsilon,z=x_2+y+\epsilon}-\left.\frac{\partial^2 \tilde{R}_{n+1}(x_1,\beta,z)}{\partial\beta\partial z}\right|_{\beta=x_0-x_1-x_2-\epsilon,z=x_2+y+\epsilon}\leq 0,
\end{aligned}$$

where the inequality follows because $\tilde{R}_{n+1}(x_1,\beta,z)$ is anti-multimodular, and thus $\frac{\partial^2 \tilde{R}_{n+1}(x_1,\beta,z)}{\partial z^2}-\frac{\partial^2 \tilde{R}_{n+1}(x_1,\beta,z)}{\partial\beta\partial z}\leq 0$ by Lemma EC.5(ii).

Subcase (iii.a.2) $y_{2n}^*(X)\leq y_{2n}^*(X+\delta e_1+\epsilon e_2)$: Because the proof of Lemma EC.8 implies $\frac{\partial y_{2n}^*(X)}{\partial x_1}\leq 0\leq\frac{\partial y_{2n}^*(X)}{\partial x_2}\leq 1$, $y_{2n}^*(X+\delta e_1+\epsilon e_2)\leq y_{2n}^*(X+\epsilon e_2)\leq y_{2n}^*(X)+\epsilon$. Thus, $y_{2n}^*(X)\leq y_{2n}^*(X+\delta e_1+\epsilon e_2)$ implies $y_{2n}^*(X)\leq y_{2n}^*(X+\delta e_1+\epsilon e_2)\leq y_{2n}^*(X)+\epsilon$. Hence, $y_{2n}^*(X+\delta e_1+\epsilon e_2)$ is a feasible sales quantity under component profile $X+\epsilon e_2$ because $y_{2n}^*(X+\delta e_1+\epsilon e_2)\leq y_{2n}^*(X+\epsilon e_2)\leq x_0$ and $y_{2n}^*(X+\delta e_1+\epsilon e_2)\leq y_{2n}^*(X)+\epsilon\leq x_2+\epsilon$. This results in

$$\begin{aligned}
&G_n\left(X+\delta e_1+\epsilon e_2,d_{2n}\right)-G_n\left(X+\epsilon e_2,d_{2n}\right)\\
\leq\ &R_{n+1}(X+\delta e_1+\epsilon e_2-y_{2n}^*(X+\delta e_1+\epsilon e_2)a_2)-R_{n+1}(X+\epsilon e_2-y_{2n}^*(X+\delta e_1+\epsilon e_2)a_2).
\end{aligned}$$

Additionally, because $y_{2n}^*(X)\leq x_0$ and $y_{2n}^*(X)\leq x_2$, $y_{2n}^*(X)$ is a feasible sales quantity under component profile $X+\delta e_1$, implying

$$G_n\left(X+\delta e_1,d_{2n}\right)-G_n\left(X,d_{2n}\right)\geq R_{n+1}(X+\delta e_1-y_{2n}^*(X)a_2)-R_{n+1}(X-y_{2n}^*(X)a_2).$$

Then, (EC.27) follows from

$$\begin{aligned}
&R_{n+1}(X+\delta e_1+\epsilon e_2-y_{2n}^*(X+\delta e_1+\epsilon e_2)a_2)-R_{n+1}(X+\epsilon e_2-y_{2n}^*(X+\delta e_1+\epsilon e_2)a_2)\\
=\ &R_{n+1}\left(x_0-y_{2n}^*(X+\delta e_1+\epsilon e_2),x_1+\delta,x_2+\epsilon-y_{2n}^*(X+\delta e_1+\epsilon e_2)\right)\\
&-R_{n+1}\left(x_0-y_{2n}^*(X+\delta e_1+\epsilon e_2),x_1,x_2+\epsilon-y_{2n}^*(X+\delta e_1+\epsilon e_2)\right)\\
\leq\ &R_{n+1}\left(x_0-y_{2n}^*(X),x_1+\delta,x_2+\epsilon-y_{2n}^*(X+\delta e_1+\epsilon e_2)\right)\\
&-R_{n+1}\left(x_0-y_{2n}^*(X),x_1,x_2+\epsilon-y_{2n}^*(X+\delta e_1+\epsilon e_2)\right)\\
\leq\ &R_{n+1}\left(x_0-y_{2n}^*(X),x_1+\delta,x_2-y_{2n}^*(X)\right)-R_{n+1}\left(x_0-y_{2n}^*(X),x_1,x_2-y_{2n}^*(X)\right)\\
=\ &R_{n+1}(X+\delta e_1-y_{2n}^*(X)a_2)-R_{n+1}(X-y_{2n}^*(X)a_2),
\end{aligned}$$

where the first inequality is because of the supermoduarity of R_{n+1} in (x_0,x_1) together with $x_0 - y^*_{2n}(X+\delta e_1+\epsilon e_2) \le x_0 - y^*_{2n}(X)$, and the second inequality is due to the induction hypothesis of the submodularity between component 1 and component 2 together with

$$x_2+\epsilon - y^*_{2n}(X+\delta e_1+\epsilon e_2) \ge x_2 - y^*_{2n}(X).$$

Case (iii.b) $y^*_{1n}(X) < y^*_{1n}(X+\delta e_1) \le y^*_{1n}(X)+\delta$: In this case, $y^*_{1n}(X)=x_1$. Recall that it suffices to prove

$$\begin{aligned}&G_n(X+\delta e_1+\epsilon e_2 - y^*_{1n}(X+\delta e_1)a_1, d_{2n}) - G_n(X+\delta e_1 - y^*_{1n}(X+\delta e_1)a_1, d_{2n})\\ \le\ &G_n(X+\epsilon e_2 - y^*_{1n}(X)a_1, d_{2n}) - G_n(X - y^*_{1n}(X)a_1, d_{2n}).\end{aligned}$$

For exposition, let $\eta_1^\delta = y^*_{1n}(X+\delta e_1)$ and $\eta_1 = y^*_{1n}(X)$. Then, it is equivalent to prove

$$\begin{aligned}&G_n(x_0-\eta_1^\delta, x_1+\delta-\eta_1^\delta, x_2+\epsilon, d_{2n}) - G_n(x_0-\eta_1^\delta, x_1+\delta-\eta_1^\delta, x_2, d_{2n})\\ \le\ &G_n(x_0-\eta_1, x_1-\eta_1, x_2+\epsilon, d_{2n}) - G_n(x_0-\eta_1, x_1-\eta_1, x_2, d_{2n}).\end{aligned}$$

We first prove that

$$\begin{aligned}&G_n(x_0-\eta_1^\delta, x_1-\eta_1, x_2+\epsilon, d_{2n}) - G_n(x_0-\eta_1^\delta, x_1-\eta_1, x_2, d_{2n})\\ \le\ &G_n(x_0-\eta_1, x_1-\eta_1, x_2+\epsilon, d_{2n}) - G_n(x_0-\eta_1, x_1-\eta_1, x_2, d_{2n}),\end{aligned}$$

which follows from $x_0-\eta_1 > x_0-\eta_1^\delta$ (as implied by the condition of this case (iii.b), i.e., $\eta_1 = y^*_{1n}(X) < \eta_1^\delta = y^*_{1n}(X+\delta e_1)$) and the supermodularity between the common component and component 2: As implied by the proof of Proposition EC.1, $G_n(X,d_{2n}) = \tilde{G}_n(x_1, x_0-x_1-x_2, x_2, d_{2n})$ is supermodular in (x_0,x_2):

$$\begin{aligned}&\frac{\partial^2 G_n(X,d_{2n})}{\partial x_0\partial x_2} = \frac{\partial^2 \tilde{G}_n(x_1, x_0-x_1-x_2, x_2, d_{2n})}{\partial x_0 \partial x_2}\\ =\ &\left.\frac{\partial^2 \tilde{G}_n(x_1,\beta,z,d_{2n})}{\partial\beta\partial z}\right|_{\beta=x_0-x_1-x_2, z=x_2} - \left.\frac{\partial^2 \tilde{G}_n(x_1,\beta,z,d_{2n})}{\partial\beta^2}\right|_{\beta=x_0-x_1-x_2, z=x_2} \ge 0,\end{aligned}$$

where the inequality follows because $\tilde{G}_n(x_1,\beta,z,d_{2n})$ is anti-multimodular, and thus $\frac{\partial^2 \tilde{G}_n(x_1,\beta,z,d_{2n})}{\partial\beta\partial z} - \frac{\partial^2 \tilde{G}_n(x_1,\beta,z,d_{2n})}{\partial\beta^2} \ge 0$ by Lemma EC.5(ii).

Now, we prove that

$$\begin{aligned}&G_n(x_0-\eta_1^\delta, x_1+\delta-\eta_1^\delta, x_2+\epsilon, d_{2n}) - G_n(x_0-\eta_1^\delta, x_1+\delta-\eta_1^\delta, x_2, d_{2n})\\ \le\ &G_n(x_0-\eta_1^\delta, x_1-\eta_1, x_2+\epsilon, d_{2n}) - G_n(x_0-\eta_1^\delta, x_1-\eta_1, x_2, d_{2n}).\end{aligned} \tag{EC.28}$$

Let $\eta_2^{\delta\epsilon}$ be the optimal sales quantity of product 2 under the inventory profile $(x_0-\eta_1^\delta, x_1+\delta-\eta_1^\delta, x_2+\epsilon)$, and η_2^δ be the optimal sales quantity of product 2 under the inventory profile $(x_0-\eta_1^\delta, x_1-\eta_1, x_2)$. Then, consider the following two subcases.

Subcase (iii.b.1) $\eta_2^\delta \le \eta_2^{\delta\epsilon}$: We have the following inequalities, which imply (EC.28). We prove each inequality as below.

$$
\begin{aligned}
&G_n(x_0-\eta_1^\delta, x_1+\delta-\eta_1^\delta, x_2+\epsilon, d_{2n}) - G_n(x_0-\eta_1^\delta, x_1-\eta_1, x_2+\epsilon, d_{2n})\\
\le\ & R_{n+1}(x_0-\eta_1^\delta-\eta_2^{\delta\epsilon}, x_1+\delta-\eta_1^\delta, x_2+\epsilon-\eta_2^{\delta\epsilon}) - R_{n+1}(x_0-\eta_1^\delta-\eta_2^{\delta\epsilon}, x_1-\eta_1, x_2+\epsilon-\eta_2^{\delta\epsilon})\\
\le\ & R_{n+1}(x_0-\eta_1^\delta-\eta_2^{\delta\epsilon}, x_1+\delta-\eta_1^\delta, x_2-\eta_2^\delta) - R_{n+1}(x_0-\eta_1^\delta-\eta_2^{\delta\epsilon}, x_1-\eta_1, x_2-\eta_2^\delta)\\
\le\ & R_{n+1}(x_0-\eta_1^\delta-\eta_2^\delta, x_1+\delta-\eta_1^\delta, x_2-\eta_2^\delta) - R_{n+1}(x_0-\eta_1^\delta-\eta_2^\delta, x_1-\eta_1, x_2-\eta_2^\delta)\\
\le\ & G_n(x_0-\eta_1^\delta, x_1+\delta-\eta_1^\delta, x_2, d_{2n}) - G_n(x_0-\eta_1^\delta, x_1-\eta_1, x_2, d_{2n}):
\end{aligned}
$$

- the first inequality follows because given d_{2n}, $\eta_2^{\delta\epsilon}$ is the optimal sales quantity of product 2 under the inventory profile $(x_0-\eta_1^\delta, x_1+\delta-\eta_1^\delta, x_2+\epsilon)$, and is also feasible under the inventory profile $(x_0-\eta_1^\delta, x_1-\eta_1, x_2+\epsilon)$, since both inventory profiles have the same number of components 0 and 2.
- the second inequality is due to the induction hypothesis that $R_{n+1}(x_0, x_1, x_2)$ is submodular in (x_1, x_2) together with the facts $x_1+\delta-\eta_1^\delta \ge x_1-\eta_1$ (or $\delta+\eta_1 \ge \eta_1^\delta$) and $x_2+\epsilon-\eta_2^{\delta\epsilon} \ge x_2-\eta_2^\delta$ (or $\epsilon+\eta_2^\delta \ge \eta_2^{\delta\epsilon}$) as proved next: $\delta+\eta_1 \ge \eta_1^\delta$ follows from the condition of this case (iii.b); To show $\epsilon+\eta_2^\delta \ge \eta_2^{\delta\epsilon}$, recall that given d_{2n}, $\eta_2^{\delta\epsilon}$ is the optimal sales quantity of product 2 under the inventory profile $(x_0-\eta_1^\delta, x_1+\delta-\eta_1^\delta, x_2+\epsilon)$, and η_2^δ is the optimal sales quantity of product 2 under the inventory profile $(x_0-\eta_1^\delta, x_1-\eta_1, x_2)$. Then, the optimal sales quantity of product 2 under the inventory profile $(x_0-\eta_1^\delta, x_1-\eta_1, x_2+\epsilon)$ satisfies $y_{2n}^*(x_0-\eta_1^\delta, x_1-\eta_1, x_2+\epsilon) \in [\eta_2^{\delta\epsilon}, \epsilon+\eta_2^\delta]$, as the proof of Lemma EC.8 implies that the optimal sales quantity of product 2 increases in the number of component 2 with an increasing speed less than 1, i.e., $\epsilon+\eta_2^\delta \ge y_{2n}^*(x_0-\eta_1^\delta, x_1-\eta_1, x_2+\epsilon) \ge \eta_2^\delta$, and the optimal sales quantity of product 2 decreases in the number of component 1, i.e., $y_{2n}^*(x_0-\eta_1^\delta, x_1-\eta_1, x_2+\epsilon) \ge \eta_2^{\delta\epsilon}$. Therefore, $\epsilon+\eta_2^\delta \ge y_{2n}^*(x_0-\eta_1^\delta, x_1-\eta_1, x_2+\epsilon) \ge \eta_2^{\delta\epsilon}$
- the third inequality holds due to $x_0-\eta_1^\delta-\eta_2^{\delta\epsilon} \le x_0-\eta_1^\delta-\eta_2^\delta$ (as implied by the condition of this subcase (iii.b.1), i.e., $\eta_2^{\delta\epsilon} \ge \eta_1$) and the supermodularity between the common component and component 2 in part (ii) of Proposition 4.
- the fourth inequality follows because given d_{2n}, η_2^δ is the optimal sales quantity of product 2 under the inventory profile $(x_0-\eta_1^\delta, x_1-\eta_1, x_2)$, and is also feasible under the inventory profile $(x_0-\eta_1^\delta, x_1+\delta-\eta_1^\delta, x_2)$, since both inventory profiles have the same number of components 0 and 2.

Subcase (iii.b.2) $\eta_2^\delta > \eta_2^{\delta\epsilon}$: We have the following inequalities, which imply (EC.28). We prove each inequality as below.

$$\begin{aligned}
&G_n(x_0-\eta_1^\delta, x_1+\delta-\eta_1^\delta, x_2+\epsilon, d_{2n}) - G_n(x_0-\eta_1^\delta, x_1+\delta-\eta_1^\delta, x_2, d_{2n})\\
\le\ & R_{n+1}(x_0-\eta_1^\delta-\eta_2^{\delta\epsilon}, x_1+\delta-\eta_1^\delta, x_2+\epsilon-\eta_2^{\delta\epsilon}) - R_{n+1}(x_0-\eta_1^\delta-\eta_2^{\delta\epsilon}, x_1+\delta-\eta_1^\delta, x_2-\eta_2^{\delta\epsilon})\\
\le\ & R_{n+1}(x_0-\eta_1^\delta-\eta_2^{\delta\epsilon}, x_1-\eta_1, x_2+\epsilon-\eta_2^{\delta\epsilon}) - R_{n+1}(x_0-\eta_1^\delta-\eta_2^{\delta\epsilon}, x_1-\eta_1, x_2-\eta_2^{\delta\epsilon})\\
\le\ & R_{n+1}(x_0-\eta_1^\delta-\eta_2^{\delta}, x_1-\eta_1, x_2+\epsilon-\eta_2^{\delta}) - R_{n+1}(x_0-\eta_1^\delta-\eta_2^{\delta}, x_1-\eta_1, x_2-\eta_2^{\delta})\\
\le\ & G_n(x_0-\eta_1^\delta, x_1-\eta_1, x_2+\epsilon, d_{2n}) - G_n(x_0-\eta_1^\delta, x_1-\eta_1, x_2, d_{2n}):
\end{aligned}$$

- the first inequality follows because given d_{2n}, η_2^δ is the optimal sales quantity of product 2 under the inventory profile $(x_0-\eta_1^\delta, x_1-\eta_1, x_2)$. Hence, $\eta_2^\delta \le x_2$ and $\eta_2^\delta \le x_0-\eta_1^\delta$. This together with the condition of this subcase (iii.b.2) implies $\eta_2^{\delta\epsilon} < \eta_2^\delta \le x_2$ and $\eta_2^{\delta\epsilon} < \eta_2^\delta \le x_0-\eta_1^\delta$. Thus, as the optimal sales quantity of product 2 under the inventory profile $(x_0-\eta_1^\delta, x_1+\delta-\eta_1^\delta, x_2+\epsilon)$, $\eta_2^{\delta\epsilon}$ is also feasible under the inventory profile $(x_0-\eta_1^\delta, x_1+\delta-\eta_1^\delta, x_2)$.
- the second inequality is due to $x_1+\delta-\eta_1^\delta \ge x_1-\eta_1$ (or $\delta+\eta_1 \ge \eta_1^\delta$ as implied by the condition of this case (iii.b)), $x_2+\epsilon-\eta_2^{\delta\epsilon} \ge x_2-\eta_2^{\delta\epsilon}$ and the induction hypothesis that $R_{n+1}(x_0,x_1,x_2)$ is submodular in (x_1,x_2).
- the third inequality follows from $x_0-\eta_1^\delta-\eta_2^{\delta\epsilon} > x_0-\eta_1^\delta-\eta_2^\delta$ (as implied by the condition of this subcase (iii.b.2), i.e., $\eta_2^{\delta\epsilon} < \eta_2^\delta$) together with the following inequality:

$$\begin{aligned}
&\frac{\partial^2 R_{n+1}(x_0-y, x_1, x_2-y)}{\partial y \partial x_2} = \frac{\partial^2 \tilde{R}_{n+1}(x_1, x_0-x_1-x_2, x_2-y)}{\partial y \partial x_2}\\
&= -\left.\frac{\partial^2 \tilde{R}_{n+1}(x_1,\beta,z)}{\partial z^2}\right|_{\beta=x_0-x_1-x_2, z=x_2-y} + \left.\frac{\partial^2 \tilde{R}_{n+1}(x_1,\beta,z)}{\partial \beta \partial z}\right|_{\beta=x_0-x_1-x_2, z=x_2-y} \ge 0
\end{aligned}$$

because $\tilde{R}_{n+1}(x_1,\beta,z)$ is anti-multimodular, and thus $-\frac{\partial^2 \tilde{R}_{n+1}(x_1,\beta,z)}{\partial z^2} + \frac{\partial^2 \tilde{R}_{n+1}(x_1,\beta,z)}{\partial \beta \partial z} \ge 0$ by Lemma EC.5(ii).
- the fourth inequality follows because given d_{2n}, η_2^δ is the optimal sales quantity of product 2 under the inventory profile $(x_0-\eta_1^\delta, x_1-\eta_1, x_2)$, and is also feasible under the inventory profile $(x_0-\eta_1^\delta, x_1-\eta_1, x_2+\epsilon)$.

Q.E.D.

D.5. Proof of Proposition 5

We first show the upper bound for the inventory of the dedicated components. For $i=1,\ldots,k$, the supermodularity of the expected optimal profit-to-go in (x_0,x_i) implies that, for any finite $x_0 \le x_1+\ldots+x_k$ and $\tilde{x}_0 \ge x_1+\ldots+x_k$,

$$\frac{\partial R_1(x_0,x_1,\ldots,x_k)}{\partial x_i} \le \frac{\partial R_1(\tilde{x}_0,x_1,\ldots,x_k)}{\partial x_i},$$

where the right-hand side is the marginal expected profit when the supply of common components is ample. It suffices to derive an upper bound for the optimal dedicated component inventory in the right-hand side system, which will serve as the upper bound for the optimal initial inventory in the left-hand side ATO system.

For the system with an ample supply of common components, i.e., $x_0 \geq \sum_{i=1}^{k} x_i$, it is optimal to fulfill orders as much as allowed by the available inventory of dedicated components. In other words, no competition for common components exists among different products. Therefore, we can decompose such a system into k separated sub-systems, indexed by $i = 1, \ldots, k$. Each sub-system i starts with x_i units of component i and an ample supply of component 0. Denote by x_0^i the number of common components in the sub-system i. That is, $x_0^i \geq x_i$ and $\sum_{i=1}^{k} x_0^i = x_0$. In sub-system i, the optimal fulfillment policy is the same as the make-to-stock policy with x_i units of sales quota. As in the proof of Lemma EC.9, by accounting for the holding costs and salvage values at the beginning of the season, the total expected profit in this single-product sub-system i is

$$\begin{aligned}
&-c_i x_i - c_0 x_0^i - N h_i x_i - N h_0 x_0^i + c_i x_i + c_0 x_0^i + E_{D_{i1}}[\bar{\bar{J}}_1^{mts,i}(x_i, D_{i1})] \\
&= -N h_i x_i - N h_0 x_0^i + E_{D_{i1}}[\bar{\bar{J}}_1^{mts,i}(x_i, D_{i1})],
\end{aligned}$$

where $-c_i x_i - c_0 x_0^i$ is the component procurement cost incurred at the beginning of the season, $-N h_i x_i - N h_0 x_0^i$ is the total component holding costs deducted at the beginning of the season, $c_i x_i + c_0 x_0^i$ is the component salvage values accounted for at the beginning of the season, and the function $\bar{\bar{J}}_1^{mts,i}(\cdot,\cdot)$ is as defined in the proof of Lemma EC.9.

