Electronic Companion

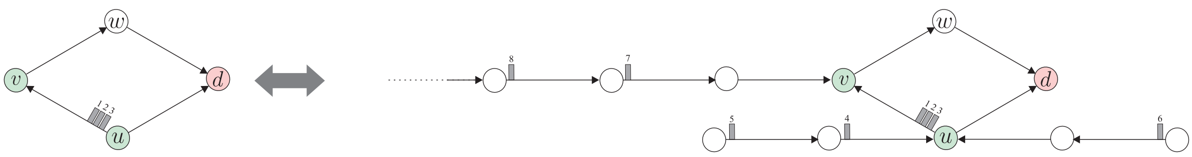
Throughout, for any directed multi-edge graph H with vertex set $\hat{V}$ and edge set $\hat{E}$, and any vertex $v \in \hat{V}$, we use $\hat{E}^+(v)$ and $\hat{E}^-(v)$ to denote the set of outgoing edges from v and the set of incoming edges to v in H, respectively. To avoid possible confusion in expressing parallel edges, for each edge $e \in \hat{E}$ with tail vertex u and head vertex v, we give u and v aliases u_e and v_e, respectively. When we write e as uv, we always mean $u = u_e$ and $v = v_e$.

For any strategy (path) profile $\boldsymbol{p} = (P_i)_{i\in\Delta}$ of game $\Gamma^{\rm N}$ and any agent subset S of Δ, the partial path profiles $(P_i)_{i\in S}$ and $(P_i)_{i\in\Delta\setminus S}$ are abbreviated to $\boldsymbol{p}_S$ and $\boldsymbol{p}_{-S}$, respectively. In particular, $\boldsymbol{p}_\emptyset$ is viewed as an empty profile.

EC.1. A technical transformation

In our model $\Gamma^{\rm N}$, the superficial difference between the agents inside the initial queues and those outside makes our presentation cumbersome. To avoid awkward descriptions and also indicate more insights into agent interactions in model $\Gamma^{\rm N}$, we introduce a new model, denoted by $\bar{\Gamma}^{\rm N}$, which we show is equivalent to $\Gamma^{\rm N}$. The three seemly different characteristics of an agent in $\Gamma^{\rm N}$, entry time, origin, and original rank, are unified to a single location feature in $\bar{\Gamma}^{\rm N}$. Studying this equivalent model not only significantly simplifies the five ranking criteria in edge-priority DQ rule (see Section 3) and many technical definitions (such as "agent preemption" in Section 4.4.1), but also substantially shortens our proofs (avoiding tedious case analyses to deal with different agent characteristics).

In $\bar{\Gamma}^{\rm N}$, all agents are located at the initial queues of the new input network $\bar{G}$, which is either a finite acyclic directed graph or some special infinite graph as illustrated in Figure EC.1: G is a subgraph of $\bar{G}$. Both games $\Gamma^{\rm N}$ and $\bar{\Gamma}^{\rm N}$ have the same set of agents $\Delta = \mathbb{Q}^0_{uv} \cup \Delta_{1,u} \cup \Delta_{2,u} \cup_{r\geq 1} \Delta_{r,v}$, and the same initial queue $\mathbb{Q}^0_{uv} = \{1,2,3\}$ in G; The sets of sequentially arriving agents in $\Gamma^{\rm N}$, i.e., $\Delta_{1,u} = \{4\}$, $\Delta_{2,u} = \{5,6\}$ with agent 5 having a higher original rank than 6, $\Delta_{1,v} = \emptyset$, and $\Delta_{r,v} = \{r+5\}$ for every $r \geq 2$, correspond to initial queues outside G in $\bar{\Gamma}^{\rm N}$.



(a) Π on G at time $t=0$ (b) $\bar{\Pi}$ on $\bar{G}$ at time $t=0$

Figure EC.1 Game $\Gamma^{\rm N}$ on G vs. game $\bar{\Gamma}^{\rm N}$ on $\bar{G}$

Recall that the input of a game instance Γ^{N} consists of network $G=(V,E)$ with initial queues $(\mathbb{Q}_e^0)_{e\in E}$ and inflows of agents $\Delta_{r,v}$ $(r\geq 1,\ v\in V)$ along with original ranks among them. Corresponding to this game instance, an instance of game $\bar{\Gamma}^{\mathrm{N}}$ is specified by the same set $\bar{\Delta}:=\Delta=(\cup_{e\in E}\mathbb{Q}_e^0)\cup(\cup_{r\geq 1,v\in V}\Delta_{r,v})$ of nonadaptive agents, but with a modified network input $\bar{G}=(\bar{V},\bar{E})$, which is obtained from G by adding a (typically infinite) number of pendant paths and specifying initial locations of the agents $\cup_{r\geq 1,v\in V}\Delta_{r,v}$ at the newly added pendant paths, as detailed below:

(T1) Network $\bar{G}$: For each vertex $v\in V$, we add a number $\max_{r\geq 1}|\Delta_{r,v}|$ (which is possibly zero) of paths $P_1^v,P_2^v,\ldots$ directed to v, each intersecting G only at v. Outside G, all added paths are mutually vertex-disjoint.

(T2) Priority preservation: For each $v\in V$, the decreasing priority ordering of the added incoming edges to v agrees with the (subscript) ordering of added paths containing them: the unique edge in P_1^v incoming to v has the highest priority, the one in P_2^v incoming to v has the second highest priority, and so on. This ordering is followed by the given ordering of incoming edges to v in G, which makes a complete priority order over all incoming edges to v in $\bar{G}$.

(T3) Rank preservation: Agent $i\in\Delta_{r,v}$ in game Γ^{N} with the hth highest original rank corresponds to agent $i\in\bar{\Delta}$ in game $\bar{\Gamma}^{\mathrm{N}}$, who is the only agent queuing at time 0 on path P_h^v at a distance r from v, where distance is measured by the number of edges. Particularly, agent i in $\bar{\Gamma}^{\mathrm{N}}$ will reach v at time r through the (added) incoming edge to v with the hth highest priority.

With the above transformation, it is easy to see that the agent set $\bar{\Delta}$ of game $\bar{\Gamma}^{\mathrm{N}}$ is simply the disjoint union of its initial queues, which we still denote as $\mathbb{Q}_e^0$, over all $e\in\bar{E}$. The game $\bar{\Gamma}^{\mathrm{N}}$ starts at time 0 with the input $\bar{G}$ and $(\mathbb{Q}_e^0)_{e\in\bar{E}}$. No entries of agents into the network $\bar{G}$ are involved throughout the game and all agents are in $\bar{G}$ from the very beginning. Therefore, no original ranks are needed to break ties. (Note that we have transformed the original ranks among $\Delta_{r,v}$ to the priorities of the added edges incoming to v.)

Due to the simpler form of the input for $\bar{\Gamma}^{\mathrm{N}}$, the edge-priority DQ rule (see Section 3) when applied to $\bar{\Gamma}^{\mathrm{N}}$ is simplified: we do not need rules (R3) and (R4) any more. Regarding any fixed edge e (with tail vertex u_e) and any pair of agents, the agent who enters e earlier has a higher queue rank at e. Ties are broken via (R2) only: higher priority is given to the agent who enters e through an incoming edge to u_e that has a higher priority according to $\prec_{u_e}$. Another simplification resulted from our transformation is that in $\bar{\Gamma}^{\mathrm{N}}$ all information about agent set is contained in the initial queues $(\mathbb{Q}_e^0)_{e\in\bar{E}}$. The chronological order of entrances into G in Γ^{N} is visualized by the lengths of paths of $\bar{G}$ in $\bar{\Gamma}^{\mathrm{N}}$, which makes the task of investigating agent interactions easier. For instance, in the example illustrated in Figure EC.1, game $\bar{\Gamma}^{\mathrm{N}}$ provides a faster way for one to find that agent 7 will preempt agent 3 at vertex v.

Each agent $i \in \bar{\Delta}$ selects a path *starting from his initial edge* (the edge where he queues at time 0) and ending at the common destination vertex d. Such paths form his strategy set, which we write as $\bar{\mathscr{P}}_i$ to distinguish it from $\mathscr{P}_i$ in game Γ^{N}. We use o_i to denote the tail vertex of initial edge of agent $i \in \bar{\Delta}$ in $\bar{G}$. The following fact is obvious by our transformation from game Γ^{N} on G to game $\bar{\Gamma}^{\text{N}}$ on $\bar{G}$.

LEMMA EC.1. *Given game Γ^{N} with input (G, Δ), let game $\bar{\Gamma}^{\text{N}}$ with input $(\bar{G}, \bar{\Delta})$ be constructed as in* (T1)–(T3)*. Then the game Γ^{N} is exactly the restriction of the game $\bar{\Gamma}^{\text{N}}$ to G: the strategies and movements of agents together with their arrival times at vertices along their paths in Γ^{N} are identical with those in the restriction of $\bar{\Gamma}^{\text{N}}$ to G.*

In view of agents' trivial movements outside G in game $\bar{\Gamma}^{\text{N}}$, the above lemma enables us to turn our attention to $\bar{\Gamma}^{\text{N}}$ when studying Γ^{N}. These two games are essentially identical. The notation and definitions introduced for game model Γ^{N} apply to game model $\bar{\Gamma}^{\text{N}}$, as the latter is simply a special case of the former.

EC.2. Algorithm for finding an IDNE

In this section, we construct an IDNE for every game $\bar{\Gamma}^{\text{N}}$ (see Definition 2). The result along with Lemma EC.1 directly yields an IDNE for every game Γ^{N}. Recall that, $[0] = \emptyset$, and for any positive integer k, $[k]$ denotes the set of all positive integers no more than k.

Algorithm. We are able to reindex the agents of $\bar{\Delta}$ as $1, 2, \ldots$ and find the associated path profile $\bar{\boldsymbol{p}} = (\bar{P}_1, \bar{P}_2, \ldots)$ such that, each agent $k \in \bar{\Delta}$ is a *dominator* in $\bar{\Delta} \backslash [k-1]$ and $\bar{P}_k$ is a *dominant path* in the following sense: under the assumption that agents in $[k-1]$ all follow $\bar{\boldsymbol{p}}_{[k-1]}$, as long as agent k takes $\bar{P}_k$, he will be among the first in $\bar{\Delta} \backslash [k-1]$ to reach every vertex of the path. Specifically, for any vertex v on $\bar{P}_k$, and partial path profile $\boldsymbol{q}_{-[k]}$ for agents in $\Delta \backslash [k]$, we have

$$\begin{aligned} t_k^v\left(\bar{\boldsymbol{p}}_{[k]}, \boldsymbol{q}_{-[k]}\right) = \min \big\{ & t_j^v(\bar{\boldsymbol{p}}_{[k-1]}, \boldsymbol{r}_{-[k-1]}) : j \in \Delta \backslash [k-1], \\ & \text{and } \boldsymbol{r}_{-[k-1]} \text{ is partial profile for } \Delta \backslash [k-1] \big\}. \end{aligned} \tag{EC.1}$$

We call such a path profile *iteratively dominant*. As explained in Section 4.1, it is actually an NE, i.e., IDNE, of game $\bar{\Gamma}^{\text{N}}$.

For completeness, we repeat the sketch of the algorithm in the context of $\bar{\Gamma}^{\text{N}}$. Our algorithm runs roughly as follows. Initially, let agent subset $[0]$ of $\bar{\Delta}$ and partial routing $\bar{\boldsymbol{p}}_{[0]}$ of agents in $[0]$ be empty. Then recursively, assuming agents in $[k-1]$ go along their paths as specified in $\bar{\boldsymbol{p}}_{[k-1]}$, we enlarge $[k-1]$ with a new agent $k \in \bar{\Delta} \backslash [k-1]$ and enlarge $\bar{\boldsymbol{p}}_{[k-1]}$ with a path $\bar{P}_k \in \bar{\mathscr{P}}_k$ in the following way. For each agent $j \in \bar{\Delta} \backslash [k-1]$ and vertex $v \in \bar{V}$, we define

$$\tau_j^v = \min\{t_j^v(\bar{\boldsymbol{p}}_{[k-1]}, R_j) \mid R_j \in \bar{\mathscr{P}}_j\}$$

as the "ideal arrival time" of agent j at vertex v, where $t_j^v(\bar{\boldsymbol{p}}_{[k-1]}, R_j)$ is j's arrival time at v assuming that all agents in $\bar{G}$ are only those in $[k-1] \cup \{j\}$ and they follow $(\bar{\boldsymbol{p}}_{[k-1]}, R_j)$. We select the candidates j and $\bar{P}_j$ for agent k and his path $\bar{P}_k$ step by step. Initially, let u denote the destination vertex d. Let $j \in \bar{\Delta} \backslash [k-1]$ and $\bar{P}_j \in \bar{\mathscr{P}}_j$ be such that

(S1) $t_j^u(\bar{\boldsymbol{p}}_{[k-1]}, \bar{P}_j)$ equals $\min_{i \in \bar{\Delta} \backslash [k-1]} \tau_i^u$, the earliest ideal arrival time at u among all agents in $\bar{\Delta} \backslash [k-1]$;

(S2) If more than one candidate $(j, \bar{P}_j)$ satisfy (S1), then the choice of $(j, \bar{P}_j)$ from the candidates is made such that the incoming edge to u on $\bar{P}_j$ has the highest possible priority;

(S3) If still more than one candidate $(j, \bar{P}_j)$ satisfy (S1) and (S2), then the paths $\bar{P}_j$ involved must share the same incoming edge e to u (whose tail vertex is denoted as u_e), and at least one such $\bar{P}_j$ satisfies that $t_j^{u_e}(\bar{\boldsymbol{p}}_{[k-1]}, \bar{P}_j)$ is the earliest ideal arrival time at vertex u_e among all agents in $\bar{\Delta} \backslash [k-1]$; we update u with u_e and go back to (S1) for further selections from the *current* candidates (i.e., those satisfying all (S1), (S2), (S3) checked), unless e is the initial edge of all candidate agents j.

The above process is repeated until either only one candidate pair $(j, \bar{P}_j)$ is left or the edge $e \in \bar{E}$ in (S3) becomes the initial edge of all the remaining candidate agents. In the former case, we set $(k, \bar{P}_k) = (j, \bar{P}_j)$. In the latter case, all candidate paths must be identical, and we choose agent k to be the head of queue $\mathbb{Q}_e^0$ and set $\bar{P}_k$ to be the identical candidate path. In either case, we enlarge $[k-1]$ by k, augment $\bar{\boldsymbol{p}}_{[k-1]}$ with $\bar{P}_k$, obtaining a larger agent subset $[k]$ and the associated path profile $\bar{\boldsymbol{p}}_{[k]}$. We then iterate the above procedure based on $[k]$ and $\bar{\boldsymbol{p}}_{[k]}$. A formal description of the process is presented in Algorithm 2 on page ec5.

Proofs. Let the game instance $\bar{\Gamma}^{\mathrm{N}}$ on $(\bar{G}, \bar{\Delta})$ be as specified in the input of Algorithm 2. To facilitate our discussions, we introduce some new notations. Given any path P in $\bar{G}$, a u-v subpath of P is often written as $P[u,v]$; furthermore, we write $P(u,v] = P[u,v] \backslash \{u\}$, $P[u,v) = P[u,v] \backslash \{v\}$ and $P(u,v) = P[u,v] \backslash \{u,v\}$.

Recall that o_i denotes the tail vertex of the initial edge of agent $i \in \bar{\Delta}$. Let agents $1, 2, \ldots$ of $\bar{\Delta}$ be indexed and path profile $\bar{\boldsymbol{p}} = (\bar{P}_k)_{k \in \bar{\Delta}}$ be computed as in Algorithm 2. Recall that $[0] = \emptyset$ and $\bar{\boldsymbol{p}}_{[0]}$ is the null profile. For any nonnegative integer k, any agent index j with $j > k$ and any vertex $v \in \bar{V}$, let

$$\ell_j^v [\![k]\!] := \min \left\{ t_j^v(\bar{\boldsymbol{p}}_{[k]}, R_j) \mid R_j \in \bar{\mathscr{P}}_j \right\}$$

denote the value ℓ_j^v computed for agent j in Step 3 at the $(k+1)$st iteration of Algorithm 2, i.e., the earliest time for agent j to reach vertex v, based only on the partial routing $\bar{\boldsymbol{p}}_{[k]}$ of agents in $[k]$. In particular, by definition, $t_j^v(\bar{\boldsymbol{p}}_{[j]}) = \ell_j^v [\![j-1]\!]$ for every agent j and every vertex $v \in \bar{P}_j$.

Algorithm 2 (Iteratively Dominant NE)

Input: game instance $\bar{\Gamma}^{\mathrm{N}}$: network $\bar{G}=(\bar{V},\bar{E})$ with initial queues $\mathbb{Q}_e^0, e\in\bar{E}$, where agent set $\bar{\Delta}=\cup_{e\in\bar{E}}\mathbb{Q}_e^0$.

Output: the special IDNE $\bar{\boldsymbol{p}}=(\bar{P}_k)_{k\in\bar{\Delta}}$ along with the corresponding agent indices 1, 2, ...

