A Proof of Theorem 2.1

We are reduced to compute :

$$E(h) = \mathbb{E}[(P_{\tau_1^{cl}} - P^{cl}_{\tau_1^{cl}})^2|\widetilde{\Omega}]$$

When N^{mm} be a Poisson process with intensity μ and $\widetilde{\Omega} = \{N^{mm}_{\tau_1^{cl}} > 0\}$. We are reduced to compute

$$E(h) = \mathbb{E}[(P_{\tau_1^{cl}} - P^{cl}_{\tau_1^{cl}})^2|\Omega].$$

Thus, recalling that $\tau_1^{op} + h = \tau_1^{cl}$, we get

$$\begin{aligned}
E(h) &= \mathbb{E}\Big[\big(\sum_{k=1}^{N^{mm}_{\tau_1^{cl}}} \frac{P_{\tau_k^{mm}} - P_{\tau_1^{cl}}}{N^{mm}_{\tau_1^{cl}}}\big)^2|\widetilde{\Omega}\Big] + \mathbb{E}\Big[\big(\sum_{k=1}^{N^{mm}_{\tau_1^{cl}}} \frac{g_k}{N^{mm}_{\tau_1^{cl}}}\big)^2|\widetilde{\Omega}\Big] + \frac{1}{K^2}\mathbb{E}\Big[\frac{I^2_{\tau_1^{cl}}}{N^{mm2}_{\tau_1^{cl}}}|\widetilde{\Omega}\Big] \\
&= \mathbb{P}(N^{mm}_{\tau_1^{cl}} > 0)^{-1}\Big(\mathbb{E}\Big[\mathbf{1}_{N^{mm}_{\tau_1^{cl}}>0}\Big\{\big(\sum_{k=1}^{N^{mm}_{\tau_1^{cl}}} \frac{P_{\tau_k^{mm}} - P_{\tau_1^{cl}}}{N^{mm}_{\tau_1^{cl}}}\big)^2 + \big(\sum_{k=1}^{N^{mm}_{\tau_1^{cl}}} \frac{g_k}{N^{mm}_{\tau_1^{cl}}}\big)^2 + \frac{1}{K^2}\frac{I^2_{\tau_1^{cl}}}{N^{mm2}_{\tau_1^{cl}}}\Big\}\Big]\Big) \\
&= \mathbb{P}(N^{mm}_{\tau_1^{cl}} > 0)^{-1} e^{\nu h}\int_h^{+\infty} \nu e^{-\nu t}\big(g(t) + \sigma^2 f(t) + \frac{1}{K^2}\ell(t)\big)\mathrm{d}t \qquad (8)
\end{aligned}$$

with

$$g(t) = \mathbb{E}[\mathbf{1}_{N_t^{mm}>0}(\sum_{k=1}^{N_t^{mm}} \frac{P_{\tau_k^{mm}} - P_t}{N_t^{mm}})^2],\ f(t) = \mathbb{E}[\frac{\mathbf{1}_{N_t^{mm}>0}}{N_t^{mm}}],\ \text{ and } \ell(t) = \mathbb{E}[I^2_{\tau_1^{cl}}]\mathbb{E}[\frac{\mathbf{1}_{N_t^{mm}>0}}{N_t^{mm2}}].$$

A direct computation gives

$$\mathbb{P}(N^{mm}_{\tau_1^{cl}} > 0) = 1 - e^{-\mu h}\frac{\nu}{\nu + \mu}. \qquad (9)$$

We now turn to the computation of the function g. We have the following lemma.

Lemma A.1. *We have for any $t > 0$*

$$g(t) = \sigma_f^2 \frac{t^2}{2}\mu\mathbb{E}[\frac{1}{(N_t^{mm}+1)^2}] + \sigma_f^2\frac{t^3}{3}\mu^2\mathbb{E}[\frac{1}{(N_t^{mm}+2)^2}]. \qquad (10)$$

Proof. Note that

$$g(t) = \sigma_f^2\mathbb{E}\Big[\mathbf{1}_{N_t^{mm}>0}\sum_{k=1}^{N_t^{mm}}\Big(\frac{W_{\tau_k^{mm}} - W_t}{N_t^{mm} - 1 + 1}\Big)^2\Big] + \sigma_f^2\mathbb{E}\Big[\mathbf{1}_{N_t^{mm}>0}\sum_{k,l=1 \text{ s.t. } k\neq l}^{N_t^{mm}} \frac{(W_{\tau_k^{mm}} - W_t)(W_{\tau_l^{mm}} - W_t)}{(N_t^{mm} - 2 + 2)^2}\Big].$$