As proved in Lemma EC.9 (i), $E_{D_{i1}}[\bar{\bar{J}}_1^{mts,i}(x_i, D_{i1})]$ is concave in x_i and

$$\frac{dE_{D_{i1}}[\bar{\bar{J}}_1^{mts,i}(x_i, D_{i1})]}{dx_i} \leq (p_i + N h_i + N h_0 - c_i - c_0)\bar{F}_i(x_i).$$

Hence, the optimal initial inventory level is bounded from above by $\bar{x}_i$, which satisfies

$$(p_i + N h_i + N h_0 - c_i - c_0)\bar{F}_i(\bar{x}_i) - N h_i = 0,$$

or equivalently, $1 - F_i(\bar{x}_i) = N h_i / (p_i + N h_i + N h_0 - c_i - c_0)$.

Next, we follow three steps to show that the optimal initial inventory for the common component in a generalized W system is bounded from above by $\bar{x}_0 = \bar{F}_0^{-1}(\frac{N h_0}{p_1 + N h_1 + N h_0 - c_1 - c_0})$.

Step 1: The supermodularity in Proposition 4 (ii) implies an upper bound, x_0^{ud}, for the optimal initial inventory of the common component, as defined below:

$$x_0^{ud} := \arg\max_{x_0} \left[R_1(x_0, I_1, \ldots, I_k) - c_0 x_0 - \sum_{i=1}^{k} c_i I_i \right],$$

where $R_n(X)$ is as defined in (1), I_i is inventory of the dedicated component i, and $I_i \geq x_0$ for all $i = 1, ..., k$. Specifically, $R_{N+1}(X) = \sum_{j=0}^{k} c_j x_j$, and for $1 \leq n \leq N$,

$$R_n(X) = E_{D_{1n},...,D_{kn}}[J_n(X, D_{1n}, ..., D_{kn})],$$

$$\text{where } J_n(X, d_{1n}, ..., d_{kn})$$

$$= \max_{y_{1n},...,y_{kn}} \left\{ \sum_{i=1}^{k} [p_i y_{in} - h_i(x_i - y_{in})] - h_0(x_0 - \sum_{i=1}^{k} y_{in}) + R_{n+1}(X - \sum_{i=1}^{k} y_{in} a_i) \right\}$$

$$\text{subject to} \quad \sum_{i=1}^{k} y_{in} \leq x_0, \quad 0 \leq y_{in} \leq x_i \wedge d_{in}, \quad i = 1, 2..., k.$$

Since $I_i \geq x_0$ for all i, order fulfillment is never constrained by the dedicated-component inventory. Thus, the dedicated-component inventory can be excluded from the set of state variables in the dynamic programming. Also, we regard the margin of product $i = 1, ..., k$ in period n as

$$p_i + (N - n + 1)h_i + (N - n + 1)h_0 - c_i - c_0$$

by deducting the holding cost, $Nh_0x_0 + N\sum_{j=1}^{k}(h_jI_j)$, and accounting for the component salvage values and procurement costs (both equal $c_0x_0 + \sum_{j=1}^{k} c_jI_j$) at the beginning of the selling season. Thus, we have:

$$x_0^{ud} = \arg\max_{x_0} \left[E_{D_{11},...,D_{k1}}[J_1^{ud}(x_0, D_{11}, ..., D_{k1})] - Nh_0x_0 - N\sum_{i=1}^{k} h_iI_i \right],$$

where $J_{N+1}^{ud}(\cdot) = 0$, and for $1 \leq n \leq N$,

$$J_n^{ud}(x_0, d_{1n}, ..., d_{kn}) = \max_{y_{1n},...,y_{kn}} \left\{ \sum_{i=1}^{k} [(p_i + (N - n + 1)h_i + (N - n + 1)h_0 - c_i - c_0)y_{in}] \right. \tag{EC.29}$$

$$\left. + E_{D_{1,n+1},...,D_{k,n+1}}[J_{n+1}^{ud}(x_0 - \sum_{i=1}^{k} y_{in}, D_{1,n+1}, ..., D_{k,n+1})] \right\}$$

$$\text{subject to} \quad \sum_{i=1}^{k} y_{in} \leq x_0, \quad 0 \leq y_{in} \leq d_{in}, \quad i = 1, 2..., k.$$

Denote the optimal fulfillment quantities by y_{in}^{ud}, $i = 1, ..., k, n = 1, ..., N$. Here, the superscript ud represents the special case of unconstrained dedicated components. We shall refer to this scenario as system UD.

Step 2: We further simplify problem (EC.29) as follows: in addition to unconstrained dedicated components, all the products have the same effective margin in all the periods, i.e., the effective margin for product $i = 1, ..., k$ in period n is $p_1 + Nh_1 + Nh_0 - c_1 - c_0$ for all i and n. That is,

$$x_0^{ud1} := \arg\max_{x_0} \left[E_{D_{11},...,D_{k1}}[J_1^{ud1}(x_0, D_{11}, ..., D_{k1})] - Nh_0x_0 - Nh_1\sum_{i=1}^{k} I_i \right],$$

where $J_{N+1}^{ud1}(\cdot)=0$, and for $1\le n\le N$,

$$J_n^{ud1}(x_0,d_{1n},...,d_{kn})=\max_{y_{1n},\dots,y_{kn}}\left\{\sum_{i=1}^{k}[(p_1+Nh_1+Nh_0-c_1-c_0)y_{in}]\right.$$
$$\left.+E_{D_{1,n+1},\dots,D_{k,n+1}}[J_{n+1}^{ud1}(x_0-\sum_{i=1}^{k}y_{in},D_{1,n+1},...,D_{k,n+1})]\right\}$$
$$\text{subject to}\quad \sum_{i=1}^{k}y_{in}\le x_0,\quad 0\le y_{in}\le d_{in},\quad i=1,2...,k.$$

Denote the optimal fulfillment quantities by y_{in}^{ud1}, $i=1,...,k,n=1,...,N$. Here, the superscript $ud1$ represents the special case of unconstrained dedicated components and all the products share the same period-independent effective margin. We shall refer to this scenario as system UD1.

In the following, we shall prove that given demand realization $(d_{11},...,d_{k1})$,

$$\frac{dJ_1^{ud}(x_0,d_{11},...,d_{k1})}{dx_0}\le\frac{dJ_1^{ud1}(x_0,d_{11},...,d_{k1})}{dx_0},$$

i.e., the marginal profit of ordering the common component in system UD is less than or equal to that in system UD1. This, together with the concavity of $J_1^{ud}(\cdot)$ and $J_1^{ud1}(\cdot)$ as implied by Lemma EC.3, leads to $x_0^{ud}\le x_0^{ud1}$.

Step 3: In system UD1, dedicated component inventory is unconstrained and all the products have the same effective margin $p_1+Nh_1+Nh_0-c_1-c_0$. Hence, it is equivalent to a single-product newsvendor problem with overage cost Nh_0 and underage cost $p_1+Nh_1-c_1-c_0$. Thus, the optimal initial inventory of common components in system UD1 is $\bar{x}_0=\bar{F}_0^{-1}(\frac{Nh_0}{p_1+Nh_1+Nh_0-c_1-c_0})$.

It remains to show
$$\frac{dJ_1^{ud}(x_0,d_{11},...,d_{k1})}{dx_0}\le\frac{dJ_1^{ud1}(x_0,d_{11},...,d_{k1})}{dx_0}$$
in Step 2, and we prove by induction that
$$\frac{dJ_n^{ud}(x_0,d_{1n},...,d_{kn})}{dx_0}\le\frac{dJ_n^{ud1}(x_0,d_{1n},...,d_{kn})}{dx_0}$$
holds for all $n\in\{1,2,...,N\}$. To this end, first note that, by

$$J_{N+1}^{ud}(x_0,d_{1,N+1},\dots,d_{k,N+1})=J_{N+1}^{ud1}(x_0,d_{1,N+1},\dots,d_{k,N+1})\equiv 0,$$
$$\frac{dJ_{N+1}^{ud}(x_0,d_{1,N+1},\dots,d_{k,N+1})}{dx_0}=\frac{dJ_{N+1}^{ud1}(x_0,d_{1,N+1},\dots,d_{k,N+1})}{dx_0}\equiv 0.$$

Now, for $n\in\{1,2,...,N\}$, suppose that the result holds for $n+1$. We shall prove that it also holds for n. In preparation, note that, as implied by Proposition 3, with unconstrained dedicated component inventory the nested allocation policy is optimal in the two systems, UD and UD1. However, due to the differences in product margins in the two systems, in system UD some demand for low-margin products may be rejected to reserve common components for future demand of high-margin products, while in system UD1, it is optimal to fulfill orders as much as requested by current demand and allowed by available inventory.

With x_0 units of common component 0 at the beginning of period n (together with $I_i \geq x_0$ units of dedicated component i), consider the following two cases:

- Case 1: $x_0 < \sum_{i=1}^k d_{in}$. In system UD1 all x_0 units of common components are exhausted while $\sum_{i=1}^k d_{in} - x_0$ units of demand realized in period n are unfilled. The marginal profit of increasing the supply of common components in system UD1 is

$$\frac{dJ_n^{ud1}(x_0, d_{1n}, ..., d_{kn})}{dx_0} = p_1 + Nh_1 + Nh_0 - c_1 - c_0$$

since additional common components will allow fulfillment of demand in the current period with an effective margin of $p_1 + Nh_1 + Nh_0 - c_1 - c_0$. In contrast, in system UD, some of x_0 units of common components may be reserved for the high-margin products in the future by rejecting the realized demand for low-margin products in period n. Because the maximum effective selling margin in system UD is $p_1 + Nh_1 + Nh_0 - c_1 - c_0$, additional common components will be carried to the future or allow fulfillment of demand in the current period, either of which will generate a maximum margin of $p_1 + Nh_1 + Nh_0 - c_1 - c_0$, so

$$\frac{dJ_n^{ud}(x_0, d_{1n}, ..., d_{kn})}{dx_0} \leq p_1 + Nh_1 + Nh_0 - c_1 - c_0.$$

Hence,

$$\frac{dJ_n^{ud}(x_0, d_{1n}, ..., d_{kn})}{dx_0} \leq \frac{dJ_n^{ud1}(x_0, d_{1n}, ..., d_{kn})}{dx_0}.$$

- Case 2: $x_0 \geq \sum_{i=1}^k d_{in}$. In system UD1, $\sum_{i=1}^k d_{in}$ units of common components are used. All the realized demands in period n are fulfilled, and there is a leftover of $x_0 - \sum_{i=1}^k d_{in}$ units of common components. Thus, the marginal profit of increasing the supply of common components in system UD1 is

$$\frac{dJ_n^{ud1}(x_0, d_{1n}, ..., d_{kn})}{dx_0} = \frac{dE_{D_{1,n+1},...,D_{k,n+1}}[J_{n+1}^{ud1}(x_0 - \sum_{i=1}^k d_{in}, D_{1,n+1}, ..., D_{k,n+1})]}{dx_0}.$$

In system UD the total sales quantity in period n is $\sum_{i=1}^k y_{in}^{ud} \leq \sum_{i=1}^k d_{in}$. By the optimality of the nested policy, there exists a threshold $\tau \in \{2, .., k\}$ in period n such that it is optimal to fill up to $y_{\tau n}^{ud} \in [0, d_{\tau n}]$ units of product τ: fill d_{in} units of product $i \leq \tau - 1$, and $y_{\tau n}^{ud} \leq d_{\tau n}$ units of product τ (which implies that it is optimal to reject $d_{\tau n} - y_{\tau n}^{ud}$ units of the demand for product τ as well as all the demand for product $i \geq \tau+1$). As a result, there is a leftover of $x_0 - \sum_{i=1}^{\tau-1} d_{in} - y_{\tau n}^{ud}$ units of common components. Then,

$$\begin{aligned}
J_n^{ud}(x_0, d_{1n}, ..., d_{kn}) = & \sum_{i=1}^{\tau-1}[(p_i + (N-n+1)h_i + (N-n+1)h_0 - c_i - c_0)d_{in}] \\
& + (p_\tau + (N-n+1)h_\tau + (N-n+1)h_0 - c_\tau - c_0)y_{\tau n}^{ud} \\
& + E_{D_{1,n+1},...,D_{k,n+1}}[J_{n+1}^{ud}(x_0 - \sum_{i=1}^{\tau-1} d_{in} - y_{\tau n}^{ud}, D_{1,n+1}, ..., D_{k,n+1})].
\end{aligned}$$

Consider the following two subcases.

—<u>Subcase 2.1: $y_{\tau n}^{ud} < d_{\tau n}$</u> This implies that $y_{\tau n}^{ud}$ is the interior solution satisfying the first order condition:

$$\begin{aligned}
0 = & p_\tau + (N-n+1)h_\tau + (N-n+1)h_0 - c_\tau - c_0 \\
& + \left[dE_{D_{1,n+1},...,D_{k,n+1}}[J_{n+1}^{ud}(x_0 - \sum_{i=1}^{\tau-1} d_{in} - y, D_{1,n+1}, ..., D_{k,n+1})]/dy \right]\Bigg|_{y=y_{\tau n}^{ud}}
\end{aligned}$$

because otherwise $y_{\tau n}^{ud}$ cannot be the optimal solution for system UD given the concavity of $E_{D_{1,n+1},...,D_{k,n+1}}[J_{n+1}^{ud}(\cdot, D_{1,n+1}, ..., D_{k,n+1})]$ in Lemma EC.3. Thus, by the envelope theorem the marginal profit of increasing the supply of common components in system UD is

$$\frac{dJ_n^{ud}(x_0, d_{1n}, ..., d_{kn})}{dx_0} = \left[dE_{D_{1,n+1},...,D_{k,n+1}}[J_{n+1}^{ud}(z, D_{1,n+1}, ..., D_{k,n+1})]/dz \right] |_{z=x_0-\sum_{i=1}^{\tau-1} d_{in} - y_{\tau n}^{ud}},$$

which is less than

$$\left[dE_{D_{1,n+1},...,D_{k,n+1}}(J_{n+1}^{ud}(z, D_{1,n+1}, ..., D_{k,n+1}))/dz \right] |_{z=x_0-\sum_{i=1}^{k} d_{in}},$$

because of $y_{\tau n}^{ud} < d_{\tau n} \leq \sum_{i=\tau}^{k} d_{in}$ and the concavity of $E_{D_{1,n+1},...,D_{k,n+1}}[J_{n+1}^{ud}(\cdot, D_{1,n+1}, ..., D_{k,n+1})$ in Lemma EC.3.

—<u>Subcase 2.2: $y_{\tau n}^{ud} = d_{\tau n}$</u> This implies either $\tau = k$ or

$$\begin{aligned}
& p_{\tau+1} + (N-n+1)h_{\tau+1} + (N-n+1)h_0 - c_{\tau+1} - c_0 \\
& \quad + \left[dE_{D_{1,n+1},...,D_{k,n+1}}[J_{n+1}^{ud}(x_0 - \sum_{i=1}^{\tau} d_{in} - y, D_{1,n+1}, ..., D_{k,n+1})]/dy \right] |_{y=0} \\
& < 0
\end{aligned}$$

because otherwise the leftover common component should have been used to fill the demand for product $\tau+1$. Thus, for any sufficiently and arbitrarily small δ, the addition of δ units of common components will be carried to the next period instead of filling the demand for product $\tau+1$. Thus, the marginal profit of increasing the supply of common components in system UD is

$$\frac{dJ_n^{ud}(x_0, d_{1n}, ..., d_{kn})}{dx_0} = \left[dE_{D_{1,n+1},...,D_{k,n+1}}[J_{n+1}^{ud}(z, D_{1,n+1}, ..., D_{k,n+1})]/dz \right] |_{z=x_0-\sum_{i=1}^{\tau} d_{in}},$$

which is less than

$$\left[dE_{D_{1,n+1},\ldots,D_{k,n+1}}[J^{ud}_{n+1}(z,D_{1,n+1},...,D_{k,n+1})]/dz\right]|_{z=x_0-\sum_{i=1}^k d_{in}}$$

because of the concavity of $E_{D_{1,n+1},\ldots,D_{k,n+1}}[J^{ud}_{n+1}(\cdot,D_{1,n+1},...,D_{k,n+1})]$ in Lemma EC.3 and $\sum_{i=1}^{\tau} d_{in}\le\sum_{i=1}^{k} d_{in}$.

Thus, combing subcases 2.1 and 2.2, with the induction hypothesis and the fact

$$\frac{dJ_n^{ud1}(x_0,d_{1n},...,d_{kn})}{dx_0}=\frac{dE_{D_{1,n+1},\ldots,D_{k,n+1}}[J^{ud1}_{n+1}(x_0-\sum_{i=1}^k d_{in},D_{1,n+1},...,D_{k,n+1})]}{dx_0}$$

as we noted earlier, the marginal profit of increasing the supply of common components in system UD is lower than in system UD1. Q.E.D.

D.6. Proof of Proposition 7

In preparation, we define five order-fulfillment policies and four initial inventory profiles, which will be used in the proof of the proposition.

Order Fulfillment for a Given Initial Inventory X

- Policy f: Selling at most $y_{in}^{(m)}(X)$ units of product i in period n, where $y_{in}^{(m)}(X)$ is the optimal solution to the following deterministic problem where demand for product i in period n is $E(D_{in}^{(m)})$ and the initial component inventory is X:

$$\begin{aligned}
&R^{Det(m)}(X) \qquad\qquad (\text{EC.30})\\
:=&\max_{y_{11},\ldots,y_{kN}}\left\{\sum_{i=1}^k\sum_{n=1}^N\left[p_iy_{in}-h_i(x_i-\sum_{j=1}^n y_{ij})\right]-h_0\sum_{n=1}^N\left[x_0-\sum_{i=1}^k\sum_{j=1}^n y_{ij}\right]+\sum_{i=1}^k c_i(x_i-\sum_{j=1}^N y_{ij})+c_0(x_0-\sum_{i=1}^k\sum_{j=1}^N y_{ij})\right\}\\
=&\left\{\max_{y_{11},\ldots,y_{kN}}\left[-N\sum_{j=0}^k(h_jx_j)+\sum_{j=0}^k(c_jx_j)+\sum_{i=1}^k\sum_{n=1}^N((p_i+(N-n+1)h_i+(N-n+1)h_0-c_i-c_0)y_{in})\right]\right\}\\
=&\left\{-N\sum_{j=0}^k(h_jx_j)+\sum_{j=0}^k(c_jx_j)+\max_{y_{11},\ldots,y_{kN}}\left[\sum_{n=1}^N\sum_{i=1}^k((p_i+(N-n+1)h_i+(N-n+1)h_0-c_i-c_0)y_{in})\right]\right\}\\
&\text{s.t.}\quad 0\le y_{in}\le E(D_{in}^{(m)}),\quad \sum_{j=1}^n y_{ij}\le x_i,\quad n=1,\ldots,N,\quad i=1,\ldots,k,\\
&\qquad \sum_{i=1}^k\sum_{j=1}^n y_{ij}\le x_0,\quad n=1,\ldots,N.
\end{aligned}$$

By definition of $y_{in}^{(m)}(X)$,

$$\begin{aligned}
&R^{Det(m)}(X)\\
=&-N\sum_{j=0}^k(h_jx_j)+\sum_{j=0}^k(c_jx_j)+\sum_{i=1}^k\sum_{n=1}^N\left[(p_i+(N-n+1)h_i+(N-n+1)h_0-c_i-c_0)(E(D_{in}^{(m)})\wedge y_{in}^{(m)}(X))\right].
\end{aligned}$$

By definition of the policy f, the expected profit-to-go of adopting the policy f for the stochastic scale-m problem is

$$\begin{aligned}
&R^{f(m)}(X)\\
:=& -N\sum_{j=0}^{k}(h_jx_j)+\sum_{j=0}^{k}(c_jx_j)\\
&+\sum_{i=1}^{k}\sum_{n=1}^{N}\Big[(p_i+(N-n+1)h_i+(N-n+1)h_0-c_i-c_0)(E(D_{in}^{(m)}\wedge y_{in}^{(m)}(X)))\Big]\\
=& \sum_{j=0}^{k}[(c_j-Nh_j)x_j]\\
&+\sum_{i=1}^{k}\sum_{n=1}^{N}\Big[(p_i+(N-n+1)h_i+(N-n+1)h_0-c_i-c_0)(E(D_{in}^{(m)}\wedge y_{in}^{(m)}(X)))\Big]. \qquad \text{(EC.31)}
\end{aligned}$$

- Naive make-to-stock policy $MTS0$: Selling at most $\sum_{n=1}^{N} y_{in}^{(m)}(X)$ units of product i in the entire selling season. That is, for each product there is no restriction on the maximum amount that can be sold in each period, except the restriction on the total amount in the entire season. This policy is in line with the make-to-stock heuristic in Gallego and van Ryzin (1997), where the firm sets a static sales quota for each product, that is, a maximum of u_i units of product i can be sold in the entire selling season. For given quotas $u_1,\dots,u_k$ that satisfy $u_i\ge 0, i=1,...,k$ and $\sum_{i=1}^{k}u_i\le x_0$, denote the profit-to-go function by

$$V_n^{mts(m)}(X,u_1,\dots,u_k):=E_{D_{1n}^{(m)},\dots,D_{kn}^{(m)}}[J_n^{mts}(X,u_1,\dots,u_k,D_{1n}^{(m)},\dots,D_{kn}^{(m)})],$$

and

$$\begin{aligned}
J_n^{mts}(X,u_1,\dots,u_k,d_{1n},\dots,d_{kn})=&\sum_{i=1}^{k}[p_iy_{in}-h_i(x_i-y_{in})]-h_0(x_0-\sum_{i=1}^{k}y_{in})\\
&+V_{n+1}^{mts(m)}(X-\sum_{i=1}^{k}y_{in}a_i,u_1-y_{1n},...,u_k-y_{kn})
\end{aligned}$$

where $y_{in}=x_i\wedge d_{in}\wedge u_i$, $i=1,\dots,k$, and $V_{N+1}^{mts}(X,u_1,\dots,u_k)=\sum_{j=0}^{k}c_jx_j$.

