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Electronic companion to *Post-trade netting and contagion*

Luitgard Anna Maria Veraart, Yuliang Zhang
Department of Mathematics, London School of Economics and Political Science, Houghton Street, London WC2A 2AE, UK, l.veraart@lse.ac.uk, y.zhang300@lse.ac.uk

Abstract. This electronic companion contains the proofs, background information and additional examples for the paper *Post-trade netting and contagion*.

Key words: Financial networks, Derivatives, Margins, Multilateral netting, Systemic risk, Post-trade risk reduction

Appendix

A. Proofs

A.1. Proofs for Section 2

COROLLARY A.1. *Let $(C, \mathcal{P}, C+R)$ be a rebalancing exercise, then $(\psi C, \mathcal{P}, \psi(C+R))$ and $(\psi C^{bi}, \mathcal{P}, \psi(C+R)^{bi})$ are PTN-exercises for all $\psi \geq 0$.*

Proof of Corollary A.1 Let $L = \psi C$ and $L^{\mathcal{P}} = \psi C^{\mathcal{P}} = \psi(C+R)$. Then,

$$\sum_{j\in\mathcal{P}}(L^{\mathcal{P}}_{ji} - L^{\mathcal{P}}_{ij}) = \psi\sum_{j\in\mathcal{P}}(C_{ji}+R_{ji}-C_{ij}-R_{ij}) = \psi\sum_{j\in\mathcal{P}}(C_{ji}-C_{ij}) + \psi\underbrace{\sum_{j\in\mathcal{P}}(R_{ji}-R_{ij})}_{=0(\text{by}(7))} = \sum_{j\in\mathcal{P}}(L_{ji}-L_{ij}).$$

Similarly, for the bilaterally netted case we set $L = \psi C^{bi}$ and $L^{\mathcal{P}} = \psi(C^{\mathcal{P}})^{bi} = \psi(C+R)^{bi}$ and then

$$\begin{aligned}
\sum_{j\in\mathcal{P}}(L^{\mathcal{P}}_{ji} - L^{\mathcal{P}}_{ij}) &= \psi \sum_{j\in\mathcal{P}}\Big((C^{\mathcal{P}})^{bi}_{ji} - (C^{\mathcal{P}})^{bi}_{ij}\Big) = \psi\left(\sum_{j\in\mathcal{P}:C^{\mathcal{P}}_{ji}\geq C^{\mathcal{P}}_{ij}}(C^{\mathcal{P}}_{ji} - C^{\mathcal{P}}_{ij}) - \sum_{j\in\mathcal{P}:C^{\mathcal{P}}_{ji}< C^{\mathcal{P}}_{ij}}(C^{\mathcal{P}}_{ij} - C^{\mathcal{P}}_{ji})\right)\\
&= \psi\sum_{j\in\mathcal{P}}(C^{\mathcal{P}}_{ji} - C^{\mathcal{P}}_{ij}) = \psi\left(\sum_{j\in\mathcal{P}}(C_{ji}-C_{ij}) + \sum_{j\in\mathcal{P}}(R_{ji}-R_{ij})\right)\\
&= \psi\sum_{j\in\mathcal{P}}(C_{ji}-C_{ij}) = \psi\left(\sum_{j\in\mathcal{P}:C_{ji}\geq C_{ij}}(C_{ji}-C_{ij}) - \sum_{j\in\mathcal{P}:C_{ji}<C_{ij}}(C_{ij}-C_{ji})\right)\\
&= \psi\sum_{j\in\mathcal{P}}(C^{bi}_{ji} - C^{bi}_{ij}) = \sum_{j\in\mathcal{P}}(L_{ji}-L_{ij}),
\end{aligned}$$

where the third line again uses (7). □

Proof of Theorem 1 1. First, we show that the K-compression-rebalancing-parity-matrix R is indeed a rebalancing matrix. By construction, we have $R_{ij} \geq 0\ \forall i,j \in \mathcal{N}$. Next, we show that R satisfies

$$\sum_{j\in\mathcal{P}} R_{ji} = \sum_{j\in\mathcal{P}} R_{ij} \quad \forall i \in \mathcal{P}.$$

By the definition of R it holds for all $i, j \in \mathcal{N}$ that

$$\begin{aligned}
R_{ij} &= \max\{0, (K_{ij}-K_{ji}) - (C_{ij}-C_{ji})\},\\
R_{ji} &= \max\{0, (K_{ji}-K_{ij}) - (C_{ji}-C_{ij})]\} = \max\{0, -[(K_{ij}-K_{ji}) - (C_{ij}-C_{ji})]\}.
\end{aligned}$$

Therefore,

$$R_{ij} = r^{+}_{ij} = \max\{0, r_{ij}\}, \tag{A.1}$$

$$R_{ji} = r^{-}_{ij} = \max\{0, -r_{ij}\}, \tag{A.2}$$

$$r_{ij} = (K_{ij}-K_{ji}) - (C_{ij}-C_{ji}). \tag{A.3}$$

Hence, for all $i \in \mathcal{P}$,

$$\sum_{j\in\mathcal{P}}(R_{ij}-R_{ji}) = \sum_{j\in\mathcal{P}}(r^{+}_{ij} - r^{-}_{ij}) = \sum_{j\in\mathcal{P}} r_{ij} = \sum_{j\in\mathcal{P}}(K_{ij}-K_{ji}) - \sum_{j\in\mathcal{P}}(C_{ij}-C_{ji}) = 0,$$

where the last equality follows from (9). Also, note that for all $(i,j)\notin\mathcal{P}\times\mathcal{P}$ it holds that $R_{ij} = R_{ji} = 0$ since $r_{ij} = 0$.

The statement that $(C+R)^{bi}_{ij} = K^{bi}_{ij}$ follows directly from the definition of R. In particular, for all $(i,j) \in \mathcal{N} \times \mathcal{N}$

$$\begin{aligned}(C+R)^{bi}_{ij} &= \max\{0, C_{ij} + R_{ij} - C_{ji} - R_{ji}\} = \max\{0, (C_{ij} - C_{ji}) + (R_{ij} - R_{ji})\} \\ &= \max\{0, (C_{ij} - C_{ji}) + r_{ij}\} = \max\{0, (C_{ij} - C_{ji}) + (K_{ij} - K_{ji}) - (C_{ij} - C_{ji})\} \\ &= \max\{0, K_{ij} - K_{ji}\} = K^{bi}_{ij}.\end{aligned}$$

Finally, let K be a super-conservative compression matrix, i.e., $0 \le K_{ij} \le C^{bi}_{ij}\ \forall i, j \in \mathcal{N}$. From (12) we obtain that the corresponding K-compression-rebalancing-parity-matrix R satisfies

$$(C^{\mathcal{P}})^{bi}_{ij} = (C+R)^{bi}_{ij} = K^{bi}_{ij} \le C^{bi}_{ij} \quad \forall i, j \in \mathcal{N}$$

and hence, R is a net-conservative rebalancing matrix.

2. Let R be a rebalancing matrix. We show that $K = (C+R)^{bi}$ is a compression matrix. K is clearly nonnegative. We only need to check that for all $i \in \mathcal{P}$,

$$\sum_{j \in \mathcal{P}} (K_{ji} - K_{ij}) = \sum_{j \in \mathcal{P}} (C_{ji} - C_{ij}).$$

Let $i \in \mathcal{P}$. Then,

$$\begin{aligned}\sum_{j \in \mathcal{P}} (K_{ji} - K_{ij}) &= \sum_{j \in \mathcal{P}} \left((C+R)^{bi}_{ji} - (C+R)^{bi}_{ij} \right) \\ &= \sum_{j \in \mathcal{P}} \left(\max\{0, C_{ji} + R_{ji} - C_{ij} - R_{ij}\} - \max\{0, C_{ij} + R_{ij} - C_{ji} - R_{ji}\} \right) \\ &= \sum_{j \in \mathcal{P}} (C_{ji} + R_{ji} - C_{ij} - R_{ij}) \\ &= \underbrace{\sum_{j \in \mathcal{P}} (R_{ji} - R_{ij})}_{=0 \text{(by (7))}} + \sum_{j \in \mathcal{P}} (C_{ji} - C_{ij}) \\ &= \sum_{j \in \mathcal{P}} (C_{ji} - C_{ij}).\end{aligned}$$

If R is a net-conservative rebalancing matrix, then it follows from above that $K = (C+R)^{bi}$ is a compression matrix. Furthermore, since R is a net-conservative rebalancing matrix we obtain that

$$K_{ij} = (C+R)^{bi}_{ij} \le C^{bi}_{ij} \quad \forall i, j \in \mathcal{P}.$$

Hence, K is a super-conservative compression matrix.

3. Let K^* be a solution to the compression optimisation problem. We show that the K^*-compression-rebalancing-parity-matrix denoted by R^* is a solution to the rebalancing optimisation problem.

We denote the set of feasible points for the compression optimisation problem by

$$F^{\text{Comp}} = \{K \in [0,\infty)^{N\times N} \mid K \text{ satisfies (9)}\}.$$

Hence, it holds that

$$\min_{K\in F^{\text{Comp}}} \sum_{i\in\mathcal{P}}\sum_{j\in\mathcal{P}} K_{ij} = \sum_{i\in\mathcal{P}}\sum_{j\in\mathcal{P}} K^*_{ij} = \sum_{i\in\mathcal{P}}\sum_{j\in\mathcal{P}} (K^*)^{bi}_{ij} = \sum_{i\in\mathcal{P}}\sum_{j\in\mathcal{P}} (C+R^*)^{bi}_{ij},$$

where the last equality follows from part 1. of this theorem and the second equality follows from the fact that if K^* solves the compression optimisation problem, then $K^*_{ij} = (K^*)^{bi}_{ij}$ for all $i, j \in \mathcal{P}$. Suppose this does not hold, then $K^*_{ij}K^*_{ji} > 0$ for some fixed $i, j \in \mathcal{P}$. We define a new matrix $\hat{K} \in [0,\infty)^{N\times N}$ by setting $\hat{K}_{\nu\mu} = K^*_{\nu\mu}$ for all $(\nu,\mu) \in \mathcal{P}\times\mathcal{P} \setminus \{(i,j),(j,i)\}$ and we set $\hat{K}_{ij} = K^{bi}_{ij}$ and $\hat{K}_{ji} = K^{bi}_{ji}$. It follows that $\hat{K} \in F^{\text{Comp}}$ and $\sum_{i,j\in\mathcal{P}} \hat{K}_{ij} < \sum_{i,j\in\mathcal{P}} K^*_{ij}$, so K^* is not a solution to the compression optimisation problem. Therefore, we must have $K^*_{ij} = (K^*)^{bi}_{ij}$ for all $i, j \in \mathcal{P}$.

From part 1. of this theorem we know that R^* is a rebalancing matrix and hence a feasible point of the rebalancing optimisation problem. We need to show that

$$\min_{R\in F^{\text{Rebal}}} \sum_{i\in\mathcal{P}}\sum_{j\in\mathcal{P}} (C+R)^{bi}_{ij} = \sum_{i\in\mathcal{P}}\sum_{j\in\mathcal{P}} (C+R^*)^{bi}_{ij}, \tag{A.4}$$

where

$$F^{\text{Rebal}} = \{R \in [0,\infty)^{N\times N} \mid R \text{ satisfies (7) }\}$$

is the set of feasible points for the rebalancing optimisation problem. We prove (A.4) by contradiction. Suppose there exists an $\tilde{R} \in F^{\text{Rebal}}$ with

$$\sum_{i\in\mathcal{P}}\sum_{j\in\mathcal{P}} (C+\tilde{R})^{bi}_{ij} < \sum_{i\in\mathcal{P}}\sum_{j\in\mathcal{P}} (C+R^*)^{bi}_{ij}.$$

Then, $\tilde{K} = (C+\tilde{R})^{bi} \in F^{\text{Comp}}$ by part 2. of this theorem. Furthermore,

$$\sum_{i\in\mathcal{P}}\sum_{i\in\mathcal{P}} \tilde{K}_{ij} = \sum_{i\in\mathcal{P}}\sum_{j\in\mathcal{P}} (C+\tilde{R})^{bi}_{ij} < \sum_{i\in\mathcal{P}}\sum_{j\in\mathcal{P}} (C+R^*)^{bi}_{ij} = \min_{K\in F} \sum_{i\in\mathcal{P}}\sum_{j\in\mathcal{P}} K_{ij}$$

which is a contradiction. Hence, (A.4) holds.