1. Initiate $\bar{\boldsymbol{p}}_{[0]}\leftarrow\emptyset$, $k\leftarrow 0$.
2. $k\leftarrow k+1$.

   (NB: Start to search for a new dominator k and his associated dominant path $\bar{P}_k$.)
3. **For** each agent $j\in\bar{\Delta}\backslash[k-1]$ and vertex $v\in\bar{V}$ **Do**
   - $\ell_j^v\leftarrow\min\left\{t_j^v(\bar{\boldsymbol{p}}_{[k-1]},R_j)\,|\,R_j\in\bar{\mathscr{P}}_j\right\}$;

     (NB: ℓ_j^v is the earliest time for j to reach vertex v, assuming that all other agents in $\bar{G}$ are those in $[k-1]$ and they go along their paths specified in $\bar{\boldsymbol{p}}_{[k-1]}$. Note that ℓ_j^v $(<\infty)$ is computable by the Dijkstra-like algorithm in Theorem EC.4 with partial path profile $\bar{\boldsymbol{p}}_{[k-1]}$ and agent j in place of $\boldsymbol{q}_{-i}$ and i over there, whose output τ^v is exactly ℓ_j^v.)
   - $\bar{\mathscr{P}}_j^v\leftarrow\left\{R_j[o_j,v]\,|\,R_j\in\bar{\mathscr{P}}_j \text{ and } t_j^v(\bar{\boldsymbol{p}}_{[k-1]},R_j)=\ell_j^v\right\}$.

     (NB: $\bar{\mathscr{P}}_j^v$ denotes the set of all paths starting with j's initial edge and ending at v along which j can reach v at time ℓ_j^v, under the above assumption. If there is no such a path in $\bar{G}$, then $\ell_j^v=\infty$ and $\bar{\mathscr{P}}_j^v=\emptyset$.)

   End-For
4. $C\leftarrow\bar{\Delta}\backslash[k-1]$, $P\leftarrow\emptyset$, $w\leftarrow d$.

   (NB: In the following while-loop, C is a set of candidates j for selecting k that will be pruned step by step; P is a subpath of $\bar{P}_j$ that will grow edge by edge starting from d; w is the latest vertex added to P; the value ℓ is strictly decreasing, which guarantees the termination of the while-loop.)
5. **While** $\ell:=\min_{j\in C}\ell_j^w\geq 1$ **Do**

   $C\leftarrow\{j\in C\,|\,\ell_j^w=\ell\}$;

   $uw\leftarrow$ the edge of the highest priority among all ending edges of paths in $\underset{j\in C}{\cup}\bar{\mathscr{P}}_j^w$;

   $P\leftarrow P\cup\{uw\}$;

   $w\leftarrow u$;

   End-While

   (NB: at the end of the while-loop the starting edge of P is the common initial edge of all agents in C.)
6. Let $k\in C$ be the agent who, at the very beginning, stands first (among all agents in C) on the starting edge of P.

   (NB: The agent k selected is called the *dominator of* $\bar{\Delta}\backslash[k-1]$.)
7. Let k be *associated* `with` $\bar{P}_k\leftarrow P$, $\bar{\boldsymbol{p}}_{[k]}\leftarrow(\bar{\boldsymbol{p}}_{[k-1]},\bar{P}_k)$.

   (NB: The algorithm outputs agent k and his associated dominant path $\bar{P}_k$.)
8. **If** $k<|\bar{\Delta}|$, **Then** go to Step 2.

LEMMA EC.2. *Let agent indices j,k and agent subset S satisfy $j \in S \subseteq \bar{\Delta} \backslash [k-1]$. Then for every vertex $v \in \bar{P}_k$ and every path profile $\boldsymbol{q} = (Q_h)_{h \in \bar{\Delta}}$ of game $\bar{\Gamma}^{\mathrm{N}}$, it holds that*

$$t_k^v(\bar{\boldsymbol{p}}_{[k]}, \boldsymbol{q}_{S \backslash \{k\}}) = \ell_k^v[\![k-1]\!] \leq t_j^v(\bar{\boldsymbol{p}}_{[k-1]}, \boldsymbol{q}_S) \tag{EC.2}$$

This lemma exhibits some invariance and dominance properties possessed by the agent order 1, 2, ... and the path profile $\bar{\boldsymbol{p}}$ computed by Algorithm 2. Specifically, for every agent index k, as long as all agents in $[k]$ follow $\bar{\boldsymbol{p}}_{[k]}$, the following properties hold, no matter what paths other agents outside $[k]$ choose (even if some or all of them are missing):

- *Invariant arrival times of agents in* $[k]$: the arrival times at any vertex of their paths in $\bar{\boldsymbol{p}}_{[k]}$ can never be affected. (This is what the equality in (EC.2) says.)
- *Universal domination of agents in* $[k]$*:* no agent $j \geq k+1$ can overtake any agent $i \in [k]$ at any vertex of his route $\bar{P}_i$, which is what the inequality in (EC.2) says. In particular, it follows that if j queues before some agent, then this agent is outside $[k]$.
- *Invariant influence of agents in* $[k]$: due to their arrival-time invariance, the agents in $[k]$ who queue at an edge at some time depends only on the time and the edge under consideration (but not on the choices of agents outside $[k]$), which implies that the agents in $[k]$ exert *invariant influence* on the movements of other agents outside $[k]$.
- *Property of no speed-up for agents in* $\bar{\Delta} \backslash [k]$: due to the invariant influence of agents in $[k]$, no agent $j \in \bar{\Delta} \backslash [k]$ can be sped up by other agent(s) in $\bar{\Delta} \backslash [k]$. More specifically, assuming j follows a path $R_j \in \bar{\mathscr{P}}_j$ and agents in $[k]$ follow $\bar{\boldsymbol{p}}_{[k]}$, the earliest arrival time of j at each vertex of R_j is attained when no other agents are involved — involving some or all agents from $\bar{\Delta} \backslash ([k] \cup \{j\})$ in the routing cannot make j's arrival time earlier at any vertex of R_j. (Note that this seemingly quite natural property does not hold in general. See Example 4 for more discussions.)

Proof of Lemma EC.2. Let e denote the incoming edge to d that has the highest priority w.r.t. $\prec_d$. For convenience, we may assume w.l.o.g. that e is the initial edge of some agent. Otherwise, we could add a dummy agent to $\mathbb{Q}_e^0$, which does not exert any influence on the original agents, nor the output of Algorithm 2 with the dummy agent and his unique path e ignored. (Note that the dummy agent would be the first output by the algorithm.)

We prove (EC.2) by induction on k. For the base case $k = 1$, under the above assumption, agent 1 is the head of $\mathbb{Q}_e^0$ and $\bar{P}_1 = e$. In any case, agent 1 reaches d at time 1, and (EC.2) is trivial. Suppose now $k \geq 2$ and (EC.2) is valid when k is smaller. This means that the invariant arrival times, universal domination and hence the *invariant influence* (resp. *no speed-up property*), as stated above, are true for agents in $[k-1]$ (resp. $\bar{\Delta} \backslash [k-1]$).

We claim $\ell_k^v[\![k-1]\!] \le \ell_j^v[\![k-1]\!]$. Otherwise, there would be a path $R_j \in \bar{\mathscr{P}}_j$ with $t_j^v(\bar{\boldsymbol{p}}_{[k-1]}, R_j) = \ell_j^v[\![k-1]\!]$ such that based on $\bar{\boldsymbol{p}}_{[k-1]}$, agent j would be able to use path $R_j[o_j, v] \cup \bar{P}_k[v, d] \in \bar{\mathscr{P}}_j$ to reach v earlier than k and subsequently reach all vertices of $\bar{P}_k(v, d]$, including d, no later than k, contradicting the choices of k and $\bar{P}_k$. The inequality part of (EC.2) thus follows from

$$\ell_k^v[\![k-1]\!] \le \ell_j^v[\![k-1]\!] \le t_j^v(\bar{\boldsymbol{p}}_{[k-1]}, Q_j) \le t_j^v(\bar{\boldsymbol{p}}_{[k-1]}, \boldsymbol{q}_S),$$

where the second inequality is by definition and the third is due to the no speed-up property of $\bar{\Delta}\backslash[k-1]$: agent j cannot be sped up by agents in $S\backslash\{j\}$.

Note that $t_k^v(\bar{\boldsymbol{p}}_{[k-1]}, \bar{P}_k) \le t_k^v(\bar{\boldsymbol{p}}_{[k-1]}, \bar{P}_k, \boldsymbol{q}_{S\backslash\{k\}})$, because k cannot be sped up by agents in $S\backslash\{k\}$. Thus, $t_k^v(\bar{\boldsymbol{p}}_{[k]}) = \ell_k^v[\![k-1]\!]$ implies

$$t_k^v(\bar{\boldsymbol{p}}_{[k]}, \boldsymbol{q}_{S\backslash\{k\}}) \ge \ell_k^v[\![k-1]\!].$$

So it remains to show that $t_k^v(\bar{\boldsymbol{p}}_{[k]}, \boldsymbol{q}_{S\backslash\{k\}}) \le \ell_k^v[\![k-1]\!]$, i.e., agents in $S\setminus\{k\}$ do not slow down agent k. Suppose on the contrary that $t_k^v(\bar{\boldsymbol{p}}_{[k]}, \boldsymbol{q}_{S\backslash\{k\}}) > \ell_k^v[\![k-1]\!] = t_k^v(\bar{\boldsymbol{p}}_{[k]})$ for some vertex $v \in \bar{P}_k$. Let v be the first such vertex encountered when traveling along $\bar{P}_k$, indicating that

(1) $t_k^w(\bar{\boldsymbol{p}}_{[k]}, \boldsymbol{q}_{S\backslash\{k\}}) = \ell_k^w[\![k-1]\!]$ for every vertex $w \in \bar{P}_k[o_k, v) = \bar{P}_k[o_k, v]\backslash\{v\}$.

In view of the invariant influence from agents in $[k-1]$ who follow $\bar{\boldsymbol{p}}_{[k-1]}$, there must exist some agent $i \in S\backslash\{k\}$ and an edge $xy \in \bar{P}_k[o_k, v]$ such that i slows down k on xy, or more precisely, under $(\bar{\boldsymbol{p}}_{[k]}, \boldsymbol{q}_{S\backslash\{k\}})$, agent i enters xy earlier than k, or enters xy at the same time as k and queues before k at xy. Let xy be the *first* such edge encountered when traveling along $\bar{P}_k[o_k, v]$. Observe that $x \in \bar{P}_k[o_k, v)$. By (1), we have

(2) under routing $(\bar{\boldsymbol{p}}_{[k]}, \boldsymbol{q}_{S\backslash\{k\}})$, agent i reaches vertex x and enters edge xy at time $t_i^x(\bar{\boldsymbol{p}}_{[k]}, \boldsymbol{q}_{S\backslash\{k\}}) \le t_k^x(\bar{\boldsymbol{p}}_{[k]}, \boldsymbol{q}_{S\backslash\{k\}}) = \ell_k^x[\![k-1]\!]$.

Construct a path $R_i := Q_i[o_i, x] \cup \bar{P}_k[x, d] \in \bar{\mathscr{P}}_i$ for agent i. Note that

(3) $t_i^x(\bar{\boldsymbol{p}}_{[k-1]}, R_i) = t_i^x(\bar{\boldsymbol{p}}_{[k-1]}, Q_i) \le t_i^x(\bar{\boldsymbol{p}}_{[k]}, \boldsymbol{q}_{S\backslash\{k\}})$,

where the equality follows from the definition of R_i and the inequality is due to the no speed-up property of $\bar{\Delta}\backslash[k-1]$: agent $i \notin [k]$ cannot be sped up by agents in $(S\cup\{k\})\backslash\{i\}$. In turn, we deduce from (3) and (2) that $t_i^x(\bar{\boldsymbol{p}}_{[k-1]}, R_i) \le \ell_k^x[\![k-1]\!] = t_k^x(\bar{\boldsymbol{p}}_{[k-1]}, \bar{P}_k)$. Consequently,

(4) $t_i^w(\bar{\boldsymbol{p}}_{[k-1]}, R_i) \le t_k^w(\bar{\boldsymbol{p}}_{[k]}) = \ell_k^w[\![k-1]\!]$ for each vertex $w \in R_i[x, d] = \bar{P}_k[x, d]$.

By the definition of agent k from Algorithm 2, we derive from (4) that

$$t_i^w(\bar{\boldsymbol{p}}_{[k-1]}, R_i) = \ell_k^w[\![k-1]\!] \text{ for each vertex } w \in R_i[x, d] = \bar{P}_k[x, d].$$

Consider $w = x$ in the above equation, we derive from (3) and (2) that

$$\ell_k^x[\![k-1]\!] = t_i^x(\bar{\boldsymbol{p}}_{[k-1]}, R_i) \le t_i^x(\bar{\boldsymbol{p}}_{[k]}, \boldsymbol{q}_{S\backslash\{k\}}) \le t_k^x(\bar{\boldsymbol{p}}_{[k]}, \boldsymbol{q}_{S\backslash\{k\}}) = \ell_k^x[\![k-1]\!].$$

The string of inequalities enforces that under $(\bar{\boldsymbol{p}}_{[k]}, \boldsymbol{q}_{S\setminus\{k\}})$, agents i and k enter xy at the same time, and therefore i queues before k at xy (recall that i slows down k on xy). So it must be the case that

- either (if R_i and $\bar{P}_k$ have different incoming edges to x) R_i has a higher priority incoming edge into x than $\bar{P}_k$ does,
- or (by the choice of edge xy, i.e., o_k's proximity to xy) $\{i,k\} \subseteq \mathbb{Q}^0_{xy}$, and agent i queues before agent k at their common initial edge xy.

However, the choice made at the kth iteration of Algorithm 2 excludes the possibilities of both cases. This completes the proof. Q.E.D.

By Lemma EC.1, profile $\bar{\boldsymbol{p}}$ is an IDNE of $\bar{\Gamma}^{\mathrm{N}}$ if and only if the restriction of $\bar{\boldsymbol{p}}$ to G is an IDNE of Γ^{N}. The following establishes Theorem 1.

Theorem EC.2. *Algorithm 2 finds an IDNE of game* $\bar{\Gamma}^{\mathrm{N}}$*.*

Proof. For any agent $j \in \Delta \setminus [k-1]$, partial path profiles $\boldsymbol{q}_{-[k]}$ and $\boldsymbol{r}_{-[k-1]}$ for $\Delta\setminus[k]$ and $\Delta\setminus[k-1]$, it is instant from (EC.2) that $t^v_k(\bar{\boldsymbol{p}}_{[k]}, \boldsymbol{q}_{-[k]}) = \ell^v_k[\![k]\!] \le t^v_j(\bar{\boldsymbol{p}}_{[k-1]}, \boldsymbol{r}_{-[k-1]})$, which shows the validity of (EC.1) and thus that $\bar{\boldsymbol{p}}$ is an IDNE of $\bar{\Gamma}^{\mathrm{N}}$. Q.E.D.

EC.3. Generalized iterative dominance

As can be seen from the proof of Lemma EC.2, our induction hypothesis only involves the equation part of (EC.2), which guarantees the critical invariant influence property. This leads us to the following generalization of Algorithm 2 (see page ec9), which computes an iteratively dominant *partial* path profile *based* on a fixed routing of some special agents.

The verbatim adaption of the proof of Lemma EC.2 gives the following generalization for iterative dominance. It plays critical roles in proving the equilibrium properties presented in Sections 4.3 and 5.3.

Lemma EC.3. *Regarding Algorithm 3, if* $j \in S \subseteq \bar{\Delta}\setminus(U \cup [i-1])$*, then for every vertex* $v \in \bar{P}_i$ *and path profile* $\boldsymbol{q}$ *of game* $\bar{\Gamma}^{\mathrm{N}}$*, it holds that*

$$t^v_i(\boldsymbol{b}, \bar{\boldsymbol{p}}_{[i]}, \boldsymbol{q}_{S\setminus[i]}) = \min_{R_i \in \bar{\mathscr{P}}_i} t^v_i(\boldsymbol{b}, \bar{\boldsymbol{p}}_{[i-1]}, R_i) \le t^v_j(\boldsymbol{b}, \bar{\boldsymbol{p}}_{[i-1]}, \boldsymbol{q}_S).$$

EC.4. Agent preemptions

This section elaborates on the notion of preemption (introduced in Section 4.4.1) for game $\bar{\Gamma}^{\mathrm{N}}$. Recall that, under some routing, if an agent does not reach a vertex, then we regard his arrival time at the vertex as infinity.

Algorithm 3 (Iteratively Dominant Partial Path Profile with a Base)

Input: game instance $\bar{\Gamma}^{\mathrm{N}}$: network $\bar{G}$ with agent set $\bar{\Delta}$, a partial path profile $\boldsymbol{b}=(B_h)_{h\in U}$ for a (possibly empty) finite subset $U\subseteq\bar{\Delta}$ that satisfies the following *arrival-time invariance*: for every agent $h\in U$ and every vertex $v\in B_h$, the arrival time $t_h^v(\boldsymbol{b},\boldsymbol{q}_S)$ of h at v is an invariant against changing partial path profile $\boldsymbol{q}_S$, i.e., it is the same over all path profiles $\boldsymbol{q}$ of $\bar{\Gamma}^{\mathrm{N}}$ and agent subsets $S\subseteq\bar{\Delta}\backslash U$.