Hence, the expected profit-to-go of adopting the policy $MTS0$ is

$$R^{mts0(m)}(X):=V_1^{mts(m)}(X,\sum_{n=1}^{N}y_{1n}^{(m)}(X),\dots,\sum_{n=1}^{N}y_{kn}^{(m)}(X)).$$

- Make-to-stock policy MTS: Same as $MTS0$, except that the static sales quota is optimized, instead of being given by the deterministic solution. Specifically, with $(u_1^*(X),\dots,u_k^*(X))$ as the optimal sales quotas defined in (8), the expected profit-to-go under the policy MTS is

$$R^{mts(m)}(X):=V_1^{mts(m)}(X,u_1^*(X),\dots,u_k^*(X)).$$

- Static nested policy q: This builds on and differs from the MTS policy by allowing a higher-priority product to use the sales quotas of a lower-priority product. For ease of exposition, we shall refer to $u_i^*(X)$ as product i's *individual quota*. As the individual quota of a lower margin product, e.g., $i+1,\ldots,k$, can be used to fulfill the demand of a higher margin product, e.g., i, the total quotas available for fulfilling demand of product i, referred to as *the available quota* for product i, equals to $\sum_{j=i}^k u_i^*(X)$. The formulation for the profit-to-go and the updating of remaining quotas under this static nested policy are defined in §4.2.1.
- Optimal nested policy o: This policy has been defined in §3. For notational convenience, in the scale-m problem we define the expected profit-to-go under policy o by

$$R^{o(m)}(X) := \bar{R}_1^{(m)}(X)$$

where, as defined in (5), $\bar{R}_{N+1}^{(m)}(X) = \sum_{j=0}^k c_j x_j$, and for $n \le N$,

$$\bar{R}_n^{(m)}(X) = E_{D_{1n}^{(m)},\ldots,D_{kn}^{(m)}}[\bar{J}_n(X, D_{1n}^{(m)}, \ldots, D_{kn}^{(m)})],$$

with

$$\bar{J}_n(X, d_{1n}^{(m)}, \ldots, d_{kn}^{(m)}) = \max_{u_1,\ldots,u_k} \left\{ \sum_{i=1}^k [p_i y_{in} - h_i(x_i - y_{in})] - h_0(x_0 - \sum_{i=1}^k y_{in}) + \bar{R}_{n+1}^{(m)}(X - \sum_{i=1}^k y_{in} a_i) \right\}$$

$$\text{subject to} \quad \sum_{i=1}^k u_i \le x_0 \text{ and } u_i \ge 0 \text{ for } i = 1, \ldots, k,$$

with

$$y_{1n} = x_1 \wedge d_{1n}^{(m)} \wedge \sum_{j=1}^k u_j,$$

and for $i = 2, \ldots, k$,

$$y_{in} = x_i \wedge d_{in}^{(m)} \wedge \left[(\sum_{j=1}^k u_j - \sum_{j=1}^{i-1} y_{jn}) \wedge (\sum_{j=2}^k u_j - \sum_{j=2}^{i-1} y_{jn}) \wedge \ldots \wedge (\sum_{j=i-1}^k u_j - y_{i-1,n}) \wedge (\sum_{j=i}^k u_j) \right].$$

For these policies, define the corresponding profit function by $\pi^{\sigma(m)}(X) := R^{\sigma(m)}(X) - \sum_{j=0}^k (c_j x_j)$, where $\sigma \in \{Det, f, MTS0, MTS, q, o\}$.

Initial Component Inventory Profile

For $i = 1, \ldots, k$, define $n_i^d = \left(\lfloor \frac{p_i - c_i - c_0}{h_0 + h_i} \rfloor + 1 \right) \wedge N$ and $E(D_i^{d(m)}) = \sum_{n=1}^{n_i^d} E(D_{in}^{(m)})$. Also recall the upper bounds for component inventories, as characterized in Proposition 5, and denote the corresponding bounds for the scale-m problem by $\bar{x}_i^{(m)}$ and $\bar{x}_0^{(m)}$.

- $X^{d(m)}$: $x_j^{d(m)} = E(D_j^{d(m)})$ for $j = 1, ..., k$, $x_0^{d(m)} = \sum_{j=1}^k x_j^{d(m)}$.
- $X^{b(m)}$: $x_j^{b(m)} = E(D_j^{d(m)}) \wedge \bar{x}_j^{(m)}$ for $j = 1, ..., k$, $x_0^{b(m)} = \sum_{j=1}^k x_j^{b(m)}$, where $\bar{x}_j^{(m)}$ is defined in Proposition 5.

- $X^{u(m)}$: $x_j^{u(m)} = E(D_j^{d(m)}) \wedge \bar{x}_j^{(m)}$, and $x_0^{u(m)} = \sum_{j=1}^k x_j^{u(m)} \wedge \bar{x}_0^{(m)}$.
- $X^{q(m)}$: $x_j^{q(m)} = E(D_j^{d(m)}) \wedge \bar{x}_j^{(m)}$, and $x_0^{q(m)} = \sum_{j=1}^k x_j^{q(m)} \wedge \tilde{x}_0^{(m)}$, where $\tilde{x}_0 = \bar{F}_0^{-1}\left(\prod_{i=1}^k (\frac{Nh_0}{p_i+Nh_i+Nh_0-c_i-c_0})\right)$. Here, $\bar{F}_0(\cdot) = 1 - F_0(\cdot)$ is the tail distribution function for the total demand of all the products in the entire season, and $\bar{F}_0^{-1}(\cdot)$ is the inverse function of $\bar{F}_0(\cdot)$.

For easy reference, the fulfillment policies and the inventory profiles are listed in Table EC.6.

Table EC.6 Heuristics for order fulfillment and component procurement

	Order Fulfillment
f	Sell at most $y_{in}(X)$ units of product i in period n, where $y_{in}(X)$ is as in (EC.30)
$mts0$	Sell at most $\sum_{n=1}^N y_{in}(X)$ units of product i in periods 1 through N
mts	Sell at most $u_i^*(X)$ units of product i in periods 1 through N, where $u_i^*(X)$ is in (8)
q	Sell at most $\sum_{j=i}^k u_j^*(X)$ units of product i in periods 1 through N, following standard nesting
o	Sell at most $\sum_{j=i}^k u_{in}^*$ units of product i in period n, following standard nesting, as in (5)
	Component Procurement
X^d	$x_j^d = E(D_j^d)$ for $j=1,...,k$, and $x_0^d = \sum_{j=1}^k x_j^d$
X^b	$x_j^b = E(D_j^d) \wedge \bar{x}_j$ for $j=1,...,k$, and $x_0^b = \sum_{j=1}^k x_j^b$
X^u	$x_j^u = E(D_j^d) \wedge \bar{x}_j$, $j=1,\ldots,k$, and $x_0^u = \bar{x}_0 \wedge \sum_{j=1}^k x_j^u$, where $\bar{x}_0 = \bar{F}_0^{-1}\left(\frac{Nh_0}{p_1+Nh_1+Nh_0-c_1-c_0}\right)$
X^q	$x_j^q = E(D_j^d) \wedge \bar{x}_j$, $j=1,\ldots,k$, and $x_0^q = \tilde{x}_0 \wedge \sum_{j=1}^k x_j^q$, where $\tilde{x}_0 = \bar{F}_0^{-1}\left(\prod_{i=1}^k (\frac{Nh_0}{p_i+Nh_i+Nh_0-c_i-c_0})\right)$

It can be proved that $X^{d(m)}$ maximizes $R^{Det(m)}(X)$. We are now ready to prove the proposition. (i) Let $R^{(m)}(X)$ and $\pi^{(m)}(X)$ denote the profit-to-go (i.e., excluding the component procurement cost) and the profit (i.e., including the component procurement cost) under the optimal fulfillment policy and a given component profile X, respectively. Denote by X^* the optimal inventory profile corresponding to the optimal fulfillment policy, i.e., $X^{*(m)} = \arg\max \pi^{(m)}(X)$.

To prove $\lim_{m\to\infty} \frac{\pi^{q(m)}(X^{q(m)})}{\pi^{(m)}(X^{*(m)})} = 1$, we will follow two steps:

$$\text{(i)}\ \frac{\pi^{f(m)}(X^{d(m)})}{\pi^{Det(m)}(X^{d(m)})} \le \frac{\pi^{q(m)}(X^{q(m)})}{\pi^{(m)}(X^{*(m)})} \le 1, \quad \text{(ii)}\ \lim_{m\to\infty} \frac{\pi^{f(m)}(X^{d(m)})}{\pi^{Det(m)}(X^{d(m)})} = 1.$$

(i.1) It suffices to show

$$\pi^{f(m)}(X^{d(m)}) \le \pi^{q(m)}(X^{q(m)}) \le \pi^{(m)}(X^{q(m)}) \le \pi^{(m)}(X^{*(m)}) \le \pi^{Det(m)}(X^{d(m)}),$$

among which $\pi^{q(m)}(X^{q(m)}) \le \pi^{(m)}(X^{q(m)})$ holds because $\pi^{(m)}(X)$ corresponds to the optimal fulfillment policy, while $\pi^{q(m)}(X)$ is for a feasible policy, the heuristic q. $\pi^{(m)}(X^{q(m)}) \le \pi^{(m)}(X^{*(m)})$ is implied by the definition of $X^{*(m)}$ as the optimal initial inventory.

To show the inequality $\pi^{(m)}(X^{*(m)}) \le \pi^{Det(m)}(X^{d(m)})$, first note that for any given component profile X, $\pi^{(m)}(X) \le \pi^{Det(m)}(X)$, i.e., the optimal profit in a stochastic system is always bounded from above by its counterpart in the corresponding deterministic system. This result is by Jensen's inequality and the concavity of the profit function $J_n(X, d_{1n}, ..., d_{kn})$ in the realized demand $(d_{1n}, ..., d_{kn})$. Therefore, $\pi^{(m)}(X^{*(m)}) \leq \pi^{Det(m)}(X^{*(m)})$. In the meanwhile, also note that $\pi^{Det(m)}(X^{*(m)}) \leq \pi^{Det(m)}(X^{d(m)})$ because $X^{d(m)}$ maximizes $\pi^{Det(m)}(\cdot)$.

We have so far proved the last three inequalities in part (i.1), i.e.,

$$\pi^{q(m)}(X^{q(m)}) \leq \pi^{(m)}(X^{q(m)}) \leq \pi^{(m)}(X^{*(m)}) \leq \pi^{Det(m)}(X^{d(m)}).$$

It remains to prove the first inequality $\pi^{f(m)}(X^{d(m)}) \leq \pi^{q(m)}(X^{q(m)})$. To this end, we will follow four sub-steps (i.1.a)-(i.1.d) by taking advantage of the two intermediate policies, $mts0$ and mts, and the inventory profile X^b. In particular, we will prove

(i.1.a) $\pi^{f(m)}(X) \leq \pi^{mts0(m)}(X)$ for given inventory X, implying $\pi^{f(m)}(X^{d(m)}) \leq \pi^{mts0(m)}(X^{d(m)})$;

(i.1.b) $\pi^{mts0(m)}(X^{d(m)}) \leq \pi^{mts0(m)}(X^{b(m)})$;

(i.1.c) $\pi^{mts0(m)}(X^{b(m)}) \leq \pi^{mts(m)}(X^{q(m)})$;

(i.1.d) $R^{mts(m)}(X) \leq R^{q(m)}(X)$ when $x_i \leq E(\sum_{n=1}^{N} D_{in}^{(m)})$ and $x_0 \leq \sum_{i=1}^{k} x_i$, implying $\pi^{mts(m)}(X^{q(m)}) \leq \pi^{q(m)}(X^{q(m)})$.

The series of inequalities proved in the four parts together imply $\pi^{f(m)}(X^{d(m)}) \leq \pi^{q(m)}(X^{q(m)})$.

(i.1.a). $R^{f(m)}(X) \leq R^{mts0(m)}(X)$: Heuristic f is dominated by the naive make-to-stock heuristic *mts0*, for any given initial inventory X.

In heuristic *f*, at most $y_{in}^{(m)}$ units of product i can be sold in period n. The $mts0$ heuristic is advantageous over heuristic f as it allows for the flexibility of using a product's total quotas (i.e., $\sum_{n=1}^{N} y_{in}^{(m)}$) across different periods. More specifically, policy $mts0$ dominates policy f because the flexibility of using quotas across periods can result in either lower holding costs or higher profits or both.

The proof for the dominance is sketched as follows: Note that under either heuristic f or heuristic *mts0*, the quotas for different products are decoupled so that the demand for each type of product is fulfilled separately. Thus, the total profit-to-go is separable, as a sum of the profit-to-go generated from each individual product. To separate the holding costs and salvage values across products, we can equivalently account for the holding costs and the salvage value in the following way: at the beginning of the season, the firm incurs the cost for holding all the initial inventory through the entire season, i.e., $-N\sum_{j=0}^{k}(h_j x_j)$, and then during the selling season, if a unit of product i is sold in period n, then the holding cost for the consumed components (i.e., a unit of component 0 and a unit of component i) from the current period till the end of the season can be "saved" and reflected as an addition to the product margin. Similarly, the firm receives a revenue equal to the total procurement cost of all the initial inventory, $\sum_{j=0}^{k}(c_j x_j)$, at the beginning of the season (i.e., as if all the components had been salvaged upfront and thus no components would be salvaged at the end of the season), and then during the selling season, if a unit of product i is sold in period n, then the components (i.e., a unit of component 0 and a unit of component i) needed to fulfill the order are paid for by the firm in the period and their costs are reflected as a subtraction of the product margin.

With this reformulation, we can decouple the k-product system into k independent single-product sub-systems, and rewrite the expected profit-to-go functions under policies f and $mts0$ as follows (for brevity, all the superscripts (m) and the argument (X) for y_{in} are omitted):

$$R^f(X) = -N\sum_{j=0}^{k}(h_j x_j) + \sum_{j=0}^{k}(c_j x_j) + \sum_{i=1}^{k} E_{D_{i1}}[\check{J}_1^{f,i}(\sum_{n=1}^{N} y_{in}, \sum_{n=1}^{N} y_{in}, y_{i1}, \dots, y_{iN}, D_{i1})],$$

where, $\check{J}_{N+1}^{f,i}(\cdot) = 0$, and for $n \le N$,

$$\begin{aligned}\check{J}_n^{f,i}(x_0^f, x_i^f, y_{in}, \dots, y_{iN}, d_{in}) &= (p_i + (N-n+1)h_i + (N-n+1)h_0 - c_i - c_0))(d_{in} \wedge y_{in}) \\ &\quad + E_{D_{i,n+1}}[\check{J}_{n+1}^{f,i}(x_0^f - (d_{in} \wedge y_{in}), x_i^f - (d_{in} \wedge y_{in}), y_{i,n+1}, \dots, y_{iN}, D_{i,n+1})].\end{aligned}$$

Here, $\check{J}_n^{f,i}(x_0^f, x_i^f, y_{in}, \dots, y_{iN}, d_{in})$ is the profit-to-go function in the sub-system with a single-product i. In particular, the first (second) argument x_0^f (x_i^f) refers to the number of component 0 (i) assigned to this sub-system, while y_{ij} is the number of sales quotas for product i in period $j = n, n+1, \dots, N$.

In the meanwhile, for the $mts0$ heuristic:

$$R^{mts0}(X) = -N\sum_{j=0}^{k}(h_j x_j) + \sum_{j=0}^{k}(c_j x_j) + \sum_{i=1}^{k} E_{D_{i1}}[\check{J}_1^{mts0,i}(\sum_{n=1}^{N} y_{in}, \sum_{n=1}^{N} y_{in}, \sum_{n=1}^{N} y_{in}, D_{i1})],$$

where $\check{J}_{N+1}^{mts0,i}(\cdot,\cdot,\cdot,\cdot) = 0$, and for $n \le N$,

$$\begin{aligned}\check{J}_n^{mts0,i}(x_0, x_i, u_i, d_{in}) &= (p_i + (N-n+1)h_i + (N-n+1)h_0 - c_i - c_0)(d_{in} \wedge u_i) \\ &\quad + E_{D_{i,n+1}}[\check{J}_{n+1}^{mts0,i}(x_0 - (d_{in} \wedge u_i), x_i - (d_{in} \wedge u_i), u_i - (d_{in} \wedge u_i), D_{i,n+1})].\end{aligned}$$

Here, $\check{J}_n^{mts0,i}(x_0, x_i, u_i, d_{in})$ is the profit-to-go function in the sub-system with a single-product i. In particular, the first argument x_0 refers to the number of component 0 assigned to this sub-system, the second argument x_i refers to the number of component i for this sub-system, while the third argument u_i refers to the number of sales quotas in this sub-system from period n to period N, the end of the season.

The proof is then completed by comparing each sub-system under either f or $mts0$ policy, as detailed below. Note that, for any arbitrarily small $\epsilon \ge 0$, we have

$$\check{J}_n^{f,i}(x_0^f, x_i^f, y_{in} + \epsilon, y_{i,n+1}, \dots, y_{iN}, d_{in}) - \check{J}_n^{f,i}(x_0^f, x_i^f, y_{in}, y_{i,n+1}, \dots, y_{iN}, d_{in}) \ge 0, \quad \text{(EC.32)}$$

that is, in heuristic f more quotas are better, and for $j \geq n$,

$$\begin{aligned}
&\check{J}_n^{f,i}(x_0^f, x_i^f, y_{in}, \ldots, y_{ij}, \ldots, y_{iN}, d_{in}) - \check{J}_n^{f,i}(x_0^f - \epsilon, x_i^f - \epsilon, y_{in}, \ldots, y_{ij} - \epsilon, \ldots, y_{iN}, d_{in}) \\
&\leq \epsilon(p_i + (N-j+1)h_i + (N-j+1)h_0 - c_i - c_0) \\
&\leq \epsilon(p_i + (N-n+1)h_i + (N-n+1)h_0 - c_i - c_0), \qquad \text{(EC.33)}
\end{aligned}$$

that is, one unit of quota for product i (and a_i unit of component resource) can at most generate a benefit of $p_i + (N-n+1)h_i + (N-n+1)h_0 - c_i - c_0$.

Next, we will use (EC.32) and (EC.33) to show

$$\begin{aligned}
\check{L}_1 &:= \check{J}_n^{mts0,i}(\sum_{j=n}^N y_{ij}, \sum_{j=n}^N y_{ij}, \sum_{j=n}^N y_{ij}, d_{in}) - \check{J}_n^{f,i}(\sum_{j=n}^N y_{ij}, \sum_{j=n}^N y_{ij}, y_{in}, \ldots, y_{iN}, d_{in}) \\
&\geq 0
\end{aligned}$$

for any $d_{in} \geq 0$ by induction.

First note $\check{J}_{N+1}^{mts0,i}(\cdot) = \check{J}_{N+1}^{f,i}(\cdot) = 0$. Now, suppose for $n \leq N$,

$$\check{J}_{n+1}^{mts0,i}(\sum_{j=n+1}^N y_{ij}, \sum_{j=n+1}^N y_{ij}, \sum_{j=n+1}^N y_{ij}, d_{i,n+1}) \geq \check{J}_{n+1}^{f,i}(\sum_{j=n+1}^N y_{ij}, \sum_{j=n+1}^N y_{ij}, y_{i,n+1}, \ldots, y_{iN}, d_{i,n+1}).$$

Next we prove that in period n, $\check{L}_1 \geq 0$.