It remains to show the statement for the super-conservative case. Let K^* be a solution to the super-conservative compression optimisation problem, then by part 1. of this theorem, the K^*-compression-rebalancing-parity-matrix R^* is a net-conservative rebalancing matrix. We can use exactly the same argument as before together with the feasible sets for the super-conservative compression optimisation problem and the net-conservative rebalancing optimisation problem given by

$$\begin{aligned} F^{\text{SC Comp}} &= \{K \in [0,\infty)^{N\times N} \mid K \text{ satisfies (9) and (10)}\},\\ F^{\text{NC Rebal}} &= \{R \in [0,\infty)^{N\times N} \mid R \text{ satisfies (7) and (8)}\}. \end{aligned} \tag{A.5}$$

Then, it follows that R^* is a solution to the net-conservative rebalancing optimisation problem.

4. Let R^* be a solution to the rebalancing optimisation problem. Then, from part 2. of this theorem $K^* = (C+R^*)^{bi}$ is a compression matrix. Furthermore,

$$\min_{R\in F^{\text{Rebal}}} \sum_{i\in\mathcal{P}}\sum_{j\in\mathcal{P}} (C+R)^{bi}_{ij} = \sum_{i\in\mathcal{P}}\sum_{j\in\mathcal{P}} (C+R^*)^{bi}_{ij} = \sum_{i\in\mathcal{P}}\sum_{j\in\mathcal{P}} K^*_{ij} = \sum_{i\in\mathcal{P}}\sum_{j\in\mathcal{P}} (K^*)^{bi}_{ij}.$$

We show that

$$\min_{K\in F^{\text{Comp}}} \sum_{i\in\mathcal{P}}\sum_{j\in\mathcal{P}} K_{ij} = \sum_{i\in\mathcal{P}}\sum_{j\in\mathcal{P}} K^*_{ij}. \tag{A.6}$$

We prove this by contradiction. Suppose there exists a $\tilde{K} \in F^{\text{Comp}}$ such that

$$\sum_{i\in\mathcal{P}}\sum_{j\in\mathcal{P}} \tilde{K}_{ij} < \sum_{i\in\mathcal{P}}\sum_{j\in\mathcal{P}} K^*_{ij}.$$

Then, the corresponding $\tilde{K}$-compression-rebalancing-parity-matrix $\tilde{R} \in F^{\text{Rebal}}$ satisfies by part 1. of this theorem

$$\sum_{i\in\mathcal{P}}\sum_{j\in\mathcal{P}} (C+\tilde{R})^{bi}_{ij} = \sum_{i\in\mathcal{P}}\sum_{j\in\mathcal{P}} \tilde{K}^{bi}_{ij} \le \sum_{i\in\mathcal{P}}\sum_{j\in\mathcal{P}} \tilde{K}_{ij} < \sum_{i\in\mathcal{P}}\sum_{j\in\mathcal{P}} K^*_{ij} = \min_{R\in F^{\text{Rebal}}} \sum_{i\in\mathcal{P}}\sum_{j\in\mathcal{P}} (C+R)^{bi}_{ij}.$$

This is a contradiction and hence such a $\tilde{K}$ does not exist and (A.6) holds.

Now, let R^* be a solution to the net-conservative rebalancing optimisation problem. Then, from part 2. of this theorem $K^* = (C+R^*)^{bi}$ is a super-conservative compression matrix. We can then use the same argument as for the general case using the feasible sets for the net- and super-conservative case $F^{\text{SC Comp}}$ and $F^{\text{NC Rebal}}$ defined in (A.5).

□

A.2. Proofs for Section 4

A.2.1. Additional properties of a PTN-exercise We use the following lemmas to prove the main results in Section 4.

LEMMA A.1. *Let $(L, \mathcal{P}, L^{\mathcal{P}})$ be a PTN-exercise. Then,*

1. $\bar{L}_i = \bar{L}_i^{\mathcal{P}}$ *for all* $i \in \mathcal{N} \setminus \mathcal{P}$*;*
2. *for all* $i \in \mathcal{N}$*,*

$$\sum_{j\in\mathcal{P}} L_{ji} - \bar{L}_i = \sum_{j\in\mathcal{P}} L_{ji}^{\mathcal{P}} - \bar{L}_i^{\mathcal{P}}. \tag{A.7}$$

Proof of Lemma A.1 1. This is a direct consequence of Definition 1.

2. This also follows directly from Definition 1, since $\forall i \in \mathcal{N}$

$$\begin{aligned} &\sum_{j\in\mathcal{P}} (L_{ji} - L_{ij}) = \sum_{j\in\mathcal{P}} (L_{ji}^{\mathcal{P}} - L_{ij}^{\mathcal{P}}) \\ \Leftrightarrow &\sum_{j\in\mathcal{P}} L_{ji} - \bar{L}_i + \sum_{j\in\mathcal{N}\setminus\mathcal{P}} L_{ji} = \sum_{j\in\mathcal{P}} L_{ji}^{\mathcal{P}} - \bar{L}_i^{\mathcal{P}} + \sum_{j\in\mathcal{N}\setminus\mathcal{P}} \underbrace{L_{ji}^{\mathcal{P}}}_{=L_{ji}} \\ \Leftrightarrow &\sum_{j\in\mathcal{P}} L_{ji} - \bar{L}_i = \sum_{j\in\mathcal{P}} L_{ji}^{\mathcal{P}} - \bar{L}_i^{\mathcal{P}}. \end{aligned}$$

□

LEMMA A.2. *Let $(L, \mathcal{P}, L^{\mathcal{P}})$ be a PTN-exercise. Set*

$$\mathcal{M} = \{i \in \mathcal{N} \mid \bar{L}_i > 0\}, \qquad \mathcal{M}^{\mathcal{P}} = \{i \in \mathcal{N} \mid \bar{L}_i^{\mathcal{P}} > 0\}.$$

Let $j \in \mathcal{M} \setminus \mathcal{M}^{\mathcal{P}}$. Then, $j \in \mathcal{P}$ and $L_{ji} = 0 \; \forall i \in \mathcal{N} \setminus \mathcal{P}$.

Proof of Lemma A.2 Let $j \in \mathcal{M} \setminus \mathcal{M}^{\mathcal{P}}$. It follows that $\bar{L}_j > 0$ and $\bar{L}_j^{\mathcal{P}} = 0$. Therefore, $\bar{L}_j \neq \bar{L}_j^{\mathcal{P}}$, which implies that $j \in \mathcal{P}$ by Lemma A.1.

Now suppose there exists an $i \in \mathcal{N} \setminus \mathcal{P}$ such that $L_{ji} > 0$. Then $\bar{L}_j^{\mathcal{P}} = \sum_{k\in\mathcal{N}} L_{jk}^{\mathcal{P}} \geq L_{ji} > 0$, which contradicts the assumption that $j \notin \mathcal{M}^{\mathcal{P}}$. □

A.2.2. Proofs of the main results in Section 4 We first introduce some additional notation useful for later proofs.

LEMMA A.3. *For all $i \in \mathcal{N}$, let $E_i^{(0)}$ and $E_i^{\mathcal{P}(0)}$ be the initial equities defined in* (15)*. For $n \in \mathbb{N}$, we define two sequences recursively by*

$$\begin{aligned} E^{(n)} &= \Phi\left(E^{(n-1)}\right), \\ E^{\mathcal{P}(n)} &= \Phi^{\mathcal{P}}\left(E^{\mathcal{P}(n-1)}\right), \end{aligned} \tag{A.8}$$

where the functions Φ and $\Phi^{\mathcal{P}}$ are defined in (13) *and* (14)*, respectively. Then*

1. $E_i^{(0)} = E_i^{\mathcal{P}(0)} \quad \forall i \in \mathcal{N}$;

2. *the sequences* $\left(E^{(n)}\right)$ *and* $\left(E^{\mathcal{P}(n)}\right)$ *are non-increasing, i.e., for all* $i \in \mathcal{N}$ *and for all* $n \in \mathbb{N}_0$*, it holds that*

$$E_i^{(n)} \geq E_i^{(n+1)}, \qquad E_i^{\mathcal{P}(n)} \geq E_i^{\mathcal{P}(n+1)};$$

3. *for all* $i \in \mathcal{N}$*, the sequences* $\left(E_i^{(n)}\right)$ *and* $\left(E_i^{\mathcal{P}(n)}\right)$ *converge to the greatest fixed points of* Φ *and* $\Phi^{\mathcal{P}}$*, respectively, i.e.,*

$$\lim_{n\to\infty} E_i^{(n)} = E_i^*, \qquad \lim_{n\to\infty} E_i^{\mathcal{P}(n)} = E_i^{\mathcal{P};*}.$$

Proof of Lemma A.3 1. We can rewrite the initial equities as

$$E_i^{(0)} = A_i^b + \sum_{j\in\mathcal{N}\setminus\mathcal{P}} (L_{ji} - L_{ij}) + \sum_{j\in\mathcal{P}} (L_{ji} - L_{ij}),$$
$$E_i^{\mathcal{P}(0)} = A_i^b + \sum_{j\in\mathcal{N}\setminus\mathcal{P}} (L_{ji}^{\mathcal{P}} - L_{ij}^{\mathcal{P}}) + \sum_{j\in\mathcal{P}} (L_{ji}^{\mathcal{P}} - L_{ij}^{\mathcal{P}}) = A_i^b + \sum_{j\in\mathcal{N}\setminus\mathcal{P}} (L_{ji} - L_{ij}) + \sum_{j\in\mathcal{P}} (L_{ji}^{\mathcal{P}} - L_{ij}^{\mathcal{P}}).$$

Finally, it follows directly from Definition 1 that $\sum_{j\in\mathcal{P}}(L_{ji} - L_{ij}) = \sum_{j\in\mathcal{P}}(L_{ji}^{\mathcal{P}} - L_{ij}^{\mathcal{P}})$.

2. We know from Veraart (2020, Lemma A.1) that the functions Φ and $\Phi^{\mathcal{P}}$ are non-decreasing, so the statement follows from Veraart (2020, Theorem 2.6).

3. The convergence of the two sequences defined by (A.8) to the greatest re-evaluated equities in the corresponding network follows from Veraart (2020, Theorem 2.6).

□

Proof of Theorem 2 Let

$$\begin{aligned} \mathcal{M} &= \{i \in \mathcal{N} \mid \bar{L}_i > 0\}, \\ \mathcal{M}^{\mathcal{P}} &= \{i \in \mathcal{N} \mid \bar{L}_i^{\mathcal{P}} > 0\}. \end{aligned} \tag{A.9}$$

1. Recall that for the financial networks $(L, A^b; \mathbb{V})$ and $(L^{\mathcal{P}}, A^b; \mathbb{V})$ we consider the functions

$$\Phi_i(E) = A_i^b + \sum_{j\in\mathcal{M}} L_{ji}\mathbb{V}\left(\frac{E_j + \bar{L}_j}{\bar{L}_j}\right) - \bar{L}_i \quad \forall i \in \mathcal{N},$$
$$\Phi_i^{\mathcal{P}}(E) = A_i^b + \sum_{j\in\mathcal{M}^{\mathcal{P}}} L_{ji}^{\mathcal{P}}\mathbb{V}\left(\frac{E_j + \bar{L}_j^{\mathcal{P}}}{\bar{L}_j^{\mathcal{P}}}\right) - \bar{L}_i^{\mathcal{P}} \quad \forall i \in \mathcal{N}$$

on $\mathcal{E} = [-\bar{L}, A^b + \bar{A} - \bar{L}]$, and $\mathcal{E}^{\mathcal{P}} = [-\bar{L}^{\mathcal{P}}, A^b + \bar{A}^{\mathcal{P}} - \bar{L}^{\mathcal{P}}]$, respectively.

It holds that $\tilde{E} \in \mathcal{E}$. Before we show that $\tilde{E}$ is a fixed point of $\Phi^{\mathcal{P}}$ we show that $\tilde{E} \in \mathcal{E}^{\mathcal{P}}$. First, note that since $L^{\mathcal{P}}$ is a PTN-exercise, the PTN-contraint (1) implies that $\mathcal{E}^{\mathcal{P}} = [-\bar{L}^{\mathcal{P}}, A^b + \bar{A} - \bar{L}]$.