Output: the special iteratively dominant partial path profile (routing) $\bar{\boldsymbol{p}}=(\bar{P}_i)_{i\in\bar{\Delta}\backslash U}$ for $\bar{\Delta}\backslash U$ along with the corresponding agent indices 1, 2, ...

1. Initiate $\bar{\boldsymbol{p}}_{[0]}\leftarrow\emptyset$, $i\leftarrow 0$.
2. $i\leftarrow i+1$

   (NB: Start to search for a new dominator i and his associated dominant path $\bar{P}_i$.)
3. **For** each agent $j\in\bar{\Delta}\backslash(U\cup[i-1])$ and vertex $v\in\bar{V}$ **Do**
   - $\ell_j^v\leftarrow\min\{t_j^v(\boldsymbol{b},\bar{\boldsymbol{p}}_{[i-1]},R_j)\,|\,R_j\in\bar{\mathscr{P}}_j\}$
   - $\bar{\mathscr{P}}_j^v\leftarrow\{R_j[o_j,v]\,|\,R_j\in\bar{\mathscr{P}}_j$ and $t_j^v(\boldsymbol{b},\bar{\boldsymbol{p}}_{[i-1]},R_j)=\ell_j^v\}$

   End-For
4. $C\leftarrow\bar{\Delta}\backslash(U\cup[i-1])$, $P\leftarrow\emptyset$, $w\leftarrow d$.
5. Run Steps 5 to 7 of Algorithm 2 to identify dominator i of $\bar{\Delta}\backslash(U\cup[i-1])$ and his associated dominant path $\bar{P}_i$.

   (NB: The algorithm returns agent i and his associated dominant path $\bar{P}_i$.)
6. Set $\bar{\boldsymbol{p}}_{[i]}\leftarrow(\bar{\boldsymbol{p}}_{[i-1]},\bar{P}_i)$.
7. **If** $i<|\bar{\Delta}|-|U|$, **Then** go to Step 2.

Throughout this section, given game $\bar{\Gamma}^{\mathrm{N}}$ on network $\bar{G}=(\bar{V},\bar{E})$ with agent set $\bar{\Delta}$, let i denote a fixed agent in $\bar{\Delta}$, and $\boldsymbol{q}_{-i}=(Q_j)_{j\in\bar{\Delta}\backslash\{i\}}$ denote a fixed partial path profile of all other agents. We consider the scenario where only agent i is allowed to change his path. For each vertex $v\in\bar{V}$, define

$$\tau^v:=\min_{P_i\in\bar{\mathscr{P}}_i}\{t_i^v(P_i,\boldsymbol{q}_{-i})\}$$

as the earliest time at which agent i can reach vertex v by unilaterally changing his path (if $\bar{\mathscr{P}}_i$ contains no path through v, then we set $\tau^v:=+\infty$). Analogously, for each agent $j\in\bar{\Delta}\backslash\{i\}$ and vertex $v\in Q_j$, define

$$\tau_j^v:=\min_{P_i\in\bar{\mathscr{P}}_i}\{t_j^v(P_i,\boldsymbol{q}_{-i})\}$$

as the earliest time at which agent j can reach vertex v when agent i unilaterally changes his path. We emphasize that j keeps following his path Q_j (specified by $\boldsymbol{q}_{-i}$) in the definition of τ_j^v.

In the following, for any non-singleton path P in $\bar{G}$ and any non-starting vertex v of P, we use $e_v(P)$ to denote the incoming edge to v on P. By virtue of the technical transformation in Section EC.1, the preempt relation defined for game Γ^{N} in Section 4.4.1 translates to the following simplified definition for preemptions in $\bar{\Gamma}^{\mathrm{N}}$.

DEFINITION EC.1 (PREEMPTION). For every agent $j \in \bar{\Delta}\backslash\{i\}$ and vertex $v \in Q_j\backslash\{o_j\}$, we say that agent i *preempts* agent j at vertex v under $\boldsymbol{q}_{-i}$ if either $\tau^v < \tau_j^v$, or $\tau^v = \tau_j^v$ and $v \neq o_i$ is on some path $P_i \in \bar{\mathscr{P}}_i$ such that $t_i^v(P_i, \boldsymbol{q}_{-i}) = \tau^v$ and $e_v(P_i) \preceq_v e_v(Q_j)$.

Define vertex subset

$$\bar{Y} := \{v \in \bar{V} \mid \tau^v < \infty\}.$$

For each $v \in \bar{Y}$, let $O_i^v \in \bar{\mathscr{P}}_i$ denote the path achieving $\tau^v = t_i^v(O_i^v, \boldsymbol{q}_{-i})$ such that the priority of $e_v(O_i^v)$ w.r.t. $\prec_v$ is as high as possible. It is clear that

$$\begin{array}{l}\text{If } i \text{ preempts } j \text{ at } v\text{, then either } t_i^v(O_i^v, \boldsymbol{q}_{-i}) < \tau_j^v, \\ \text{or } t_i^v(O_i^v, \boldsymbol{q}_{-i}) = \tau_j^v \text{ and } e_v(O_i^v) \preceq_v e_v(Q_j).\end{array} \tag{EC.3}$$

For each vertex $v \in \bar{V}$, we denote $\mathcal{A}_v \subset \bar{\Delta}$ as the set of agents j other than i whose arrival times at v can be *affected by* i (with his unilateral path change), i.e., there exist $P_i, P_i' \in \bar{\mathscr{P}}_i$ such that $t_j^v(P_i, \boldsymbol{q}_{-i}) < t_j^v(P_i', \boldsymbol{q}_{-i})$.

LEMMA EC.4. *If $\mathcal{A}_v \neq \emptyset$, then v is on some path in $\bar{\mathscr{P}}_i$, i.e., $\tau^v < \infty$.*

Proof. Suppose $j \in \mathcal{A}_v$ and agent j's arrival time at v can be influenced. Let $e_v(Q_j) = uv$ be the incoming edge to v on Q_j. If $\mathcal{A}_u = \emptyset$ and uv is not contained in any path in $\bar{\mathscr{P}}_i$, then no matter which path agent i switches to, the arrival times at u of all agents in $\bar{\Delta}\backslash\{i\}$ and hence j's queuing time at edge uv remain the same as those under $\boldsymbol{q}$, which shows a contradiction to $j \in \mathcal{A}_v$. Therefore, either uv and hence v are contained in some path in $\bar{\mathscr{P}}_i$, in which case we are done, or $\mathcal{A}_u \neq \emptyset$, to which we can apply backward induction (as $\bar{G}$ is acyclic) to derive a path $P \in \bar{\mathscr{P}}_i$ that contains u, giving $v \in P[o_i, u] \cup \{uv\} \cup Q_j[v, d] \in \bar{\mathscr{P}}_i$, as desired. Q.E.D.

LEMMA EC.5. *For any agent $j \in \bar{\Delta}\backslash\{i\}$ and vertex $v \in Q_j$, if there exist paths $P_i, P_i' \in \bar{\mathscr{P}}_i$ such that $t_j^v(P_i, \boldsymbol{q}_{-i}) \neq t_j^v(P_i', \boldsymbol{q}_{-i})$, then $v \in Q_j\backslash\{o_j\}$ and i preempts j at v under $\boldsymbol{q}_{-i}$.*

Proof. Recall from the Unit Assumption that all edges of network $\bar{G}$ have a unit capacity and a unit length. Apparently, if $j \in \mathcal{A}_v$, then it must be the case that $v \in Q_j(o_j, d]$. The lemma can be restated as: agent i preempts all agents of $\mathcal{A}_v$ at vertex v. Notice from Lemma EC.4 that $\{v \mid \mathcal{A}_v \neq \emptyset\} \subseteq \bar{Y}$. To prove the lemma, it suffices to prove that

$$\text{For any vertex } v \in \bar{Y}\text{, agent } i \text{ preempts every agent } j \in \mathcal{A}_v \text{ at vertex } v. \tag{EC.4}$$

Since $\bar{G}$ is acyclic, there exists a complete order on the vertices in $\bar{Y}$ which is *acyclic* in that for each edge with both end-vertices in $\bar{Y}$, its tail vertex has an order smaller than its head vertex. We will verify (EC.4) by induction on the order of the vertices in $\bar{Y}$.

Suppose that o_ia is the initial edge of agent i, which is contained in every path in $\bar{\mathscr{P}}_i$. Therefore, $a \in \bar{Y}$. Apparently, the order of vertex a is the smallest, and the base case where $v = a$ is trivial because of $\mathcal{A}_a = \emptyset$. To proceed inductively, assume that (EC.4) is true for all vertices in $\bar{Y}$ with orders smaller than v.

Since the case $\mathcal{A}_v = \emptyset$ is trivial, we suppose now $\mathcal{A}_v \neq \emptyset$ and consider an arbitrary agent $j \in \mathcal{A}_v$ with $e_v(Q_j) = uv$. In the following, we prove first that agent i preempts agent j at vertex u, then show the preemption at vertex v.

If $j \in \mathcal{A}_u$, since u has a smaller order than v, then by induction hypothesis, agent i preempts agent j at vertex u. If $j \notin \mathcal{A}_u$, then no matter how i changes his path, agent j's arrival time at u cannot be influenced by i. On the other hand, since $j \in \mathcal{A}_v$, there exist $P_i, P_i' \in \bar{\mathscr{P}}_i$ such that $t_j^v(P_i, \boldsymbol{q}_{-i}) < t_j^v(P_i', \boldsymbol{q}_{-i})$. Then, combining $j \notin \mathcal{A}_u$ and $j \in \mathcal{A}_v$, we deduce that one of the two following cases must happen:

(a) There exists agent $h \in \mathcal{A}_u$ with $uv \in Q_h \cap Q_j$ such that $t_h^u(P_i', \boldsymbol{q}_{-i}) < t_j^u(P_i', \boldsymbol{q}_{-i})$, or $t_h^u(P_i', \boldsymbol{q}_{-i}) = t_j^u(P_i', \boldsymbol{q}_{-i})$ and $e_u(Q_h) \prec_u e_u(Q_j)$, i.e., j queues at uv under $(P_i', \boldsymbol{q}_{-i})$ for a longer time than he does under $(P_i, \boldsymbol{q}_{-i})$ due to h's presence (resp. absence) at uv at the time j reaches u under $(P_i', \boldsymbol{q}_{-i})$ (resp. $(P_i, \boldsymbol{q}_{-i})$).

(b) Edge $uv \in P_i' \cap Q_j$ and $t_i^u(P_i', \boldsymbol{q}_{-i}) < t_j^u(P_i', \boldsymbol{q}_{-i})$, or $t_i^u(P_i', \boldsymbol{q}_{-i}) = t_j^u(P_i', \boldsymbol{q}_{-i})$ and $e_u(P_i') \prec_u e_u(Q_j)$, i.e., the role of h in the above case is played by i here.

In case (a), by the induction hypothesis, i preempts all agents in $\mathcal{A}_u$ and in particular h at vertex u. Thus by (EC.3), we have $\tau^u = t_i^u(O_i^u, \boldsymbol{q}_{-i}) \le \tau_h^u \le t_h^u(P_i', \boldsymbol{q}_{-i}) \le t_j^u(P_i', \boldsymbol{q}_{-i})$ and the inequalities hold with equalities only if $e_u(O_i^u) \prec_u e_u(Q_h) \prec_u e_u(Q_j)$. Since $j \notin \mathcal{A}_u$, it follows that $t_j^u(P_i', \boldsymbol{q}_{-i}) = \tau_j^u$, and further that agent i preempts agent j at vertex u.

In case (b), $\tau^u = t_i^u(O_i^u, \boldsymbol{q}_{-i}) \le t_i^u(P_i', \boldsymbol{q}_{-i}) \le t_j^u(P_i', \boldsymbol{q}_{-i}) = \tau_j^u$ and the inequalities hold with equalities only if $e_u(O_i^u) \preceq_u e_u(P_i') \prec_u e_u(Q_j)$, which shows that i preempts agent j at vertex u. Hence, no matter whether j belongs to $\mathcal{A}_u$ or not, agent i always preempts agent j at vertex u.

Next we prove i preempts j at vertex v. Suppose that path $R_i \in \bar{\mathscr{P}}_i$ satisfies $\tau_j^v = t_j^v(R_i, \boldsymbol{q}_{-i})$. Notice that $\breve{O}_i := O_i^u[o_i, u] \cup Q_j[u, d] \in \bar{\mathscr{P}}_i$. Under the path profile $(\breve{O}_i, \boldsymbol{q}_{-i})$, consider first the case where i moves along edge uv immediately after he reaches u, i.e., there is no queue before him over there. In this case, $t_i^v(\breve{O}_i, \boldsymbol{q}_{-i}) = \tau^u + 1 \le \tau_j^u + 1 \le t_j^u(R_i, \boldsymbol{q}_{-i}) + 1 \le t_j^v(R_i, \boldsymbol{q}_{-i}) = \tau_j^v$. Combining this with the facts that $\tau^v \le t_i^v(\breve{O}_i, \boldsymbol{q}_{-i})$ and $e_v(\breve{O}_i) = uv = e_v(Q_j)$, we can deduce that i preempts j at v. Now we are left with the case where under $(\breve{O}_i, \boldsymbol{q}_{-i})$ agent i spends at least one time unit queuing at uv, i.e., there is a nonempty queue before him at the time he reaches u. Let $\mathcal{B}$ be the set of agents in this queue and those who pass through uv earlier than that queue. Let $h \in \mathcal{B}$ be the last agent in that queue, i.e., he queues at uv right before i: $t_i^v(\breve{O}_i, \boldsymbol{q}_{-i}) = t_h^v(\breve{O}_i, \boldsymbol{q}_{-i}) + 1$. Since $t_i^u(\breve{O}_i, \boldsymbol{q}_{-i}) = \tau^u$ (by the definition of $\breve{O}_i$), it follows from Definition EC.1 that i cannot preempt any agent in $\mathcal{B}$ at vertex u. Now as i preempts all agents in $\mathcal{A}_u$ at u by the inductive hypothesis, we see that $\mathcal{B} \cap \mathcal{A}_u = \emptyset$ and further that, no matter how i changes his path, every agent in $\mathcal{B}$ travels along uv at the same time and his arrival time at v is not affected, which gives $\mathcal{B} \cap \mathcal{A}_v = \emptyset$. Thus, $t_h^v(R_i, \boldsymbol{q}_{-i}) = \tau_h^v = t_h^v(\breve{O}_i, \boldsymbol{q}_{-i})$. Recall that i preempts j at vertex u and $uv \in Q_h \cap Q_j$. Therefore, no matter how i chooses his path, agent j will always arrive at vertex v at least one time unit later than h. So, by the definition of path R_i, we have $\tau_j^v = t_j^v(R_i, \boldsymbol{q}_{-i}) \geq t_h^v(R_i, \boldsymbol{q}_{-i}) + 1 = t_h^v(\breve{O}_i, \boldsymbol{q}_{-i}) + 1 = t_i^v(\breve{O}_i, \boldsymbol{q}_{-i})$. This along with the facts that $\tau^v \leq t_i^v(\breve{O}_i, \boldsymbol{q}_{-i})$ and $e_v(\breve{O}_i) = e_v(Q_j)$ implies that agent i preempts agent j at vertex v, as desired. Q.E.D.

Note that what the last paragraph of the above proof does is to derive agent i's preemption over agent j at vertex v from his preemption at vertex u, where uv is an edge of Q_j. This particularly gives the following stronger result.

COROLLARY EC.1. *Given i and $\boldsymbol{q}_{-i}$, if agent i preempts agent $j \in \bar{\Delta} \backslash \{i\}$ at vertex $v \in Q_j$, then i preempts j at all vertices on the subpath $Q_j[v, d]$.*

REMARK EC.1. Edge priorities play an important role in defining the preemption and validating Lemma EC.5 (equivalently, Lemma 1 in Section 4.4.1) and several results that follow from it. The properties implied by Lemma 1 might be invalid if global priorities were placed on agents (as in Scarsini et al. 2018). For example, consider a modification of the game presented in Example 3, where the edge $y_2 d$ is subdivided by a newly added vertex. Suppose that the path profile $(P_h, \boldsymbol{q}_{-h})$ is such that agents g and i both choose their upper paths and agent h chooses his lower path. Under this path profile, agent g reaches destination d at time 5, one time unit after agent i. Note that agent h is able to affect g's arrival time at vertex d (decrease it to 4) by switching to his upper path P_h'. However, fixing $\boldsymbol{q}_{-h}$ (i.e., the upper path choices of g and i), agent h is unable to reach d at time 4 or earlier in any case.

EC.5. Computation of EE best-responses

By virtue of Lemma EC.5 established for agent preemptions, we prove in this section the correctness of the Dijkstra-like algorithm presented in Section 4.4.2 for computing EE best-responses.