- **Case i.1.a.1:** $d_{in} \leq y_{in}$ d_{in} units of product i will be sold in both heuristic f and the make-to-stock heuristic, but $y_{in} - d_{in}$ units of quotas are discarded in heuristic f. In contrast, the make-to-stock heuristic carries these quotas for future periods, which can be used to fulfill the demand of product i in the future:

$$\begin{aligned}
\check{L}_1 &= E_{D_{i,n+1}}[\check{J}_{n+1}^{mts0,i}(\sum_{j=n}^N y_{ij} - d_{in}, \sum_{j=n}^N y_{ij} - d_{in}, \sum_{j=n}^N y_{ij} - d_{in}, D_{i,n+1})] \\
&\quad - E_{D_{i,n+1}}[\check{J}_{n+1}^{f,i}(\sum_{j=n}^N y_{ij} - d_{in}, \sum_{j=n}^N y_{ij} - d_{in}, y_{i,n+1}, \ldots, y_{iN}, D_{i,n+1})] \\
&\geq E_{D_{i,n+1}}[\check{J}_{n+1}^{mts0,i}(\sum_{j=n}^N y_{ij} - d_{in}, \sum_{j=n}^N y_{ij} - d_{in}, \sum_{j=n}^N y_{ij} - d_{in}, D_{i,n+1})] \\
&\quad - E_{D_{i,n+1}}[\check{J}_{n+1}^{f,i}(\sum_{j=n}^N y_{ij} - d_{in}, \sum_{j=n}^N y_{ij} - d_{in}, y_{i,n+1} + y_{in} - d_{in}, y_{i,n+2}, \ldots, y_{iN}, D_{i,n+1})] \geq 0,
\end{aligned}$$

where the first inequality is due to (EC.32), and the second inequality is implied from the induction hypothesis.

- **Case i.1.a.2: $d_{in} > y_{in}$** Only $d_{in} \wedge y_{in} = y_{in}$ units of product i will be sold in heuristic f. However, in the make-to-stock heuristic $y_{in}^{mts0} := d_{in} \wedge \sum_{j=n}^{N} y_{ij} \geq y_{in}$ units of product i will be sold. To prove

$$
\begin{aligned}
\check{L}_1 =& (p_i + (N-n+1)h_i + (N-n+1)h_0 - c_i - c_0)(y_{in}^{mts0} - y_{in}) \\
&+ E_{D_{i,n+1}}[\check{J}_{n+1}^{mts0,i}(\sum_{j=n}^{N} y_{ij} - y_{in}^{mts0}, \sum_{j=n}^{N} y_{ij} - y_{in}^{mts0}, \sum_{j=n}^{N} y_{ij} - y_{in}^{mts0}, D_{i,n+1})] \\
&- E_{D_{i,n+1}}[\check{J}_{n+1}^{f,i}(\sum_{j=n}^{N} y_{ij} - y_{in}, \sum_{j=n}^{N} y_{ij} - y_{in}, y_{i,n+1}, \ldots, y_{iN}, D_{i,n+1})] \geq 0
\end{aligned}
$$

we consider two subcases: $y_{in} < y_{in}^{mts0} \leq y_{iN} + y_{in}$ and $y_{in}^{mts0} > y_{iN} + y_{in}$.

Subcase i.1.a.2.1: $y_{in} < y_{in}^{mts0} \leq y_{iN} + y_{in}$ Let $\check{y}_{iN} = y_{iN} + y_{in} - y_{in}^{mts0}$, then $y_{in}^{mts0} - y_{in} = y_{iN} - \check{y}_{iN}$ and

$$
\begin{aligned}
&\check{J}_{n+1}^{f,i}(\sum_{j=n+1}^{N} y_{ij}, \sum_{j=n+1}^{N} y_{ij}, y_{i,n+1}, \ldots, y_{iN}, d_{i,n+1}) \\
\leq& (p_i + h_i + h_0 - c_i - c_0)(y_{iN} - \check{y}_{iN}) \\
&+ \check{J}_{n+1}^{f,i}(\sum_{j=n+1}^{N} y_{ij} - (y_{iN} - \check{y}_{iN}), \sum_{j=n+1}^{N} y_{ij} - (y_{iN} - \check{y}_{iN}), y_{i,n+1}, \ldots, y_{i,N-1}, \check{y}_{iN}, d_{i,n+1}) \\
=& (p_i + h_i + h_0 - c_i - c_0)(y_{in}^{mts0} - y_{in}) \\
&+ \check{J}_{n+1}^{f,i}(\sum_{j=n}^{N} y_{ij} - y_{in}^{mts0}, \sum_{j=n}^{N} y_{ij} - y_{in}^{mts0}, y_{i,n+1}, \ldots, y_{i,N-1}, \check{y}_{iN}, d_{i,n+1}) \\
\leq& (p_i + (N-n+1)h_i + (N-n+1)h_0 - c_i - c_0)(y_{in}^{mts0} - y_{in}) \\
&+ \check{J}_{n+1}^{f,i}(\sum_{j=n}^{N} y_{ij} - y_{in}^{mts0}, \sum_{j=n}^{N} y_{ij} - y_{in}^{mts0}, y_{i,n+1}, \ldots, y_{i,N-1}, \check{y}_{iN}, d_{i,n+1}) \\
\leq& (p_i + (N-n+1)h_i + (N-n+1)h_0 - c_i - c_0)(y_{in}^{mts0} - y_{in}) \\
&+ \check{J}_{n+1}^{mts0,i}(\sum_{j=n}^{N} y_{ij} - y_{in}^{mts0}, \sum_{j=n}^{N} y_{ij} - y_{in}^{mts0}, \check{y}_{iN} + \sum_{j=n+1}^{N-1} y_{ij}, d_{i,n+1}) \\
=& (p_i + (N-n+1)h_i + (N-n+1)h_0 - c_i - c_0)(y_{in}^{mts0} - y_{in}) \\
&+ \check{J}_{n+1}^{mts0,i}(\sum_{j=n}^{N} y_{ij} - y_{in}^{mts0}, \sum_{j=n}^{N} y_{ij} - y_{in}^{mts0}, \sum_{j=n}^{N} y_{ij} - y_{in}^{mts0}, d_{i,n+1}),
\end{aligned}
$$

where the first inequality is by (EC.33) with $\epsilon = y_{iN} - \check{y}_{iN}$: in the left-hand side of this inequality the quota for the last period is y_{iN}, while in its right-hand side the quota for the last period is $\check{y}_{iN}$. The second inequality is due to $n \leq N$. The third inequality is implied by the induction hypothesis

$$
\begin{aligned}
&\check{J}_{n+1}^{f,i}(\sum_{j=n}^{N} y_{ij} - y_{in}^{mts0}, \sum_{j=n}^{N} y_{ij} - y_{in}^{mts0}, y_{i,n+1}, \ldots, y_{i,N-1}, \check{y}_{iN}, d_{i,n+1}) \\
\leq& \check{J}_{n+1}^{mts0,i}(\sum_{j=n}^{N} y_{ij} - y_{in}^{mts0}, \sum_{j=n}^{N} y_{ij} - y_{in}^{mts0}, \check{y}_{iN} + \sum_{j=n+1}^{N-1} y_{ij}, d_{i,n+1})
\end{aligned}
$$

for any $d_{i,n+1}$. The two equalities follow from $y_{in}^{mts0} - y_{in} = y_{iN} - \check{y}_{iN}$. Thus, $\check{L}_1 \geq 0$.

Subcase i.1.a.2.2: $y_{in}^{mts0} > y_{iN} + y_{in}$ If $n = N-1$, this case does not exist because when $n = N-1$, according to the definition of y_{in}^{mts0}, $y_{i,N-1}^{mts0} \leq \sum_{j=N-1}^{N} y_{ij}$. So, this case is only valid when $n \leq N-2$. Furthermore, define $\check{j} \in \{n+1, \dots, N-1\}$ as the value satisfying

$$\sum_{j=\check{j}+1}^{N} y_{ij} + y_{in} < y_{in}^{mts0} \leq \sum_{j=\check{j}}^{N} y_{ij} + y_{in}, \quad \text{and} \check{y}_{i,\check{j}} = \sum_{j=\check{j}}^{N} y_{ij} + y_{in} - y_{in}^{mts0}.$$

That is, fulfilling the demand in period n uses all the quotas in periods $\check{j}+1, \dots, N$ and n as well as a portion of the quotas in period $\check{j}$. Similar to the subcase $y_{in} < y_{in}^{mts0} \leq y_{iN} + y_{in}$ above, we have

$$\begin{aligned}
&\check{J}_{n+1}^{f,i}\Big(\sum_{j=n+1}^{N} y_{ij}, \sum_{j=n+1}^{N} y_{ij}, y_{i,n+1}, \dots, y_{i,\check{j}-1}, y_{i\check{j}}, y_{i,\check{j}+1}, \dots, y_{iN}, d_{i,n+1}\Big) \\
\leq &\sum_{j=\check{j}+1}^{N} [(p_i + (N-j+1)h_i + (N-j+1)h_0 - c_i - c_0)y_{ij}] + (p_i + (N-\check{j}+1)h_i + (N-\check{j}+1)h_0 - c_i - c_0)(y_{i\check{j}} - \check{y}_{i\check{j}}) \\
&+\check{J}_{n+1}^{f,i}\Big(\sum_{j=n+1}^{\check{j}-1} y_{ij} + \check{y}_{i\check{j}}, \sum_{j=n+1}^{\check{j}-1} y_{ij} + \check{y}_{i\check{j}}, y_{i,n+1}, \dots, y_{i,\check{j}-1}, \check{y}_{i\check{j}}, 0, \dots, 0, d_{i,n+1}\Big) \\
\leq &(p_i + (N-n+1)h_i + (N-n+1)h_0 - c_i - c_0)\Big(\sum_{j=\check{j}+1}^{N} y_{ij} + y_{i\check{j}} - \check{y}_{i\check{j}}\Big) \\
&+\check{J}_{n+1}^{f,i}\Big(\sum_{j=n+1}^{\check{j}-1} y_{ij} + \check{y}_{i\check{j}}, \sum_{j=n+1}^{\check{j}-1} y_{ij} + \check{y}_{i\check{j}}, y_{i,n+1}, \dots, y_{i,\check{j}-1}, \check{y}_{i\check{j}}, 0, \dots, 0, d_{i,n+1}\Big),
\end{aligned}$$

where the first inequality is obtained from applying (EC.33) $\check{j}+1$ times (with $\epsilon = y_{ij}$ for $j = \check{j}+1, \dots, N$ and $\epsilon = y_{i\check{j}} - \check{y}_{i\check{j}}$ for $j = \check{j}$): in the left-hand side of this inequality the quotas for period $j = \check{j}, \check{j}+1, \dots, N$ is y_{ij}, while in its right-hand side the quota for period $\check{j}$ is $\check{y}_{i\check{j}}$, and for period $j = \check{j}+1, \dots, N$ is 0. The second inequality is due to $\check{j} \geq n+1$.

Also, the induction hypothesis implies

$$\begin{aligned}
&\check{J}_{n+1}^{f,i}\Big(\sum_{j=n+1}^{\check{j}-1} y_{ij} + \check{y}_{i\check{j}}, \sum_{j=n+1}^{\check{j}-1} y_{ij} + \check{y}_{i\check{j}}, y_{i,n+1}, \dots, y_{i,\check{j}-1}, \check{y}_{i\check{j}}, 0, \dots, 0, d_{i,n+1}\Big) \\
\leq\ &\check{J}_{n+1}^{mts0,i}\Big(\sum_{j=n+1}^{\check{j}-1} y_{ij} + \check{y}_{i\check{j}}, \sum_{j=n+1}^{\check{j}-1} y_{ij} + \check{y}_{i\check{j}}, \sum_{j=n+1}^{\check{j}-1} y_{ij} + \check{y}_{i\check{j}}, d_{i,n+1}\Big) \\
=\ &\check{J}_{n+1}^{mts0,i}\Big(\sum_{j=n}^{N} y_{ij} - y_{in}^{mts0}, \sum_{j=n}^{N} y_{ij} - y_{in}^{mts0}, \sum_{j=n}^{N} y_{ij} - y_{in}^{mts0}, d_{i,n+1}\Big).
\end{aligned}$$

As such, $\check{L}_1 \geq 0$.

(i.1.b). $\pi^{mts0(m)}(X^{d(m)}) \leq \pi^{mts0(m)}(X^{b(m)})$.

First note that $X_j^{d(m)} \geq X_j^{b(m)}$ for $j = 0, 1, ..., k$. Hence, it suffices to show that the amount of component inventory in $X^{d(m)}$ beyond that in $X^{b(m)}$, i.e., $X^{d(m)} - X^{b(m)}$, does not lead to an increase in the expected total profit. Note that both $X^{d(m)}$ and $X^{b(m)}$ are balanced, i.e., the common component inventory equals to the sum of the dedicated component inventories. Hence, under either $X^{d(m)}$ or $X^{b(m)}$, the k-product balanced system can be decoupled into k single-product balanced systems, where the inventory of the common component matches with that of the dedicated component in each single-product system. Noting $x_0 = x_i$ in system i, we define the expected profit in system i under the MTS0 policy as a univariate function of x_i and show that it is nonincreasing for $x_i \geq x_i^{b(m)}$.

The detailed proof is as follows. Because the fulfillment decision for each product in the system under heuristic *mts0* is handled independently, we can decouple the k-product system (under heuristic *mts0*) into k independent single-product sub-systems. Thus, the profit for each product can be evaluated separately to obtain the total profit-to-go for the k-product system. More importantly, regarding the optimal solutions of the deterministic problem, $y_{11}, \ldots, y_{kN}$, with the initial inventory X^d all the components will be used so that $\sum_{n=1}^N y_{in} = x_i^d = E(D_i^d)$ for $i = 1, ..., k$. As such, the number of sales quotas for product $i = 1, ..., k$ under heuristic *mts0* is $x_i^d = E(D_i^d)$. Therefore, the sub-system with product $i = 1, ..., k$ has $E(D_i^d)$ units of common component and $E(D_i^d)$ units of dedicated component i as well as $E(D_i^d)$ units of sales quotas for product i. Similarly, in the deterministic problem with the initial inventory X^b, as implied by the fact that X^b is less than X^d, all the components will be used so that $\sum_{n=1}^N y_{in} = x_i^b$ for $i = 1, ..., k$.

Then, the profit-to-go for the k-product system obtained from heuristic *mts0* with the initial inventory X^σ for $\sigma \in \{b, d\}$ is:

$$\pi^{mts0}(X^\sigma) = \sum_{i=1}^k \pi^{mts0,i}(x_i^\sigma),$$

where

$$\begin{aligned}\pi^{mts0,i}(x_i^\sigma) &= -N(h_0+h_i)x_i^\sigma + (c_0+c_i)x_i^\sigma + E_{D_{i1}}[\check{J}_1^{mts0,i}(x_i^\sigma, x_i^\sigma, x_i^\sigma, D_{i1})] - (c_0+c_i)x_i^\sigma \\ &= -N(h_0+h_i)x_i^\sigma + E_{D_{i1}}[\check{J}_1^{mts0,i}(x_i^\sigma, x_i^\sigma, x_i^\sigma, D_{i1})]\end{aligned}$$

Denote by MTS0 the set $\{i = 1, ..., k | \bar{x}_i < E(D_i^d)\}$, that is, $x_i^b = \bar{x}_i \wedge E(D_i^d) = \bar{x}_i < x_i^d = E(D_i^d)$ for $i \in$ MTS0, and $x_i^b = x_i^d = E(D_i^d)$ for $i \notin$ MTS0. Then, for $i \notin$ MTS0, $\pi^{mts0,i}(x_i^b) = \pi^{mts0,i}(E(D_i^d))$, while for $i \in$ MTS0, $\pi^{mts0,i}(x_i^b) \geq \pi^{mts0,i}(E(D_i^d))$ because, given x_i as the quota for the product-i sub-system, $\check{J}_1^{mts0,i}(x_i, x_i, x_i, D_{i1}) = \bar{\bar{J}}_1^{mts,i}(x_i, D_{i1})$ and proof of Lemma EC.9(i) implies that

$$0 \leq \frac{dE_{D_{i1}}[\bar{\bar{J}}_1^{mts,i}(x_i, D_{i1})]}{dx_i} \leq (p_i + Nh_i + Nh_0 - c_i - c_0)\bar{F}_i(x_i)$$

so that $\frac{d\pi^{mts0,i}(x_i)}{dx_i} \leq (p_i + Nh_i + Nh_0 - c_i - c_0)\bar{F}_i(x_i) - N(h_0+h_i)$. Hence, $\frac{d\pi^{mts0,i}(x_i)}{dx_i} \leq Nh_i - N(h_0 + h_i) = -Nh_0 \leq 0$ when $x_i \geq \bar{x}_i$, where $\bar{x}_i$ is defined in Proposition 5 with $\bar{F}_i(\bar{x}_i) = (Nh_i)/(p_i + Nh_i + Nh_0 - c_i - c_0)$.

Therefore, $\pi^{mts0,i}(x_i^b) \geq \pi^{mts0,i}(E(D_i^d))$ for either $i \notin$ MTS0 or $i \in$ MTS0, which implies $\pi^{mts0}(X^b) \geq \pi^{mts0}(X^d)$.

(i.1.c). $\pi^{mts0(m)}(X^{b(m)}) \leq \pi^{mts(m)}(X^{b(m)}) \leq \pi^{mts(m)}(X^{q(m)})$.

The first inequality follows from the fact that the naive make-to-stock heuristic *mts0* is dominated by the make-to-stock heuristic *mts*, as the latter adopts the optimal static sales quota.

To show $\pi^{mts(m)}(X^{b(m)}) \leq \pi^{mts(m)}(X^{q(m)})$, first recall that $X^{b(m)}$ and $X^{q(m)}$ are identical, except that $x_0^{q(m)} = x_0^{b(m)} \wedge \tilde{x}_0$. If $x_0^{b(m)} \leq \tilde{x}_0$, the proof is trivial. If $x_0^{b(m)} > \tilde{x}_0$, it suffices to show that, fixing the dedicated components, $\pi_b^{mts}(x_0) := \pi^{mts(m)}(x_0, x_1^{b(m)}, ..., x_k^{b(m)})$ decreases in $x_0 \in [\tilde{x}_0, x_0^{b(m)}]$. The monotonicity of $\pi_b^{mts}(x_0)$ is shown by contradiction. Suppose $\frac{d\pi_b^{mts}(x_0)}{dx_0} > 0$ for some $x_0' \in [\tilde{x}_0, x_0^{b(m)}]$. We partition the product set $(1, ..., k)$ into three subsets based on the optimal sell-up-to levels $(u_1^*, u_2^*, ..., u_k^*)$ under the inventory profile $(x_0', x_1^{b(m)}, ..., x_k^{b(m)})$, where $(u_1^*, u_2^*, ..., u_k^*)$ is as defined in (8) and we omitted its arguments for brevity. We then derive necessary conditions for the optimal sell-up-to levels in each product subset, and subsequently, apply these necessary conditions to show that x_0' must satisfy

$$\prod_{i=1}^{k}\left(\frac{Nh_0}{p_i + Nh_i + Nh_0 - c_i - c_0}\right) < \mathcal{P}\left(\sum_{i=1}^{k}\sum_{n=1}^{N} D_{in}^{(m)} \geq x_0'\right).$$

This, however, contradicts with the fact that

$$\prod_{i=1}^{k}\left(\frac{Nh_0}{p_i + Nh_i + Nh_0 - c_i - c_0}\right) = \bar{F}_0(\tilde{x}_0) \geq \bar{F}_0(x_0') = \mathcal{P}\left(\sum_{i=1}^{k}\sum_{n=1}^{N} D_{in}^{(m)} \geq x_0'\right)$$

as $x_0' \geq \tilde{x}_0$.