Hence, $\mathcal{E}$ and $\mathcal{E}^{\mathcal{P}}$ have the same upper bound but different lower bounds. To see that $\tilde{E} \in \mathcal{E}^{\mathcal{P}}$, note that for all $i \in \mathcal{N} \setminus \mathcal{P}$ it holds that $\bar{L}_i = \bar{L}_i^{\mathcal{P}}$ and hence the corresponding lower bound for $\tilde{E}_i$ is the same in $\mathcal{E}$ and $\mathcal{E}^{\mathcal{P}}$. Furthermore, for all $i \in \mathcal{P}$ by assumption (16) $\tilde{E}_i \geq 0$ and hence the lower bound does not matter. Hence, indeed $\tilde{E} \in \mathcal{E}^{\mathcal{P}}$.

Since $\tilde{E}$ is a fixed point of Φ it holds that for all $i \in \mathcal{N}$

$$\tilde{E}_i = \Phi_i(\tilde{E}) = A_i^b + \sum_{j \in \mathcal{M}} L_{ji} \mathbb{V}\left(\frac{\tilde{E}_j + \bar{L}_j}{\bar{L}_j}\right) - \bar{L}_i.$$

We show that $\tilde{E}$ is also a fixed point of $\Phi^{\mathcal{P}}$. Let $i \in \mathcal{N} \setminus \mathcal{P}$. Then,

$$\begin{aligned}
\tilde{E}_i &= A_i^b + \sum_{j \in \mathcal{M}} L_{ji} \mathbb{V}\left(\frac{\tilde{E}_j + \bar{L}_j}{\bar{L}_j}\right) - \bar{L}_i \\
&= A_i^b + \sum_{j \in \mathcal{M}} L_{ji}^{\mathcal{P}} \mathbb{V}\left(\frac{\tilde{E}_j + \bar{L}_j}{\bar{L}_j}\right) - \bar{L}_i^{\mathcal{P}} \qquad (\text{since } i \in \mathcal{N} \setminus \mathcal{P}) \\
&= A_i^b + \sum_{j \in \mathcal{M} \setminus \mathcal{P}} L_{ji}^{\mathcal{P}} \mathbb{V}\left(\frac{\tilde{E}_j + \bar{L}_j}{\bar{L}_j}\right) + \sum_{j \in \mathcal{M} \cap \mathcal{P}} L_{ji}^{\mathcal{P}} \underbrace{\mathbb{V}\left(\frac{\tilde{E}_j + \bar{L}_j}{\bar{L}_j}\right)}_{=1 (\text{no defaults in } \mathcal{P})} - \bar{L}_i^{\mathcal{P}} \\
&= A_i^b + \sum_{j \in \mathcal{M}^{\mathcal{P}} \setminus \mathcal{P}} L_{ji}^{\mathcal{P}} \mathbb{V}\left(\frac{\tilde{E}_j + \bar{L}_j^{\mathcal{P}}}{\bar{L}_j^{\mathcal{P}}}\right) + \sum_{j \in \mathcal{M} \cap \mathcal{P}} L_{ji}^{\mathcal{P}} - \bar{L}_i^{\mathcal{P}} \\
&= A_i^b + \sum_{j \in \mathcal{M}^{\mathcal{P}} \setminus \mathcal{P}} L_{ji}^{\mathcal{P}} \mathbb{V}\left(\frac{\tilde{E}_j + \bar{L}_j^{\mathcal{P}}}{\bar{L}_j^{\mathcal{P}}}\right) + \sum_{j \in \mathcal{M}^{\mathcal{P}} \cap \mathcal{P}} L_{ji}^{\mathcal{P}} \underbrace{\mathbb{V}\left(\frac{\tilde{E}_j + \bar{L}_j^{\mathcal{P}}}{\bar{L}_j^{\mathcal{P}}}\right)}_{=1 (\text{no defaults in } \mathcal{P})} - \bar{L}_i^{\mathcal{P}} \\
&= A_i^b + \sum_{j \in \mathcal{M}^{\mathcal{P}}} L_{ji}^{\mathcal{P}} \mathbb{V}\left(\frac{\tilde{E}_j + \bar{L}_j^{\mathcal{P}}}{\bar{L}_j^{\mathcal{P}}}\right) - \bar{L}_i^{\mathcal{P}} \\
&= \Phi_i^{\mathcal{P}}(\tilde{E}).
\end{aligned}$$

Let $i \in \mathcal{P}$. Then,

$$\begin{aligned}
\tilde{E}_i &= A_i^b + \sum_{j \in \mathcal{M}} L_{ji} \mathbb{V}\left(\frac{\tilde{E}_j + \bar{L}_j}{\bar{L}_j}\right) - \bar{L}_i \\
&= A_i^b + \sum_{j \in \mathcal{M} \setminus \mathcal{P}} \underbrace{L_{ji}}_{=L_{ji}^{\mathcal{P}}} \mathbb{V}\left(\frac{\tilde{E}_j + \bar{L}_j}{\bar{L}_j}\right) + \sum_{j \in \mathcal{M} \cap \mathcal{P}} L_{ji} \underbrace{\mathbb{V}\left(\frac{\tilde{E}_j + \bar{L}_j}{\bar{L}_j}\right)}_{=1 (\text{no defaults in } \mathcal{P})} - \bar{L}_i
\end{aligned}$$

$$
\begin{aligned}
&= A_i^b + \sum_{j\in\mathcal{M}^{\mathcal{P}}\setminus\mathcal{P}} L_{ji}^{\mathcal{P}} \mathbb{V}\left(\frac{\tilde{E}_j + \bar{L}_j^{\mathcal{P}}}{\bar{L}_j^{\mathcal{P}}}\right) + \underbrace{\sum_{j\in\mathcal{P}} L_{ji} - \bar{L}_i}_{=\sum_{j\in\mathcal{P}} L_{ji}^{\mathcal{P}} - \bar{L}_i^{\mathcal{P}} \text{ (by (A.7))}} \\
&= A_i^b + \sum_{j\in\mathcal{M}^{\mathcal{P}}\setminus\mathcal{P}} L_{ji}^{\mathcal{P}} \mathbb{V}\left(\frac{\tilde{E}_j + \bar{L}_j^{\mathcal{P}}}{\bar{L}_j^{\mathcal{P}}}\right) + \sum_{j\in\mathcal{M}^{\mathcal{P}}\cap\mathcal{P}} L_{ji}^{\mathcal{P}} - \bar{L}_i^{\mathcal{P}} \\
&= A_i^b + \sum_{j\in\mathcal{M}^{\mathcal{P}}\setminus\mathcal{P}} L_{ji}^{\mathcal{P}} \mathbb{V}\left(\frac{\tilde{E}_j + \bar{L}_j^{\mathcal{P}}}{\bar{L}_j^{\mathcal{P}}}\right) + \sum_{j\in\mathcal{M}^{\mathcal{P}}\cap\mathcal{P}} L_{ji}^{\mathcal{P}} \underbrace{\mathbb{V}\left(\frac{\tilde{E}_j + \bar{L}_j^{\mathcal{P}}}{\bar{L}_j^{\mathcal{P}}}\right)}_{=1 \text{(no defaults in } \mathcal{P})} - \bar{L}_i^{\mathcal{P}} \\
&= A_i^b + \sum_{j\in\mathcal{M}^{\mathcal{P}}} L_{ji}^{\mathcal{P}} \mathbb{V}\left(\frac{\tilde{E}_j + \bar{L}_j^{\mathcal{P}}}{\bar{L}_j^{\mathcal{P}}}\right) - \bar{L}_i^{\mathcal{P}} \\
&= \Phi_i^{\mathcal{P}}(\tilde{E}).
\end{aligned}
$$

Hence, $\tilde{E}$ is also a fixed point of $\Phi^{\mathcal{P}}$.

Furthermore, $\mathcal{D}(\tilde{E}, L, A^b; \mathbb{V}) = \{i \in \mathcal{N} \mid \tilde{E}_i < 0\} = \mathcal{D}(\tilde{E}, L^{\mathcal{B}}, A^b; \mathbb{V})$ and hence $\mathcal{D}(\tilde{E}, L^{\mathcal{B}}, A^b; \mathbb{V}) \subseteq \mathcal{D}(\tilde{E}, L, A^b; \mathbb{V})$, i.e., systemic risk is reduced but $\mathcal{D}(\tilde{E}, L^{\mathcal{B}}, A^b; \mathbb{V}) \not\subsetneq \mathcal{D}(\tilde{E}, L, A^b; \mathbb{V})$, i.e., there is no strong reduction of systemic risk.

2. Let $\tilde{E}$ be the greatest fixed point of Φ satisfying (16). We show that it is also the greatest fixed point of $\Phi^{\mathcal{P}}$. By part 1. of this theorem, $\tilde{E}$ is also a fixed point of $\Phi^{\mathcal{P}}$. Let $\tilde{E}^{\mathcal{P}}$ be the greatest fixed point of $\Phi^{\mathcal{P}}$. Since $\tilde{E}$ is a fixed point of $\Phi^{\mathcal{P}}$, we have $\tilde{E} \leq \tilde{E}^{\mathcal{P}}$ and in particular $0 \leq \tilde{E}_i \leq \tilde{E}_i^{\mathcal{P}}$ for all $i \in \mathcal{P}$ by (16) and hence $\{i \in \mathcal{P} \mid \tilde{E}_i^{\mathcal{P}} < 0\} = \emptyset$. Therefore, it follows from part 1. of Proposition 1 that $\tilde{E}^{\mathcal{P}}$ is also a fixed point of Φ, implying $\tilde{E}^{\mathcal{P}} \leq \tilde{E}$. This leads to $\tilde{E} = \tilde{E}^{\mathcal{P}}$.

3. We consider the network presented in Figure A.1 for $\mathbb{V} = \mathbb{V}^{\mathrm{RV}}$ with $\beta = 0.1$, where

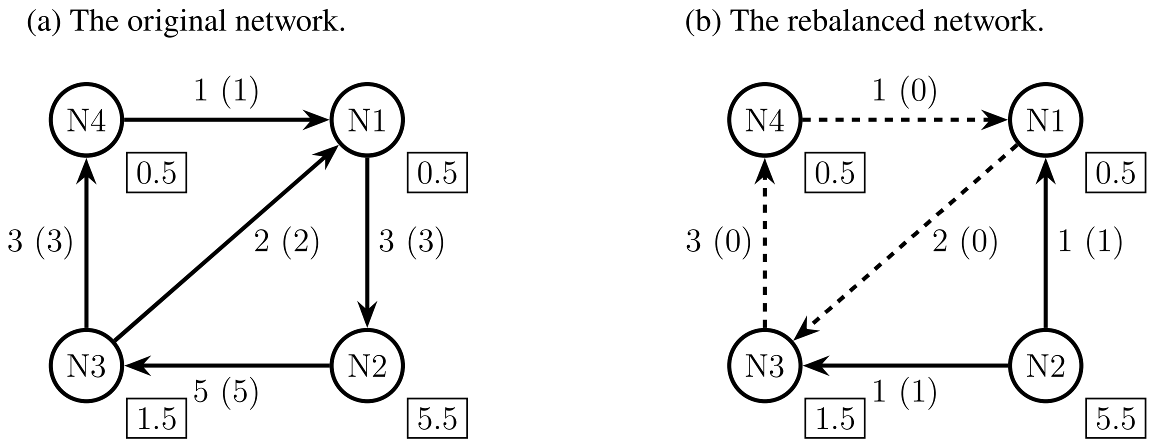
$$
\mathbb{V}^{\mathrm{RV}}(y) = \begin{cases} 1, & \text{if } y \geq 1, \\ \beta y^+, & \text{if } y < 1, \end{cases}
$$

which corresponds to a special case of the model by Rogers and Veraart (2013).

In particular, here $N = 4$, $\mathcal{P} = \{1, 2, 3\}$, and

$$
A^b = \begin{pmatrix} 0.5 \\ 5 \\ 1.5 \\ 0.5 \end{pmatrix}, L = \begin{pmatrix} 0 & 3 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 2 & 0 & 0 & 3 \\ 1 & 0 & 0 & 0 \end{pmatrix}, L^{\mathcal{P}} = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \\ 1 & 0 & 0 & 0 \end{pmatrix}.
$$

Figure A.1 Networks of variation margin payments corresponding to the least fixed point.



Note. The figure shows a harmful rebalancing exercise with respect to the least fixed point. The liabilities are next to the arrows, and the numbers in boxes and brackets represent the liquid assets and actual payments, respectively. The dashed lines indicate the liabilities that are not settled.