Given an arbitrarily fixed agent $i \in \bar{\Delta}$ and an arbitrarily fixed partial path profile $\boldsymbol{q}_{-i}$ for other agents in game $\bar{\Gamma}^{\mathrm{N}}$, the EE best-response of agent i to $\boldsymbol{q}_{-i}$ is defined as in Definition 5 with $\bar{\mathscr{P}}_i$ in place of $\mathscr{P}_i$. The agent sets $\mathcal{Q}_e^r$ and $\mathcal{Q}_{e,e'}^r$ given in Definition 6 are now defined w.r.t. $(\bar{G}, \bar{\Delta})$ instead of (G, Δ). We have denoted, for each vertex $v \in \bar{V}$, agent i's earliest achievable arrival time at v as $\tau^v := \min\{t_i^v(P_i, \boldsymbol{q}_{-i}) \,|\, P_i \in \bar{\mathscr{P}}_i\}$. As in Section EC.4, there exists an acyclic complete order on the vertices of $\bar{Y} = \{v \in \bar{V} \,|\, \tau^v < \infty\}$ such that for each edge of $\bar{E}$ with both end-vertices in $\bar{Y}$, its tail vertex has an order smaller than its head vertex. Recalling the transformation in Section EC.1, it is apparent that the vertices in $\bar{Y}\backslash V$ have smaller orders (if any) than those in $\bar{Y} \cap V = Y = \{v \in V \,|\, \tau^v < \infty\}$, which is defined in Section 4.4.2. When $\bar{Y}\backslash V \neq \emptyset$, there is only one edge between $\bar{Y}\backslash V$ and Y, i.e., the one incoming to i's origin vertex at G. So, it is clear from Lemma EC.1 that the correctness of the Dijkstra-like algorithm for $\bar{\Gamma}^{\mathrm{N}}$ implies directly its correctness for Γ^{N}.

Since all agents are inside $\bar{G}$ at time 0, the initial setting of our dynamic program is now simplified: If $e = uv$ is the initial edge of agent i, then trivially $\tau^u = 0$, and we initially use the symbol $\mathcal{Q}^0_{e,\hat{e}_u}$ to denote the set of agents in $\mathbb{Q}^0_e$ who queue after i. The following result shows the correctness of the Dijkstra-like algorithm (Algorithm 1) for computing EE best-responses.

THEOREM EC.4. *Let E' denote the set of edges on paths in $\bar{\mathscr{P}}_i$. For any vertex $v \in \bar{Y}$ that is not i's starting vertex, it holds that*

$$\tau^v = \min_{u:uv\in E'} \left\{\tau^u + \left|\mathcal{Q}^{\tau^u}_{uv} \backslash \mathcal{Q}^{\tau^u}_{uv,\hat{e}_u}\right| + 1\right\}, \tag{EC.5}$$

where, when u is not i's starting vertex, $\hat{e}_u$ is the edge wu in $\operatorname{argmin}_{wu\in E'} \left\{\tau^w + 1 + \left|\mathcal{Q}^{\tau^w}_{wu} \backslash \mathcal{Q}^{\tau^w}_{wu,\hat{e}_w}\right|\right\}$ *that has the highest priority (w.r.t. $\prec_u$).*

Proof. We prove (EC.5) by induction on the order of those vertices in $\bar{Y}$. The base case where v is the head of i's initial edge is trivial. Let us consider the case where v is not the head of i's initial edge, and suppose (EC.5) is true for vertices $u \in Y$ with orders smaller than v.

We claim that, for every edge $uv \in E'$, no matter how i chooses his path, the arrival times of agents in $\mathcal{Q}^{\tau^u}_{uv} \setminus \mathcal{Q}^{\tau^u}_{uv,\hat{e}_u}$ at vertex u will never be influenced. Suppose the contrary. Then, by Lemma EC.5, agent i preempts at least one agent $j \in \mathcal{Q}^{\tau^u}_{uv} \setminus \mathcal{Q}^{\tau^u}_{uv,\hat{e}_u}$ at vertex u under $\boldsymbol{q}_{-i}$. Note first from the definition of $\mathcal{Q}^{\tau^u}_{uv}$ that $\tau^u_j \leq \tau^u$, where τ^u_j is the earliest time j can reach u when i changes his path. By Definition EC.1, it can only be the case that $\tau^u_j = \tau^u$ and i is able to arrive at u at time τ^u via an edge e' that has a priority no lower than the one taken by j. By induction hypothesis, $\tau^u = \min_{wu\in E'} \left\{\tau^w + 1 + \left|\mathcal{Q}^{\tau^w}_{wu} \backslash \mathcal{Q}^{\tau^w}_{wu,\hat{e}_w}\right|\right\}$; in turn the definition of $\hat{e}_u$ implies that the priority of $\hat{e}_u$ is not lower than that of e', and hence not lower than that of the edge taken by j. However, this is impossible because $j \notin \mathcal{Q}^{\tau^u}_{uv,\hat{e}_u}$. Hence the claim is valid. Therefore, regardless of i's choice, all agents in $\mathcal{Q}^{\tau^u}_{uv} \setminus \mathcal{Q}^{\tau^u}_{uv,\hat{e}_u}$ arrive at u no later than τ_u and those arriving at time τ^u (if any) use incoming edges to u with priorities higher than edge $\hat{e}_u$. It follows from the definition of $\tau^u = \min\{t_i^u(P_i, \boldsymbol{q}_{-i}) \,|\, P_i \in \bar{\mathscr{P}}_i\}$, induction hypothesis on u and definition of $\hat{e}_u$ that i cannot move along uv until all agents in $\mathcal{Q}^{\tau^u}_{uv} \setminus \mathcal{Q}^{\tau^u}_{uv,\hat{e}_u}$ exit uv.

Consequently, if agent i uses edge $uv \in E'$ to reach v, his arrival time at v is at least $\tau_u + |\mathcal{Q}_{uv}^{\tau^u} \setminus \mathcal{Q}_{uv,\hat{e}_u}^{\tau^u}| + 1$. On the other hand, by the induction hypothesis, this value is obtainable by reaching u at time τ^u via $\hat{e}_u$. It follows that the earliest time i can reach v via edge uv is exactly $\tau_u + |\mathcal{Q}_{uv}^{\tau^u} \setminus \mathcal{Q}_{uv,\hat{e}_u}^{\tau^u}| + 1$. Since i must use one edge $uv \in E'$ to reach v, the correctness of (EC.5) is established. Q.E.D.

The subgraph of $\bar{G}$ spanned by all edges $\hat{e}_v$, $v \in \bar{Y} \setminus \{o_i\}$ defined in Theorem EC.4 contains a unique o_i-d path. By Definition 5, it is the EE best-response of agent i to $\boldsymbol{q}_{-i}$.

EC.6. Characterization of NEs

In this section, we first make some observations on agent interactions, then establish the iterative batch-dominance characterization of all NEs of game $\bar{\Gamma}^{\mathrm{N}}$ and hence Γ^{N}.

EC.6.1. Agent precedence

We investigate the precedence relations between agents under the same (partial) routing of $\bar{\Gamma}^{\mathrm{N}}$. These relations are much more direct and visible than the preemption relations (see Definition EC.1), which generally involve two different routings.

Given game $\bar{\Gamma}^{\mathrm{N}}$ on $(\bar{G}, \bar{\Delta})$, every (partial) path profile $\boldsymbol{q}_S = (Q_i)_{i \in S}$ of $\bar{\Gamma}^{\mathrm{N}}$ for agents in $S \subseteq \bar{\Delta}$ is often considered as a routing for the game restricted to agents in S, where each agent $i \in S$ follows Q_i. For any agent $i \in S$ and vertex $v \in \bar{G}$, we use $t_i^v(\boldsymbol{q}_S)$ to denote agent i's arrival time at v under routing $\boldsymbol{q}_S$.

Definition EC.2 (Precedence). Given a (partial) path profile $\boldsymbol{q}_S$ of game $\bar{\Gamma}^{\mathrm{N}}$, and agents $i, j \in S$, we say that agent i *strongly precedes* agent j through vertex v under $\boldsymbol{q}_S$ at time $t_i^v(\boldsymbol{q}_S)$ if under routing $\boldsymbol{q}_S$ they both pass v and i reaches v earlier than j. We say that i *precedes* j through vertex v under $\boldsymbol{q}_S$ at time $t_i^v(\boldsymbol{q}_S)$ if either i strongly precedes j through vertex v, or i and j reach v at the same time but i comes from an edge (incoming to v) with a higher priority than the edge from which agent j comes.

Observe from the above definition that if agent i precedes agent j through a vertex u and both i and j choose to enter the same edge uv, then i strongly precedes j through vertex v. It is possible that agent i strongly precedes agent j through some vertex and j strongly precedes i through another vertex, even under NEs (see Example EC.2 in Section EC.12). We emphasize again that while the notion of preemption (Definition EC.1) compares the arrival times of two agents at the same vertex under possibly *different* path profiles, precedence compares two arrival times under the *same* (partial) path profile. Unlike the Braess-like paradox presented in Example 4, as far as precedence is concerned, the following lemma accords with the intuition that fewer agents lead to faster travel.

Lemma EC.6. *Let S and T be agent subsets with $\emptyset \neq S \subset T \subseteq \bar{\Delta}$, and $\boldsymbol{q}_T$ be a partial path profile for agents in T. If under $\boldsymbol{q}_T$ some agent in $T\backslash S$ precedes an agent in S at some time τ, then there exists agent $i \in T\backslash S$ such that under $\boldsymbol{q}_{S\cup\{i\}}$ agent i precedes some agent in S no later than τ.*

Proof. Suppose that under $\boldsymbol{q}_T$, agent $i \in T\backslash S$ precedes agent $j \in S$ through some vertex v, and further that $t_i^v(\boldsymbol{q}_T)$ is as small as possible. The minimality implies that $t_i^v(\boldsymbol{q}_T) \leq \tau$, and under $\boldsymbol{q}_T$ no agent in $T\backslash(S\cup\{i\})$ precedes any agent in S before time $t_i^v(\boldsymbol{q}_T)$. So removing $\boldsymbol{q}_{T\backslash(S\cup\{i\})}$ (i.e., removing agents of $T\backslash(S\cup\{i\})$ and their paths) from routing $\boldsymbol{q}_T$ can only possibly reduce i's queuing time before the time when he reaches v, accelerating his arrival time at v, which implies $t_i^v(\boldsymbol{q}_{S\cup\{i\}}) \leq t_i^v(\boldsymbol{q}_T) \leq \tau$. Moreover, since under $\boldsymbol{q}_T$ before time $t_i^v(\boldsymbol{q}_{S\cup\{i\}}) \leq t_i^v(\boldsymbol{q}_T)$, all agents of $T\backslash(S\cup\{i\})$ run after or reach no common vertices with all agents of S, we see that removing $\boldsymbol{q}_{T\backslash(S\cup\{i\})}$ does not change the routing status of agents in S before time $t_i^v(\boldsymbol{q}_{S\cup\{i\}})$. Therefore, i precedes j through v under $\boldsymbol{q}_{S\cup\{i\}}$ at time $t_i^v(\boldsymbol{q}_{S\cup\{i\}}) \leq \tau$. Q.E.D.

EC.6.2. Characterization of iterative batch-dominance

Building on the lemmas (established in Sections EC.4 and EC.6.1) for agent preemption and precedence, we prove the NE characterization in this subsection. The notation and definitions presented in Section 4.3 apply directly to $\bar{\Gamma}^{\mathrm{N}}$, with the only symbolic replacement of Δ by $\bar{\Delta}$ to indicate that we are in the setting of $\bar{\Gamma}^{\mathrm{N}}$. For example, the kth batch of a routing $\boldsymbol{q}$ for $\bar{\Gamma}^{\mathrm{N}}$ is written as $\bar{\Delta}(\boldsymbol{q}, k)$.

Lemma EC.7. *Let $\boldsymbol{p} = (P_h)_{h\in\bar{\Delta}}$ be an NE of game $\bar{\Gamma}^{\mathrm{N}}$. For every $k \geq 1$ and every agent $j \in \bar{\Delta} \setminus \bar{\Delta}(\boldsymbol{p}, [k])$, agent j cannot preempt any agent $i \in \bar{\Delta}(\boldsymbol{p}, [k])$ at any vertex of path P_i under $\boldsymbol{p}_{-j}$.*

Proof. Suppose on the contrary that agent $j \in \bar{\Delta}\backslash\bar{\Delta}(\boldsymbol{p}, [k])$ preempts agent $i \in \bar{\Delta}(\boldsymbol{p}, [k])$ at some vertex of P_i under $\boldsymbol{p}_{-j}$. Then from Corollary EC.1 (with i and j switching their roles over there), we deduce that under $\boldsymbol{p}_{-j}$ agent j also preempts agent i at vertex d. This means that there exists a path $P_j^* \in \bar{\mathscr{P}}_j$ such that

$$t_j^d(P_j^*, \boldsymbol{p}_{-j}) \leq \min_{R_j \in \bar{\mathscr{P}}_j} \{t_i^d(R_j, \boldsymbol{p}_{-j})\} \leq t_i^d(\boldsymbol{p}) \leq \tau(\boldsymbol{p}, k).$$

However, $t_j^d(\boldsymbol{p}) > \tau(\boldsymbol{p}, k)$ due to $j \in \bar{\Delta}\backslash\bar{\Delta}(\boldsymbol{p}, [k])$, indicating that j has an incentive to switch to P_j^*, which violates the fact that $\boldsymbol{p}$ is an NE. Q.E.D.

Given a partial path profile $\boldsymbol{q}_S = (Q_i)_{i\in S}$ of $\bar{\Gamma}^{\mathrm{N}}$ on agent set $S \subseteq \bar{\Delta}$, for every agent $j \in S$ and vertex $v \in Q_j$, we consider $(Q_j[o_j, v], \boldsymbol{q}_{S\backslash\{j\}})$ as the (incomplete) routing in which j follows $Q_j[o_j, v]$ and agents in $S\backslash\{j\}$ follow $\boldsymbol{q}_{S\backslash\{j\}} = (Q_i)_{i\in S\backslash\{j\}}$. It is clear that for every vertex $u \in Q_j[o_j, v]$, the arrival time of agent j at u under $(Q_j[o_j, v], \boldsymbol{q}_{S\backslash\{j\}})$, denoted as $t_j^u(Q_j[o_j, v], \boldsymbol{q}_{S\backslash\{j\}})$, is the same as that under $\boldsymbol{q}_S$, i.e., $t_j^u(\boldsymbol{q}_S)$.

Lemma EC.8. *Let $\boldsymbol{p}=(P_h)_{h\in\bar{\Delta}}$ be an NE of game $\bar{\Gamma}^{\mathrm{N}}$. For any batch index $k\geq 1$, agent $i\in\Omega:=\bar{\Delta}(\boldsymbol{p},[k])$, vertex $v\in P_i$, agent $j\in\bar{\Delta}\backslash\Omega$, and partial path profile $\boldsymbol{q}_{-\Omega}$ for agents in $\bar{\Delta}\backslash\Omega$, the following hold:*

$$t_i^v(\boldsymbol{p})=t_i^v(\boldsymbol{p}_\Omega)=t_i^v(\boldsymbol{p}_\Omega,\boldsymbol{q}_{-\Omega})\leq t_j^v(\boldsymbol{p}_\Omega,\boldsymbol{q}_{-\Omega}), \tag{EC.6}$$

$$t_j^d(\boldsymbol{p}_\Omega,\boldsymbol{q}_{-\Omega})\geq\tau(\boldsymbol{p},k+1)>t_i^d(\boldsymbol{p}_\Omega,\boldsymbol{q}_{-\Omega}). \tag{EC.7}$$

Proof. For each agent $j\in\bar{\Delta}\backslash\Omega$, define r_j as the earliest time when j can precede (recall Definition EC.2) some agent of Ω under (partial) path profile $(\boldsymbol{p}_\Omega,R_j)$ among all paths $R_j\in\bar{\mathscr{P}}_j$. If for any $R_j\in\bar{\mathscr{P}}_j$, under $(\boldsymbol{p}_\Omega,R_j)$ agent j can never precede any agent in Ω, we set $r_j:=\infty$. Define

$$r_*:=\min\{r_j\mid j\in\bar{\Delta}\backslash\Omega\}.$$

It follows from Lemma EC.6 that for any agent subset $S\subseteq\bar{\Delta}\backslash\Omega$ and any partial path profile $\boldsymbol{x}_S$ of agents in S,

$$\text{Under } (\boldsymbol{p}_\Omega,\boldsymbol{x}_S) \text{ no agent of } S \text{ can precede any agent of } \Omega \text{ before time } r_*. \tag{EC.8}$$

Validity of (EC.6) is implied by $r_*=\infty$. Indeed, if $r_*=\infty$, then applying (EC.8) with $S=\bar{\Delta}\backslash\Omega$ and $\boldsymbol{x}_S=\boldsymbol{q}_{-\Omega}$, Definition EC.2 directly gives the inequality in (EC.6). The equalities in (EC.6) will also be valid, because as long as agents in Ω follow $\boldsymbol{p}_\Omega$, they are not affected by the remaining agents, none of whom can precede agents in Ω.