The detailed proof is as follows. First recall that under make-to-stock policy, as defined in Lemma EC.9, when $\sum_{i=1}^{k} u_i \leq x_0$ and $0 \leq u_i \leq x_i$, $i = 1, ...k$, we can decouple the system into k independent single-product sub-systems:

$$V_1^{mts}(X, u_1, \ldots, u_k) = -N\sum_{j=0}^{k}(h_j x_j) + \sum_{j=0}^{k}(c_j x_j) + \sum_{i=1}^{k} E_{D_{i1}}[\bar{\bar{J}}_1^{mts,i}(u_i, D_{i1})],$$

where $\bar{\bar{J}}_{N+1}^{mts,i}(\cdot,\cdot) = 0$, and for $n \in \{1, ..., N\}$,

$$\bar{\bar{J}}_n^{mts,i}(u_i, d_{in}) = (p_i + (N-n+1)h_i + (N-n+1)h_0 - c_i - c_0)(d_{in} \wedge u_i) + E_{D_{i,n+1}}[\bar{\bar{J}}_{n+1}^{mts,i}(u_i - (d_{in} \wedge u_i), D_{i,n+1})].$$

We now prove $\pi^{mts}(X^b) \leq \pi^{mts}(X^q)$. Recall that X^b is identical to X^q, except that x_0^q equals to $x_0^b \wedge \tilde{x}_0 = \sum_{j=1}^{k} x_j^b \wedge \tilde{x}_0$. If $\tilde{x}_0 \geq \sum_{j=1}^{k} x_j^b$, $X^b = X^q$ and the proof is trivial. If $\tilde{x}_0 < \sum_{j=1}^{k} x_j^b$, given

$$x_j^b = \bar{x}_j \wedge E(D_j^d) \leq E(D_j^d) = \sum_{n=1}^{n_j^d} E(D_{jn}) \leq \sum_{n=1}^{N} E(D_{jn})$$

and $\pi_b^{mts}(x_0) := R_b^{mts}(x_0) - c_0 x_0 - \sum_{j=1}^k (c_j x_j^b)$ with $R_b^{mts}(x_0) := V_1^{mts}(x_0, x_1^b, \ldots, x_k^b, u_1^*, \ldots, u_k^*)$, we shall show $\frac{d\pi_b^{mts}(x_0)}{dx_0} \leq 0$ (or $\frac{dR_b^{mts}(x_0)}{dx_0} \leq c_0$) for all $x_0 \in [\tilde{x}_0, \sum_{j=1}^k x_j^b)$. In the meanwhile, since

$$x_i^b = E(D_i^d) \wedge \bar{x}_i \leq E(D_i^d) \leq E(\sum_{n=1}^N D_{in})$$

for all $i = 1, ..., k$, by Lemma EC.9(ii), $R_b^{mts}(x_0)$ is concave in $x_0 \leq \sum_{j=1}^k x_j^b$. Thus, it suffices to show $\frac{dR_b^{mts}(x_0)}{dx_0} \leq c_0$ for all $x_0 \in [\tilde{x}_0, \sum_{j=1}^k x_j^b)$. We prove it by contradiction. Suppose $\frac{dR_b^{mts}(x_0)}{dx_0}|_{x_0 = x_0'} > c_0$ for some $x_0' \in [\tilde{x}_0, \sum_{j=1}^k x_j^b)$. That is,

$$\begin{aligned}
c_0 \quad &< \frac{dR_b^{mts}(x_0)}{dx_0}|_{x_0=x_0'} \\
&= \frac{\partial V_1^{mts}(x_0, x_1^b, ..., x_k^b, u_1, \ldots, u_k)}{\partial x_0}\bigg|_{x_0=x_0',\ (u_1,\ldots,u_k)=(u_1^*,\ldots,u_k^*)} \\
&\quad + \sum_{i=1}^k \left[\frac{\partial V_1^{mts}(x_0', x_1^b, ..., x_k^b, u_1, \ldots, u_k)}{\partial u_i}\bigg|_{(u_1,\ldots,u_k)=(u_1^*,\ldots,u_k^*)} \frac{du_i^*}{dx_0}\bigg|_{x_0=x_0'} \right] \\
&= c_0 - Nh_0 + \sum_{i=1}^k \left[\frac{\partial V_1^{mts}(x_0', x_1^b, ..., x_k^b, u_1, \ldots, u_k)}{\partial u_i}\bigg|_{(u_1,\ldots,u_k)=(u_1^*,\ldots,u_k^*)} \frac{du_i^*}{dx_0}\bigg|_{x_0=x_0'} \right] \\
\Rightarrow Nh_0 &< \sum_{i=1}^k \left[\frac{\partial V_1^{mts}(x_0', x_1^b, ..., x_k^b, u_1, \ldots, u_k)}{\partial u_i}\bigg|_{(u_1,\ldots,u_k)=(u_1^*,\ldots,u_k^*)} \frac{du_i^*}{dx_0}\bigg|_{x_0=x_0'} \right].
\end{aligned} \tag{EC.34}$$

In preparation for the proof by contradiction, note that with

$$x_i^b = E(D_i^d) \wedge \bar{x}_i \leq E(D_i^d) \leq E(\sum_{n=1}^N D_{in})$$

for all $i = 1, ..., k$ and $x_0' < \sum_{i=1}^k x_i^b$, Lemma EC.9(i) implies that $0 \leq u_i^* \leq x_i^b$ for all $i = 1, ..., k$. Denote by $\mathcal{MTS}$ the set $\{i = 1, ..., k | u_i^* = x_i^b\}$, $\mathcal{MTSA}$ the set $\{i = 1, ..., k | u_i^* = 0\}$ and $\mathcal{MTSB}$ the set $\{i = 1, ..., k | 0 < u_i^* < x_i^b\}$. These three sets are disjoint. Furthermore,

$$1 = \sum_{i=1}^k \frac{du_i^*}{dx_0}\bigg|_{x_0=x_0'} = \sum_{i \notin \mathcal{MTS}} \frac{du_i^*}{dx_0}\bigg|_{x_0=x_0'}, \tag{EC.35}$$

where the first equality is due to $\sum_{i=1}^k u_i^* = x_0'$ in Lemma EC.9(iv), and the second equality is from $\frac{du_i^*}{dx_0}\Big|_{x_0=x_0'} = 0$ for all $i \in \mathcal{MTS}$, since $\frac{du_i^*}{dx_0} \geq 0$ in Lemma EC.9(iv) implies that, with an arbitrarily small increase in the common component (e.g., from x_0' to $x_0' + \epsilon$), u_i^* remains x_i^b for all $i \in \mathcal{MTS}$, as we have shown $u_i^* \leq x_i^b$ for all $i = 1, ...k$.[1]

Next we will show contradiction in two cases: $\mathcal{MTSA} = \emptyset$ and $\mathcal{MTSA} \neq \emptyset$.

- $\underline{\mathcal{MTSA} = \emptyset}$: In this case, $u_j^* > 0$ for all $j = 1, ..., k$. Furthermore, Lemma EC.9(iii-1) implies that, at the point $(u_1, \ldots, u_k) = (u_1^*, \ldots, u_k^*)$,

$$\frac{\partial V_1^{mts}(x_0', x_1^b, ...x_k^b, u_1, \ldots, u_k)}{\partial u_{i_1}} = \frac{\partial V_1^{mts}(x_0', x_1^b, ...x_k^b, u_1, \ldots, u_k)}{\partial u_{i_2}}$$

[1] If there is an arbitrarily small decrease in the common component, it is optimal to first reduce the quotas of those products which are not in the set $\mathcal{MTS}$, as implied by Lemma EC.9(iii-1).

for $i_1, i_2 \notin \mathcal{MTS}$. Hence, by (EC.34) and (EC.35), we have

$$\begin{aligned} Nh_0 &< \sum_{i \notin \mathcal{MTS}} \left[\frac{\partial V_1^{mts}(x_0', x_1^b, ...x_k^b, u_1, \ldots, u_k)}{\partial u_i} \bigg|_{(u_1,\ldots,u_k)=(u_1^*,\ldots,u_k^*)} \frac{du_i^*}{dx_0}\bigg|_{x_0=x_0'} \right] \\ &= \frac{\partial V_1^{mts}(x_0', x_1^b, ...x_k^b, u_1, \ldots, u_k)}{\partial u_i} \bigg|_{(u_1,\ldots,u_k)=(u_1^*,\ldots,u_k^*)} \sum_{i \notin \mathcal{MTS}} \left(\frac{du_i^*}{dx_0}\bigg|_{x_0=x_0'} \right) \\ &= \frac{\partial V_1^{mts}(x_0', x_1^b, ...x_k^b, u_1, \ldots, u_k)}{\partial u_i} \bigg|_{(u_1,\ldots,u_k)=(u_1^*,\ldots,u_k^*)}. \end{aligned}$$

This, together with Lemma EC.9 (iii-1) and (EC.19), implies

$$Nh_0 < \frac{\partial V_1^{mts}(x_0, x_1^b, ...x_k^b, u_1, \ldots, u_k)}{\partial u_i}|_{(x_0,u_1,\ldots,u_k)=(x_0',u_1^*,\ldots,u_k^*)} \leq (p_i + Nh_i + Nh_0 - c_i - c_0)\bar{F}_i(u_i^*)$$

for all $i = 1, ..., k$. Given $\bar{F}_i(u_i^*) = \mathcal{P}\left(\sum_{n=1}^N D_{in} \geq u_i^*\right)$, for $i = 1, ..., k$,

$$\frac{Nh_0}{p_i + Nh_i + Nh_0 - c_i - c_0} < \mathcal{P}\left(\sum_{n=1}^N D_{in} \geq u_i^*\right).$$

Thus,

$$\prod_{i=1}^k \left(\frac{Nh_0}{p_i + Nh_i + Nh_0 - c_i - c_0} \right) < \prod_{i=1}^k \mathcal{P}\left(\sum_{n=1}^N D_{in} \geq u_i^*\right) \leq \mathcal{P}\left(\sum_{i=1}^k \sum_{n=1}^N D_{in} \geq x_0'\right),$$

where the last inequality is due to that $\sum_{n=1}^N D_{in} \geq u_i^*$ for all $i = 1, ..., k$ implies $\sum_{i=1}^k \sum_{n=1}^N D_{in} \geq \sum_{i=1}^k u_i^* = x_0'$. This, however, contradicts with the fact that

$$\prod_{i=1}^k \left(\frac{Nh_0}{p_i + Nh_i + Nh_0 - c_i - c_0} \right) = \bar{F}_0(\tilde{x}_0) \geq \bar{F}_0(x_0') = \mathcal{P}\left(\sum_{i=1}^k \sum_{n=1}^N D_{in} \geq x_0'\right)$$

for $x_0' \in [\tilde{x}_0, \sum_{j=1}^k x_j^b)$. Thus, $\frac{d\pi_b^{mts}(x_0)}{dx_0} \leq 0$ for $x_0 \in [\tilde{x}_0, \sum_{j=1}^k x_j^b]$.

- $\underline{\mathcal{MTSA} \neq \emptyset}$: In this case, $u_j^* = 0$ for some product index j. Denote

$$D_f = \max_{j \in \mathcal{MTSA}} \left[\frac{\partial V_1^{mts}(x_0', x_1^b, ...x_k^b, u_1, \ldots, u_k)}{\partial u_j}|_{(u_1,\ldots,u_k)=(u_1^*,\ldots,u_k^*)} \right].$$

Consider two sub-cases as below.

If $D_f > Nh_0$, by Lemma EC.9(iii-2),

$$\frac{\partial V_1^{mts}(x_0', x_1^b, ...x_k^b, u_1, \ldots, u_k)}{\partial u_i} \bigg|_{(u_1,\ldots,u_k)=(u_1^*,\ldots,u_k^*)} > Nh_0$$

for all $i \notin \mathcal{MTSA}$.

On the other hand, if $D_f \leq Nh_0$, so that

$$\frac{\partial V_1^{mts}(x_0', x_1^b, ...x_k^b, u_1, \ldots, u_k)}{\partial u_i} \bigg|_{(u_1,\ldots,u_k)=(u_1^*,\ldots,u_k^*)} \leq Nh_0$$

for all $i \in \mathcal{MTSA}$, to prove

$$\left.\frac{\partial V_1^{mts}(x_0', x_1^b, ...x_k^b, u_1, \ldots, u_k)}{\partial u_i}\right|_{(u_1,\ldots,u_k)=(u_1^*,\ldots,u_k^*)} > Nh_0$$

for all $i \notin \mathcal{MTSA}$, it suffices to prove that

$$\left.\frac{\partial V_1^{mts}(x_0', x_1^b, ...x_k^b, u_1, \ldots, u_k)}{\partial u_i}\right|_{(u_1,\ldots,u_k)=(u_1^*,\ldots,u_k^*)} > Nh_0$$

for all $i \in \mathcal{MTSB}$: because, according to Lemma EC.9(iii-1), all these derivatives for the product indexes in set $\mathcal{MTS}$ are greater than or equal to those in set $\mathcal{MTSB}$, i.e.

$$\frac{\partial V_1^{mts}(x_0^{'}, x_1^b, ...x_k^b, u_1, \ldots, u_k)}{\partial u_{i_1}} \geq \frac{\partial V_1^{mts}(x_0^{'}, x_1^b, ...x_k^b, u_1, \ldots, u_k)}{\partial u_{i_2}}$$

for $i_1 \in \mathcal{MTS}, i_2 \in \mathcal{MTSB}$ at the point $(u_1, \ldots, u_k) = (u_1^*, \ldots, u_k^*)$. Given that all these derivatives for the product indexes in set $\mathcal{MTSB}$ are the same (Lemma EC.9(iii-1)), i.e.,

$$\frac{\partial V_1^{mts}(x_0^{'}, x_1^b, ...x_k^b, u_1, \ldots, u_k)}{\partial u_{i_3}} = \frac{\partial V_1^{mts}(x_0^{'}, x_1^b, ...x_k^b, u_1, \ldots, u_k)}{\partial u_{i_4}}$$

for $i_3, i_4 \in \mathcal{MTSB}$ at the point $(u_1, \ldots, u_k) = (u_1^*, \ldots, u_k^*)$, suppose for all $i \in \mathcal{MTSB}$,

$$\left.\frac{\partial V_1^{mts}(x_0', x_1^b, ...x_k^b, u_1, \ldots, u_k)}{\partial u_i}\right|_{(u_1,\ldots,u_k)=(u_1^*,\ldots,u_k^*)} \leq Nh_0.$$

Thus, because $\sum_{i \notin \mathcal{MTS}} \left.\frac{du_i^*}{dx_0}\right|_{x_0=x_0'} = 1$,

$$\begin{aligned} Nh_0 &= Nh_0 \sum_{i \notin \mathcal{MTS}} \left.\frac{du_i^*}{dx_0}\right|_{x_0=x_0'} \\ &\geq \sum_{i \notin \mathcal{MTS}} \left[\left.\frac{\partial V_1^{mts}(x_0', x_1^b, ...x_k^b, u_1, \ldots, u_k)}{\partial u_i}\right|_{(u_1,\ldots,u_k)=(u_1^*,\ldots,u_k^*)} \left.\frac{du_i^*}{dx_0}\right|_{x_0=x_0'} \right] \\ &= \sum_{i=1}^{k} \left[\left.\frac{\partial V_1^{mts}(x_0', x_1^b, ...x_k^b, u_1, \ldots, u_k)}{\partial u_i}\right|_{(u_1,\ldots,u_k)=(u_1^*,\ldots,u_k^*)} \left.\frac{du_i^*}{dx_0}\right|_{x_0=x_0'} \right], \end{aligned} \tag{EC.36}$$

where the first equality is from (EC.35), the last equality holds since $\left.\frac{du_i^*}{dx_0}\right|_{x_0=x_0'} = 0$ for $i \in \mathcal{MTS}$, and the inequality follow from $D_f \leq Nh_0$ (i.e.,

$$\left.\frac{\partial V_1^{mts}(x_0', x_1^b, ...x_k^b, u_1, \ldots, u_k)}{\partial u_i}\right|_{(u_1,\ldots,u_k)=(u_1^*,\ldots,u_k^*)} \leq Nh_0$$

for all $i \in \mathcal{MTSA}$) and the hypothesis above that

$$\left.\frac{\partial V_1^{mts}(x_0', x_1^b, ...x_k^b, u_1, \ldots, u_k)}{\partial u_i}\right|_{(u_1,\ldots,u_k)=(u_1^*,\ldots,u_k^*)} \leq Nh_0$$

for all $i \in \mathcal{MTSB}$. (EC.36), however, contradicts with (EC.34).

Summarizing the two sub-cases, when $\mathcal{MTSA} \neq \emptyset$, for all $i \notin \mathcal{MTSA}$,

$$Nh_0 < \left.\frac{\partial V_1^{mts}(x_0', x_1^b, ...x_k^b, u_1, \ldots, u_k)}{\partial u_i}\right|_{(u_1,\ldots,u_k)=(u_1^*,\ldots,u_k^*)} \leq (p_i + Nh_i + Nh_0 - c_i - c_0)\bar{F}_i(u_i^*),$$

where the latter inequality is by (EC.19).

Hence, for all $i \notin \mathcal{MTSA}$,

$$\frac{Nh_0}{p_i + Nh_i + Nh_0 - c_i - c_0} < \mathcal{P}\left(\sum_{n=1}^N D_{in} \geq u_i^*\right).$$

Because $c_0 \leq c_0 + c_i \leq p_i \leq p_i + Nh_i$ implies $Nh_0/(p_i + Nh_i + Nh_0 - c_i - c_0) \leq 1$ for all $i = 1, ..., k$,

$$\prod_{i=1}^k \left(\frac{Nh_0}{p_i + Nh_i + Nh_0 - c_i - c_0}\right) \leq \prod_{i \notin \mathcal{MTSA}} \left(\frac{Nh_0}{p_i + Nh_i + Nh_0 - c_i - c_0}\right)$$
$$< \prod_{i \notin \mathcal{MTSA}} \mathcal{P}\left(\sum_{n=1}^N D_{in} \geq u_i^*\right) \leq \mathcal{P}\left(\sum_{i \notin \mathcal{MTSA}} \sum_{n=1}^N D_{in} \geq x_0'\right) \leq \mathcal{P}\left(\sum_{i=1}^k \sum_{n=1}^N D_{in} \geq x_0'\right)$$

where the last two inequalities are due to that $\sum_{n=1}^N D_{in} \geq u_i^*$ for all $i \notin \mathcal{MTSA}$ implies

$$\sum_{i \notin \mathcal{MTSA}} \sum_{n=1}^N D_{in} \geq \sum_{i \notin \mathcal{MTSA}} u_i^* = \sum_{i=1}^k u_i^* = x_0',$$

which further implies $\sum_{i=1}^k \sum_{n=1}^N D_{in} \geq x_0'$. This, however, contradicts with the fact that

$$\prod_{i=1}^k \left(\frac{Nh_0}{p_i + Nh_i + Nh_0 - c_i - c_0}\right) = \bar{F}_0(\tilde{x}_0) \geq \bar{F}_0(x_0') = \mathcal{P}\left(\sum_{i=1}^k \sum_{n=1}^N D_{in} \geq x_0'\right)$$

for $x_0' \in [\tilde{x}_0, \sum_{j=1}^k x_j^b)$. Thus, $\frac{d\pi_b^{mts}(x_0)}{dx_0} \leq 0$ for $x_0 \in [\tilde{x}_0, \sum_{j=1}^k x_j^b)$.

(i.1.d). $R^{mts(m)}(X) \leq R^{q(m)}(X)$ when $x_i \leq E(\sum_{n=1}^N D_{in}^{(m)})$ and $x_0 \leq \sum_{i=1}^k x_i$: Under the same initial inventory X when $x_i \leq E(\sum_{n=1}^N D_{in}^{(m)})$ and $x_0 \leq \sum_{i=1}^k x_i$, say $X^{q(m)}$, the make-to-stock heuristic *mts* is dominated by the heuristic q.

The make-to-stock heuristic allows quota-sharing across periods for the same product, and the heuristic q further allows sharing quotas across products, i.e., filling the demand for a higher margin product can use the quotas for a lower margin product.

In the following, we prove that the profit-to-go under this nested policy with quotas is higher than that under any make-to-stock policy. The intuition is as follows: under the make-to-stock heuristic, if in period n a demand for product i is rejected as product i exhausts its individual quota, no demand for product i is filled in the remaining selling season. Meanwhile, if there is still individual quota for a product $j = i+1, \ldots, k$, the demand for product j is filled. This fulfillment policy is dominated by heuristic q, because under the heuristic q, filling the demand for product

i can use (individual) quotas for product j, and accepting the demand for product i dominates filling the demand for product j: the margin can be improved from $p_j - c_j - c_0$ to $p_i - c_i - c_0$, with a saving of the holding cost (due to $h_j \le h_i$).

Formally, when $x_i \le E(\sum_{n=1}^{N} D_{in}^{(m)})$ and $x_0 \le \sum_{i=1}^{k} x_i$ we use induction to prove

$$V_1^q(X, \sum_{j=1}^{k} u_j, \sum_{j=2}^{k} u_j, \ldots, u_k) \ge V_1^{mts}(X, u_1, u_2, \ldots, u_k).$$

To this end, we re-formulate both $V_1^q(\cdot)$ and $V_1^{mts}(\cdot)$ to explicitly account for the sequence in filling orders of different products. Also, following a same approach as in (i.1.a) of this proof, we reformulate the problem by equivalently accounting for the inventory costs and the salvage values at the beginning of the season.

For heuristic q, we rewrite

$$\begin{aligned} &V_1^q(X, \sum_{j=1}^{k} u_j, \sum_{j=2}^{k} u_j, \ldots, \sum_{j=k-1}^{k} u_j, u_k) \\ &= -N\sum_{j=0}^{k}(h_j x_j) + \sum_{j=0}^{k}(c_j x_j) + E_{D_{11},\ldots,D_{k1}}[\tilde{J}_1^{q,1}(X, \sum_{j=1}^{k} u_j, \sum_{j=2}^{k} u_j, \ldots, \sum_{j=k-1}^{k} u_j, u_k, D_{11}, \ldots, D_{k1})], \end{aligned}$$

and for $i = 1, \ldots, k$ and $1 \le n \le N$,

$$\begin{aligned} &\tilde{J}_n^{q,i}(X, u_1, \ldots, u_k, d_{in}, \ldots, d_{kn}) \\ &= (p_i + (N-n+1)h_i + (N-n+1)h_0 - c_i - c_0)y_{in}^q + \tilde{J}_n^{q,i+1}(X - y_{in}^q a_i, u_1', \ldots, u_k', d_{i+1,n}, \ldots, d_{kn}), \end{aligned}$$

where $y_{in}^q = x_i \wedge d_{in} \wedge u_i$, $u_j' = u_j - y_{in}^q$ for $j = 1, \ldots, i$, and $u_j' = u_j \wedge u_{j-1}'$ for $j = i+1, \ldots, k$, and for $1 \le n < N$,

$$\tilde{J}_n^{q,k+1}(X, u_1, \ldots, u_k) = E_{D_{1,n+1},\ldots,D_{k,n+1}}[\tilde{J}_{n+1}^{q,1}(X, u_1, \ldots, u_k, D_{1,n+1}, \ldots, D_{k,n+1})].$$

The boundary condition is $\tilde{J}_N^{q,k+1}(X, u_1, \ldots, u_k) = 0$, and $\tilde{J}_{N+1}^{q,i}(X, u_1, \ldots, u_k) = 0$ for $i = 1, ..., k, k+1$.