One can check that $E^* = (0.5, 3.5, 1.5, 2.5)^\top = E_*$ is the greatest and least fixed point in the original network, hence there are no defaults in the original network. Furthermore, $E^{\mathcal{P};*} = E^*$ is the greatest fixed point in the PTN-network (no defaults). But the least fixed point in the PTN-network is given by $E_*^{\mathcal{P}} = (-47/111, 3.5, -38/111, -26/111)^\top \ngeq E_*$, i.e., banks $1,3,4$ default and hence under the least fixed point this PTN-exercise is harmful.

4. Let $\tilde{E}$ be the least fixed point of Φ. Let $E_*^{\mathcal{P}}$ be the least fixed point of $\Phi^{\mathcal{P}}$. Since the conditions of Theorem 3 are satisfied, we can conclude with Theorem 3 result (19) that $\tilde{E} \leq E_*^{\mathcal{P}}$. From part 1. of this theorem it follows that $\tilde{E}$ is also a fixed point of $\Phi^{\mathcal{P}}$ and since $E_*^{\mathcal{P}}$ is the least fixed point of $\Phi^{\mathcal{P}}$ it follows that $\tilde{E} = E_*^{\mathcal{P}}$.

□

Proof of Corollary 1 Since (16) holds, $\tilde{E}_i \geq 0\ \forall i \in \mathcal{P}$, and hence,

$$\mathbb{V}\left(\frac{\tilde{E}_i + \bar{L}_i}{\bar{L}_i}\right) = 1 = \mathbb{V}\left(\frac{\tilde{E}_i + \bar{L}_i^{\mathcal{P}}}{\bar{L}_i^{\mathcal{P}}}\right) \quad \forall i \in \mathcal{P}. \tag{A.10}$$

The payments in the original network are

$$p_{ij}(\tilde{E}) = \mathbb{V}\left(\frac{\tilde{E}_i + \bar{L}_i}{\bar{L}_i}\right) L_{ij} \quad \forall i, j \in \mathcal{N},$$

and because of (A.10)

$$p_{ij}(\tilde{E}) = L_{ij} \quad \forall i \in \mathcal{P}, \forall j \in \mathcal{N}.$$

The payments in the PTN-network are

$$p_{ij}^{\mathcal{P}}(\tilde{E}) = \mathbb{V}\left(\frac{\tilde{E}_i + \bar{L}_i^{\mathcal{P}}}{\bar{L}_i^{\mathcal{P}}}\right) L_{ij}^{\mathcal{P}} \quad \forall i, j \in \mathcal{N},$$

which reduces to (using the fact that nonparticipants have the same payment obligations in both networks and (A.10))

$$\begin{aligned} p_{ij}^{\mathcal{P}}(\tilde{E}) &= \mathbb{V}\left(\frac{\tilde{E}_i + \bar{L}_i}{\bar{L}_i}\right) L_{ij} && \forall i \in \mathcal{N} \setminus \mathcal{P}, \forall j \in \mathcal{N}, \\ p_{ij}^{\mathcal{P}}(\tilde{E}) &= L_{ij}^{\mathcal{P}} && \forall i \in \mathcal{P}, \forall j \in \mathcal{N}. \end{aligned}$$

Hence,

$$\begin{aligned} p_{ij}(\tilde{E}) - p_{ij}^{\mathcal{P}}(\tilde{E}) &= 0 = L_{ij} - L_{ij}^{\mathcal{P}}, && \forall i \in \mathcal{N} \setminus \mathcal{P}, \forall j \in \mathcal{N}, \\ p_{ij}(\tilde{E}) - p_{ij}^{\mathcal{P}}(\tilde{E}) &= L_{ij} - L_{ij}^{\mathcal{P}} && \forall i \in \mathcal{P}, \forall j \in \mathcal{N}. \end{aligned}$$

□

We use Lemma A.4 (whose second and third statements are similar to Veraart (2022, Lemma B.4)) to prove Proposition 2.

LEMMA A.4. *Let $\mathcal{F} = \{i \in \mathcal{N} \mid E_i^{(0)} < 0\}$ and $\mathcal{F}^{\mathcal{P}} = \{i \in \mathcal{N} \mid E_i^{\mathcal{P}(0)} < 0\}$ be the fundamental default set in the original network and in the PTN-network, respectively. Let $\tilde{E}$ be a fixed point of Φ and let $\tilde{E}^{\mathcal{P}}$ be a fixed point of $\Phi^{\mathcal{P}}$.*
Then 1.) $\mathcal{F}^{\mathcal{P}} = \mathcal{F}$, 2.) $\mathcal{F} \subseteq \mathcal{D}(\tilde{E}, L, A^b; \mathbb{V})$, and 3.) $\mathcal{F}^{\mathcal{P}} \subseteq \mathcal{D}(\tilde{E}^{\mathcal{P}}, L^{\mathcal{B}}, A^b; \mathbb{V})$.

Proof of Lemma A.4 1. First, note that the definition of $\mathcal{F}$ is indeed the same as our earlier Definition 9. We have $\mathcal{F} = \{i \in \mathcal{N} \mid E_i^{(0)} < 0\}$ and $\mathcal{F}^{\mathcal{P}} = \{i \in \mathcal{N} \mid E_i^{\mathcal{P}(0)} < 0\}$. Since $E_i^{(0)} = E_i^{\mathcal{P}(0)}$ for all $i \in \mathcal{N}$ by Lemma A.3, we obtain $\mathcal{F}^{\mathcal{P}} = \mathcal{F}$.

2. We consider the sequences $\left(E^{(n)}\right)$ and $\left(E^{\mathcal{P}(n)}\right)$ defined by (A.8). Fix $i \in \mathcal{F}$. Lemma A.3 implies that $\forall m \in \mathbb{N}$, $0 > E_i^{(0)} \geq E_i^{(m)} \geq \lim_{n\to\infty} E_i^{(n)} = E_i^*$. Therefore, $i \in \mathcal{D}(E^*, L, A^b; \mathbb{V})$ where E^* is the greatest fixed point of Φ. Since, $\tilde{E}_i \leq E_i^* < 0$, this implies that $i \in \mathcal{D}(\tilde{E}, L, A^b; \mathbb{V})$.

3. Fix $i \in \mathcal{F}^{\mathcal{P}}$. Again by Lemma A.3, $\forall m \in \mathbb{N}$, $0 > E_i^{\mathcal{P}(0)} \geq E_i^{\mathcal{P}(m)} \geq \lim_{n\to\infty} E_i^{\mathcal{P}(n)} = E_i^{\mathcal{P};*}$, hence $i \in \mathcal{D}(E^{\mathcal{P};*}, L^{\mathcal{B}}, A^b; \mathbb{V})$, where $E^{\mathcal{P};*}$ is the greatest fixed point of $\Phi^{\mathcal{P}}$. Since, $\tilde{E}_i^{\mathcal{P}} \leq E_i^{\mathcal{P};*} < 0$, this implies that $i \in \mathcal{D}(\tilde{E}^{\mathcal{P}}, L^{\mathcal{B}}, A^b; \mathbb{V})$.

□

Proof of Proposition 2 Let $\tilde{E}$ be a fixed point of Φ and let $\tilde{E}^{\mathcal{P}}$ be a fixed point of $\Phi^{\mathcal{P}}$. We know from Lemma A.4 that $\mathcal{F}^{\mathcal{P}} \subseteq \mathcal{D}(\tilde{E}^{\mathcal{P}}, L, A^b; \mathbb{V})$. We prove $\mathcal{F}^{\mathcal{P}} = \mathcal{D}(\tilde{E}^{\mathcal{P}}, L, A^b; \mathbb{V})$ by showing that $\mathcal{D}(\tilde{E}^{\mathcal{P}}, L, A^b; \mathbb{V}) \subseteq \mathcal{F}^{\mathcal{P}}$. Let $i \in \mathcal{D}(\tilde{E}^{\mathcal{P}}, L, A^b; \mathbb{V})$. Then

$$0 > \tilde{E}_i^{\mathcal{P}} = A_i^b + \sum_{j \in \mathcal{M}^{\mathcal{P}}} L_{ji}^{\mathcal{P}} \mathbb{V}\left(\frac{\tilde{E}_j^{\mathcal{P}} + \bar{L}_j^{\mathcal{P}}}{\bar{L}_j^{\mathcal{P}}}\right) - \bar{L}_i^{\mathcal{P}}.$$

Hence, $\bar{L}_i^{\mathcal{P}} > 0$. Since the graph corresponding to the optimal PTN-network (when $\mathcal{P} = \mathcal{N}$) is bipartite by D'Errico and Roukny (2021, Lemma 1), this implies that $L_{ji}^{\mathcal{P}} = 0$ for all $j \in \mathcal{M}^{\mathcal{P}}$. To be more precise, we can apply D'Errico and Roukny (2021, Lemma 1) because the optimal PTN-optimisation problem is identical to the non-conservative compression problem in D'Errico and Roukny (2021) when $\mathcal{P} = \mathcal{N}$. It follows that

$$\sum_{j \in \mathcal{M}^{\mathcal{P}}} L_{ji}^{\mathcal{P}} \mathbb{V}\left(\frac{\tilde{E}_j^{\mathcal{P}} + \bar{L}_j^{\mathcal{P}}}{\bar{L}_j^{\mathcal{P}}}\right) = 0 = \sum_{j \in \mathcal{M}^{\mathcal{P}}} L_{ji}^{\mathcal{P}},$$

and hence

$$0 > \tilde{E}_i^{\mathcal{P}} = A_i^b - \bar{L}_i^{\mathcal{P}} = E_i^{\mathcal{P}(0)}.$$

This implies that $i \in \mathcal{F}^{\mathcal{P}}$, therefore $\mathcal{F}^{\mathcal{P}} = \mathcal{D}(\tilde{E}^{\mathcal{P}}, L^{\mathcal{B}}, A^b; \mathbb{V})$. Hence, $\mathcal{F}^{\mathcal{P}} = \mathcal{D}(\tilde{E}^{\mathcal{P}}, L^{\mathcal{B}}, A^b; \mathbb{V}) \subseteq \mathcal{D}(\tilde{E}, L, A^b; \mathbb{V})$ by Lemma A.4. □

We will use Lemma A.5 to prove Theorem 3. The statement of the Lemma and its proofs is only a small modification of ideas used in the proof of Veraart (2022, Proposition 4.12).

LEMMA A.5. *Consider a PTN-exercise $(L, \mathcal{P}, L^{\mathcal{P}})$ that satisfies (3). Let the valuation function be $\mathbb{V} = \mathbb{V}^{zero}$. Let $E^{(n)} \in \mathcal{E}, E^{\mathcal{P}(n)} \in \mathcal{E}^{\mathcal{P}}$ be such that $E^{\mathcal{P}(n)} \geq E^{(n)}$. Then,*

$$\Phi_i(E^{\mathcal{P}(n)}) = \Phi_i(E^{\mathcal{P}(n)}; \mathbb{V}^{zero}) \geq \Phi_i^{\mathcal{P}}(E^{(n)}; \mathbb{V}^{zero}) = \Phi_i^{\mathcal{P}}(E^{(n)}) \quad \forall i \in \mathcal{N}.$$

Proof of Lemma A.5 Let $E^{\mathcal{P}(n)} \geq E^{(n)}$ and $\mathbb{V} = \mathbb{V}^{\text{zero}}$. Then, for any $\bar{L}, \bar{L}^{\mathcal{P}} \in [0, \infty)^N$

$$\mathbb{V}^{\text{zero}}\left(\frac{E_j^{\mathcal{P}(n)} + \bar{L}_j^{\mathcal{P}}}{\bar{L}_j^{\mathcal{P}}}\right) = \mathbb{I}_{\left\{E_i^{\mathcal{P}(n)} \geq 0\right\}} \geq \mathbb{I}_{\left\{E_i^{(n)} \geq 0\right\}} = \mathbb{V}^{\text{zero}}\left(\frac{E_j^{(n)} + \bar{L}_j}{\bar{L}_j}\right). \tag{A.11}$$