Consider X_t the Poisson scatter made of the event times of N^{mm} between time 0 and t. Then we have

$$g(t) = \sigma_f^2\mathbb{E}\Big[\sum_{x\in X_t}\frac{(W_x - W_t)^2}{(\#\{X_t\backslash\{x\}\}+1)^2}\Big] + \sigma_f^2\mathbb{E}\Big[\sum_{x,y\in X_t \text{ s.t. } x\neq y}\frac{(W_x - W_t)(W_y - W_t)}{(\#\{X_t\backslash\{x,\ y\}\}+2)^2}\Big].$$

Since $P_t = \sigma_f W_t$ is independent of N^{mm}, we get

$$g(t) = \sigma_f^2 \mathbb{E}\Big[\sum_{x\in X_t} \frac{(t-x)^2}{(\#\{X_t\backslash\{x\}\}+1)^2}\Big] + \sigma_f^2 \mathbb{E}\Big[\sum_{x,y\in X_t \text{ s.t. } x\neq y} \frac{(t-x)\wedge(t-y)}{(\#\{X_t\backslash\{x,\ y\}\}+2)^2}\Big].$$

Finally using Palm's Formula, see for example Coeurjolly et al. (2017), we get

$$g(t) = \sigma_f^2 \mathbb{E}\big[\frac{1}{(N_t^{mm}+1)^2}\big]\int_0^t (t-u)\mu \mathrm{d}u + \sigma_f^2 \mathbb{E}\big[\frac{1}{(N_t^{mm}+2)^2}\big]\int_0^t\int_0^t (t-u)\wedge(t-v)\mu^2 \mathrm{d}u\mathrm{d}v,$$

and (10) follows.

□

To compute explicitly f, ℓ and g from Lemma A.1, we need the following additional results.

Lemma A.2. *Let N be a general inhomogeneous Poisson process with intensity measure λ. The following equalities hold:*

$$\mathbb{E}[\frac{\mathbf{1}_{N_t>0}}{N_t}] = e^{-m_t}\int_0^{m_t} \frac{e^s-1}{s}\mathrm{d}s, \quad \textit{and} \quad \mathbb{E}[\frac{\mathbf{1}_{N_t>0}}{N_t^2}] = e^{-m_t}\int_0^{m_t}\frac{1}{s}\int_0^s \frac{e^u-1}{u}\mathrm{d}u\mathrm{d}s, \qquad (11)$$

$$\mathbb{E}[\frac{1}{(1+N_t)^2}] = \frac{e^{-m_t}}{m_t}\int_0^{m_t}\frac{e^s-1}{s}\mathrm{d}s, \quad \textit{and} \quad \mathbb{E}[\frac{1}{(2+N_t)^2}] = \frac{1}{m_t^2}\big(1-e^{-m_t}-e^{-m_t}\int_0^{m_t}\frac{e^s-1}{s}\mathrm{d}s\big), \qquad (12)$$

with $m_t = \int_0^t \lambda(\mathrm{d}s)$.

Proof of (11). Note that

$$\mathbb{E}[\frac{\mathbf{1}_{N_t>0}}{N_t}] = \sum_{n=1}^{+\infty}\frac{1}{n}\frac{m_t^n}{n!}e^{-m_t} \text{ and } \mathbb{E}[\frac{\mathbf{1}_{N_t>0}}{N_t^2}] = \sum_{n=1}^{+\infty}\frac{1}{n^2}\frac{m_t^n}{n!}e^{-m_t}.$$

The functions e_1 and e_2 defined by

$$e_1(x) = \sum_{n=1}^{+\infty}\frac{1}{n}\frac{x^n}{n!} \text{ and } e_2(x) = \sum_{n=1}^{+\infty}\frac{1}{n^2}\frac{x^n}{n!}$$

are continuously differentiable function, so that

$$e_1'(x) = \sum_{n=1}^{+\infty}\frac{x^{n-1}}{n!} = \frac{e^x-1}{x} \text{ and } xe_2'(x) = \sum_{n=1}^{+\infty}\frac{1}{n}\frac{x^n}{n!} = e_1(x).$$

By integrating these functions, we get (11).