Similarly, for the mts heuristic we rewrite

$$\begin{aligned} &V_1^{mts}(X, u_1, u_2, \ldots, u_k, d_{1n}, \ldots, d_{kn}) \\ &= -N\sum_{j=0}^{k}(h_j x_j) + \sum_{j=1}^{k}(c_j x_j) + E_{D_{11},\ldots,D_{k1}}[\tilde{J}_1^{mts,1}(X, u_1, u_2, \ldots, u_k, D_{11}, \ldots, D_{k1})], \end{aligned}$$

and for $i = 1, \ldots, k$ and $1 \le n \le N$,

$$\begin{aligned} &\tilde{J}_n^{mts,i}(X, u_1, \ldots, u_i, \ldots, u_k, d_{in}, \ldots, d_{kn}) \\ &= (p_i + (N-n+1)h_i + (N-n+1)h_0 - c_i - c_0)y_{in}^{mts} + \tilde{J}_n^{mts,i+1}(X - y_{in}^{mts} a_i, u_1, \ldots, u_i - y_{in}^{mts}, \ldots, u_k, d_{i+1,n}, \ldots, d_{kn}), \end{aligned}$$

where $y_{in}^{mts} = x_i \wedge d_{in} \wedge u_i$, and for $1 \le n < N$,

$$\tilde{J}_n^{mts,k+1}(X, u_1, \ldots, u_k) = E_{D_{1,n+1},\ldots,D_{k,n+1}}[\tilde{J}_{n+1}^{mts,1}(X, u_1, \ldots, u_k, D_{1,n+1}, \ldots, D_{k,n+1})].$$

The boundary condition is $\tilde{J}_N^{mts,k+1}(X, u_1, \ldots, u_k) = 0$, and $\tilde{J}_{N+1}^{mts,i}(X, u_1, \ldots, u_k) = 0$ for $i = 1, ..., k, k+1$.

We then follow induction to prove that for $n = 1, ..., N$ and $i = 1, ..., k+1$,

$$\tilde{J}_n^{q,i}(X, \sum_{j=1}^{k} u_j, \sum_{j=2}^{k} u_j, \ldots, u_k, d_{1n}, \ldots, d_{kn}) \ge \tilde{J}_n^{mts,i}(X, u_1, \ldots, u_k, d_{1n}, \ldots, d_{kn}). \quad \text{(EC.37)}$$

Note that, when $n = N+1$, (EC.37) holds trivially with all the product indexes $i = 1, ..., k+1$. Suppose (EC.37) holds for period $n+1$ with any product index $i = 1, ..., k+1$. We shall prove that (EC.37) holds for period n and all the product indexes $i = 1, ..., k+1$. To establish it, we first note three useful properties for the profit function $\tilde{J}_n^{mts,i}(X, u_1, \ldots, u_k, d_{in}, \ldots, d_{kn})$, as specified in equations (EC.38), (EC.39), and (EC.40) below.

Three preliminary properties:

When $x_i \le E(\sum_{n=1}^{N} D_{in}^{(m)})$ and $x_0 \le \sum_{i=1}^{k} x_i$, Lemma EC.9(i) implies that it is without loss of optimality to consider $u_j \le x_j$ for $j = 1, \ldots, k$. Note that given $u_j \le x_j$ for $j = 1, \ldots, k$ and $\sum_{j=1}^{k} u_j \le x_0$, in the make-to-stock policy, for $i = 1, \ldots, k$, $j > i$, and $\epsilon \in [0, u_j]$, we have

$$\begin{aligned}
&\tilde{J}_n^{mts,i+1}(X - \epsilon a_j, u_1, \ldots, u_{j-1}, u_j - \epsilon, u_{j+1}, \ldots, u_k, d_{i+1,n}, \ldots, d_{kn}) \\
=&\tilde{J}_n^{mts,i+1}(x_0 - \epsilon, x_1, \ldots, x_{j-1}, x_j - \epsilon, x_{j+1}, \ldots, x_k, u_1, \ldots, u_{j-1}, u_j - \epsilon, u_{j+1}, \ldots, u_k, d_{i+1,n}, \ldots, d_{kn}) \\
=&\tilde{J}_n^{mts,i+1}(x_0 - \epsilon, x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_k, u_1, \ldots, u_{j-1}, u_j - \epsilon, u_{j+1}, \ldots, u_k, d_{i+1,n}, \ldots, d_{kn}). \quad \text{(EC.38)}
\end{aligned}$$

The reason for (EC.38) is that adding ϵ units of component j does not increase the sales of product j due to the shortage of quotas characterized by $u_j - \epsilon \le x_j - \epsilon \le x_j$. Furthermore, regarding the effect of adding ϵ units of component j upon the salvage value and holding costs, the component holding costs from period n to period N have already been taken into account at the beginning of the season. So does the component salvage value. Thus, the second equality of (EC.38) holds.

Similarly, when $u_i = 0$, adding $\epsilon \in [0, x_i]$ units of component i does not increase the sales of product i. As such, with $j > i$, we have

$$\begin{aligned}
&\tilde{J}_n^{mts,i+1}(X - \epsilon a_i, u_1, \ldots, u_{j-1}, u_j - \epsilon, u_{j+1}, \ldots, u_k, d_{i+1,n}, \ldots, d_{kn}) \\
=&\tilde{J}_n^{mts,i+1}(x_0 - \epsilon, x_1, \ldots, x_{i-1}, x_i - \epsilon, x_{i+1}, \ldots, x_k, u_1, \ldots, u_{j-1}, u_j - \epsilon, u_{j+1}, \ldots, u_k, d_{i+1,n}, \ldots, d_{kn}) \\
=&\tilde{J}_n^{mts,i+1}(x_0 - \epsilon, x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_k, u_1, \ldots, u_{j-1}, u_j - \epsilon, u_{j+1}, \ldots, u_k, d_{i+1,n}, \ldots, d_{kn}), \quad \text{(EC.39)}
\end{aligned}$$

where the second equality holds because the component salvage value and the component holding costs from period n to period N have already been taken into account at the beginning of the season.

Moreover, in period n under the *mts* policy, after filling the demand for product i, when the leftover quota for product i is $u_i = 0$, there will not be any sales of product i in the future periods $n+1,...,N$. In this case of $u_i = 0$, with $\epsilon \in [0, x_i \wedge u_j]$, we characterize the change of $\tilde{J}_n^{mts,i+1}(X, u_1, \ldots, u_{j-1}, u_j, u_{j+1}, \ldots, u_k, d_{i+1,n}, \ldots, d_{kn})$ by using ϵ units of the sales quotas for product $j > i$:

$$\begin{aligned}
&\tilde{J}_n^{mts,i+1}(X, u_1, \ldots, u_{j-1}, u_j, u_{j+1}, \ldots, u_k, d_{i+1,n}, \ldots, d_{kn}) \\
\le\ & \epsilon(p_j + (N-n+1)h_j + (N-n+1)h_0 - c_j - c_0) \\
&+ \tilde{J}_n^{mts,i+1}(X - \epsilon a_j, u_1, \ldots, u_{j-1}, u_j - \epsilon, u_{j+1}, \ldots, u_k, d_{i+1,n}, \ldots, d_{kn}) \\
\le\ & \epsilon(p_i + (N-n+1)h_i + (N-n+1)h_0 - c_i - c_0) \\
&+ \tilde{J}_n^{mts,i+1}(X - \epsilon a_j, u_1, \ldots, u_{j-1}, u_j - \epsilon, u_{j+1}, \ldots, u_k, d_{i+1,n}, \ldots, d_{kn}), \\
=\ & \epsilon(p_i + (N-n+1)h_i + (N-n+1)h_0 - c_i - c_0) \\
&+ \tilde{J}_n^{mts,i+1}(X - \epsilon a_i, u_1, \ldots, u_{j-1}, u_j - \epsilon, u_{j+1}, \ldots, u_k, d_{i+1,n}, \ldots, d_{kn}),
\end{aligned} \tag{EC.40}$$

where the first inequality holds because, with $u_j \le x_j$ and $\epsilon \le x_i \wedge u_j \le x_i \wedge x_j \wedge u_j$, in period n ϵ units of component 0, ϵ units of component j, and ϵ units of quotas for product j can at most generate a profit as $\epsilon(p_j + (N-n+1)h_j + (N-n+1)h_0 - c_j - c_0)$. The second inequality follows from $h_j \le h_i$. The last equality of (EC.40) follows from

$$\begin{aligned}
&\tilde{J}_n^{mts,i+1}(X - \epsilon a_j, u_1, \ldots, u_{j-1}, u_j - \epsilon, u_{j+1}, \ldots, u_k, d_{i+1,n}, \ldots, d_{kn}) \\
=\ & \tilde{J}_n^{mts,i+1}(x_0 - \epsilon, x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_k, u_1, \ldots, u_{j-1}, u_j - \epsilon, u_{j+1}, \ldots, u_k, d_{i+1,n}, \ldots, d_{kn}) \\
=\ & \tilde{J}_n^{mts,i+1}(X - \epsilon a_i, u_1, \ldots, u_{j-1}, u_j - \epsilon, u_{j+1}, \ldots, u_k, d_{i+1,n}, \ldots, d_{kn}),
\end{aligned}$$

where the first equality is due to (EC.38), and the second equality follows from (EC.39).

The induction for a general period index n:

Suppose that for period $n+1$ and any product index $i = 1, ..., k+1$,

$$\tilde{J}_{n+1}^{q,i}(X, \sum_{j=1}^{k} u_j, \sum_{j=2}^{k} u_j, \ldots, u_k, d_{i,n+1}, \ldots, d_{k,n+1}) \ge \tilde{J}_{n+1}^{mts,i}(X, u_1, \ldots, u_k, d_{i,n+1}, \ldots, d_{k,n+1})$$

holds. Then, for period n, we shall prove (EC.37) for all the product indexes $i = 1, ..., k+1$. First, according to the definitions, we note that

$$\begin{aligned}
&\tilde{J}_n^{q,k+1}(X, \sum_{j=1}^{k} u_j, \sum_{j=2}^{k} u_j, \ldots, u_k) \\
=\ & E_{D_{1,n+1},\ldots,D_{k,n+1}}[\tilde{J}_{n+1}^{q,1}(X, \sum_{j=1}^{k} u_j, \sum_{j=2}^{k} u_j, \ldots, u_k, D_{1,n+1}, \ldots, D_{k,n+1})] \\
\ge\ & E_{D_{1,n+1},\ldots,D_{k,n+1}}[\tilde{J}_{n+1}^{mts,1}(X, u_1, \ldots, u_k, D_{1,n+1}, \ldots, D_{k,n+1})] \\
=\ & \tilde{J}_n^{mts,k+1}(X, u_1, \ldots, u_k).
\end{aligned}$$

This further implies that

$$\tilde{J}_n^{q,k}(X, \sum_{j=1}^k u_j, \sum_{j=2}^k u_j, \ldots, u_k, d_{kn}) \geq \tilde{J}_n^{mts,k}(X, u_1, \ldots, u_k, d_{kn})$$

since $y_{kn}^q = y_{kn}^{mts} = x_k \wedge u_k \wedge d_{kn}$.

Next, we follow induction on the product index. Given n, suppose that for $i = 1, ..., k-1$,

$$\tilde{J}_n^{q,i+1}(X, \sum_{j=1}^k u_j, \sum_{j=2}^k u_j, \ldots, u_k, d_{i+1,n}, \ldots, d_{kn}) \geq \tilde{J}_n^{mts,i+1}(X, u_1, \ldots, u_k, d_{i+1,n}, \ldots, d_{kn}), \quad \text{(EC.41)}$$

we shall prove that

$$\tilde{L}_2 := \tilde{J}_n^{q,i}(X, \sum_{j=1}^k u_j, \sum_{j=2}^k u_j, \ldots, u_k, d_{in}, \ldots, d_{kn}) - \tilde{J}_n^{mts,i}(X, u_1, \ldots, u_k, d_{in}, \ldots, d_{kn}) \geq 0.$$

Recall that when $x_i \leq E(\sum_{n=1}^N D_{in}^{(m)})$ and $x_0 \leq \sum_{i=1}^k x_i$, without loss of optimality to consider $u_j \leq x_j$ for all $j = 1, ..., k$. It suffices to consider two cases:

- $d_{in} \leq x_i \wedge u_i = u_i$: $y_{in}^q = y_{in}^{mts} = d_{in}$, so the remaining quotas under the nested policy with quotas can be characterized by

$$(\sum_{j=1}^k u_j - y_{in}^q, \sum_{j=2}^k u_j - y_{in}^q, \ldots, \sum_{j=i-1}^k u_j - y_{in}^q, \sum_{j=i}^k u_j - y_{in}^q, \sum_{j=i+1}^k u_j, \sum_{j=i+2}^k u_j, \ldots, u_k).$$

The number of available quotas that product $1, \ldots, i$ can use is reduced by y_{in}^q, while the available quota for product $i+1, \ldots, k$ remains the same. On the other hand, those remaining individual quotas under the make-to-stock policy can be characterized by $(u_1, \ldots, u_{i-1}, u_i - y_{in}^{mts}, u_{i+1}, u_{i+2}, \ldots, u_k)$. As such, because of the induction hypothesis (EC.41),

$$\begin{aligned}\tilde{L}_2 = {}& \tilde{J}_n^{q,i+1}(X - y_{in}^q a_i, \sum_{j=1}^k u_j - y_{in}^q, \ldots, \sum_{j=i-1}^k u_j - y_{in}^q, \sum_{j=i}^k u_j - y_{in}^q, \sum_{j=i+1}^k u_j, \sum_{j=i+2}^k u_j, \ldots, u_k, d_{i+1,n}, \ldots, d_{kn}) \\ & - \tilde{J}_n^{mts,i+1}(X - y_{in}^{mts} a_i, u_1, \ldots, u_{i-1}, u_i - y_{in}^{mts}, u_{i+1}, u_{i+2}, \ldots, u_k, d_{i+1,n}, \ldots, d_{kn}) \geq 0.\end{aligned}$$

- $d_{in} > x_i \wedge u_i = u_i$: If $u_i = x_i$ or $u_{i+1} = ... = u_k = 0$, then $y_{in}^{mts} = y_{in}^q = u_i$, and $\tilde{L}_2 \geq 0$ can be proved similarly as in the case $d_{in} \leq x_i \wedge u_i = u_i$. Now, consider the case that $u_i < x_i$ and at least one of u_j, $j = i+1, ..., k$, is positive, i.e., $\sum_{j=i+1}^k u_j > 0$. In such a case, $y_{in}^{mts} = u_i < y_{in}^q \leq d_{in} \wedge (\sum_{j=i}^k u_j)$. Then, there exists a critical value $\tilde{j} \in \{i+1, \ldots, k\}$ such that $\sum_{j=i}^{\tilde{j}-1} u_j < y_{in}^q \leq \sum_{j=i}^{\tilde{j}} u_j$. That is, fulfilling the demand for product i uses all of the individual quotas for product $i, i+1, \ldots, \tilde{j}-1$ as well as a portion of the individual quotas for product $\tilde{j}$. Accordingly, the remaining available quotas under the static nested policy are

$$\begin{aligned}(&\sum_{j=1}^k u_j - y_{in}^q, \sum_{j=2}^k u_j - y_{in}^q, \ldots, \sum_{j=i-1}^k u_j - y_{in}^q, \sum_{j=i}^k u_j - y_{in}^q, \\ & \sum_{j=i}^k u_j - y_{in}^q, \ldots, \sum_{j=i}^k u_j - y_{in}^q, \sum_{j=\tilde{j}+1}^k u_j, \sum_{j=\tilde{j}+2}^k u_j, \ldots, u_k).\end{aligned}$$

The available quota for product $1,\ldots,i$ is reduced by y_{in}^q, while the available quota for product $i+1,\ldots,\tilde{j}$ becomes $\sum_{j=i}^k u_j - y_{in}^q$. The available quota for product $\tilde{j}+1,\ldots,k$ remains the same. On the other hand, those remaining individual quotas under the make-to-stock policy are $(u_1,\ldots,u_{i-1},u_i - y_{in}^{mts},u_{i+1},\ldots,u_k) = (u_1,\ldots,u_{i-1},0,u_{i+1},\ldots,u_k)$. As such,

$$\begin{aligned}\tilde{L}_2 =& (p_i + (N-n+1)h_i + (N-n+1)h_0 - c_i - c_0)(y_{in}^q - y_{in}^{mts})\\ &+ \tilde{J}_n^{q,i+1}(X - y_{in}^q a_i, \sum_{j=1}^k u_j - y_{in}^q, \ldots, \sum_{j=i}^k u_j - y_{in}^q, \sum_{j=i}^k u_j - y_{in}^q, \ldots, \sum_{j=i}^k u_j - y_{in}^q, \sum_{j=\tilde{j}+1}^k u_j, \ldots, u_k, d_{i+1,n}, \ldots, d_{kn})\\ &- \tilde{J}_n^{mts,i+1}(X - y_{in}^{mts} a_i, u_1, \ldots, 0, u_{i+1}, \ldots, u_{\tilde{j}}, u_{\tilde{j}+1}, \ldots, u_k, d_{i+1,n}, \ldots, d_{kn}).\end{aligned}$$

Also,

$$\begin{aligned}&\tilde{J}_n^{mts,i+1}(X - y_{in}^{mts} a_i, u_1, \ldots, u_{i-1}, 0, u_{i+1}, u_{i+2}, \ldots, u_k, d_{i+1,n}, \ldots, d_{kn})\\ =\ & \tilde{J}_n^{mts,i+1}(X - u_i a_i, u_1, \ldots, u_{i-1}, 0, u_{i+1}, u_{i+2}, \ldots, u_k, d_{i+1,n}, \ldots, d_{kn})\\ \le\ & (p_i + (N-n+1)h_i + (N-n+1)h_0 - c_i - c_0)u_{i+1}\\ & +\tilde{J}_n^{mts,i+1}(X - (u_i + u_{i+1})a_i, u_1, \ldots, u_{i-1}, 0, 0, u_{i+2}, \ldots, u_k, d_{i+1,n}, \ldots, d_{kn})\\ \le\ & (p_i + (N-n+1)h_i + (N-n+1)h_0 - c_i - c_0)(u_{i+1} + u_{i+2})\\ & +\tilde{J}_n^{mts,i+1}(X - (u_i + u_{i+1} + u_{i+2})a_i, u_1, \ldots, u_{i-1}, 0, 0, 0, \ldots, u_k, d_{i+1,n}, \ldots, d_{kn})\\ \le\ & \cdots\\ \le\ & (p_i + (N-n+1)h_i + (N-n+1)h_0 - c_i - c_0)\sum_{j=i+1}^{\tilde{j}-1} u_j\\ & +\tilde{J}_n^{mts,i+1}(X - (\sum_{j=i}^{\tilde{j}-1} u_j)a_i, u_1, \ldots, u_{i-1}, 0, \ldots, 0, u_{\tilde{j}}, u_{\tilde{j}+1}, \ldots, u_k, d_{i+1,n}, \ldots, d_{kn})\\ \le\ & (p_i + (N-n+1)h_i + (N-n+1)h_0 - c_i - c_0)(y_{in}^q - u_i)\\ & +\tilde{J}_n^{mts,i+1}(X - y_{in}^q a_i, u_1, \ldots, u_{i-1}, 0, \ldots, 0, u_{\tilde{j}} - (y_{in}^q - \sum_{j=i}^{\tilde{j}-1} u_j), u_{\tilde{j}+1}, \ldots, u_k, d_{i+1,n}, \ldots, d_{kn})\\ =\ & (p_i + (N-n+1)h_i + (N-n+1)h_0 - c_i - c_0)(y_{in}^q - y_{in}^{mts})\\ & +\tilde{J}_n^{mts,i+1}(X - y_{in}^q a_i, u_1, \ldots, u_{i-1}, 0, \ldots, 0, u_{\tilde{j}} - (y_{in}^q - \sum_{j=i}^{\tilde{j}-1} u_j), u_{\tilde{j}+1}, \ldots, u_k, d_{i+1,n}, \ldots, d_{kn})\end{aligned}$$

where the two equalities follow from $y_{in}^{mts} = u_i$ and the inequalities hold due to (EC.40) with $j = i+1, i+2, \ldots, \tilde{j}-1, \tilde{j}$ by setting $\epsilon = u_{i+1}, u_{i+2}, \ldots, u_{\tilde{j}-1}, (y_{in}^q - \sum_{j=i}^{\tilde{j}-1} u_j)$. Here, for the first inequality (EC.40) is applied since $\epsilon = u_{i+1} \le x_i - u_i$ as proved next:

$$u_i + u_{i+1} \le \sum_{j'=i}^{\tilde{j}-1} u_{j'} \le \sum_{j'=i}^{\tilde{j}-1} u_{j'} + (y_{in}^q - \sum_{j'=i}^{\tilde{j}-1} u_{j'}) = y_{in}^q \le x_i,$$

where the first two inequalities follow from the definition of $\tilde{j}$ above. Similarly, for the inequality with $\epsilon = u_j$, $j = i+2, \ldots, \tilde{j}-1$, $\epsilon = u_j \le x_i - \sum_{j'=i}^{j-1} u_{j'}$ holds because

$$\sum_{j'=i}^{j} u_{j'} \le \sum_{j'=i}^{\tilde{j}-1} u_{j'} \le \sum_{j'=i}^{\tilde{j}-1} u_{j'} + (y_{in}^q - \sum_{j'=i}^{\tilde{j}-1} u_{j'}) = y_{in}^q \le x_i.$$