Suppose on the contrary that $r_*<\infty$. By the definition of r_*, there exist agent $i\in\Omega$, agent $j_*\in\bar{\Delta}\backslash\Omega$, path $R_{j_*}\in\bar{\mathscr{P}}_{j_*}$ and vertex $v\in P_i\cap R_{j_*}$ such that under $(\boldsymbol{p}_\Omega,R_{j_*})$ agent j_* precedes agent i through vertex v at time

$$t_j^v(\boldsymbol{p}_\Omega,R_{j_*})=r_*.$$

Therefore, there exists vertex $u\in P_i[o_i,v]$ such that under $(\boldsymbol{p}_\Omega,R_{j_*})$ agent i reaches u at time $t_i^u(\boldsymbol{p}_\Omega,R_{j_*})=r_*$. Moreover, applying (EC.8) with $\boldsymbol{x}_S=R_{j_*}$ and $\boldsymbol{x}_S=\boldsymbol{q}_{-\Omega}$, respectively, we derive

$$t_i^u(\boldsymbol{p}_\Omega,R_{j_*})=r_*=t_i^u(\boldsymbol{p}_\Omega)\ \text{ and }\ t_i^u(\boldsymbol{p}_\Omega)=r_*=t_i^u(\boldsymbol{p}_\Omega,\boldsymbol{q}_{-\Omega}).$$

The trivial relation $t_i^v(\boldsymbol{p}_\Omega,\boldsymbol{q}_{-\Omega})\geq t_i^u(\boldsymbol{p}_\Omega,\boldsymbol{q}_{-\Omega})$ (as $u\in P_i[o_i,v]$) and the precedence of j_* over i through v give the following:

$$\begin{array}{l} t_i^v(\boldsymbol{p}_\Omega,\boldsymbol{q}_{-\Omega})\geq r_* \text{ for any partial path profile } \boldsymbol{q}_{-\Omega} \text{ of agents in } \bar{\Delta}\backslash\Omega, \text{ and} \\ e_v(R_{j_*})\prec e_v(P_i) \text{ if } u=v. \end{array} \tag{EC.9}$$

Moreover, notice from (EC.8) that as long as agents in Ω follow $\boldsymbol{p}_\Omega$, from time 0 till time r_*, the arrival times of all agents in Ω at the corresponding vertices are invariant against route changes of agents outside Ω. These invariant arrival times lead to invariant influence of agents in Ω on agents in

$\bar{\Delta}\backslash\Omega$ till time r_*. Therefore, we may define $j \in \bar{\Delta}\backslash\Omega$, using an adaptation of Algorithm 3 with vertex v (resp. Ω and $\boldsymbol{p}_\Omega$) in place of destination d (resp. U and $\boldsymbol{b}$) over there, as the "dominator" agent of $\bar{\Delta}\backslash\Omega$ (the first agent output by the adaptation) who is associated with an o_j-v path $\bar{Q}$ starting with the initial edge of j. Recalling that $t^v_{j_*}(\boldsymbol{p}_\Omega, R_{j_*}) = r_*$, the dominance of j gives $t^v_j(\boldsymbol{p}_\Omega, \bar{Q}) \le r_*$. In turn, the minimality of r_* enforces $t^v_j(\boldsymbol{p}_\Omega, \bar{Q}) = r_*$, which along with the dominance of j implies

$$e_v(\bar{Q}) \preceq e_v(R_{j_*}). \tag{EC.10}$$

Combining $t^v_j(\boldsymbol{p}_\Omega, \bar{Q}) = r_*$ and Ω's invariant influence on $\bar{\Delta}\backslash\Omega$ till time r_*, we deduce as in Lemma EC.3 that, assuming $\boldsymbol{p}_\Omega$, the "dominator" agent j is not preceded by any agent in $\bar{\Delta}\backslash(\Omega \cup \{j\})$ when he travels along $\bar{Q}$, regardless of the choices of agents in $\bar{\Delta}\backslash(\Omega \cup \{j\})$. In particular, we have $t^v_j(\bar{Q}, \boldsymbol{p}_{-j}) = r_*$. Define path $Q_j := \bar{Q} \cup P_i[v,d] \in \bar{\mathscr{P}}_j$. Then $t^v_j(Q_j, \boldsymbol{p}_{-j}) = r_*$, and it follows from (EC.9) and (EC.10) that under $(Q_j, \boldsymbol{p}_{-j})$ agent j precedes agent i through vertex v at time r_*. Thus j arrives at d no later than i under routing profile $(Q_j, \boldsymbol{p}_{-j})$, i.e., $t^d_j(Q_j, \boldsymbol{p}_{-j}) \le t^d_i(Q_j, \boldsymbol{p}_{-j})$, because of $Q_j[v,d] = P_i[v,d]$. (Note equation $t^d_j(Q_j, \boldsymbol{p}_{-j}) = t^d_i(Q_j, \boldsymbol{p}_{-j})$ holds only when $v = d$.)

Now we turn our attention from precedence (Definition EC.2) to preemption (Definition EC.1). If $t^w_i(Q_j, \boldsymbol{p}_{-j}) \ne t^w_i(\boldsymbol{p})$ for some vertex $w \in P_i$, then by Lemma EC.5 agent j preempts i at w under $\boldsymbol{p}_{-j}$, which is a contradiction to Lemma EC.7. We are left with the case where $t^w_i(Q_j, \boldsymbol{p}_{-j}) = t^w_i(\boldsymbol{p})$ holds for all vertices $w \in P_i$. It follows that $t^d_j(Q_j, \boldsymbol{p}_{-j}) \le t^d_i(Q_j, \boldsymbol{p}_{-j}) = t^d_i(\boldsymbol{p}) = \tau(\boldsymbol{p}, k) < t^d_j(\boldsymbol{p})$, where the last inequality follows from $j \notin \Omega$. However, $t^d_j(Q_j, \boldsymbol{p}_{-j}) < t^d_j(\boldsymbol{p})$ contradicts the fact that $\boldsymbol{p}$ is an NE. This proves the correctness of (EC.6).

Now let us prove (EC.7). Once the agents in Ω have chosen their paths as specified by $\boldsymbol{p}_\Omega$, thanks to (EC.6) about the invariant influence of Ω on $\bar{\Delta}\backslash\Omega$, we can apply Algorithm 3 with $U := \Omega$ and $\boldsymbol{b} := \boldsymbol{p}_\Omega$, which provides us a dominator $f \in \bar{\Delta}\backslash\Omega$ and his associated path $\bar{P}_f \in \bar{\mathscr{P}}_f$ such that (by Lemma EC.3) $t^d_f(\boldsymbol{p}_{-f}, \bar{P}_f) = t^d_f(\boldsymbol{p}_\Omega, \bar{P}_f) \le t^d_f(\boldsymbol{p})$ and $t^d_j(\boldsymbol{p}_\Omega, \boldsymbol{q}_{-\Omega}) \ge t^d_f(\boldsymbol{p}_\Omega, \bar{P}_f)$ for any $j \in \bar{\Delta}\backslash\Omega$ and partial path profile $\boldsymbol{q}_{-\Omega}$ of $\bar{\Delta}\backslash\Omega$.

Since $\boldsymbol{p}$ is an NE of $\bar{\Gamma}^{\mathrm{N}}$, we have $t^d_f(\boldsymbol{p}_{-f}, \bar{P}_f) \ge t^d_f(\boldsymbol{p})$, and hence $t^d_f(\boldsymbol{p}_\Omega, \bar{P}_f) = t^d_f(\boldsymbol{p}) \ge \tau(\boldsymbol{p}, k+1)$, where the last inequality is due to $f \notin \Omega$. On the other hand, $t^d_f(\boldsymbol{p}_\Omega, \bar{P}_f) \le \min\{t^d_h(\boldsymbol{p}) \,|\, h \in \bar{\Delta}\backslash\Omega\} = \tau(\boldsymbol{p}, k+1)$, from which we deduce that $t^d_j(\boldsymbol{p}_\Omega, \boldsymbol{q}_{-\Omega}) \ge t^d_f(\boldsymbol{p}_\Omega, \bar{P}_f) = \tau(\boldsymbol{p}, k+1)$, yielding the first inequality in (EC.7). The second inequality in (EC.7) follows from $t^d_i(\boldsymbol{p}_\Omega, \boldsymbol{q}_{-\Omega}) \le \tau(\boldsymbol{p}, k)$, which is guaranteed by the equalities in (EC.6). Q.E.D.

We are ready to prove Theorem 2 in the language of game $\bar{\Gamma}^{\mathrm{N}}$ (recalling Lemma EC.1).

THEOREM EC.5. *A path profile is an NE for $\bar{\Gamma}^{\mathrm{N}}$ if and only if it is iteratively batch-dominant.*

Proof. By Lemma EC.8, it suffices to prove the "if" part. Suppose $\boldsymbol{q}$ is an iteratively batch-dominant path profile of $\bar{\Gamma}^{\mathrm{N}}$ as specified in Definition 3. Consider an arbitrary agent $j \in \bar{\Delta}$ and suppose he belongs to the kth batch $\bar{\Delta}(\boldsymbol{q}, k)$, i.e., $t_j^d(\boldsymbol{q}) = \tau(\boldsymbol{q}, k)$. For any $R_j \in \bar{\mathscr{P}}_j$, it follows from Definition 3 that $t_j^d(\boldsymbol{q}_{-j}, R_j) \geq \tau(\boldsymbol{q}, k) = t_j^d(\boldsymbol{q})$, which states that $\boldsymbol{q}$ is indeed an NE of $\bar{\Gamma}^{\mathrm{N}}$. Q.E.D.

EC.7. More NE properties

In this section, we first verify that every NE of game $\bar{\Gamma}^{\mathrm{N}}$ (equivalently game Γ^{N}) possesses the properties that have been mentioned in Section 4.3. Then we discuss more NE properties implied by the EE best-response and global FIFO.

THEOREM EC.6. *Let $\boldsymbol{p}$ be an NE of game $\bar{\Gamma}^{\mathrm{N}}$. The following properties are satisfied.*

(i) Hierarchical independence. *If agents in a batch and those in earlier batches all follow their equilibrium strategies as in $\boldsymbol{p}$, then their arrival times at any vertex are independent of other agents' strategies.*

(ii) Hierarchal optimality. *The arrival time of each agent in the first batch $\bar{\Delta}(\boldsymbol{p}, 1)$ is the smallest among the arrival times of all agents under any routing of $\bar{\Gamma}^{\mathrm{N}}$. In general, for all $k \geq 2$, the arrival time of each agent in the kth batch $\bar{\Delta}(\boldsymbol{p}, k)$ is the smallest among the arrival times of all agents outside the first $k-1$ batches (i.e., those in $\bar{\Delta} \setminus \bar{\Delta}(\boldsymbol{p}, [k-1])$) under any routing of $\bar{\Gamma}^{\mathrm{N}}$ in which agents in the first $k-1$ batches $\bar{\Delta}(\boldsymbol{p}, [k-1])$ follow their routes specified by $\boldsymbol{p}$.*

(iii) General FIFO. *Under $\boldsymbol{p}$, if agent i precedes agent j through some vertex (see Definition EC.2), then i reaches the destination d no later than j. (Apparently, the property of general FIFO includes the global FIFO as a special case.)*

(iv) Strong NE. *Profile $\boldsymbol{p}$ is a strong NE of game $\bar{\Gamma}^{\mathrm{N}}$, and thus it is weakly Pareto optimal.*

Proof. (i) The hierarchical independence is simply an interpretation of $t_i^v(\boldsymbol{p}) = t_i^v(\boldsymbol{p}_\Omega, \boldsymbol{q}_{-\Omega})$ with $\Omega = \bar{\Delta}(\boldsymbol{p}, [k])$ for each $k \geq 1$ in Lemma EC.8.

(ii) For each $k \geq 1$, let $\Omega := \bar{\Delta}(\boldsymbol{p}, [k-1])$, and let r^* denote the *earliest* time an agent in $\bar{\Delta} \backslash \Omega$ reaches d among all routings of $\bar{\Gamma}^{\mathrm{N}}$ in which agents in Ω take their routes as in $\boldsymbol{p}_\Omega$. We need to verify the hierarchical optimality that $\tau(\boldsymbol{p}, k) = r^*$. Clearly,

$$r^* \leq \tau(\boldsymbol{p}, k).$$

The equalities in (EC.6) (i.e., Ω's invariant influences on $\bar{\Delta} \backslash \Omega$) enable us to apply Algorithm 3 and Lemma EC.3, which provides us a dominator agent $i \in \bar{\Delta} \backslash \Omega$, who is associated with a path $\bar{P}_i \in \bar{\mathscr{P}}_i$, provided the agents in Ω follow their routes as in $\boldsymbol{p}_\Omega$. It follows from Lemma EC.3 that $t_i^d(\boldsymbol{p}_{-i}, \bar{P}_i) = \min\{t_i^v(\boldsymbol{p}_\Omega, R_i) \mid R_i \in \bar{\mathscr{P}}_i\} \leq r^*$. On the other hand, since i cannot be better off by switching to $\bar{P}_i$, we have $t_i^d(\boldsymbol{p}_{-i}, \bar{P}_i) \geq t_i^d(\boldsymbol{p}) \geq r^*$. Therefore, $t_i^d(\boldsymbol{p}) = r^*$, which along with $r^* \leq \tau(p, k) \leq t_i^d(\boldsymbol{p})$ enforces $\tau(\boldsymbol{p}, k) = r^*$ as desired.

(iii) If under $\boldsymbol{p} = (P_h)_{h\in\bar{\Delta}}$ agent i reaches d later than agent j, i.e., $i \notin \bar{\Delta}(\boldsymbol{p}, k) \ni j$ for some k, then by Lemma EC.8 we claim that

$$\text{For every } v \in P_j \text{ and every } Q_i \in \bar{\mathscr{P}}_i, \text{ it holds that } t_j^v(\boldsymbol{p}) = t_j^v(\boldsymbol{p}_{-i}, Q_i) \le t_i^v(Q_i, \boldsymbol{p}_{-i}).$$

As i precedes j at some vertex u, it must be the case that $t_i^u(\boldsymbol{p}) = t_j^u(\boldsymbol{p})$ and $e_u(P_i) \prec_u e_u(P_j)$. As $t_i^d(\boldsymbol{p}) > t_j^d(\boldsymbol{p})$, we see that $u \neq d$ and suppose that uw is the outgoing edge from u on P_j. If $uw \in P_i$, then at time $t_i^u(\boldsymbol{p})$ agent i queues before agent j at edge uw, yielding $t_j^w(\boldsymbol{p}) \ge t_i^w(\boldsymbol{p}) + 1$, a contradiction to the above claim. So we are left with the case of $uw \notin P_i$. Considering $Q_i := P_i[o_i, u] \cup P_j[u, d] \in \bar{\mathscr{P}}_i$ with $w \in Q_i$, we have $t_j^w(Q_i, \boldsymbol{p}_{-i}) = t_i^w(Q_i, \boldsymbol{p}_{-i}) + 1$ (because the capacity of edge uw is 1, and i comes from the edge $e_u(Q_i) = e_u(P_i)$ with a higher priority than edge $e_u(P_j)$), which is a contradiction to the above claim. So we have proved the general FIFO property.

(iv) Suppose on the contrary that there exists a set $S \subseteq \bar{\Delta}$ of agents who can be strictly better off through collectively deviating from an NE $\boldsymbol{p}$ of game $\bar{\Gamma}^{\text{N}}$. Let k be the smallest batch index such that $S \cap \bar{\Delta}(\boldsymbol{p}, k) \neq \emptyset$. Due to the hierarchical optimality stated in (ii), all agents in $\bar{\Delta}(\boldsymbol{p}, k)$ obtain their earliest arrival times since no agent in $\bar{\Delta}(\boldsymbol{p}, [k-1])$ deviates from $\boldsymbol{p}$, a contradiction. Q.E.D.

According to Theorem EC.4, in game $\bar{\Gamma}^{\text{N}}$ every agent possesses a best response that is a path for the earliest arrival. This implies the following NE property.

Corollary EC.2 **(Weak earliest arrival)**. *Any NE for a new game building on $\bar{\Gamma}^{\text{N}}$ with the additional restriction that all agents take earliest-arrival paths is still an NE of the game $\bar{\Gamma}^{\text{N}}$ that does not have this restriction.*

For ease of exposition, in the next corollary, we restrict our attention to game Γ^{N} on network $G = (V, E)$ with a single origin. Recalling the original ranks defined in Section 3, let agents in Δ be indexed as $1, 2, \ldots$ according to their entry times into G and their original ranks (smaller indices correspond to earlier entry times and higher ranks in the case of equal entry time). We have the following straightforward corollary of the global FIFO property stated in Theorem 3 or in Theorem EC.6(iii).

Corollary EC.3. *If $\boldsymbol{p}$ is an NE of Γ^{N} with a single origin o, then the following properties are satisfied:*

(i) Consecutive exiting. *The indices of agents within the same batch under $\boldsymbol{p}$ are consecutive. That is, if $i, j \in \Delta(\boldsymbol{p}, k)$ with $i < j$, then $h \in \Delta(\boldsymbol{p}, k)$ for all $i \le h \le j$.*

(ii) Temporal overtaking. *If under $\boldsymbol{p}$ agent j strongly precedes agent i $(< j)$ at some vertex $v \in V \setminus \{o\}$, i.e., j reaches v earlier than i, then under $\boldsymbol{p}$ they reach the destination d at the same time.*

When focusing on agents originating from the same origin, the above two properties can be extended to networks with multiple origins.