First, let $i \in \mathcal{N} \setminus \mathcal{P}$. Then,

$$\Phi_i^{\mathcal{P}}(E^{\mathcal{P}(n)}) = \Phi_i^{\mathcal{P}}(E^{\mathcal{P}(n)}; \mathbb{V}^{\text{zero}}) = A_i^b + \sum_{j \in \mathcal{N}: \bar{L}_j^{\mathcal{P}} > 0} L_{ji}^{\mathcal{P}} \mathbb{V}^{\text{zero}}\left(\frac{E_j^{\mathcal{P}(n)} + \bar{L}_j^{\mathcal{P}}}{\bar{L}_j^{\mathcal{P}}}\right) - \bar{L}_i^{\mathcal{P}}$$

$$
\begin{aligned}
&= A_i^b + \sum_{j\in\mathcal{N}:\bar{L}_j^{\mathcal{P}}>0} L_{ji}\mathbb{V}^{\text{zero}}\left(\frac{E_j^{\mathcal{P}(n)}+\bar{L}_j^{\mathcal{P}}}{\bar{L}_j^{\mathcal{P}}}\right) - \bar{L}_i \quad (\text{since } i\notin\mathcal{P})\\
&= A_i^b + \sum_{j\in\mathcal{N}} L_{ji}\mathbb{I}_{\left\{E_i^{\mathcal{P}(n)}\geq 0\right\}} - \bar{L}_i\\
&\geq A_i^b + \sum_{j\in\mathcal{N}} L_{ji}\mathbb{I}_{\left\{E_i^{(n)}\geq 0\right\}} - \bar{L}_i \quad (\text{by (A.11)})\\
&= A_i^b + \sum_{j\in\mathcal{N}:\bar{L}_j>0} L_{ji}\mathbb{V}^{\text{zero}}\left(\frac{E_j^{(n)}+\bar{L}_j}{\bar{L}_j}\right) - \bar{L}_i\\
&= \Phi_i(E^{(n)};\mathbb{V}^{\text{zero}}) = \Phi_i(E^{(n)}).
\end{aligned}
$$

Second, let $i\in\mathcal{P}$. Then,

$$
\begin{aligned}
\Phi_i^{\mathcal{P}}(E^{\mathcal{P}(n)}) &= \Phi_i^{\mathcal{P}}(E^{\mathcal{P}(n)};\mathbb{V}^{\text{zero}}) = A_i^b + \sum_{j\in\mathcal{N}:\bar{L}_j^{\mathcal{P}}>0} L_{ji}^{\mathcal{P}}\mathbb{V}^{\text{zero}}\left(\frac{E_j^{\mathcal{P}(n)}+\bar{L}_j^{\mathcal{P}}}{\bar{L}_j^{\mathcal{P}}}\right) - \bar{L}_i^{\mathcal{P}}\\
&= A_i^b + \sum_{j\in\mathcal{N}} L_{ji}^{\mathcal{P}}\mathbb{I}_{\left\{E_i^{\mathcal{P}(n)}\geq 0\right\}} - \bar{L}_i^{\mathcal{P}}\\
&\geq A_i^b + \sum_{j\in\mathcal{N}} L_{ji}^{\mathcal{P}}\mathbb{I}_{\left\{E_i^{(n)}\geq 0\right\}} - \bar{L}_i^{\mathcal{P}} \quad (\text{by(A.11)})\\
&= A_i^b + \sum_{j\in\mathcal{N}\setminus\mathcal{P}} \underbrace{L_{ji}^{\mathcal{P}}}_{=L_{ji}}\mathbb{I}_{\left\{E_i^{(n)}\geq 0\right\}} + \sum_{j\in\mathcal{P}} L_{ji}^{\mathcal{P}}\mathbb{I}_{\left\{E_i^{(n)}\geq 0\right\}} - \bar{L}_i^{\mathcal{P}}\\
&= A_i^b + \sum_{j\in\mathcal{N}\setminus\mathcal{P}} L_{ji}\mathbb{I}_{\left\{E_i^{(n)}\geq 0\right\}} + \sum_{j\in\mathcal{P}} L_{ji}^{\mathcal{P}}\mathbb{I}_{\left\{E_i^{(n)}\geq 0\right\}} - \bar{L}_i + \sum_{j\in\mathcal{P}} L_{ji} - \sum_{j\in\mathcal{P}} L_{ji}^{\mathcal{P}} \quad (\text{by(A.7)})\\
&= A_i^b + \sum_{j\in\mathcal{N}} L_{ji}\mathbb{I}_{\left\{E_i^{(n)}\geq 0\right\}} - \sum_{j\in\mathcal{P}} L_{ji}\mathbb{I}_{\left\{E_i^{(n)}\geq 0\right\}} + \sum_{j\in\mathcal{P}} L_{ji}^{\mathcal{P}}\mathbb{I}_{\left\{E_i^{(n)}\geq 0\right\}} - \bar{L}_i + \sum_{j\in\mathcal{P}} L_{ji} - \sum_{j\in\mathcal{P}} L_{ji}^{\mathcal{P}}\\
&= A_i^b + \sum_{j\in\mathcal{N}} L_{ji}\mathbb{I}_{\left\{E_i^{(n)}\geq 0\right\}} - \bar{L}_i + \sum_{j\in\mathcal{P}} \underbrace{(L_{ji}-L_{ji}^{\mathcal{P}})}_{\geq 0(\text{by (3)})}\underbrace{(1-\mathbb{I}_{\left\{E_i^{(n)}\geq 0\right\}})}_{\geq 0}\\
&\geq A_i^b + \sum_{j\in\mathcal{N}} L_{ji}\mathbb{I}_{\left\{E_i^{(n)}\geq 0\right\}} - \bar{L}_i = \sum_{j\in\mathcal{N}:\bar{L}_j>0} L_{ji}\mathbb{V}^{\text{zero}}\left(\frac{E_j^{(n)}+\bar{L}_j}{\bar{L}_j}\right) - \bar{L}_i\\
&= \Phi_i(E^{(n)};\mathbb{V}^{\text{zero}}) = \Phi_i(E^{(n)}).
\end{aligned}
$$

□

Proof of Theorem 3 1. The result for the greatest fixed points, i.e., (18) has already been shown in Veraart (2022, Proposition 4.12, Proposition 4.13). As shown there, one can again consider a fixed point iteration. For the sequences $\left(E^{(n)}\right)$ and $\left(E^{\mathcal{P}(n)}\right)$ defined by (A.8) one can show by induction that if $\mathbb{V} = \mathbb{V}^{\text{zero}}$, then $E_i^{\mathcal{P}(n)} \geq E_i^{(n)}\ \forall i \in \mathcal{N}$ holds for all $n \in \mathbb{N}_0$. In particular, the statement for $n = 0$, $E_i^{\mathcal{P}(0)} = E_i^{(0)}$ for all $i \in \mathcal{N}$ holds by Lemma A.3. If we assume that $E_i^{\mathcal{P}(n)} \geq E_i^{(n)}$ holds for a fixed $n \in \mathbb{N}_0$, then, by Lemma A.5 $E_i^{\mathcal{P}(n+1)} = \Phi_i^{\mathcal{P}}\left(E^{\mathcal{P}(n)}; \mathbb{V}^{\text{zero}}\right) \geq \Phi_i^{\mathcal{P}}\left(E^{\mathcal{P}(n)}; \mathbb{V}^{\text{zero}}\right) = E_i^{\mathcal{P}(n+1)}$ which completes the proof by induction. Therefore, $E_i^{\mathcal{P};*} = \lim_{n\to\infty} E_i^{\mathcal{P}(n)} \geq \lim_{n\to\infty} E_i^{(n)} = E_i^* \quad \forall i \in \mathcal{N}$ by Lemma A.3 which proves (18).

Next, we prove the result for the least fixed point, i.e., that (19) holds. We set

$$
\begin{aligned}
(E_{(0)})_i &= A_i^b - \bar{L}_i, \quad \forall i \in \mathcal{N}, \\
(E_{\mathcal{P}(0)})_i &= A_i^b - \bar{L}_i^{\mathcal{P}} \quad \forall i \in \mathcal{N},
\end{aligned}
$$

and we define the sequences

$$
\begin{aligned}
(E_{(n+1)})_i &= \Phi_i(E_{(n)}; \mathbb{V}^{\text{zero}}) \quad \forall i \in \mathcal{N}, \\
(E_{\mathcal{P}(n+1)})_i &= \Phi_i^{\mathcal{P}}(E_{\mathcal{P}(n)}; \mathbb{V}^{\text{zero}}) \quad \forall i \in \mathcal{N},
\end{aligned}
$$

for $n \geq 0$. We show by induction that $(E_{(n)})_{n\in\mathbb{N}_0}$ and $(E_{\mathcal{P}(n)})_{\mathbb{N}_0}$ are non-decreasing sequences, i.e.,

$$
\begin{aligned}
(E_{(n+1)})_i &\geq (E_{(n)})_i, \quad \forall i \in \mathcal{P} \\
(E_{\mathcal{P}(n+1)})_i &\geq (E_{\mathcal{P}(n)})_i \quad \forall i \in \mathcal{P}.
\end{aligned}
$$

Let $n = 0$. It follows directly from the definition of Φ and $\Phi^{\mathcal{P}}$ that $\forall i \in \mathcal{N}$

$$
(E_{(1)})_i = \Phi_i(E_{(0)}; \mathbb{V}^{\text{zero}}) = A_i^b + \underbrace{\sum_{j\in\mathcal{N}:\bar{L}_i>0} L_{ij} \mathbb{V}^{\text{zero}}\left(\frac{(E_{(0)})_j + \bar{L}_j}{\bar{L}_j}\right)}_{\geq 0} - \bar{L}_i \geq A_i^b - \bar{L}_i = (E_{(0)})_i
$$

$$
(E_{\mathcal{P}(1)})_i = \Phi_i^{\mathcal{P}}(E_{\mathcal{P}(0)}; \mathbb{V}^{\text{zero}}) = A_i^b + \underbrace{\sum_{j\in\mathcal{N}:\bar{L}_i^{\mathcal{P}}>0} L_{ij}^{\mathcal{P}} \mathbb{V}^{\text{zero}}\left(\frac{(E_{\mathcal{P}(0)})_j + \bar{L}_j^{\mathcal{P}}}{\bar{L}_j^{\mathcal{P}}}\right)}_{\geq 0} - \bar{L}_i^{\mathcal{P}} \geq A_i^b - \bar{L}_i^{\mathcal{P}} = (E_{\mathcal{P}(0)})_i.
$$

Now fix $n \in \mathbb{N}_0$ and assume that $E_{(n)} \geq E_{(n-1)}$ and $E_{\mathcal{P}(n)} \geq E_{\mathcal{P}(n-1)}$ holds. Then,

$$
(E_{(n+1)})_i = \Phi_i(E_{(n)}; \mathbb{V}^{\text{zero}}) = A_i^b + \sum_{j\in\mathcal{N}:\bar{L}_i>0} L_{ij} \underbrace{\mathbb{V}^{\text{zero}}\left(\frac{(E_{(n)})_j + \bar{L}_j}{\bar{L}_j}\right)}_{\geq \mathbb{V}^{\text{zero}}\left(\frac{(E_{(n-1)})_j + \bar{L}_j}{\bar{L}_j}\right)} - \bar{L}_i
$$

$$\geq \Phi_i(E_{(n-1)}; \mathbb{V}^{\text{zero}}) = (E_{(n)})_i$$

$$(E_{\mathcal{P}(n+1)})_i = \Phi_i^{\mathcal{P}}(E_{\mathcal{P}(n)}; \mathbb{V}^{\text{zero}}) = A_i^b + \sum_{j \in \mathcal{N}: \bar{L}_i^{\mathcal{P}} > 0} L_{ij}^{\mathcal{P}} \underbrace{\mathbb{V}^{\text{zero}}\left(\frac{(E_{\mathcal{P}(n)})_j + \bar{L}_j^{\mathcal{P}}}{\bar{L}_j^{\mathcal{P}}}\right)}_{\geq \mathbb{V}^{\text{zero}}\left(\frac{(E_{\mathcal{P}(n-1)})_j + \bar{L}_j^{\mathcal{P}}}{\bar{L}_j^{\mathcal{P}}}\right)} - \bar{L}_i^{\mathcal{P}}$$

$$\geq \Phi_i^{\mathcal{P}}(E_{\mathcal{P}(n-1)}; \mathbb{V}^{\text{zero}}) = (E_{\mathcal{P}(n)})_i$$

which completes the proof by induction.

Hence, $(E_{(n)})_{n \in \mathbb{N}_0}$ and $(E_{(n)})_{\mathbb{N}_0}$ are non-decreasing sequences. They are also bounded from above, since for all $i \in \mathcal{N}$ $(E_{(n)})_i \leq A_i^b + \sum_{j \in \mathcal{N}} L_{ij} - \bar{L}_i$ and $(E_{\mathcal{P}(n)})_i \leq A_i^b + \sum_{j \in \mathcal{N}} L_{ij}^{\mathcal{P}} - \bar{L}_i^{\mathcal{P}}$. Hence, both sequences converge to a limit.