Additionally, for the last inequality with $\epsilon = y_{in}^q - \sum_{j=i}^{\tilde{j}-1} u_j$,

$$\epsilon = y_{in}^q - \sum_{j=i}^{\tilde{j}-1} u_j \leq x_i - \sum_{j=i}^{\tilde{j}-1} u_j.$$

Therefore, we obtain $\tilde{L}_2 \geq 0$ from the induction hypothesis

$$\begin{aligned}
&\tilde{J}_n^{mts,i+1}(X - y_{in}^q a_i, u_1, \ldots, u_{i-1}, 0, \ldots, 0, u_{\tilde{j}} - (y_{in}^q - \sum_{j=i}^{\tilde{j}-1} u_j), u_{\tilde{j}+1}, u_{\tilde{j}+2}, \ldots, u_k, d_{i+1,n}, \ldots, d_{kn}) \\
&\leq \tilde{J}_n^{q,i+1}(X - y_{in}^q a_i, \sum_{j=1}^k u_j - y_{in}^q, \ldots, \sum_{j=i-1}^k u_j - y_{in}^q, \sum_{j=i}^k u_j - y_{in}^q, \ldots, \\
&\qquad \sum_{j=i}^k u_j - y_{in}^q, \sum_{j=i}^k u_j - y_{in}^q, \sum_{j=\tilde{j}+1}^k u_j, \sum_{j=\tilde{j}+2}^k u_j, \ldots, u_k, d_{i+1,n}, \ldots, d_{kn}).
\end{aligned}$$

(i.2) It can be proved that

$$y_{in}^{(m)}(X^{d(m)}) = \begin{cases} 0 & \text{if } \; n > n_i^d \\ E(D_{in}^{(m)}) = mE(D_{in}) & \text{if } \; n \leq n_i^d \end{cases}$$

Thus, $y_{in}^{(m)}(X^{d(m)})/m = y_{in}(X^d)$. By convergence in distribution,

$$\begin{aligned}
&m^{-1} \sum_{n=1}^N \sum_{i=1}^k [(p_i + (N-n+1)h_i + (N-n+1)h_0 - c_i - c_0)(D_{in}^{(m)} \wedge y_{in}^{(m)}(X^{d(m)}))] \\
&= \sum_{n=1}^N \sum_{i=1}^k [(p_i + (N-n+1)h_i + (N-n+1)h_0 - c_i - c_0)(m^{-1} D_{in}^{(m)} \wedge m^{-1} y_{in}^{(m)}(X^{d(m)}))] \\
&\overset{\mathscr{D}}{\to} \sum_{n=1}^N \sum_{i=1}^k [(p_i + (N-n+1)h_i + (N-n+1)h_0 - c_i - c_0)(E(D_{in}) \wedge y_{in}(X^d))]. \qquad \text{(EC.42)}
\end{aligned}$$

In the meanwhile, note that the optimal initial inventory for the deterministic problem is scalable: $X^{d(m)}/m = X^d$. Therefore,

$$\begin{aligned}
&\lim_{m\to\infty} \frac{\pi^{f(m)}(X^{d(m)})}{\pi^{Det(m)}(X^{d(m)})} = \lim_{m\to\infty} \frac{m^{-1}[R^{f(m)}(X^{d(m)}) - \sum_{j=0}^k (c_j x_j^{d(m)})]}{m^{-1}[R^{Det(m)}(X^{d(m)}) - \sum_{j=0}^k (c_j x_j^{d(m)})]} \\
&= \lim_{m\to\infty} \frac{-N\sum_{j=0}^k h_j x_j^{d(m)} + E[m^{-1}\sum_{n=1}^N \sum_{i=1}^k ((p_i + (N-n+1)h_i + (N-n+1)h_0 - c_i - c_0)(D_{in}^{(m)} \wedge y_{in}^{(m)}(X^{d(m)})))]}{-N\sum_{j=0}^k h_j x_j^{d(m)} + \sum_{n=1}^N \sum_{i=1}^k [(p_i + (N-n+1)h_i + (N-n+1)h_0 - c_i - c_0)(E(D_{in}^{(m)}) \wedge y_{in}^{(m)}(X^{d(m)}))]}
\end{aligned}$$

By the convergence in distribution in (EC.42), $\lim_{m\to\infty} \dfrac{\pi^{f(m)}(X^{d(m)})}{\pi^{Det(m)}(X^{d(m)})} = 1$.

(ii) First note that $\frac{\pi^{f(m)}(X^{d(m)})}{\pi^{Det(m)}(X^{d(m)})} \leq \frac{\pi^{q(m)}(X^{q(m)})}{\pi^{(m)}(X^{*(m)})} \leq 1$ in part (i) implies

$$1 - \frac{\pi^{q(m)}(X^{q(m)})}{\pi^{(m)}(X^{*(m)})} \leq \frac{\pi^{Det(m)}(X^{d(m)}) - \pi^{f(m)}(X^{d(m)})}{\pi^{Det(m)}(X^{d(m)})}.$$

Then, it suffices to show

$$\frac{\pi^{Det(m)}(X^{d(m)}) - \pi^{f(m)}(X^{d(m)})}{\pi^{Det(m)}(X^{d(m)})} \leq 1 - O(m^{-1/2})$$

when the demand in the scale-m problem is the sum of m random variables which are independently and identically distributed as the corresponding demand in the scale-1 problem, which implies $\pi^{q(m)}(X^{q(m)})$ converges to $\pi^{(m)}(X^{*(m)})$ with a speed faster than $m^{-1/2}$.

Next, we establish

$$\frac{\pi^{Det(m)}(X^{d(m)}) - \pi^{f(m)}(X^{d(m)})}{\pi^{Det(m)}(X^{d(m)})} \leq 1 - O(m^{-1/2}):$$

because the ATO system under the initial inventory $X^{d(m)}$ is balanced, the allocation problem for k products is decoupled. Then, in this case (of applying the heuristic f with the initial inventory $X^{d(m)}$), in period $n \leq n_i^d$ the sales quantity of product i under heuristic f is at most $y_{in}^{(m)}(X^{d(m)}) = E(D_{in}^{(m)})$ which is the optimal sales quantity of the corresponding deterministic problem (where demand for product i in period n is $E(D_{in}^{(m)})$ and the initial component inventory is $X^{d(m)}$), so the profit earned from product i in period $n \leq n_i^d$ under heuristic f (by accounting for the inventory cost and salvage value at the beginning of the season) is

$$\begin{aligned}
R_{in}^{f(m)} &= (p_i + (N-n+1)h_i + (N-n+1)h_0 - c_i - c_0)E\left(D_{in}^{(m)} \wedge y_{in}^{(m)}(X^{d(m)})\right) \\
&= (p_i + (N-n+1)h_i + (N-n+1)h_0 - c_i - c_0)E\left(D_{in}^{(m)} - (D_{in}^{(m)} - y_{in}^{(m)}(X^{d(m)}))^+\right) \\
&= (p_i + (N-n+1)h_i + (N-n+1)h_0 - c_i - c_0)E(D_{in}^{(m)}) \\
&\quad -(p_i + (N-n+1)h_i + (N-n+1)h_0 - c_i - c_0)E\left((D_{in}^{(m)} - y_{in}^{(m)}(X^{d(m)}))^+\right) \\
&\geq (p_i + (N-n+1)h_i + (N-n+1)h_0 - c_i - c_0)\left(E(D_{in}^{(m)}) - \frac{SD(D_{in}^{(m)})}{2} - \left(ED_{in}^{(m)} - y_{in}^{(m)}(X^{d(m)})\right)^+\right) \\
&= (p_i + (N-n+1)h_i + (N-n+1)h_0 - c_i - c_0)\left(E(D_{in}^{(m)}) - \frac{SD(D_{in}^{(m)})}{2}\right)
\end{aligned}$$

where, with $SD(D_{in}^{(m)}) = \sqrt{m}SD(D_{in}^{(1)}) = \sqrt{m}SD(D_{in})$ as the standard deviation of the random demand for product i in period n in the scale-m problem, the inequality is from

$$\begin{aligned}
E\left[(D_{in}^{(m)} - y)^+\right] &\leq \frac{\sqrt{SD^2(D_{in}^{(m)}) + (y - E(D_{in}^{(m)}))^2} - \left(y - E(D_{in}^{(m)})\right)}{2} \\
&\leq \frac{SD(D_{in}^{(m)})}{2} + \frac{|y - E(D_{in}^{(m)})| - (y - E(D_{in}^{(m)}))}{2} \\
&= \frac{SD(D_{in}^{(m)})}{2} + (E(D_{in}^{(m)}) - y)^+
\end{aligned}$$

by applying $E[(D-y)^+] \leq \frac{\sqrt{SD^2(D) + (y - E(D))^2} - (y - E(D))}{2}$ for a random variable D and any real number y (Gallego and van Ryzin 1997). In addition, we have $R_{in}^{f(m)} = 0$ for $n > n_i^d$.

Meanwhile, the profit earned from product i in period n in the corresponding deterministic problem (by accounting for the inventory cost and salvage value at the beginning of the season) is

$$R_{in}^{Det(m)} = (p_i + (N-n+1)h_i + (N-n+1)h_0 - c_i - c_0)E(D_{in}^{(m)})$$

for $n \le n_i^d$, and $R_{in}^{Det(m)} = 0$ for $n > n_i^d$. Given

$$\begin{aligned}
\pi^{f(m)}(X^{d(m)}) &= \sum_{i=1}^{k}\sum_{n=1}^{n_i^d} R_{in}^{f(m)} + \sum_{j=0}^{k}(c_j x_j^{d(m)}) - N\sum_{j=0}^{k}(h_j x_j^{d(m)}) - \sum_{j=0}^{k}(c_j x_j^{d(m)}) \\
&= \sum_{i=1}^{k}\sum_{n=1}^{N} R_{in}^{f(m)} + \sum_{j=0}^{k}(c_j x_j^{d(m)}) - N\sum_{j=0}^{k}(h_j x_j^{d(m)}) - \sum_{j=0}^{k}(c_j x_j^{d(m)}) \\
&= \sum_{i=1}^{k}\sum_{n=1}^{N} R_{in}^{f(m)} - N\sum_{j=0}^{k}(h_j x_j^{d(m)}) \\
\pi^{Det(m)}(X^{d(m)}) &= \sum_{i=1}^{k}\sum_{n=1}^{n_i^d} R_{in}^{Det(m)} + \sum_{j=0}^{k}(c_j x_j^{d(m)}) - N\sum_{j=0}^{k}(h_j x_j^{d(m)}) - \sum_{j=0}^{k}(c_j x_j^{d(m)}) \\
&= \sum_{i=1}^{k}\sum_{n=1}^{N} R_{in}^{Det(m)} + \sum_{j=0}^{k}(c_j x_j^{d(m)}) - N\sum_{j=0}^{k}(h_j x_j^{d(m)}) - \sum_{j=0}^{k}(c_j x_j^{d(m)}) \\
&= \sum_{i=1}^{k}\sum_{n=1}^{N} R_{in}^{Det(m)} - N\sum_{j=0}^{k}(h_j x_j^{d(m)}),
\end{aligned}$$

we have

$$\begin{aligned}
&\frac{\pi^{Det(m)}(X^{d(m)}) - \pi^{f(m)}(X^{d(m)})}{\pi^{Det(m)}(X^{d(m)})} = \frac{\sum_{i=1}^{k}\sum_{n=1}^{N}[R_{in}^{Det(m)} - R_{in}^{f(m)}]}{\sum_{i=1}^{k}\sum_{n=1}^{N} R_{in}^{Det(m)} - N\sum_{j=0}^{k}(h_j x_j^{d(m)})} \\
\le\ & \frac{\sum_{i=1}^{k}\sum_{n=1}^{n_i^d}[(p_i + (N-n+1)h_i + (N-n+1)h_0 - c_i - c_0)SD(D_{in}^{(m)})/2]}{\sum_{i=1}^{k}\sum_{n=1}^{n_i^d} R_{in}^{Det(m)} - N\sum_{j=0}^{k}(h_j x_j^{d(m)})} \\
=\ & \frac{\sqrt{m}\sum_{i=1}^{k}\sum_{n=1}^{n_i^d}[(p_i + (N-n+1)h_i + (N-n+1)h_0 - c_i - c_0)SD(D_{in})/2]}{m[\sum_{i=1}^{k}\sum_{n=1}^{n_i^d}((p_i + (N-n+1)h_i + (N-n+1)h_0 - c_i - c_0)E(D_{in})) - N\sum_{j=0}^{k}(h_j x_j^d)]} \\
=\ & \frac{\sum_{i=1}^{k}\sum_{n=1}^{n_i^d}[(p_i + (N-n+1)h_i + (N-n+1)h_0 - c_i - c_0)SD(D_{in})/2]}{\sqrt{m}[\sum_{i=1}^{k}\sum_{n=1}^{n_i^d}((p_i + (N-n+1)h_i + (N-n+1)h_0 - c_i - c_0)E(D_{in})) - N\sum_{j=0}^{k}(h_j x_j^d)]}.
\end{aligned}$$

Therefore, $\pi^{f(m)}(X^{d(m)})/\pi^{Det(m)}(X^{d(m)}) \ge 1 - O(m^{-1/2})$. Q.E.D.

D.7. Proofs of Other Results in Section 4

Proof of Proposition 6. As in the proof of Proposition 7, to establish the asymptotical optimality of heuristic o (i.e., the optimal nested policy) with the initial inventory X^u, we follow two steps to prove $\lim_{m\to\infty}\frac{\pi^{o(m)}(X^{u(m)})}{\pi^{(m)}(X^{*(m)})}=1$:

$$\text{(i)}\ \frac{\pi^{f(m)}(X^{d(m)})}{\pi^{Det(m)}(X^{d(m)})}\le\frac{\pi^{o(m)}(X^{u(m)})}{\pi^{(m)}(X^{*(m)})}\le 1,\quad \text{(ii)}\ \lim_{m\to\infty}\frac{\pi^{f(m)}(X^{d(m)})}{\pi^{Det(m)}(X^{d(m)})}=1.$$

where for the balanced inventory profile $X^{d(m)}$, $x_j^{d(m)}=E(D_j^{d(m)})$ for $j=1,...,k$ and $x_0^{d(m)}=\sum_{j=1}^k x_j^{d(m)}$. For the inventory profile $X^{u(m)}$, $x_j^{u(m)}=[E(D_j^{d(m)})]\wedge\bar{x}_j^{(m)}$, $j=1,\dots,k$, and $x_0^{u(m)}=(\sum_{j=1}^k x_j^{b(m)})\wedge\bar{x}_0^{(m)}$. $\pi^{f(m)}(X^{d(m)})$ is the profit-to-go of adopting the heuristic f with the initial inventory $X^{d(m)}$. $\pi^{Det(m)}(X^{d(m)})$ is the profit obtained from the corresponding deterministic problem, with the initial inventory $X^{d(m)}$.

(i) It suffices to show

$$\pi^f(X^d)\le\pi^{mts0}(X^d)\le\pi^{mts0}(X^b)\le\pi^o(X^b)=\pi(X^b)\le\pi(X^u)=\pi^o(X^u)\le\pi(X^*)\le\pi^{Det}(X^d).$$

All the subscripts (m) are omitted for brevity.

$\pi^f(X^d)\le\pi^{mts0}(X^d)$ is from (i.1.a) in the proof of Proposition 7, while $\pi^{mts0}(X^d)\le\pi^{mts0}(X^b)$ is from (i.1.b) in the proof of Proposition 7. $\pi^{mts0}(X^b)\le\pi(X^b)=\pi^o(X^b)$ under $k=2$ is trivial since in a two-product system, the nested policy is optimal. Similarly, $\pi(X^u)\le\pi(X^*)$ is trivial because X^* is the optimal initial inventory under the optimal allocation policy. On the other hand, the proof of $\pi(X^*)\le\pi^{Det}(X^*)\le\pi^{Det}(X^d)$ is exactly the same as in the proof of Proposition 7.

It remains to show $\pi(X^b)\le\pi(X^u)$: first note that regarding these two vectors of the initial inventory, X^b is identical to X^u, except that x_0^u equals to x_0^b bounded by $\bar{x}_0$ from above, i.e., for $j=1,\dots,k$, $x_j^b=E(D_j^d)\wedge\bar{x}_j=x_j^u$, and $x_0^b=\sum_{j=1}^k x_j^b\ge x_0^u=(\sum_{j=1}^k x_j^b)\wedge\bar{x}_0$. As in the proof of Proposition 5, given I_i as a large enough inventory of the dedicated component under which there will be no shortage of dedicated components, the supermodularity in Proposition 4 (ii) implies that $\frac{\partial\pi(x_0,x_1^b,\dots,x_k^b)}{\partial x_0}\le\frac{\partial\pi(x_0,I_1,\dots,I_k)}{\partial x_0}$. Then, the definition of $\bar{x}_0$ in Proposition 5 implies that $\frac{\partial\pi(x_0,I_1,\dots,I_k)}{\partial x_0}\le 0$ for $x_0>\bar{x}_0$. Thus, $\frac{\partial\pi(x_0,x_1^b,\dots,x_k^b)}{\partial x_0}\le 0$ for $x_0>\bar{x}_0$, implying $\pi(X^b)\le\pi(X^u)$.

(ii) The proof is exactly the same as in the proof of Proposition 7. Q.E.D.

Proof of Proposition 8. We first formulate the firm's problem for System G. In period n, for given on-hand inventory $\hat{X}=(\hat{x}_0,\hat{x}_1,\dots,\hat{x}_k)$, the firm's dynamic program is as follows:

$$\hat{R}_n(\hat{X})=E_{\hat{D}_{1n},\dots,\hat{D}_{kn}}[\hat{J}_n(\hat{X},\hat{D}_{1n},...,\hat{D}_{kn})],\tag{EC.43}$$

where

$$\begin{aligned}&\hat{J}_n(\hat{X},\hat{d}_{1n},...,\hat{d}_{kn})\\ =&\max_{\hat{y}_{1n},\dots,\hat{y}_{kn}}\left\{\sum_{i=1}^k\left[\hat{p}_i\hat{y}_{in}-\hat{h}_i(\hat{x}_i-\hat{z}_i\hat{y}_{in})\right]-\hat{h}_0(\hat{x}_0-\sum_{i=1}^k(\hat{z}_i^0\hat{y}_{in}))\right.\end{aligned}$$

$$+\hat{R}_{n+1}(\hat{x}_0 - \sum_{i=1}^{k}(\hat{z}_i^0\hat{y}_{in}), \hat{x}_1 - \hat{z}_1\hat{y}_{1n}, ..., \hat{x}_k - \hat{z}_k\hat{y}_{kn})\Bigg\}$$

$$= \max_{\hat{y}_{1n},...,\hat{y}_{kn}} \Bigg[\sum_{i=1}^{k}((\hat{p}_i + \hat{z}_i\hat{h}_i + \hat{z}_i^0\hat{h}_0)\hat{y}_{in}) - \sum_{j=0}^{k}\hat{h}_j\hat{x}_j + \hat{R}_{n+1}(\hat{x}_0 - \sum_{i=1}^{k}(\hat{z}_i^0\hat{y}_{in}), \hat{x}_1 - \hat{z}_1\hat{y}_{1n}, ..., \hat{x}_k - \hat{z}_k\hat{y}_{kn})\Bigg]$$

$$\text{subject to} \quad \sum_{i=1}^{k}\hat{z}_i^0\hat{y}_{in} \le \hat{x}_0, \quad 0 \le \hat{z}_i\hat{y}_{in} \le \hat{x}_i, \quad \hat{y}_{in} \le \hat{d}_{in}, \quad i = 1, 2..., k,$$

with $\hat{R}_{N+1}(\hat{X}) = \sum_{j=0}^{k}\hat{c}_j\hat{x}_j$.

As defined, one unit of demand for product i in System-G is converted into $\hat{z}_i^0$ orders of one unit of product i in System-S, and thus, an order of one unit of product i in System-S consumes one unit of common component 0. Meanwhile, $\hat{z}_i/\hat{z}_i^0$ unit of dedicated component i in System-G is combined into one unit of component i in System-S so that each unit of demand for product i in System-S consumes one unit of component i.

To illustrate the transformation, consider a three-product system with $(\hat{z}_1^0, \hat{z}_2^0, \hat{z}_3^0) = (4, 2, 1)$ and $(\hat{z}_1, \hat{z}_2, \hat{z}_3) = (8, 4, 2)$ in System-G. Then, one unit of demand for product 1/2/3 in System-G is converted into 4/2/1 units of demand for product 1/2/3 in the equivalent System-S, and accordingly each unit of demand for product 1/2/3 in System-S consumes one unit of common component 0. Meanwhile, 2 units of dedicated component $i = 1, 2, 3$ in System-G is combined into one unit of component i in System-S so that each unit of demand for product i in System-S consumes one unit of component i. An initial inventory $\hat{X} = (\hat{x}_0, \hat{x}_1, \hat{x}_2, \hat{x}_3) = (4, 4, 4, 4)$ in System-G achieves the same profit-to-go (for System-G) as $X = (x_0, x_1, x_2, x_3) = (4, 2, 2, 2)$ in System-S when $d_{in} = \hat{d}_{in}\hat{z}_i^0$.