EC.8. Actions and consecutive configurations in game Γ^{A}

Given configuration $\mathsf{c}_r = (\mathbb{Q}_e^r)_{e\in E}$, the *action set* of agent $i \in \Delta(\mathsf{c}_r) = (\cup_{e\in E}\mathbb{Q}_e^r) \cup (\cup_{v\in V}\Delta_{r+1,v})$, denoted by $E(i,\mathsf{c}_r)$, is defined as follows. If $i\in\Delta_{r+1,v}$, then $E(i,\mathsf{c}_r)=E^+(v)$. Suppose $i\in\mathbb{Q}_e^r$, where $e=uv$.

- If $v=d$ and i queues first in $\mathbb{Q}_e^r$, then $E(i,\mathsf{c}_r):=\emptyset$, i.e., i simply exits G at time $r+1$ (from d).
- If $v\neq d$ and i queues first in $\mathbb{Q}_e^r$, then $E(i,\mathsf{c}_r):=E^+(v)$, i.e., agent i selects the next edge that is available at v.
- Otherwise (i.e., i is not the head of $\mathbb{Q}_e^r$), agent i has to stay at e with $E(i,\mathsf{c}_r):=\{e\}$.

Given a configuration c_r and an action profile $\boldsymbol{a}=(a_i)_{i\in\Delta(\mathsf{c}_r)}$ with $a_i\in E(i,\mathsf{c}_r)$, the edge-priority DQ rule leads to a new configuration $\mathsf{c}_{r+1}=(\mathbb{Q}_e^{r+1})_{e\in E}$ at time $r+1$, referred to as a *consecutive configuration* of c_r:

- As a set, $\mathbb{Q}_e^{r+1}=\{i\in\Delta(\mathsf{c}_r)\,|\,a_i=e\}$ consists of agents choosing e in action profile $\boldsymbol{a}$.
- As a sequence, $\mathbb{Q}_e^{r+1}$ is obtained from $\mathbb{Q}_e^r$ by removing its head and making its tail followed by agents in $\mathbb{Q}_e^{r+1}\backslash\mathbb{Q}_e^r$ whose positions are determined according to the priority order $\prec_u$ at the tail vertex u of edge $e=uv$.

EC.9. Construction of a special SPE

This section is devoted to proving the SPE existence in game Γ^{A}, which has been discussed in Section 5.2. We call the normal-form game $\Gamma^{\mathrm{N}}(\mathsf{c}_r)$ introduced in Section 5.2 the *intermediary game of* Γ^{N} *starting from* c_r. For any time point $r\geq 0$, let C_r denote the set of all possible configurations at time r; in particular $\mathsf{C}_0=\{\mathsf{c}_0\}$ consists of the unique initial configuration given by initial queues in G at time 0.

Proof of Theorem 5. Given any history $\mathsf{h}_r=(\mathsf{c}_0,\ldots,\mathsf{c}_r)\in\mathsf{H}_r$ for any time point $r\geq 0$, recall that $\mathscr{D}(\mathsf{c}_r)=\Delta(\mathsf{c}_r)\cup(\cup_{s\geq r+2,v\in V}\Delta_{s,v})$ is the agent set of game $\Gamma^{\mathrm{N}}(\mathsf{c}_r)$. According to Lemma EC.1, let $\bar{\Gamma}^{\mathrm{N}}(\mathsf{c}_r)$ denote the game instance of model $\bar{\Gamma}^{\mathrm{N}}$ transformed from game $\Gamma^{\mathrm{N}}(c_r)$ using (T1)–(T3). So the agent set of $\bar{\Gamma}^{\mathrm{N}}(\mathsf{c}_r)$ is $\mathscr{D}(\mathsf{c}_r)$, and the restriction of $\bar{\Gamma}^{\mathrm{N}}(\mathsf{c}_r)$ to G is $\Gamma^{\mathrm{N}}(\mathsf{c}_r)$. Suppose that the agents in $\mathscr{D}(\mathsf{c}_r)$ are named as $1_r,2_r,\ldots$ such that agent i_r is the ith agent added to D in Step 13 of Algorithm 2 with input being the game instance $\bar{\Gamma}^{\mathrm{N}}(\mathsf{c}_r)$. For each agent $i_r\in\mathscr{D}(\mathsf{c}_r)$, let $\bar{P}_{i_r}^{\mathsf{c}_r}$ denote the dominant path associated to i_r in Algorithm 2.

Consider any agent $i_r\in\Delta(\mathsf{c}_r)=(\cup_{e\in E}\mathbb{Q}_e^r)\cup(\cup_{v\in V}\Delta_{r+1,v})$. Note that at the beginning of $\bar{\Gamma}^{\mathrm{N}}(\mathsf{c}_r)$, agent i_r queues at the first edge of $\bar{P}_{i_r}^{\mathsf{c}_r}$. If $i_r\in\cup_{e\in E}\mathbb{Q}_e^r$, then $\bar{P}_{i_r}^{\mathsf{c}_r}$ is a path in G; otherwise, the first edge of $\bar{P}_{i_r}^{\mathsf{c}_r}$ is the only edge of $\bar{P}_{i_r}^{\mathsf{c}_r}$ that is outside G and i_r is the only agent queuing, at the beginning of $\bar{\Gamma}^{\mathrm{N}}(\mathsf{c}_r)$, at that edge (i.e., he will enter G at the next time point). A configuration in C_{r+1} will result from c_r according to action profile $\boldsymbol{a}^{\mathsf{c}_r}$ defined as follows:

$$\text{The action of } i_r = \begin{cases} \text{the first edge of } \bar{P}_{i_r}^{\mathsf{c}_r}, & \text{if } i \text{ queues after another agent;} \\ \emptyset, & \text{if } i \text{ will exit } G \text{ from } d \text{ at the next time point;} \\ \text{the second edge of } \bar{P}_{i_r}^{\mathsf{c}_r}, & \text{otherwise;} \end{cases}$$

where the second condition is equivalent to i queuing first at the last edge of $\bar{P}_{i_r}^{\mathsf{c}_r}$. Observe that in any case the action defined above (if not $\emptyset$) is an edge of graph G. The set $\cup_{r\geq 0}\cup_{\mathsf{c}_r\in\mathsf{C}_r}\boldsymbol{a}^{\mathsf{c}_r}$ of action profiles defines a strategy profile $\sigma^*=(\sigma_i^*)_{i\in\Delta}$ of Γ^{A}. We will prove that σ^* is an SPE of game Γ^{A}.

Let $(\mathsf{c}_r,\mathsf{c}_{r+1},\ldots)$ be the list of configurations and $(P_i^*)_{i\in\mathscr{D}(\mathsf{c}_r)}$ be the path profile induced by h_r and σ^*. It can be deduced from Lemma EC.2 and Algorithm 2 that

- For any $s\geq r+1$, agent sequence $(1_s,2_s,\ldots)$ is a subsequence of $(1_{s-1},2_{s-1},\ldots)$ such that $\mathscr{D}(\mathsf{c}_{s-1})\backslash\mathscr{D}(\mathsf{c}_s)$ consists of the first $|\mathscr{D}(\mathsf{c}_{s-1})\backslash\mathscr{D}(\mathsf{c}_s)|$ agents of $1_{s-1},2_{s-1},\ldots$;
- For any $s\geq r+1$ and $i\in\Delta(\mathsf{c}_s)\backslash\cup_{v\in V}\Delta_{s+1,v}$, $\bar{P}_i^{\mathsf{c}_s}$ is a subpath of $\bar{P}_i^{c_{s-1}}$ ($\bar{P}_i^{\mathsf{c}_s}$ is either $\bar{P}_i^{c_{s-1}}$ or $\bar{P}_i^{c_{s-1}}$ with its first vertex and edge removed).

Therefore, the path P_i^* formed by the actions of each agent $i\in\mathscr{D}(\mathsf{c}_r)$ is exactly the restriction of $\bar{P}_i^{\mathsf{c}_r}$ to G. According to Lemma EC.2 (the equation in (EC.2)), we have

$$t_{i_r}(\sigma^*|\mathsf{h}_r)=r+\min\left\{t_{i_r}^d(\bar{P}_{1_r}^{\mathsf{c}_r},\ldots,\bar{P}_{(i-1)_r}^{\mathsf{c}_r},R_{i_r})\,|\,R_{i_r}\in\bar{\mathscr{P}}_{i_r}\right\}\text{ for every } i\geq 1.$$

Moreover, for any $j\geq 1$ and any strategy profile σ' of Γ^{A} with $\sigma'_{i_r}=\sigma^*_{i_r}$ for all $i\in[j]$, considering the path profile $(P'_i)_{i\in\mathscr{D}(\mathsf{c}_r)}$ induced by h_r and σ', we can deduce from an inductive argument that for each $i=1,\ldots,j$, P'_{i_r} is exactly $P^*_{i_r}$, i.e., the restriction of $\bar{P}_{i_r}^{\mathsf{c}_r}$ to G.

Now given any $k\geq 1$ and any $\sigma'_{k_r}\in\Sigma_{k_r}$, we consider strategy profile $\sigma'=(\sigma'_{k_r},\sigma^*_{-k_r})$ and the path profile $\boldsymbol{p}'=(P'_i)_{i\in\mathscr{D}(\mathsf{c}_r)}$ induced by h_r and σ'. We have $P'_{i_r}=P^*_{i_r}$ for all $i\in[k-1]$, and

$$t_{k_r}(\sigma'_{k_r},\sigma^*_{-k_r}|\mathsf{h}_r)=r+t_{k_r}^d(\boldsymbol{p}')=r+t_{k_r}^d(P^*_{1_r},\ldots,P^*_{(k-1)_r},\boldsymbol{p}'_{-\{1_r,\ldots,(k-1)_r\}}).$$

It follows from Lemma EC.2 (the inequality in (EC.2)) that

$$t_{k_r}(\sigma'_{k_r},\sigma^*_{-k_r}|\mathsf{h}_r)\geq r+\min\left\{t_{k_r}^d(\bar{P}_{1_r}^{\mathsf{c}_r},\ldots,\bar{P}_{(k-1)_r}^{\mathsf{c}_r},R_{k_r})\,|\,R_{k_r}\in\bar{\mathscr{P}}_{k_r}\right\}=t_{k_r}(\sigma^*|\mathsf{h}_r).$$

The arbitrary choices of k and σ'_{k_r} imply that σ^* is an SPE of game Γ^{A}. Q.E.D.

[1] That is, i queues first at the last edge of $\bar{P}_{i_r}^{\mathsf{c}_r}$.

EC.10. Realization of NEs from SPEs

In this section, we establish by construction that with the same input, each NE outcome of game Γ^{N} is a certain SPE outcome of game Γ^{A}.

Recall from Section 5.2 that each configuration c_r of the extensive-form game Γ^{A} corresponds to a normal-form game $\Gamma^{\mathrm{N}}(\mathsf{c}_r)$ on network $G=(V,E)$ with agent set $\mathscr{D}(\mathsf{c}_r)$, i.e., the intermediary game of Γ^{N} starting from c_r at time r. For every agent $i\in\mathscr{D}(\mathsf{c}_r)$, let $\mathscr{P}_i^{\mathsf{c}_r}$ denote his strategy set in $\Gamma^{\mathrm{N}}(\mathsf{c}_r)$, i.e., the set of paths in G along which i could travel (during a time period no earlier than r) given his position specified by c_r and $\cup_{s\geq r+1, v\in V}\Delta_{s,v}$.

Given any path profile $\boldsymbol{q}=(Q_i)_{i\in\mathscr{D}(\mathsf{c}_r)}$ of game $\Gamma^{\mathrm{N}}(\mathsf{c}_r)$, agent $j\in\mathscr{D}(\mathsf{c}_r)$ and vertex $v\in Q_j$, we use $t_j^v(\boldsymbol{q})_{\mathsf{c}_r}$ to denote the time when j reaches v under $\boldsymbol{q}$.

EC.10.1. Outline

Given any NE profile $\boldsymbol{p}$ of game Γ^{N}, we construct, for every history $\mathsf{h}_r=(\mathsf{c}_0,\ldots,\mathsf{c}_r)$ of game Γ^{A}, an NE $\boldsymbol{p}(\mathsf{h}_r)$ of game $\Gamma^{\mathrm{N}}(\mathsf{c}_r)$, i.e., the intermediary game of Γ^{N} starting from c_r with agent set $\mathscr{D}(\mathsf{c}_r)$. In particular, we set $\boldsymbol{p}(h_0):=\boldsymbol{p}$. Then, we construct an SPE of Γ^{A} by assembling these NEs such that starting from any history h_r ($r\geq 0$) the outcome of the SPE is exactly the NE $\boldsymbol{p}(\mathsf{h}_r)$. Note that the reference of each NE constructed is a history instead of a configuration. Since different histories may have the same ending configuration c_r, we may construct multiple NEs for the same intermediary game $\Gamma^{\mathrm{N}}(\mathsf{c}_r)$.

Such an NE-based assembling is more complicated than the one discussed in Sections 5.2 and EC.9, which aims at producing nothing more than an SPE. What is more complicated here is that we are unable to design a Markovian SPE. In particular, the natural idea of constructing the NEs $\boldsymbol{p}(\mathsf{h}_r)$, $r\geq 1$, directly using Algorithm 2 does not work anymore. For example, an agent outside the first batch under $\boldsymbol{p}$ may have an incentive to deviate at the game tree root of Γ^{A} to another child node for which the special IDNE computed by Algorithm 2 chooses different routes (with unchanged arrival times) for agents in earlier batches, which creates room for the agent to minimize his own arrival time.

EC.10.2. Inductive construction of history-based NEs

Our (inductive) construction of the NEs $\boldsymbol{p}(\mathsf{h}_r)$ is done iteratively on the game tree of Γ^{A} starting from the root $\mathsf{h}_0=(\mathsf{c}_0)$. Initially, the constructed NE $\boldsymbol{p}(\mathsf{h}_0)$ for h_0 is simply the given NE $\boldsymbol{p}$. For each $r\geq 1$, suppose inductively that for a history $\mathsf{h}_{r-1}=(\mathsf{c}_0,\ldots,\mathsf{c}_{r-1})\in\mathsf{H}_{r-1}$, the NE $\boldsymbol{p}(\mathsf{h}_{r-1})$ of game $\Gamma^{\mathrm{N}}(\mathsf{c}_{r-1})$, written for convenience as $\boldsymbol{\alpha}=(A_i)_{i\in\mathscr{D}(\mathsf{c}_{r-1})}$, has been constructed. We construct in two steps the NE $\boldsymbol{p}(\mathsf{h}_r)$, denoted $\boldsymbol{\beta}=(B_i)_{i\in\mathscr{D}(\mathsf{c}_r)}$, for each child history $\mathsf{h}_r=(\mathsf{c}_0,\ldots,\mathsf{c}_{r-1},\mathsf{c}_r)$ of h_{r-1}. In the first step, we identify a subset U of $\mathscr{D}(\mathsf{c}_r)$ and let B_i, for each $i\in U$, be the subpath of A_i that i has not visited until time r under $\boldsymbol{\alpha}$. In the second step, based on $\boldsymbol{\beta}_U$ determined, we find an iteratively dominant partial path profile $\boldsymbol{\beta}_{\mathscr{D}(\mathsf{c}_r)\setminus U}$ for the remaining agents, who can by no means affect the agents in U provided the latter follow $\boldsymbol{\beta}_U$.

The first step. Let $(a_i)_{i\in\Delta(\mathsf{c}_{r-1})}$ be the action profile at game tree node h_{r-1} determined by $\boldsymbol{\alpha}$, i.e, no action in the profile deviates from $\boldsymbol{\alpha}$. More specifically,

- a_i is the first edge of A_i if under $\boldsymbol{\alpha}$ agent i does not move during time period $[r-1,r]$;
- a_i is the second edge of A_i if under $\boldsymbol{\alpha}$ agent i queues at the second edge of A_i at time r;
- a_i is the null action ϕ if under $\boldsymbol{\alpha}$ agent i exits G at time r.

Let $(b_i)_{i\in\Delta(\mathsf{c}_{r-1})}$ be the action profile that leads history h_{r-1} to its child history h_r (or equivalently leads c_{r-1} to c_r).

Recall that $\Delta(\boldsymbol{\alpha},[k])$ denotes the first k batches of agents reaching d under routing $\boldsymbol{\alpha}$, where $k\geq 0$. Define $\Bbbk\geq 0$ to be the maximum nonnegative integer k such that the action of each agent of $\Delta(\boldsymbol{\alpha},[k])\cap\Delta(\mathsf{c}_{r-1})$ under $(a_i)_{i\in\Delta(\mathsf{c}_{r-1})}$ is the same as that under $(b_i)_{i\in\Delta(\mathsf{c}_{r-1})}$, i.e., we set

$$\Bbbk := \sup\{k \mid a_i = b_i \text{ for all } i\in\Delta(\boldsymbol{\alpha},[k])\cap\Delta(\mathsf{c}_{r-1})\}. \tag{EC.11}$$

It is possible that $\Bbbk=0$ with $\Delta(\boldsymbol{\alpha},[0])=\emptyset$ or $\Bbbk=\infty$ with $\Delta(\boldsymbol{\alpha},[\infty])=\mathscr{D}(\mathsf{c}_{r-1})$. Define

$$\Omega := \Delta(\boldsymbol{\alpha},[\Bbbk]) \text{ and } U := \Omega\cap\mathscr{D}(\mathsf{c}_r). \tag{EC.12}$$

The set U consists of agents who under $\boldsymbol{\alpha}$ are in the first $\Bbbk$ batches and will not exit G from d by time r.