Next, we show that $\forall i \in \mathcal{N}$ and $\forall n \in \mathcal{N}_0$

$$(E_{\mathcal{P}(n)})_i \geq (E_{(n)})_i. \tag{A.12}$$

This follows directly by induction. For $n = 0$, since $(L, \mathcal{P}, L^{\mathcal{P}})$ satisfies (3) it holds that

$$(E_{\mathcal{P}(0)})_i = A_i^b - \bar{L}_i^{\mathcal{P}} \geq A_i^b - \bar{L}_i = (E_{(0)})_i \quad \forall i \in \mathcal{N}$$

and the induction step follows directly with Lemma A.5.

Hence, we obtain that

$$\lim_{n\to\infty} (E_{\mathcal{P}(n)})_i \geq \lim_{n\to\infty} (E_{(n)})_i \quad \forall i \in \mathcal{N}.$$

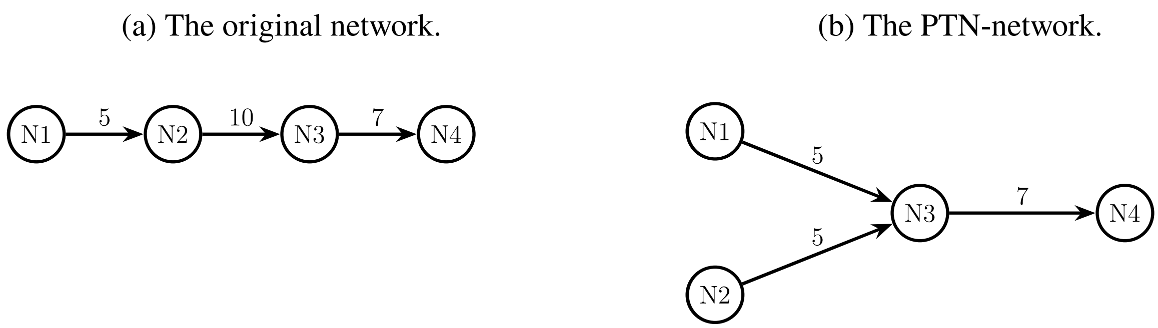
If $\lim_{n\to\infty}(E_{\mathcal{P}(n)})$ is a fixed point of $\Phi^{\mathcal{P}}$ and $\lim_{n\to\infty}(E_{(n)})$ is a fixed point of $\Phi^{\mathcal{P}}$, then there is nothing left to show.

But since $\mathbb{V}^{\text{zero}}$ is not left-continuous, there is no guarantee that $\lim_{n\to\infty}(E_{(n)})$ is a fixed point of Φ or that $\lim_{n\to\infty}(E_{\mathcal{P}(n)})$ is a fixed point of $\Phi^{\mathcal{P}}$. Then, as discussed in Rogers and Veraart (2013, Section 3.1), if at least one of these limits is not a fixed point, then one will need to restart the iteration from these limits, i.e., set for $n \in \mathbb{N}_0$

$$\hat{E}_{(0)} = \lim_{m\to\infty} (E_{(m)}), \quad \hat{E}_{(n+1)} = \Phi(\hat{E}_{(n)}),$$
$$\hat{E}_{\mathcal{P}(0)} = \lim_{m\to\infty} (E_{\mathcal{P}(m)}), \quad \hat{E}_{\mathcal{P}(n+1)} = \Phi(\hat{E}_{\mathcal{P}(n)}),$$

and repeat the previous arguments. If the initial vector of such a sequence is a fixed point, then the sequence is just constant.

Figure A.2 Harmful PTN-exercise for zero recovery rates.



Note. The figure shows a harmful PTN-exercise (that does not satisfy (3)) for $\mathbb{V} = \mathbb{V}^{\text{zero}}$. The liquidity buffers are $A^b = (1, 10, 1, 0)^\top$. The liabilities L_{ij} and $L^{\mathcal{P}}_{ij}$ are next to the arrows.

The situation that the limit is not a fixed point can only occur at a point where a bank just becomes solvent in the limit. This can happen at most N times since there are N banks, meaning at most $N - 1$ restarts of this fixed point iteration could become necessary as discussed in Rogers and Veraart (2013). Then, after at most $N - 1$ restarts, the limits of the iterations are indeed the least fixed points. If we need to restart the iteration, the same argument can be used to show the equivalence of (A.12) for the next two sequences. This sequence of arguments can be repeated until the fixed points are obtained.

2. In the following we provide an example for the situation mentioned in the statement. We consider a PTN-exercise with

$$L = \begin{pmatrix} 0 & 5 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 \end{pmatrix}, L^{\mathcal{P}} = \begin{pmatrix} 0 & 0 & 5 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and $A^b = (1, 10, 1, 0)^\top$. An illustration is provided in Figure A.2.

One can check that $E^* = (-4, 0, 4, 7)^\top = E_*$, $E^{\mathcal{P};*} = (-4, 5, -1, 0)^\top = E^{\mathcal{P}}_*$. Hence, neither (18) nor (19) are satisfied and this PTN-exercise is harmful under the greatest and the least re-evaluated equity. In particular, in the original network, bank N1 is the only bank in fundamental default, which does not trigger further defaults. In the PTN-network, bank N1 remains the only one in fundamental default, but it now triggers the contagious default of bank N3. This difference is due to the fact that bank N3 now faces bank N1 directly and receives no payment from N1.

□

A.2.3. Clearing algorithms Before proceeding to the proof of Theorem 4, we first introduce the clearing equilibrium for the Rogers and Veraart (2013) model. Given a financial network (L, A^b), the relative liabilities matrix Π is defined by $\Pi_{ij} = L_{ij}/\bar{L}_i$ if $\bar{L}_i > 0$, and $\Pi_{ij} = 0$ otherwise for all $i, j \in \mathcal{N}$. A *clearing vector* in the Rogers and Veraart (2013) model is a vector $L \in [0, \bar{L}]$ satisfying

$$L = \Psi^{RV}(L),$$

where the function $\Psi^{RV} : [0, \bar{L}] \to [0, \bar{L}]$ is given by

$$\Psi_i^{RV}(L) = \begin{cases} \bar{L}_i, & \text{if } A_i^b + \sum_{j\in\mathcal{N}} \Pi_{ji} L_j \geq \bar{L}_i, \\ \alpha A_i^b + \beta \sum_{j\in\mathcal{N}} \Pi_{ji} L_j, & \text{otherwise,} \end{cases}$$

with default cost parameters $\alpha, \beta \in [0, 1]$.

In Figure A.3, we present the two clearing algorithms that are related to the modelling assumption in Section 4.1.1.

Algorithm 1 corresponds to the Full Payment Algorithm (FPA) by Bardoscia et al. (2019). It computes a vector $\tilde{l}_*$ that corresponds to the payments made by all banks in the network. The mechanism in the algorithm can be understood as follows. At the time t, $e(t)$ consists of the available liquid assets including received payments, and $\mathcal{A}(t)$ comprises banks that can pay in full. The assumption that banks either make full payment or pay nothing is incorporated into step 7. Note that an important difference between the FPA and the hard default in Paddrik et al. (2020) is that in the former setting, it is assumed that—unlike the Eisenberg and Noe (2001) model in finding an equilibrium payment vector—there is no coordination among banks in the FPA to determine the payments. Therefore, the modelling assumption results in a sequence of payments, and banks can only pay in full if they have received sufficient liquidity.

Algorithm 2 considers the least clearing vector in the Rogers and Veraart (2013) model with $\alpha = \beta = 0$. It corresponds to the case in which the defaulting banks make zero payments. We refer to Algorithm 2 as the Least Clearing Vector Algorithm (LA). The algorithm starts by assuming that initially there is no solvent bank that would be able to make any payment. $\mathcal{S}^{(0)}$ is the set of banks that would be able to pay liabilities in full even if all other banks did not meet their obligations. Similar to the construction in Rogers and Veraart (2013, Theorem 3.7), as the algorithm terminates, the output is the least clearing vector.

We use Lemma A.6 to prove Theorem 4.

Figure A.3 Clearing algorithms.

Algorithm 1 Full Payment Algorithm (FPA) in Bardoscia et al. (2019)

1: Set $e(0) := A^b$, $l(0) := \mathbf{0}$, and $\mathcal{A}(0) := \emptyset$. Set $t = 1$.

2: For all $i \in \mathcal{N}$, set

$$e_i(t) = e_i(t-1) + \sum_{j \in \mathcal{N}} l_j(t-1)\Pi_{ji} - l_i(t-1). \tag{A.13}$$

3: Determine

$$\mathcal{A}(t) = \{i \in \mathcal{N} \mid e_i(t) \geq \bar{L}_i\} \setminus \bigcup_{s=0}^{t-1} \mathcal{A}(s). \tag{A.14}$$

4: **if** $\mathcal{A}(t) \equiv \emptyset$ **then**

5: **return** $\tilde{l}_* = \sum_{s=0}^{t-1} l(s)$.

6: **else**

7: set $l_i(t) = \bar{L}_i$ for all $i \in \mathcal{A}(t)$, and $l_i(t) = 0$ otherwise.

8: **end if**

9: Set $t = t + 1$ and go back to step 2.

Algorithm 2 Least Clearing Vector Algorithm (LA) for Rogers and Veraart (2013) model with $\alpha = \beta = 0$

1: Set $t = 0$, $l^{(0)} := \mathbf{0}$, and $\mathcal{D}^{(-1)} := \mathcal{N}$.

2: For all $i \in \mathcal{N}$, determine

$$v_i^{(t)} := A_i^b + \sum_{j \in \mathcal{N}} l_j^{(t)}\Pi_{ji} - \bar{L}_i. \tag{A.15}$$

3: Define

$$\mathcal{D}^{(t)} := \{i \in \mathcal{N} \mid v_i^{(t)} < 0\} \text{ and } \mathcal{S}^{(t)} := \{i \in \mathcal{N} \mid v_i^{(t)} \geq 0\}. \tag{A.16}$$

4: **if** $\mathcal{D}^{(t)} \equiv \mathcal{D}^{(t-1)}$ **then**

5: **return** $l_* = l^{(t-1)}$.

6: **else**

7: set $l_i^{(t+1)} = \bar{L}_i$ for all $i \in \mathcal{S}^{(t)}$, and $l_i^{(t+1)} = 0$ for all $i \in \mathcal{D}^{(t)}$.

8: **end if**

9: Set $t = t + 1$ and go back to step 2.

LEMMA A.6. *Consider the FPA (Algorithm 1) and the LA (Algorithm 2) described in Figure A.3. Fix an iteration $t \in \mathbb{N}_0$. Then*

$$\bigcup_{s=0}^{t+1} \mathcal{A}(s) = \mathcal{S}^{(t)}. \tag{A.17}$$

In particular, the banks that make payments in the FPA up to time $t+1$ are identical to those that make payments in the LA up to time $t+1$.

Proof of Lemma A.6 We prove the result by induction. Let $t = 0$. By plugging the initial values of Algorithm 1 into (A.13) and (A.14), we obtain that $\bigcup_{s=0}^{1} \mathcal{A}(s) = \mathcal{A}(0) \cup (\mathcal{A}(1) \setminus \mathcal{A}(0)) = \mathcal{A}(1) = \{i \in \mathcal{N} \mid e_i(1) \geq \bar{L}_i\} = \{i \in \mathcal{N} \mid A_i^b \geq \bar{L}_i\}$ and $\mathcal{S}^{(0)} = \{i \in \mathcal{N} \mid A_i^b - \bar{L}_i \geq 0\}$. Therefore, $\bigcup_{s=0}^{1} \mathcal{A}(s) = \mathcal{S}^{(0)}$.

Now suppose (A.17) holds for a fixed t. We show that it also holds for $t+1$, i.e., $\bigcup_{s=0}^{t+2} \mathcal{A}(s) = \mathcal{S}^{(t+1)}$. We rewrite

$$\bigcup_{s=0}^{t+2} \mathcal{A}(s) = \bigcup_{s=0}^{t+1} \mathcal{A}(s) \cup \mathcal{A}(t+2) \text{ and } \mathcal{S}^{(t+1)} = \mathcal{S}^{(t)} \cup \left(\mathcal{S}^{(t+1)} \setminus \mathcal{S}^{(t)}\right),$$

so it is sufficient to prove that $\mathcal{A}(t+2) = \mathcal{S}^{(t+1)} \setminus \mathcal{S}^{(t)}$.