Now we follow induction to prove equivalence of the two systems. Regarding the salvage value,

$$\begin{aligned}\hat{R}_{N+1}(\hat{x}_0, \hat{x}_1, ..., \hat{x}_k) &= \hat{c}_0\hat{x}_0 + \sum_{j=1}^{k}\hat{c}_j\hat{x}_j = c_0x_0 + \sum_{j=1}^{k}\hat{c}_j\hat{x}_j \\ &= c_0x_0 + \sum_{j=1}^{k}\frac{c_j\hat{z}_j^0}{\hat{z}_j}\frac{x_j\hat{z}_j}{\hat{z}_j^0} = \sum_{j=0}^{k}c_jx_j = R_{N+1}(x_0, x_1, ..., x_k).\end{aligned}$$

Thus, when $x_0 = \hat{x}_0$ and $x_j = \hat{z}_j^0\hat{x}_j/\hat{z}_j$ for $j = 1, \ldots, k$, $\hat{R}_{N+1}(\hat{X}) = R_{N+1}(X)$.

In the last period, in System-G, we have

$$\hat{J}_N(\hat{X}, \hat{d}_{1N}, ..., \hat{d}_{kN})$$

$$= \max_{\hat{y}_{1N},...,\hat{y}_{kN}} \Bigg[\sum_{i=1}^{k}((\hat{p}_i + \hat{z}_i\hat{h}_i + \hat{z}_i^0\hat{h}_0)\hat{y}_{iN}) - \sum_{j=0}^{k}\hat{h}_j\hat{x}_j + \hat{R}_{N+1}(\hat{x}_0 - \sum_{i=1}^{k}(\hat{z}_i^0\hat{y}_{iN}), \hat{x}_1 - \hat{z}_1\hat{y}_{1N}, \ldots, \hat{x}_k - \hat{z}_k\hat{y}_{kN})\Bigg]$$

$$\text{subject to} \quad \sum_{i=1}^{k}\hat{z}_i^0\hat{y}_{iN} \le \hat{x}_0, \quad 0 \le \hat{z}_i\hat{y}_{iN} \le \hat{x}_i, \quad \hat{y}_{iN} \le \hat{d}_{iN}, \quad i = 1, 2..., k.$$

Regarding the constraints, under the state transformation

$$\sum_{i=1}^{k}\hat{z}_i^0\hat{y}_{iN}\le\hat{x}_0\Leftrightarrow\sum_{i=1}^{k}y_{iN}=\sum_{i=1}^{k}\hat{z}_i^0\hat{y}_{iN}\le x_0=\hat{x}_0,$$

and for $i=1,...,k$,

$$\hat{z}_i\hat{y}_{iN}\le\hat{x}_i\Leftrightarrow y_{iN}\le\frac{\hat{z}_i^0\hat{x}_i}{\hat{z}_i}=x_i,$$
$$\hat{y}_{iN}\le\hat{d}_{iN}\Leftrightarrow y_{iN}\le\hat{z}_i^0\hat{d}_{iN}=d_{iN},$$
$$\hat{y}_{iN}\ge 0\Leftrightarrow y_{iN}=\hat{z}_i^0\hat{y}_{iN}\ge 0.$$

This together with

$$\begin{aligned}
&\sum_{i=1}^{k}[(\hat{p}_i+\hat{z}_i\hat{h}_i+\hat{z}_i^0\hat{h}_0)\hat{y}_{iN}]-\sum_{j=0}^{k}\hat{h}_j\hat{x}_j+\hat{R}_{N+1}(\hat{x}_0-\sum_{i=1}^{k}(\hat{z}_i^0\hat{y}_{iN}),\hat{x}_1-\hat{z}_1\hat{y}_{1N},\dots,\hat{x}_k-\hat{z}_k\hat{y}_{kN})\\
=&\sum_{i=1}^{k}[(\hat{z}_i^0p_i+\hat{z}_i^0h_i+\hat{z}_i^0h_0)\frac{y_{iN}}{\hat{z}_i^0}]-h_0x_0-\sum_{j=1}^{k}(\frac{z_jz_j^0h_jx_j}{z_jz_j^0})\\
&+\hat{R}_{N+1}(x_0-\sum_{i=1}^{k}(\hat{z}_i^0\frac{y_{iN}}{\hat{z}_i^0}),\frac{x_1\hat{z}_1}{\hat{z}_1^0}-\frac{\hat{z}_1y_{1N}}{\hat{z}_1^0},\dots,\frac{x_k\hat{z}_k}{\hat{z}_k^0}-\frac{\hat{z}_ky_{kN}}{\hat{z}_k^0})\\
=&\sum_{i=1}^{k}[(p_i+h_i+h_0)y_{iN}]-\sum_{j=0}^{k}h_jx_j+\hat{R}_{N+1}(x_0-\sum_{i=1}^{k}y_{iN},\frac{\hat{z}_1(x_1-y_{1N})}{\hat{z}_1^0},\dots,\frac{\hat{z}_k(x_k-y_{kN})}{\hat{z}_k^0})\\
=&\sum_{i=1}^{k}[(p_i+h_i+h_0)y_{iN}]-\sum_{j=0}^{k}h_jx_j+R_{N+1}(x_0-\sum_{i=1}^{k}y_{iN},x_1-y_{1N},\dots,x_k-y_{kN})
\end{aligned}$$

implies that

$$\begin{aligned}
&\hat{J}_N(\hat{X},\hat{d}_{1N},...,\hat{d}_{kN})\\
=&\max_{\hat{y}_{1N},\dots,\hat{y}_{kN}}\left[\sum_{i=1}^{k}((\hat{p}_i+\hat{z}_i\hat{h}_i+\hat{z}_i^0\hat{h}_0)\hat{y}_{iN})-\sum_{j=0}^{k}\hat{h}_j\hat{x}_j+\hat{R}_{N+1}(\hat{x}_0-\sum_{i=1}^{k}(\hat{z}_i^0\hat{y}_{iN}),\hat{x}_1-\hat{z}_1\hat{y}_{1N},\dots,\hat{x}_k-\hat{z}_k\hat{y}_{kN})\right]\\
&\text{subject to}\quad\sum_{i=1}^{k}\hat{z}_i^0\hat{y}_{iN}\le\hat{x}_0,\quad 0\le\hat{z}_i\hat{y}_{iN}\le\hat{x}_i,\quad\hat{y}_{iN}\le\hat{d}_{iN}\quad i=1,2...,k,
\end{aligned}$$

is the same as

$$\begin{aligned}
&J_N(X,d_{1N},...,d_{kN})\\
=&\max_{y_{1N},\dots,y_{kN}}\left[(\sum_{i=1}^{k}(p_i+h_i+h_0)y_{iN})-\sum_{j=0}^{k}h_jx_j+R_{N+1}(x_0-\sum_{i=1}^{k}y_{iN},x_1-y_{1N},\dots,x_k-y_{kN})\right]\\
&\text{subject to}\quad\sum_{i=1}^{k}y_{iN}\le x_0,\quad 0\le y_{iN}\le x_i\wedge d_{iN},\quad i=1,2...,k.
\end{aligned}$$

Thus, $\hat{R}_N(\hat{X})=E_{\hat{D}_{1N},\dots,\hat{D}_{kN}}[\hat{J}_N(\hat{X},\hat{D}_{1N},...,\hat{D}_{kN})]=E_{D_{1N},\dots,D_{kN}}[J_N(X,D_{1N},...,D_{kN})]=R_N(X)$.

Assume that for $n \le N-1$, $\hat{R}_{n+1}(\hat{X}) = R_{n+1}(X)$, i.e.,

$$\hat{R}_{n+1}(\hat{x}_0, \hat{x}_1, \ldots, \hat{x}_k) = \hat{R}_{n+1}(x_0, \hat{z}_1 x_1/\hat{z}_1^0, \ldots, \hat{z}_k x_k/\hat{z}_k^0) = R_{n+1}(x_0, x_1, \ldots, x_k).$$

Next we will prove $\hat{R}_n(\hat{X}) = R_n(X)$. Recall that the optimal profit-to-go in System-G is

$$\hat{R}_n(\hat{X}) = E_{\hat{D}_{1n},\ldots,\hat{D}_{kn}}[\hat{J}_n(\hat{X}, \hat{D}_{1n}, ..., \hat{D}_{kn})],$$

where

$$\begin{aligned} \hat{J}_n(\hat{X}, \hat{d}_{1n}, ..., \hat{d}_{kn}) = & \max_{\hat{y}_{1n},\ldots,\hat{y}_{kn}} [\sum_{i=1}^{k}((\hat{p}_i + \hat{z}_i \hat{h}_i + \hat{z}_i^0 \hat{h}_0)\hat{y}_{in}) - \sum_{j=0}^{k} \hat{h}_j \hat{x}_j \\ & + \hat{R}_{n+1}(\hat{x}_0 - \sum_{i=1}^{k}(\hat{z}_i^0 \hat{y}_{in}), \hat{x}_1 - \hat{z}_1 \hat{y}_{1n}, \ldots, \hat{x}_k - \hat{z}_k \hat{y}_{kn})] \\ \text{subject to} \quad & \sum_{i=1}^{k} \hat{z}_i^0 \hat{y}_{in} \le \hat{x}_0, \quad 0 \le \hat{z}_i \hat{y}_{in} \le \hat{x}_i, \quad \hat{y}_{in} \le \hat{d}_{in}, \quad i = 1, 2..., k, \end{aligned}$$

with $\hat{R}_{N+1}(\hat{X}) = \sum_{j=0}^{k} \hat{c}_j \hat{x}_j$. Because $\hat{y}_{in} = y_{in}/\hat{z}_i^0$ for $i = 1, ..., k$, and $\hat{X} = (\hat{x}_0, \hat{x}_1, \ldots, \hat{x}_k) = (x_0, \hat{z}_1 x_1/\hat{z}_1^0, \ldots, \hat{z}_k x_k/\hat{z}_k^0)$,

$$\begin{aligned} & \hat{R}_{n+1}(\hat{x}_0 - \sum_{i=1}^{k}(\hat{z}_i^0 \hat{y}_{in}), \hat{x}_1 - \hat{z}_1 \hat{y}_{1n}, \ldots, \hat{x}_k - \hat{z}_k \hat{y}_{kn}) \\ = & \hat{R}_{n+1}(x_0 - \sum_{i=1}^{k} y_{in}, \frac{\hat{z}_1(x_1 - y_{1n})}{\hat{z}_1^0}, \ldots, \frac{\hat{z}_k(x_k - y_{kn})}{\hat{z}_k^0}) \\ = & R_{n+1}(x_0 - \sum_{i=1}^{k} y_{in}, x_1 - y_{1n}, \ldots, x_k - y_{kn}), \end{aligned}$$

where the last equality is because of the induction hypothesis. Then, by substituting this into the expression of $\hat{J}_n(\hat{X}, \hat{d}_{1n}, ..., \hat{d}_{kn})$ and rewriting it, we obtain

$$\begin{aligned} & \hat{J}_n(\hat{X}, \hat{d}_{1n}, ..., \hat{d}_{kn}) \\ = & \max_{y_{1n},\ldots,y_{kn}} [\sum_{i=1}^{k}((p_i + h_i + h_0)y_{in}) - \sum_{j=0}^{k} h_j x_j + R_{n+1}(x_0 - \sum_{i=1}^{k} y_{in}, x_1 - y_{1n}, \ldots, x_k - y_{kn})] \\ & \text{subject to} \quad \sum_{i=1}^{k} y_{in} \le x_0, \quad 0 \le y_{in} \le x_i \wedge d_{in}, \quad i = 1, 2..., k. \end{aligned}$$

Thus, $\hat{J}_n(\hat{X}, \hat{d}_{1n}, ..., \hat{d}_{kn}) = J_n(X, d_{1n}, ..., d_{kn})$. Then, $R_n(X) = \hat{R}_n(\hat{X})$. Q.E.D.

D.8. Proofs of Results in Appendices A and B

Proof of Proposition A.2. Denote the policy as stated in the proposition by policy ϖ_1, and by $\bar{\Pi}^{\varpi_1}$ the long-run average profit achieved by following policy ϖ_1. Clearly, ϖ_1 is feasible.

By (EC.1), under any feasible policy ϖ_2,

$$\bar{\Pi}^{\varpi_2}(X_1^0) = \liminf_{T\to\infty} \frac{\Pi_T^{\varpi_2}(X_1^0)}{T} = \liminf_{T\to\infty} \frac{1}{T} E_{\varpi_2}\left[\sum_{t=1}^{T}\left(R(X_t^{\varpi_2}, Y_t^{\varpi_2}) - \sum_{j=0}^{k} c_j x_{jt}^{\varpi_2}\right) + \sum_{j=0}^{k} c_j x_{j1}^0\right]$$
$$\leq \lim_{T\to\infty} \frac{1}{T}\left[\sum_{t=1}^{T} \pi(X^*) + \sum_{j=0}^{k} c_j x_{j1}^0\right] = \pi(X^*),$$

where the inequality holds because of (EC.2). Furthermore, since $X_1^0 = 0$, $\bar{\Pi}^{\varpi_1}(X_1^0) = \pi(X^*)$. Hence, $\bar{\Pi}^{\varpi_1}(X_1^0) = \bar{\Pi}^* = \pi(X^*)$, which implies the optimality of ϖ_1. Q.E.D.

Proof of Proposition A.3. By (EC.4), given $X_1^0 = 0$, $\Pi_T^q(X_1^0) = T\pi^q(X^q)$. Thus,

$$\bar{\Pi}^q = \liminf_{T\to\infty} \frac{\Pi_T^q(X_1^0)}{T} = \lim_{T\to\infty} \frac{T\pi^q(X^q)}{T} = \pi^q(X^q)$$

The asymptotic optimality of the policy q thus follows from $1 \geq \frac{\bar{\Pi}^q}{\bar{\Pi}^*} = \frac{\pi^q(X^q)}{\pi(X^*)}$ and $\lim_{m\to\infty} \frac{\pi^{q(m)}(X^{q(m)})}{\pi^{(m)}(X^{*(m)})} = 1$ by the single-season analysis. Q.E.D.

Proof of Lemma B.1. From the constraints, we know $L_{in} \geq (d_{in} - x_i)^+$. Since $p_i + (N - n + 1)h_i + (N - n + 1)h_0 - c_i - c_0$ is nonnegative, the optimal L_{in}'s are the lowest L_{in}'s that satisfy both $L_{in} \geq (d_{in} - x_i)^+$ for all $i = 1, ..., k$ and $\sum_{i=1}^k L_{in} \geq \sum_{i=1}^k d_{in} - x_0$. Thus, if

$$\sum_{i=1}^{k} (d_{in} - x_i)^+ \geq \sum_{i=1}^{k} d_{in} - x_0,$$

$L_{in}^* = (d_{in} - x_i)^+$ for all $i = 1, ..., k$. Otherwise, noting that $p_i + (N - n + 1)h_i + (N - n + 1)h_0 - c_i - c_0$ decreases in i, $L_{in}^* = (d_{in} - x_i)^+$ for $i = 1, ..., k - 1$ and

$$L_{kn}^* = (d_{kn} - x_k)^+ + \left[\left(\sum_{i=1}^{k} d_{in} - x_0\right) - \sum_{i=1}^{k} (d_{in} - x_i)^+\right]^+.$$

Q.E.D.

Proof of Lemma B.2. For $i = 1, ..., k - 1$, by (EC.8) through (EC.10),

$$y_{in}^{SP} \leq \min(d_{in}, x_i),\ i = 1, ..., k - 1 \tag{EC.44}$$
$$\min(\min(d_{in}, x_i) - y_{in}^{SP}, x_0 - \sum_{j=1}^{i} y_{jn}^{SP}, x_i - y_{in}^{SP}) = 0,\ i = 2, ..., k - 1 \tag{EC.45}$$

Starting from product 1, we have

$$y_{1n}^{SP} = \min(d_{1n}, x_0, x_1),\ y_{2n}^{SP} = \min(d_{2n}, x_0 - y_{1n}^{SP}, x_2), ...,\ y_{k-1,n}^{SP} = \min(d_{k-1,n}, x_0 - \sum_{i=1}^{k-2} y_{in}^{SP}, x_{k-1}).$$

For product k, (EC.10) implies

$$\min\left((d_{kn} - L_{kn}^*)^+ - y_{kn}^{SP}, x_0 - \sum_{i=1}^{k} y_{in}^{SP}, x_k - y_{kn}^{SP}\right) = 0.$$

By Lemma B.1,

$$L^*_{kn} = (d_{kn} - x_k)^+ + \left[\left(\sum_{i=1}^{k} d_{in} - x_0\right) - \sum_{i=1}^{k}(d_{in} - x_i)^+\right]^+.$$

Consider two cases below:

• If $(\sum_{i=1}^k d_{in} - x_0) - \sum_{i=1}^k (d_{in} - x_i)^+ \le 0$, $L^*_{kn} = (d_{kn} - x_k)^+$ and $(d_{kn} - L^*_{kn})^+ = \min(d_{kn}, x_k)$. Thus,

$$\min\left(\min(d_{kn}, x_k) - y^{SP}_{kn}, x_0 - \sum_{i=1}^{k} y^{SP}_{in}, x_k - y^{SP}_{kn}\right) = 0$$

implies $y^{SP}_{kn} = \min(d_{kn}, x_k, x_0 - \sum_{i=1}^{k-1} y^{SP}_{in})$. As noted above, $x_0 \ge \sum_{i=1}^{k-1} y^{SP}_{in}$. Hence, equivalently, we can write

$$y^{SP}_{kn} = \min(d_{kn}, x_k, x_0 - \sum_{i=1}^{k-1} y^{SP}_{in})^+.$$

• Consider the case of $(\sum_{i=1}^k d_{in} - x_0) - \sum_{i=1}^k (d_{in} - x_i)^+ > 0$.

$$\begin{aligned} L^*_{kn} &= (d_{kn} - x_k)^+ + \left[\left(\sum_{i=1}^{k} d_{in} - x_0\right) - \sum_{i=1}^{k}(d_{in} - x_i)^+\right]^+ \\ &= (d_{kn} - x_k)^+ + \left(\sum_{i=1}^{k} d_{in} - x_0\right) - \sum_{i=1}^{k}(d_{in} - x_i)^+ \\ &= d_{kn} - x_0 + \sum_{i=1}^{k-1} d_{in} - \sum_{i=1}^{k-1}(d_{in} - x_i)^+ \\ &= d_{kn} - x_0 + \sum_{i=1}^{k-1}(x_i \wedge d_{in}), \end{aligned}$$

Thus, $d_{kn} - L^*_{kn} = x_0 - \sum_{i=1}^{k-1} x_i \wedge d_{in}$, and (EC.10) implies

$$\min\left(\left(x_0 - \sum_{i=1}^{k-1}(x_i \wedge d_{in})\right)^+ - y^{SP}_{kn}, x_0 - \sum_{j=1}^{k} y^{SP}_{jn}, x_k - y^{SP}_{kn}\right) = 0.$$

That is,

$$y^{SP}_{kn} = \min\left(\left(x_0 - \sum_{i=1}^{k-1} x_i \wedge d_{in}\right)^+, x_0 - \sum_{j=1}^{k-1} y^{SP}_{jn}, x_k\right) \tag{EC.46}$$

Consider the following two subcases:

- if $\sum_{i=1}^{k-1}(y^{SP}_{in}) = x_0$, $y^{SP}_{kn} = x_0 - \sum_{i=1}^{k-1}(y^{SP}_{in}) = 0 = \min\left(d_{kn}, \left(x_0 - \sum_{j=1}^{k} y^{SP}_{jn}\right)^+, x_k\right)$;

- otherwise, $\sum_{i=1}^{k-1}(y^{SP}_{in}) < x_0$ implies that for product $i = 1, 2, ..., k-1$, $y^{SP}_{in} = x_i \wedge d_{in}$. By (EC.46), $y^{SP}_{kn} = \min\left(x_0 - \sum_{i=1}^{k-1}(x_i \wedge d_{in}), x_k\right)$. Furthermore, the condition $(\sum_i d_{in} - x_0) - \sum_i (d_{in} - x_i)^+ > 0$ implies

$$d_{kn} > x_0 - \sum_{i=1}^{k-1} d_{in} + \sum_{i}(d_{in} - x_i)^+ = x_0 - \sum_{i=1}^{k-1}(x_i \wedge d_{in}) + (d_{kn} - x_k)^+ \ge x_0 - \sum_{i=1}^{k-1}(x_i \wedge d_{in}).$$

Thus, $y_{kn}^{SP} = \min\left(x_0 - \sum_{i=1}^{k-1}(x_i \wedge d_{in}), x_k\right) = \min\left(d_{kn}, \left(x_0 - \sum_{i=1}^{k-1} y_{in}^{SP}\right)^+, x_k\right)$.

Q.E.D.

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