In the following construction of B_i for each $i\in U$, we let i "keep" his path under $\boldsymbol{\alpha}$, which yields an invariance of arrival times as specified below in Lemma EC.9. For each agent $i\in\Delta(\mathsf{c}_{r-1})=(\cup_{e\in E}\mathbb{Q}_e^r)\cup(\cup_{v\in V}\Delta_{r+1,v})$, let e_i denote the first edge of A_i.

Construction I: (Construction of $\boldsymbol{\beta}_U$ with Invariant Arrival Times)

For each agent $i\in U$, set

$$B_i := \begin{cases} A_i\setminus\{e_i\}, & \text{if } a_i=b_i \text{ is the second edge of } A_i \text{ (which implies } i\in\cup_{e\in E}\mathbb{Q}_e^r\subseteq\Delta(\mathsf{c}_{r-1}));\\ A_i, & \text{otherwise.}\end{cases}$$

(NB: The if-condition in the above construction is equivalent to stating that when configuration c_{r-1} changes to configuration c_r, from time $r-1$ to time r, agent $i\in U$ travels along the edge e_i in G whose tail vertex is not the destination d, i.e., at time r agent i queues at the second edge of A_i. When the condition is satisfied, we set B_i to be $A_i\setminus\{e_i\}$, which is the path obtained from A_i by deleting its starting vertex and first edge e_i.)

The NE paths $B_i \in \{A_i, A_i \backslash \{e_i\}\}$ kept for agents i in $U = \Delta(\boldsymbol{\alpha}, [\Bbbk]) \cap \mathscr{D}(\mathsf{c}_r)$ particularly guarantee invariant arrival times at any vertex for these agents regardless of other agents' choices. To be specific, with the hierarchical independence of $\boldsymbol{\alpha}$ (as an NE of game $\Gamma^{\mathrm{N}}(\mathsf{c}_{r-1})$) stated in Section 4.3 and Theorem EC.6(i), we see that, as long as the chosen paths of agents in $\Omega = \Delta(\boldsymbol{\alpha}, [\Bbbk])$ remain as in $\boldsymbol{\alpha}_\Omega$, no matter what paths the agents in $\mathscr{D}(\mathsf{c}_{r-1}) \backslash \Omega$ choose, the latter agents have no impact on the arrival times of the former agents at any vertex. This along with Construction I above implies the following lemma, which is the base of our construction of $\boldsymbol{\beta}_{\mathscr{D}(\mathsf{c}_r) \backslash U}$ in the second step.

For notational convenience, for all $i \in \mathscr{D}(\mathsf{c}_{r-1}) \backslash \Delta(\mathsf{c}_{r-1})$ (i.e., agents who enter G at times later than r), we set e_i to be the null element ϕ.

Lemma EC.9 **(Invariant Arrival Times)**. *For any agent $i \in U$, any vertex $v \in B_i \subseteq A_i$, and any partial path profile $\boldsymbol{q}_{\mathscr{D}(\mathsf{c}_r) \backslash U} = (Q_j)_{j \in \mathscr{D}(\mathsf{c}_r) \backslash U}$ in game $\Gamma^{\mathrm{N}}(\mathsf{c}_r)$, where $Q_j \in \mathscr{P}_j^{\mathsf{c}_r}$ for every $j \in \mathscr{D}(\mathsf{c}_r) \setminus U$, it holds that*

$$t_i^v(\boldsymbol{\alpha})_{\mathsf{c}_{r-1}} = t_i^v(\boldsymbol{\alpha}_\Omega, (\{e_j\} \cup Q_j)_{j \in \mathscr{D}(\mathsf{c}_r) \backslash U})_{\mathsf{c}_{r-1}} = t_i^v((B_j)_{j \in U}, \boldsymbol{q}_{\mathscr{D}(\mathsf{c}_r) \backslash U})_{\mathsf{c}_r}.$$

Before proving the lemma, we make some observations. For any agent $j \in \mathscr{D}(\mathsf{c}_r)$ and any path $Q_j \in \mathscr{P}_j^{\mathsf{c}_r}$, it is clear that $\{e_j\} \cup Q_j \in \mathscr{P}_j^{\mathsf{c}_{r-1}}$. Observe that either $\mathscr{D}(\mathsf{c}_{r-1}) = \mathscr{D}(\mathsf{c}_r)$, or $\mathscr{D}(\mathsf{c}_{r-1}) \backslash \mathscr{D}(\mathsf{c}_r) \neq \emptyset$ and each agent in $\mathscr{D}(\mathsf{c}_{r-1}) \backslash \mathscr{D}(\mathsf{c}_r)$ exits G at time r, giving $\mathscr{D}(\mathsf{c}_{r-1}) \backslash \mathscr{D}(\mathsf{c}_r) = \Delta(\boldsymbol{\alpha}, 1) \subseteq \Delta(\boldsymbol{\alpha}, [\Bbbk])$. In any case we have

$$\mathscr{D}(\mathsf{c}_{r-1}) \backslash \mathscr{D}(\mathsf{c}_r) \subseteq \Omega = \Delta(\boldsymbol{\alpha}, [\Bbbk]) \text{ and } \mathscr{D}(\mathsf{c}_r) \backslash U = \mathscr{D}(\mathsf{c}_r) \backslash \Omega = \mathscr{D}(\mathsf{c}_{r-1}) \backslash \Omega.$$

Therefore, $(\boldsymbol{\alpha}_\Omega, (\{e_j\} \cup Q_j)_{j \in \mathscr{D}(\mathsf{c}_r) \backslash U})$ in Lemma EC.9 is simply $(\boldsymbol{\alpha}_\Omega, (\{e_j\} \cup Q_j)_{j \in \mathscr{D}(\mathsf{c}_{r-1}) \backslash \Omega})$, a strategy profile of game $\Gamma^{\mathrm{N}}(\mathsf{c}_{r-1})$, in which the agents, including i, of the first $\Bbbk$ batches (defined w.r.t. $\boldsymbol{\alpha}$) follow their paths as in $\boldsymbol{\alpha}$.

Proof of Lemma EC.9. The first equality of the conclusion follows from the hierarchical independence in Theorem EC.6(i). The second equality is straightforward from Construction I and the fact that each agent in $\Omega \backslash U = \Omega \backslash \mathscr{D}(\mathsf{c}_r) \subseteq \mathscr{D}(\mathsf{c}_{r-1}) \backslash \mathscr{D}(\mathsf{c}_r)$ (if any) exits G at time r, and he only has the null action under c_{r-1}, which has no effect on other agents. Q.E.D.

The second step. Based on the partial path profile $\boldsymbol{\beta}_U$ constructed (i.e., inherited from α_U) in the first step, we call Algorithm 3 to find an iteratively dominant path profile $(B_i)_{i \in \mathscr{D}(\mathsf{c}_r) \backslash U}$ for the remaining agents.

Recalling Lemma EC.1, let $\bar{\Gamma}^{\mathrm{N}}(\mathsf{c}_r)$ be the game on $\bar{G}$ whose restriction to G is the game $\Gamma^{\mathrm{N}}(\mathsf{c}_r)$. The partial path profile $(B_i)_{i \in U}$ constructed in Construction I naturally extends to a partial path profile $(\bar{B}_i)_{i \in U}$ of $\bar{\Gamma}^{\mathrm{N}}(\mathsf{c}_r)$, where the restriction of each $\bar{B}_i$ to G is B_i.

Construction II: (Construction of Iteratively Dominant $\boldsymbol{\beta}_{\mathscr{D}(\mathsf{c}_r) \backslash U}$)

1. Run Algorithm 3 with input $\bar{\Gamma}^{\mathrm{N}}(\mathsf{c}_r)$ and $\boldsymbol{b}=(\bar{B}_i)_{i\in U}$, which outputs $(\bar{P}_i)_{i\in\mathscr{D}(\mathsf{c}_r)\setminus\Omega}$.
2. For each agent $i\in\mathscr{D}(\mathsf{c}_r)\setminus U$, set B_i to be the restriction of $\bar{P}_i$ to G.

For easy expression of the null actions B_j of agents in $j\in\mathscr{D}(\mathsf{c}_{r-1})\setminus\mathscr{D}(\mathsf{c}_r)=\Delta(\mathsf{c}_{r-1})\setminus\Delta(\mathsf{c}_r)$, we reserve symbol $\boldsymbol{\phi}$ for the profile $(B_j)_{j\in\mathscr{D}(\mathsf{c}_{r-1})\setminus\mathscr{D}(\mathsf{c}_r)}$ of the null actions.

LEMMA EC.10. *Profile $\boldsymbol{\beta}$ is an NE of game $\Gamma^{\mathrm{N}}(\mathsf{c}_r)$.*

Proof. We need to prove that $t_i^d(\boldsymbol{\beta})_{\mathsf{c}_r}\le t_i^d(B_i',\boldsymbol{\beta}_{\mathscr{D}(\mathsf{c}_r)\setminus\{i\}})_{\mathsf{c}_r}$ for every agent $i\in\mathscr{D}(\mathsf{c}_r)$ and every path $B_i'\in\mathscr{P}_i^{\mathsf{c}_r}$.

Case 1: $i\in U\subseteq\Omega$. Suppose $i\in\Delta(\boldsymbol{\alpha},k)$ for some $k\le\Bbbk$. Then for any path profile $\boldsymbol{q}=(Q_j)_{j\in\mathscr{D}(\mathsf{c}_r)}$ of $\Gamma^{\mathrm{N}}(\mathsf{c}_r)$, with $\mho:=\Delta(\boldsymbol{\alpha},[k-1])\subset\Omega$, we have

$$t_i^d(\boldsymbol{\beta})_{\mathsf{c}_r}=t_i^d(\boldsymbol{\alpha})_{\mathsf{c}_{r-1}}\le t_i^d(\boldsymbol{\alpha}_{\mho},(\{e_j\}\cup Q_j)_{j\in\mathscr{D}(\mathsf{c}_r)\setminus\mho},\boldsymbol{\phi}_{\mathscr{D}(\mathsf{c}_{r-1})\setminus\mathscr{D}(\mathsf{c}_r)\setminus\mho})_{\mathsf{c}_{r-1}}=t_i^d(\boldsymbol{\beta}_{\mathscr{D}(\mathsf{c}_r)\cap\mho},\boldsymbol{q}_{\mathscr{D}(\mathsf{c}_r)\setminus\mho})_{\mathsf{c}_r},$$

where the first equality is by Lemma EC.9, the inequality is from hierarchical optimality in Theorem EC.6(ii), and the last equality is due to Construction I. In particular, when taking $Q_i=B_i'$ (noting $i\notin\mho$) and $Q_j=B_j$ for every $j\in\mathscr{D}(\mathsf{c}_r)\setminus\mho\setminus\{i\}$, we obtain $t_i^d(\boldsymbol{\beta})_{\mathsf{c}_r}\le t_i^d(B_i',\boldsymbol{\beta}_{\mathscr{D}(\mathsf{c}_r)\setminus\{i\}})_{\mathsf{c}_r}$ as desired.

Case 2: $i\in\mathscr{D}(\mathsf{c}_r)\setminus U=\mathscr{D}(\mathsf{c}_r)\setminus\Omega$. By Construction II, we deduce from Lemma EC.3 that the path B_i is i's best response to other agents' choices, giving $t_i^d(\boldsymbol{\beta})_{\mathsf{c}_r}\le t_i^d(B_i',\boldsymbol{\beta}_{\mathscr{D}(\mathsf{c}_r)\setminus\{i\}})_{\mathsf{c}_r}$. Q.E.D.

With Lemma EC.10, we complete our inductive constructions of history-based NEs $\boldsymbol{p}(\mathsf{h}_r)$ for all histories h_r of game Γ^{A}.

EC.10.3. Assembling an SPE from NEs

The partial hierarchical independence and iterative dominance guaranteed by Constructions I and II enable us to accomplish our task of assembling all the NEs $\boldsymbol{p}(\mathsf{h}_r)$, $\mathsf{h}_r\in\mathsf{H}_r$, $r\ge 0$, constructed in Section EC.10.2 into an SPE of Γ^{A}.

Let $\sigma=(\sigma_i)_{i\in\Delta}$ be a strategy profile of Γ^{A} defined as follows: at each history $\mathsf{h}_r=(\mathsf{c}_0,\dots,\mathsf{c}_r)$, agents in $\mathscr{D}(\mathsf{c}_r)$ take actions as specified by the NE $\boldsymbol{p}(\mathsf{h}_r)$ constructed in Section EC.10.2 for h_r, where $\boldsymbol{p}(\mathsf{c}_0)$ is the given NE $\boldsymbol{p}$ of game Γ^{N}.

THEOREM EC.7. *The strategy profile σ is an SPE of game Γ^{A} such that the path profile induced by the initial history h_0 and σ is exactly $\boldsymbol{p}$.*

Proof. Similar to the proof of Theorem 5 (see Section EC.9), it can be deduced from Constructions I and II (and Lemma EC.3) that, for each history h_r, the path profile induced by h_r and σ is exactly $\boldsymbol{p}(\mathsf{h}_r)$.

To see that σ is an SPE of Γ^{A}, we fix an arbitrary $r \geq 0$ and an arbitrary history $\mathsf{h}_r = (\mathsf{c}_0, \ldots, \mathsf{c}_r) \in \mathsf{H}_r$. Let $\boldsymbol{\beta} = (B_i)_{i \in \mathscr{D}(\mathsf{c}_r)}$ denote the NE $\boldsymbol{p}(\mathsf{h}_r)$ of $\Gamma^{\mathrm{N}}(\mathsf{c}_r)$ we have constructed for h_r. In the case of $r = 0$, we set $\boldsymbol{\beta} := \boldsymbol{p}$. Moreover, we consider any $i \in \mathscr{D}(\mathsf{c}_r)$, any $\sigma'_i \in \Sigma_i$, and the path profile $\boldsymbol{q} = (Q_j)_{j \in \mathscr{D}(\mathsf{c}_r)}$ induced by h_r and $\sigma' := (\sigma'_i, \sigma_{-i})$. We need to verify that $t_i(\sigma|\mathsf{h}_r) \leq t_i(\sigma'|\mathsf{h}_r)$.

If $r = 0$, then we suppose that $i \in \Delta(\boldsymbol{p}, k)$ and write $\mho = \Delta(\boldsymbol{p}, [k-1])$. By the hierarchical independence of $\boldsymbol{p}$ (Theorem EC.6(i)), no action change of agent i can alter the batch index of any agent in $\mho$. Therefore, using an inductive argument, we deduce from Construction I that at each history node $\mathsf{h}_s = (\mathsf{c}_0, \ldots, \mathsf{c}_s)$ on the path (in the game tree of Γ^{A}) induced by σ', all agents of $\mathscr{D}(\mathsf{c}_s) \cap \mho$ belong to the set Ω defined w.r.t. $\boldsymbol{p}(\mathsf{h}_s)$ (cf. (EC.12), where Ω is defined w.r.t. $\boldsymbol{\alpha}$). It follows that $Q_j = P_j = B_j$ for all $j \in \mho$. In turn, $\boldsymbol{p}$'s hierarchical optimality (Theorem EC.6(ii)) states that $t_i(\sigma|\mathsf{h}_0) = t_i^d(\boldsymbol{p}) \leq t_i^d(\boldsymbol{p}_{\mho}, \boldsymbol{q}_{\Delta \setminus \mho})_{\mathsf{c}_0} = t_i^d(\boldsymbol{q})_{\mathsf{c}_0} = t_i(\sigma'|\mathsf{h}_0)$.

So we assume now $r \geq 1$. Then h_r is a child history of some (unique) history $\mathsf{h}_{r-1} = (\mathsf{c}_0, \ldots, \mathsf{c}_{r-1}) \in \mathsf{H}_{r-1}$. Let $\boldsymbol{\alpha}$ denote the NE $\boldsymbol{p}(\mathsf{h}_{r-1})$ of $\Gamma^{\mathrm{N}}(\mathsf{c}_{r-1})$, and let $\Bbbk$ and $\Omega = \Delta(\boldsymbol{\alpha}, [\Bbbk])$ be defined as in (EC.11) and (EC.12).

If $i \in \Delta(\boldsymbol{\alpha}, k) \subseteq \Omega$ for some $k \leq \Bbbk$, then Construction I implies that $Q_j = B_j$ for all $j \in \mathscr{D}(\mathsf{c}_r) \cap \mho$, where $\mho := \Delta(\boldsymbol{\alpha}, [k-1])$. As in Case 1 of the proof of Lemma EC.10, we deduce that $t_i(\sigma|\mathsf{h}_r) = t_i^d(\boldsymbol{\beta})_{\mathsf{c}_r} \leq t_i^d(\boldsymbol{\beta}_{\mathscr{D}(\mathsf{c}_r) \cap \mho}, \boldsymbol{q}_{\mathscr{D}(\mathsf{c}_r) \setminus \mho})_{\mathsf{c}_r} = t_i^d(\boldsymbol{q})_{\mathsf{c}_r} = t_i(\sigma'|\mathsf{h}_r)$.