First, note that according to the definition of Algorithm 1, for all $i \in \mathcal{N}$, we can write $e_i(t+1)$ as

$$\begin{aligned}
e_i(t+1) &= e_i(t) + \sum_{j \in \mathcal{N}} l_j(t)\Pi_{ji} - l_i(t) \\
&= \underbrace{e_i(t-1) + \sum_{j \in \mathcal{N}} l_j(t-1)\Pi_{ji} - l_i(t-1)}_{=e_i(t)} + \sum_{j \in \mathcal{N}} l_j(t)\Pi_{ji} - l_i(t) \\
&= \cdots \\
&= \underbrace{A_i^b}_{=e_i(0)} + \sum_{j \in \mathcal{N}} \Pi_{ji} \sum_{s=0}^{t} l_j(s) - \sum_{s=0}^{t} l_i(s) \\
&\overset{(\star)}{=} A_i^b + \sum_{j \in \bigcup_{s=0}^{t} \mathcal{A}(s)} \Pi_{ji} \sum_{s=0}^{t} l_j(s) - \sum_{s=0}^{t} l_i(s) \\
&\overset{(\star\star)}{=} A_i^b + \sum_{j \in \bigcup_{s=0}^{t} \mathcal{A}(s)} \bar{L}_j \Pi_{ji} - \sum_{s=0}^{t} l_i(s),
\end{aligned} \tag{A.18}$$

where $(\star)$ follows from the fact that $\sum_{s=0}^{t} l_j(s) > 0$ implies $j \in \bigcup_{s=0}^{t} \mathcal{A}(s)$ (see step 3-7 in Algorithm 1), and $(\star\star)$ holds since $\sum_{s=0}^{t} l_j(s) = \bar{L}_j$ for all $j \in \bigcup_{s=0}^{t} \mathcal{A}(s)$.

In addition, we can rewrite (A.15) at $t+1$ as

$$v_i^{(t+1)} = A_i^b + \sum_{j\in\mathcal{N}} l_j^{(t+1)}\Pi_{ji} - \bar{L}_i = A_i^b + \sum_{j\in\mathcal{S}^{(t)}} \bar{L}_j\Pi_{ji} - \bar{L}_i. \tag{A.19}$$

First, we show that $\mathcal{S}^{(t+1)} \setminus \mathcal{S}^{(t)} \subseteq \mathcal{A}(t+2)$. Let $i \in \mathcal{S}^{(t+1)} \setminus \mathcal{S}^{(t)}$, then $i \notin \bigcup_{s=0}^{t+1}\mathcal{A}(s)$ by the induction hypothesis $\mathcal{S}^{(t)} = \bigcup_{s=0}^{t+1}\mathcal{A}(s)$. From the definition of $l_i(s)$, $s\in\{0,\dots,t+1\}$, this implies that $\sum_{s=0}^{t+1} l_i(s) = 0$. Combining this with equation (A.19) gives

$$\begin{aligned}0 \le v_i^{(t+1)} &= A_i^b + \sum_{j\in\mathcal{S}^{(t)}} \bar{L}_j\Pi_{ji} - \bar{L}_i + \underbrace{0}_{=\sum_{s=0}^{t+1} l_i(s)} = A_i^b + \sum_{j\in\bigcup_{s=0}^{t+1}\mathcal{A}(s)} \bar{L}_j\Pi_{ji} + \sum_{s=0}^{t+1} l_i(s) - \bar{L}_i \\ &= e_i(t+2) - \bar{L}_i.\end{aligned}$$

Hence, $i \in \mathcal{A}(t+2)$.

Second, we show that $\mathcal{A}(t+2) \subseteq \mathcal{S}^{(t+1)} \setminus \mathcal{S}^{(t)}$. Let $i \in \mathcal{A}(t+2)$, then

$$\begin{aligned}\bar{L}_i \le e_i(t+2) &= A_i^b + \sum_{j\in\bigcup_{s=0}^{t+1}\mathcal{A}(s)} \bar{L}_j\Pi_{ji} - \sum_{s=0}^{t+1} l_i(s) \overset{\text{ind.hyp.}}{=} A_i^b + \sum_{j\in\mathcal{S}^{(t)}} \bar{L}_j\Pi_{ji} - \underbrace{\sum_{s=0}^{t+1} l_i(s)}_{\overset{(\diamond)}{=}0} \\ &= A_i^b + \sum_{j\in\mathcal{S}^{(t)}} \bar{L}_j\Pi_{ji},\end{aligned} \tag{A.20}$$

where $(\diamond)$ holds because the definition of $\mathcal{A}(t+2)$ implies that $i \notin \bigcup_{s=0}^{t+1}\mathcal{A}(s) \overset{\text{ind.hyp}}{=} \mathcal{S}^{(t)}$, which implies that $\sum_{s=0}^{t+1} l_i(s) = 0$. Combining this with (A.20) implies that $0 \le A_i^b + \sum_{j\in\mathcal{S}^{(t)}} \bar{L}_j\Pi_{ji} - \bar{L}_i = v_i^{(t+1)}$ and hence $i \in \mathcal{S}^{(t+1)} \setminus \mathcal{S}^{(t)}$. □

Now we can prove the result that the outcomes of Algorithms 1 and 2 in Figure A.3 coincide.

Proof of Theorem 4 Suppose for a fixed $t > 0$ at one iteration it holds that $\mathcal{A}(t) = \emptyset$. Then $\tilde{l}_\star = \sum_{s=0}^{t-1} l(s)$, where $\tilde{l}_{\star,i} = \bar{L}_i$ if $i \in \bigcup_{s=0}^{t-1}\mathcal{A}(s)$ and 0 otherwise. In addition, since $\bigcup_{s=0}^{t-1}\mathcal{A}(s) = \mathcal{S}^{(t-2)}$ by Lemma A.6, $\mathcal{A}(t) = \emptyset$ is equivalent to $\mathcal{D}^{(t)} = \mathcal{D}^{(t-1)}$ in the LA. Furthermore, the LA returns $l_\star = l^{(t-1)}$, where $l_{\star,i} = \bar{L}_i$ if $i \in \mathcal{S}(t-2)$ and 0 otherwise.

Therefore, both algorithms terminate when the same banks are selected, and all their payments are identical. According to Rogers and Veraart (2013), the LA generates a sequence of vectors increasing to the least clearing vector, so the statement follows immediately. □

B. More background on PTRR services

This section provides more background on post-trade netting services drawn from various sources. It is worth highlighting that the services develop fast and update frequently; we refer interested readers to the website of service providers such as OSTTRA[1] for the latest information. At the time of writing, CME's TriOptima—a leading PTRR service provider—is part of OSTTRA, a joint venture formed on 1st September 2021 between IHS Markit and CME Group. (IHS Markit was acquired by S&P Global on 1st March 2022.)

We split the development into three stages (which may have some overlapping). First, before the Global Financial Crisis, the volumes of outstanding derivatives contracts grew rapidly. In particular, concerns about counterparty risk drove the increase in the CDS market near the crisis; the notional amount then fell significantly, which according to Vause (2010), can be partly attributed to portfolio compression. The second stage follows the mandatory clearing of standardised derivatives contracts. For example, TriOptima introduced its triReduce service in 2003 for the interest rate swap (IRS) market, which now compresses bilateral swaps as well as products in centrally cleared markets. TriOptima collaborates with LCH.Clearnet on SwapClear.[2] ISDA (2012) reports that the progress on eliminating outstanding IRS notional positions since 2011 is significant.

While regulatory reforms are intended to make the financial markets more resilient, the implementation is costly. As a result, the most recent stage contributing to the development of post-trade netting services is attributable to those reforms that introduce higher costs, such as the Uncleared Margin Rules (UMRs), capital requirements (for example, the SA-CCR and G-SIBs' capital surcharges), Leverage Ratio (LR) requirements, and so on. [3] With the leverage ratio that uses gross exposures being an exception, margin and capital costs are often aligned with risk metrics based on net exposures. Therefore, portfolio rebalancing that optimises counterparty exposures could potentially reduce the all-in cost of trading derivatives.[4] Figure A.4 shows the example of portfolio rebalancing provided in ESMA (2020, Annex 1).

In summary, the regulatory regime spurs the development of post-trade netting services. Which feature leads to the wide use of post-trade netting services today, in addition to incentives from

[1] See `https://osttra.com/`.

[2] See `https://www.lch.com/services/swapclear/enhancements/compression`.

[3] See `https://osttra.com/press_releases/osttras-compression-service-unlocks-additional-compression-potential-for-g-sibs/`.

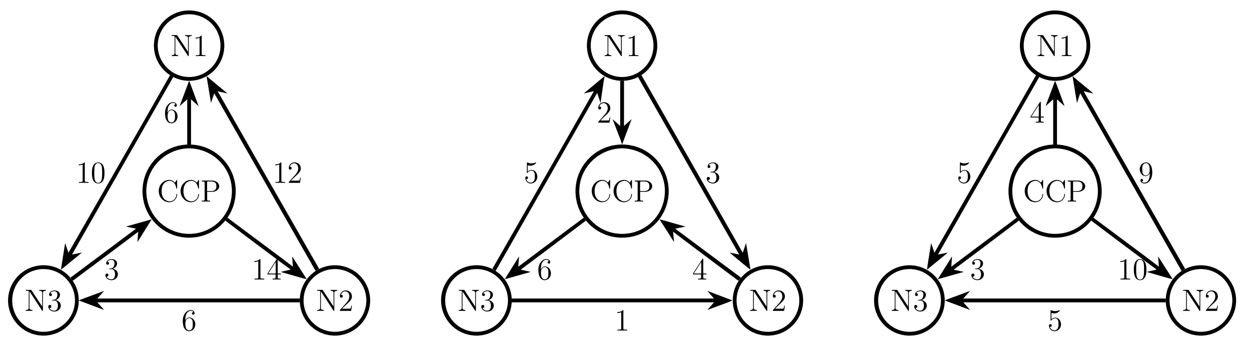
[4] For OSTTRA's triBalance, see `https://osttra.com/articles/managing-ccr-to-reduce-the-all-in-cost-of-otc-derivatives-portfolios/`.

Figure A.4 An illustration of portfolio rebalancing given in ESMA (2020, Annex 1) with three counterparties and a CCP.

(a) The original notional matrix C.

(b) The rebalancing notional matrix R.

(c) The bilaterally netted rebalanced matrix $(C+R)^{bi}$.



Note. Note that the initial network in the example only involves a portfolio for risk reduction purposes (i.e., a subset of all transactions) made up of four counterparties. Therefore, the CCP does not need to have a matched book.

regulations? One answer is technology. Equipped with advanced optimisation techniques and data processing skills, the Fintech vendors who have access to all information submitted by a large network of market participants could, in principle, achieve desirable outcomes more efficiently[5] compared to the fairs in pre-industrial Europe which rely on the decentralised searching and matching procedure, see the descriptions in Börner and Hatfield (2017).

C. Additional examples

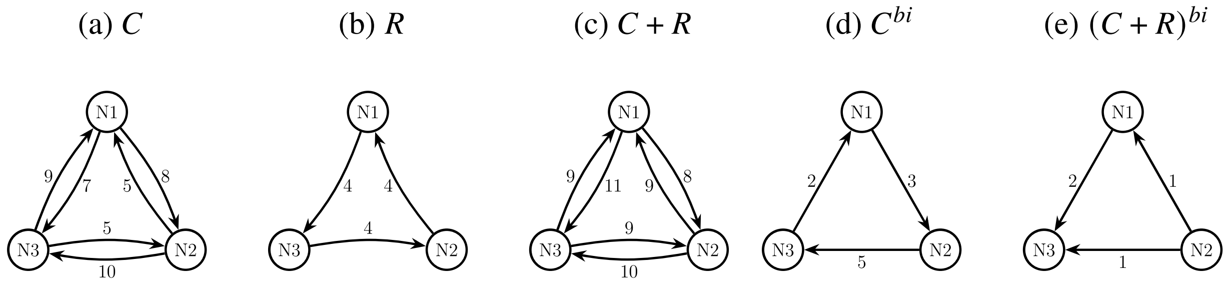
EXAMPLE A.1 ((OPTIMAL) PORTFOLIO REBALANCING). Figure A.5 illustrates an example of portfolio rebalancing in a network of three banks. The notional matrix C, the rebalancing notional positions R, the rebalanced notional matrix $C+R$, and the bilaterally netted notional matrices C^{bi} and $(C+R)^{bi}$ are given, respectively, by

$$C = \begin{pmatrix} 0 & 8 & 7 \\ 5 & 0 & 10 \\ 9 & 5 & 0 \end{pmatrix}, \quad R = \begin{pmatrix} 0 & 0 & 4 \\ 4 & 0 & 0 \\ 0 & 4 & 0 \end{pmatrix}, \quad C+R = \begin{pmatrix} 0 & 8 & 11 \\ 9 & 0 & 10 \\ 9 & 9 & 0 \end{pmatrix}, \quad C^{bi} = \begin{pmatrix} 0 & 3 & 0 \\ 0 & 0 & 5 \\ 2 & 0 & 0 \end{pmatrix}, \quad (C+R)^{bi} = \begin{pmatrix} 0 & 0 & 2 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

In line with the definition of rebalancing, the net exposures of the three banks in the original network are given by $C^\top 1 - C1 = (-1,-2,3)^\top$, and they coincide with the net exposures $(C+R)^\top 1 - (C+R)1$ after portfolio rebalancing.

[5] See `https://osttra.com/press_releases/osttra-streamlines-trade-reconciliation-with-connectivity-between-markitwire-and-triresolve/`.

Figure A.5 Example of portfolio rebalancing.



Note. Starting from the notional matrix C (Figure A.5a), injecting the rebalancing notional matrix R (Figure A.5b) results in the rebalanced notional matrix $C+R$ (Figure A.5c). The bilaterally netted positions prior to rebalancing are C^{bi} (Figure A.5d) and the bilaterally netted positions after rebalancing are $(C+R)^{bi}$ (Figure A.5e).

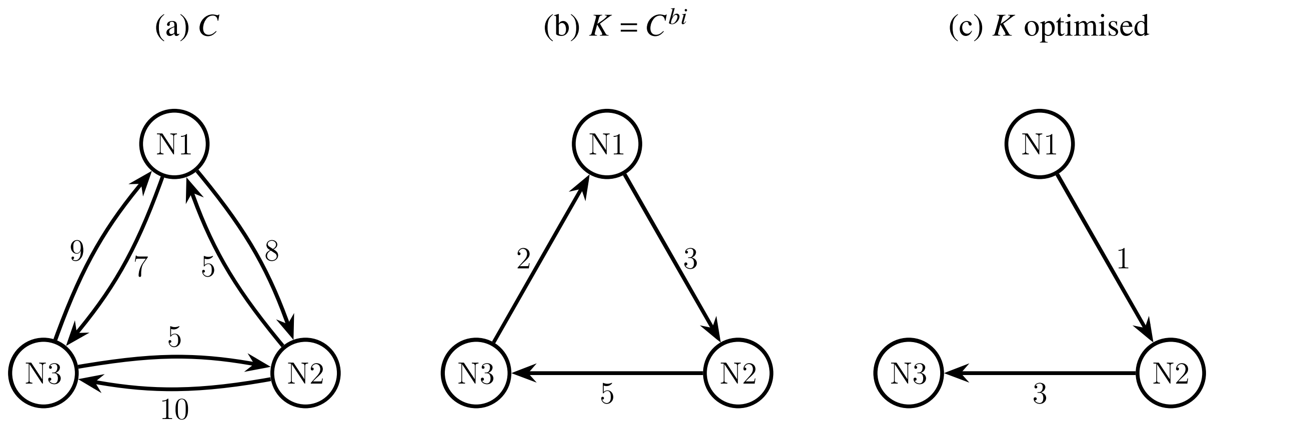
Furthermore, portfolio rebalancing increases the gross exposures of the notional positions from $1^\top(C^\top 1 + C1) = 88$ to $1^\top((C+R)^\top 1 + (C+R)1) = 112$, but it decreases the gross exposures of the bilaterally netted positions from $1^\top((C^{bi})^\top 1 + C^{bi}1) = 20$ to $1^\top((C+R)^{bi})^\top 1 + ((C+R)^{bi})1) = 8$.

This rebalancing exercise is not an optimal rebalancing exercise. It is easy to check that an optimal rebalancing exercise can be attained by choosing R such that $R_{13} = R_{21} = R_{32} = 3$, and $R_{ij} = 0$ otherwise. In this case, the sum of the gross exposures in the resulting network is equal to 6, which is strictly less than the previous outcome. We also see that this matrix R is not a net-conservative rebalancing matrix since for example $1 = (C+R)^{bi}_{13} > C^{bi}_{13} = 0$.

It is easy to check that an optimal net-conservative rebalancing exercise can be attained by choosing R such that $R_{13} = R_{21} = R_{32} = 2$, and $R_{ij} = 0$ otherwise. In this case, the sum of the bilaterally netted gross exposures in the resulting network is equal to 8, which is the same as in Figure A.5e where R was such that $R_{13} = R_{21} = R_{32} = 4$ and $R_{ij} = 0$ otherwise. The difference now is that counterparty relationships are controlled, which was not the case in Figure A.5 where for example bank $N1$ was a net borrower from bank $N2$ before rebalancing but became a net lender to bank $N2$ after the exercise.

EXAMPLE A.2 ((OPTIMAL) PORTFOLIO COMPRESSION). To provide intuition about the idea of compression, we present an example in Figure A.6. This example follows the idea of O'Kane (2017), who proposes a loop compression algorithm by finding and eliminating all closed loops on the bilaterally netted notional matrix. Here, Figure A.6a shows the original network which is the same as in Figure A.5a, Figure A.6b shows the bilaterally compressed network and Figure A.6c is then the outcome after removing all closed loops in the bilaterally compressed network which is in this example a solution to the super-conservative compression optimisation problem.

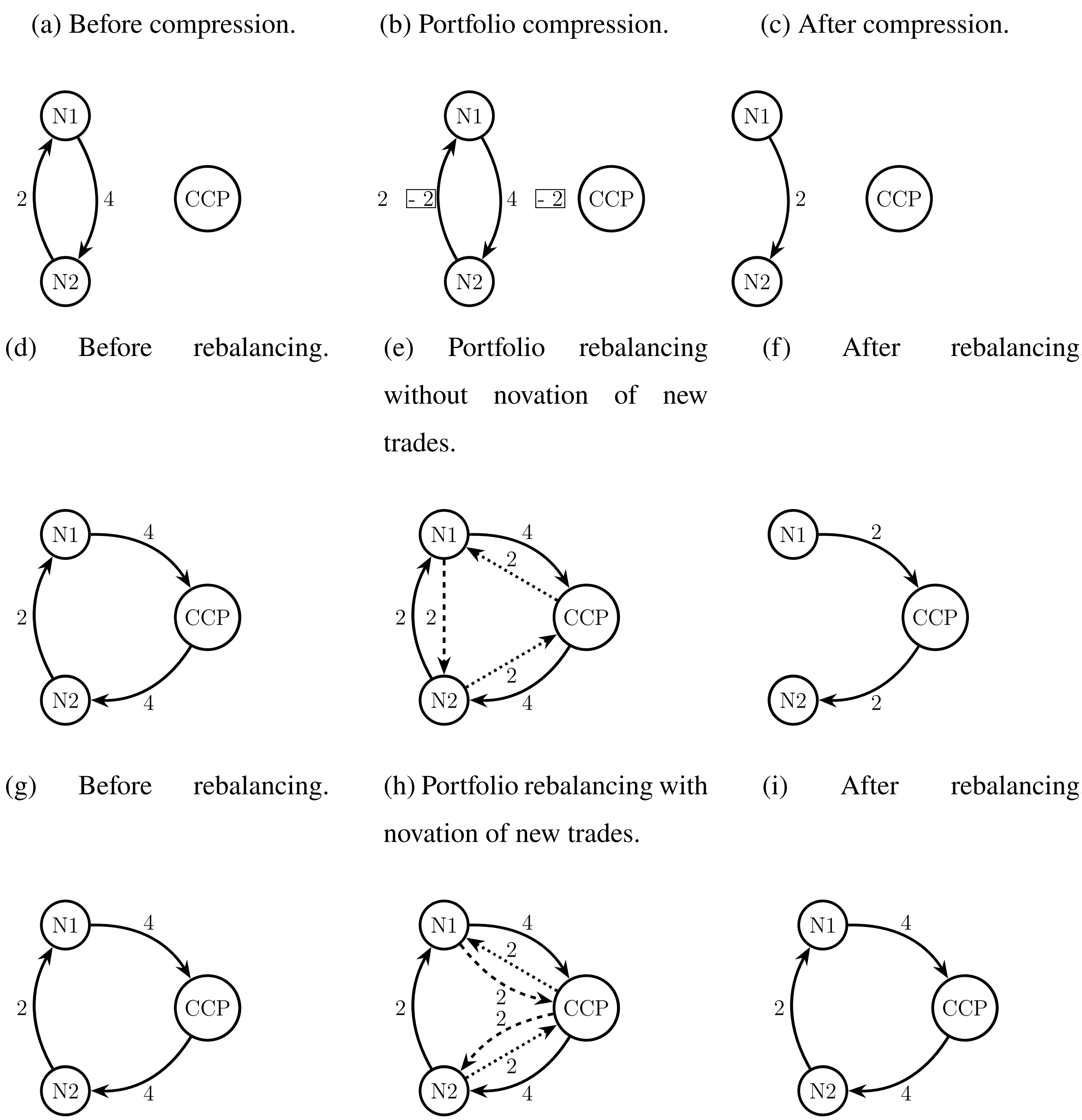
Figure A.6 Examples of portfolio compression.



Note. The numbers in the figures represent notional positions. Figure A.6a shows an illustration of the notional matrix C. Figure A.6b shows an illustration of $K = C^{bi}$, i.e., the network obtained by removing cycles between pairs of counterparties. Figure A.6c shows another example of multilateral compression where the cycle in C^{bi} between all three counterparties has been removed. This is also a solution to the super-conservative compression optimisation problem.

EXAMPLE A.3 (DIFFERENT USE CASES OF PORTFOLIO COMPRESSION AND REBALANCING). We illustrate in Figure A.7 different use cases of portfolio compression and portfolio rebalancing. In particular, it shows how a portfolio that can be compressed in the non-centrally cleared space (Figure A.7a), can no longer be compressed in the partially centrally cleared space (Figure A.7d). Still, portfolio rebalancing (Figure A.7f) can achieve a comparable outcome to portfolio compression (Figure A.7c) if new trades established as part of the rebalancing exercise are not subject to central clearing. Otherwise, Figure A.7i shows that novation prevents the netting benefit in the non-cleared part of the portfolio and hence the overall netting benefit.

Figure A.7 Different use cases of portfolio compression and portfolio rebalancing.



Note. The first row shows a portfolio that is not centrally cleared and subject to portfolio compression. The second and third row show the same positions under partial-central clearing and subject to portfolio rebalancing. The second row assumes that new trades are not novated to the CCP and the third row assumes that they are novated to the CCP.

References

Bardoscia M, Ferrara G, Vause N, Yoganayagam M (2019) Full payment algorithm. Available at SSRN: https://dx.doi.org/10.2139/ssrn.3344580.

Börner L, Hatfield JW (2017) The design of debt-clearing markets: Clearinghouse mechanisms in preindustrial Europe. *Journal of Political Economy* 125(6):1991–2037.

D'Errico M, Roukny T (2021) Compressing over-the-counter markets. *Operations Research* 69(6):1660–1679.

Eisenberg L, Noe TH (2001) Systemic risk in financial systems. *Management Science* 47(2):236–249.

ESMA (2020) Report to the European Commission, Report on post trade risk reduction services with regards to the clearing obligation (EMIR Article 85(3a)). Available at https://www.esma.europa.eu/sites/default/files/library/esma70-156-3351_report_on_ptrr_services_with_regards_to_the_clearing_obligation_0.pdf.

ISDA (2012) Interest Rate Swaps Compression: A Progress Report. Available at https://www.isda.org/a/BeiDE/irs-compression-progress-report-feb-2012.pdf.

O'Kane D (2017) Optimising the multilateral netting of fungible OTC derivatives. *Quantitative Finance* 17(10):1523–1534.

Paddrik M, Rajan S, Young HP (2020) Contagion in derivatives markets. *Management Science* 66(8):3603–3616.

Rogers LCG, Veraart LAM (2013) Failure and rescue in an interbank network. *Management Science* 59(4):882–898.

Vause N (2010) Counterparty risk and contract volumes in the credit default swap market, BIS Quarterly Review, December.

Veraart LAM (2020) Distress and default contagion in financial networks. *Mathematical Finance* 30(3):705–737.

Veraart LAM (2022) When does portfolio compression reduce systemic risk? *Mathematical Finance* 32(3):727–778.