It remains to consider the case of $i \in \mathscr{D}(\mathsf{c}_r) \setminus \Omega = \mathscr{D}(\mathsf{c}_r) \setminus U$. Assume w.l.o.g. that i is exactly the ith agent in the ordering $1, 2, \ldots$ of agents in $\mathscr{D}(\mathsf{c}_r) \setminus \Omega$ associated with the iteratively dominant path profile constructed in Construction II. Again Construction I guarantees $\boldsymbol{q}_U = \boldsymbol{\beta}_U$. It follows from Lemma EC.3 (i.e., the iterative dominance) that $\boldsymbol{q}_{[i-1]} = \boldsymbol{\beta}_{[i-1]}$, and $t_i(\sigma|\mathsf{h}_r) = t_i^d(\boldsymbol{\beta})_{\mathsf{c}_r} \leq t_i^d(\boldsymbol{\beta}_U, \boldsymbol{\beta}_{[i-1]}, \boldsymbol{q}_{\mathscr{D}(\mathsf{c}_r) \setminus U \setminus [i-1]})_{\mathsf{c}_r} = t_i^d(\boldsymbol{q})_{\mathsf{c}_r} = t_i(\sigma'|\mathsf{h}_r)$, which completes the proof. Q.E.D.

EC.11. NE existence: edge priorities vs. agent priorities

We have proved that our game Γ^{N} admits an NE, where edge priorities play a crucial role. In contrast, as Example 3 shows, an NE may not exist in the multi-origin case under the model of Scarsini et al. (2018), where priorities are placed on agents. On the other hand, in the case of single origin, their model does guarantee the NE existence. In this section, we explain why the NE existence result on single-origin networks extends to the multi-origin case in our model, but not in the model of Scarsini et al. (2018).

The critical reason lies in whether we are able to order all agents in some way such that former agents in this order have absolute advantages over latter ones, using their heterogeneities, such as initial priorities, entering times, and different origins, etc. This is possible in the single-origin case of Scarsini et al. (2018), because a proper combination of the agents' entry times into the network and their initial priorities works. In this combination, entry times play a dominant role over initial priorities and hence the two factors are actually combined in a lexicographical way. To be more specific, since there is a single origin, agents entering the network earlier are always ordered before later ones; for agents entering the network at the same time, priorities associated with them can be used to break ties. Along with the local FIFO principle, we have seen that this ordering, a lexicographical combination of entry times and initial priorities, is decisive in that as long as an agent has some advantage over another at the origin, he will have advantages at all subsequent vertices. This idea is the essence of almost all related NE existence results in atomic dynamic routing games.

As Example 3 demonstrates, the above idea does not extend to the multi-origin case for the model of Scarsini et al. (2018), because Rock-Paper-Scissor relationships may occur. When agents enter the network from different origins, the same two factors, entry times and agent priorities, are still important. But they cannot be reconciled so well as in the single-origin case. The power of entry times is significantly weakened: when two agents come into the network from different origins, their entry times might not so important, while the locations of their entry points matter. However, the original locations and agent priorities cannot work together in a lexicographical way to determine a decisive ordering: sometimes original locations are more powerful and some other times agent priorities are more powerful, and this may lead to cyclic phenomenon as demonstrated in Example 3, making the existence of an NE impossible. To be more specific, we have shown in Example 3 that the first prioritized agent g may be blocked by the last prioritized agent i in every possible path for him (due to i's original location advantage); the last prioritized agent i may be blocked by the second prioritized agent h (due to h's priority over i), and the second prioritized agent h may be blocked by the first prioritized agent g (due to g's priority over h). The three agents form a Rock-Paper-Scissor cycle, destroying the existence of NE.

One advantage of our model is that we introduce edge priorities, which may be viewed as a tool of space, to help us untangle the complicated relationships among all agents. (Note that this kind of space information is ignored in the model of Scarsini et al. (2018).) We have seen that the combination of time and space plays a decisive role in the routing from a new perspective: as long as an agent is able to reach the destination earlier than another, he is able to do so for any intermediary vertex. To be more specific, the location of an agent's origin and fixed edge priorities of the network under our model can induce a space advantage for the agent, while the entry time of an agent can be viewed as his time advantage. The agents can be linearly ordered according to a kind of "combination" of their space and time advantages so that an agent with a higher order can find a path from his origin to the destination such that he dominates all agents with lower orders all the way along the path. Intuitively, the agent priorities (though consistent with the time advantages) in the model of Scarsini et al. (2018) may not reconcile with the space advantages, while the edge priories in our model, which define parts of space advantages, make possible the reconciliation with time and space advantages.

EC.12. Supplementary examples

In this section, we present several supplementary examples under our game model Γ^{N}, which demonstrate a Braess-like paradox (involving route changes due to routing environment improvement or deterioration), absence of the earliest arrival, and presence of overtaking. (Recall from Section 4.1 that IDNEs are earliest arrival and no overtaking.)

A paradox involving route changes. We illustrate the counter-intuitive phenomenon that the *route changes* resulting from removing initial queues (or removing agents or shortening path lengths) in a series-parallel network may slow the system performance. This kind of paradox was discovered by Scarsini et al. (2018) under their model. Example 3 presented in Scarsini et al. (2018) is an extension-parallel network adjusted from the one in Macko et al. (2013) for showing a classical Braess's paradox in nonatomic dynamic flow games. Our example below is a direct adaptation of the example in Scarsini et al. (2018).

Example EC.1. Consider a game instance Γ^{N} on the series-parallel network illustrated in Figure EC.2, where o is the single origin, d is the single destination, e_1 has a higher priority than e_2, and at e_3 there is an initial queue of three agents. At each time point $r \geq 1$, three agents of $\Delta_{r,o} = \{1_r, 2_r, 3_r\}$ enter the network from origin o. Regarding the original ranks, 1_r's rank is higher than 2_r's, and 2_r's is higher than 3_r's. The agents in $\cup_{r\geq 1}\Delta_{r,o}$ may choose one of the five o-d paths $R_1 := ou_1u_2d$, $R_2 := ou_1u_2u_3d$, $R_3 := ovu_2d$, $R_4 := ovu_2u_3d$ and $R_5 := ow_1w_2w_3d$.

(E1) It is easy to verify that, with the presence of the initial queue at e_3, every NE of the game Γ^{N} incurs a travel cost 4 to each agent outside the initial queue. For example, that agents $1_r, 2_r, 3_r$ (for all $r \geq 1$) follow R_1, R_4, R_5 respectively gives an NE.

(E2) Removal of the initial queue (i.e., the three agents) at e_3 may lead the system to a less efficient NE. While agent 1_1 still follows R_1, which incurs him the smallest travel cost 3, agent 2_1 (resp. 3_1) may *change* his route to R_1 (resp. R_4) along which he pays the smallest possible travel cost 4 (given the choice of 1_1). Building on the best choices R_1, R_1, R_3 of agents $1_1, 2_1, 3_1$, it is routine to verify that for every $r = 2, 3, \ldots$, the sequential route changes of agents $1_r, 2_r, 3_r$ to paths R_3, R_5, R_2 incur them sequentially smallest possible costs 4, 4, 5. These paths indeed form an NE.

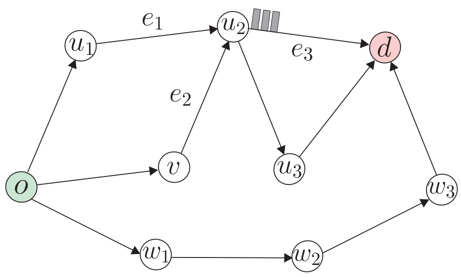


Figure EC.2 Removal of an initial queue may slow down the system performance

It is worth noting that the role of the initial queue in the above example can be played by some agents who enter the network earlier or by decreasing the length of a certain u_2-d path.

An NE that is not earliest arrival. The NE specified in (E2) of Example EC.1 is not earliest arrival, since, given other agents' choices, the earliest time agent 2_1 could reach vertex u_2 is 3, one time unit earlier than his arrival time at u_2 under the NE.

An NE that is temporally overtaking. The following example shows that an NE of game Γ^{N} is not necessarily no-overtaking.

Example EC.2. Consider a game Γ^{N} on the single-origin single-destination network in Figure EC.3, where at edge wx (resp. wy) there is an initial queue of three agents. In addition to the six agents, there are two agents, 1 and 2, entering the network from origin o at times 1 and 2, respectively. If agents 1 and 2 go through paths $ouvwxd$ and $owyd$, respectively, then they both reach destination d at the earliest possible time 6, yielding an NE of the game. Under this NE, agent 2 overtakes agent 1 at vertex w.

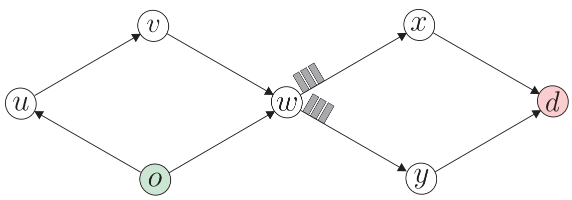


Figure EC.3 A temporal overtaking NE

EC.13. The hybrid game model

In this section, we consider "hybrid" agents, whose behaviors lie between adaptive and nonadaptive. An *agent* used without specification is meant a hybrid agent in this section. The corresponding game model is referred to as hybrid.

EC.13.1. Model description

For every agent i and every vertex v that is neither i's origin nor the destination d, we are given a probability $\theta_{i,v}$ that agent i contemplates switching to other paths at v. Let $\boldsymbol{\theta}$ denote the vector of these probabilities. We use $\Gamma^\sharp(\boldsymbol{\theta})$ to denote the hybrid game with parameter vector $\boldsymbol{\theta}$.

While adaptive agents make routing decisions at every nonterminal vertex they reach as to which *edge* to take next, hybrid agents make decisions at every nonterminal vertex as to which *path* to take in the future if they are given the chances (by Nature) to reconsider their plans, and just follow their previous plans otherwise. Intuitively, each agent always holds a plan (a path from his current edge to the destination) and may update it with a new one when chances are given. A precise definition of a strategy is presented as follows.

Definition EC.3 (Strategy). A *strategy* of agent $i \in \Delta$ is a mapping $\sigma_i^\sharp$ that maps each history $\mathsf{h}_r = (\mathsf{c}_0, \ldots, \mathsf{c}_r)$ till time r with $i \in \Delta(\mathsf{c}_r)$ to $\sigma_i^\sharp(\mathsf{h}_r)$ such that, based on c_r and the edge-priority DQ rule, either $\sigma_i^\sharp(\mathsf{h}_r)$ is a path from the current edge where i stays to the destination d, or $\sigma_i^\sharp(\mathsf{h}_r)$ is a null element when under c_r agent i will exit G at time $r+1$.

The *strategy set* of agent i is denoted as $\Sigma_i^\sharp$. A vector $\sigma^\sharp = (\sigma_i^\sharp)_{i\in\Delta}$ is called a *strategy profile* of the hybrid game $\Gamma^\sharp(\boldsymbol{\theta})$. Note that this game is typically a stochastic model. We use $\mathbb{E}[t_i(\sigma^\sharp|\mathsf{h}_r)]$ to denote the expected arrival time of agent i at the destination under strategy profile $\sigma^\sharp$ starting from history h_r.

Definition EC.4 (SPE in the hybrid game). A strategy profile $\sigma^\sharp = (\sigma_i)_{i\in\Delta}$ is a *subgame perfect equilibrium* (SPE) of $\Gamma^\sharp(\boldsymbol{\theta})$ if for any time $r \geq 0$ and any history $\mathsf{h}_r \in \mathsf{H}_r$, $\mathbb{E}[t_i(\sigma^\sharp|\mathsf{h}_r)] \leq \mathbb{E}[t_i(\sigma_i^{\sharp\prime}, \sigma_{-i}^\sharp|\mathsf{h}_r)]$ holds for all $i \in \Delta(\mathsf{c}_r)$ and all $\sigma_i^{\sharp\prime} \in \Sigma_i^\sharp$ such that $(\sigma_i^{\sharp\prime}, \sigma_{-i}^\sharp)$ still leads to history h_r, where $\sigma_{-i}^\sharp$ is the partial strategy profile of $\sigma^\sharp$ for agents in $\Delta\backslash\{i\}$.

EC.13.2. Results

As intuitively expected, we have the following observation.

Lemma EC.11. *For the hybrid model $\Gamma^\sharp(\boldsymbol{\theta})$, the case $\boldsymbol{\theta} = \mathbf{0}$ corresponds to the nonadaptive model Γ^{N} and the case $\boldsymbol{\theta} = \mathbf{1}$ corresponds to the adaptive model Γ^{A}.*

Proof. In fact, when $\theta = 0$, all the plans at the non-origin vertices will never be used and hence a strategy for a hybrid agent reduces to a strategy of a nonadaptive agent. On the other hand, when $\theta = 1$, all the plans at the non-origin vertices will always be given the chances to realize and hence only the immediate next edges are meaningful for the plans and the set of these immediate next edges is equivalent to a strategy of the adaptive agent. Q.E.D.

Suppose that we are given an SPE σ for game Γ^{A} that is constructed from an NE $\boldsymbol{p}$ of game Γ^{N}, as discussed in Sections 5.3 and EC.10. We construct a strategy profile $\sigma^\sharp$ for the hybrid model $\Gamma^\sharp(\boldsymbol{\theta})$ as follows. For each history $\mathsf{h}_r = (\mathsf{c}_0, \ldots, \mathsf{c}_r)$, if all players carry out their strategies in σ, then for each player i, a path from his current edge to the destination will be determined. We set $\sigma_i^\sharp(\mathsf{h}_r)$ as this path. This defines a strategy profile $\sigma^\sharp$ for the hybrid model $\Gamma^\sharp(\boldsymbol{\theta})$.

THEOREM EC.8. *The strategy profile $\sigma^\sharp$ constructed above is an SPE for the hybrid game $\Gamma^\sharp(\boldsymbol{\theta})$.*

Proof. By definition, it suffices to prove that, for any time $r \geq 0$ and any history $\mathsf{h}_r \in \mathsf{H}_r$, $\mathbb{E}[t_i(\sigma^\sharp|\mathsf{h}_r)] \leq \mathbb{E}[t_i(\sigma_i^{\sharp\prime}, \sigma_{-i}^\sharp|\mathsf{h}_r)]$ holds for all $i \in \Delta(\mathsf{c}_r)$ and all $\sigma_i^{\sharp\prime} \in \Sigma_i^\sharp$ such that $(\sigma_i^{\sharp\prime}, \sigma_{-i}^\sharp)$ still leads to history h_r, where $\sigma_{-i}^\sharp$ is the partial strategy profile of $\sigma^\sharp$ for agents in $\Delta\backslash\{i\}$.

Consider the subgame starting from history h_r. At the starting time r, all agents i at their initial positions in the subgame hold $\sigma_i^\sharp(\mathsf{h}_r)$ their initial plans. Then all agents i act during time $[r, r+1]$ according to $\sigma_i^\sharp(\mathsf{h}_r)$, which leads to a history h_{r+1}. By our construction presented in Section EC.10, agent i's new plan $\sigma_i^\sharp(\mathsf{h}_{r+1})$ at time $r+1$ is consistent with his old plan $\sigma_i^\sharp(\mathsf{h}_r)$ at time r, i.e., he does not switch his path even if he is given the chance to do so. Inductively, we see that the realized path profiles of the two strategy profiles $\sigma^\sharp$ (in game $\Gamma^\sharp(\boldsymbol{\theta})$) and σ (in game Γ^{A}) are the same. Therefore, the arrival time $t_i(\sigma^\sharp|\mathsf{h}_r)$ of i at the destination is also deterministic, and equals $t_i(\sigma|\mathsf{h}_r)$.

Suppose that i is in the kth batch in the routings determined by $\sigma^\sharp$ and h_r. Consider the single-deviation of agent i. By the construction of the SPE σ, all agents in the first $k-1$ batches will keep their plans unchanged in the histories following h_r. In other words, regardless of the chances given by Nature, all agents in the first $k-1$ batches will always follow their paths in the corresponding NE $\boldsymbol{p}(\mathsf{h}_r)$ of game $\Gamma^{\mathrm{N}}(\mathsf{c}_r)$ (see Section EC.10). Recalling the hierarchically optimality property for any NE of game $\Gamma^{\mathrm{N}}(\mathsf{c}_r)$, we see that $t_i(\sigma^\sharp|\mathsf{h}_r) = t_i(\sigma|\mathsf{h}_r)$, the exit time of agent i, is the smallest among all the exit times of all agents outside the first $k-1$ batches under any routing in which agents in the first $k-1$ batches follow their NE routes (see Sections 4.3 and EC.7). This proves that i cannot be better off by a unilateral deviation in game $\Gamma^\sharp(\boldsymbol{\theta})$ and hence the constructed $\sigma^\sharp$ is an SPE. Q.E.D.