Proof of (12). Note that

$$\mathbb{E}[\frac{1}{(1+N_t)^2}] = \sum_{n=0}^{+\infty} \frac{1}{(1+n)^2} \frac{m_t^n}{n!} e^{-m_t} \text{ and } \mathbb{E}[\frac{1}{(2+N_t)^2}] = \sum_{n=0}^{+\infty} \frac{1}{(2+n)^2} \frac{m_t^n}{n!} e^{-m_t}.$$

Consider, for $i > 0$, the functions

$$r_i(x) = \sum_{n=0}^{+\infty} \frac{1}{(i+n)^2} \frac{x^{n+i}}{n!} \text{ and } s_i(x) = \sum_{n=0}^{+\infty} \frac{1}{i+n} \frac{x^{n+i}}{n!}.$$

We have

$$r_i'(x) = \sum_{n=0}^{+\infty} \frac{1}{i+n} \frac{x^{n+i-1}}{n!} \text{ hence } r_i(x) = \int_0^x \frac{s_i(s)}{s} \mathrm{d}s.$$

Since

$$s_i'(x) = \sum_{n=0}^{+\infty} \frac{x^{n+i-1}}{n!} = x^{i-1} e^x \text{ we get } r_i(x) = \int_0^x \frac{1}{s} \int_0^s u^{i-1} e^u \mathrm{d}u \mathrm{d}s.$$

Taking $i = 1$ and $i = 2$ we get (12). □

Injecting Equations (11) and (9) into f and ℓ and Equation (12) into g in view of (10), using (8) we obtain the formulas stated in Theorem 2.1.

B Computation of the expected square imbalance in the Poisson case

We want to compute $\mathbb{E}[I^2_{\tau_1^{op}+h}]$ when N^a and N^b are independent Poisson processes with intensity $\nu/2$. We have

$$\mathbb{E}[I^2_{\tau_1^{op}+h}] = v^2 \mathbb{E}[\big((N^a_{\tau_1^{op}+h} - N^a_{\tau_1^{op}} + N^a_{\tau_1^{op}}) - (N^b_{\tau_1^{op}+h} - N^b_{\tau_1^{op}} + N^b_{\tau_1^{op}})\big)^2].$$

Using the strong Markov property of Poisson process and taking conditional expectation with respect to τ_1^{op} we get

$$\mathbb{E}[I^2_{\tau_1^{op}+h}] = v^2(\nu h + 1),$$

where we use $\mathbb{E}[N^a_{\tau_1^{op}}] = \mathbb{E}[(N^a_{\tau_1^{op}})^2] = 1/2$.

C Existence of a Nash equilibrium

In this section, we set $h > 0$ as a terminal time of the auction to investigate the game played by the market takers.

C.1 Nash equilibrium

We are interested in finding a Nash equilibrium to the game between buyers and sellers. Starting at $(N^a_0, N^b_0) = (\alpha, \beta) \in \mathbb{N}^2$, we set[6]

$$V_h^{a,\alpha,\beta}(\lambda_a, \lambda_b) = \mathbb{E}^{\mathbb{P}^{\lambda_a,\lambda_b}}[N^a_h(N^a_h - N^b_h)] \tag{13}$$

$$V_h^{b,\alpha,\beta}(\lambda_a, \lambda_b) = \mathbb{E}^{\mathbb{P}^{\lambda_a,\lambda_b}}[N^b_h(N^b_h - N^a_h)]. \tag{14}$$

Formally, we can thus compute the optimal P&L of market takers for buy orders and sell orders by solving the following coupled system

$$\begin{cases} \inf\limits_{\lambda^a \in \mathcal{U}} V_h^{a,\alpha,\beta}(\lambda_a, \lambda_b^\star) &= \mathbb{E}^{\mathbb{P}^{\lambda_a^\star,\lambda_b^\star}}[N^a_h(N^a_h - N^b_h)] \\ \inf\limits_{\lambda^b \in \mathcal{U}} V_h^{b,\alpha,\beta}(\lambda_a^\star, \lambda_b) &= \mathbb{E}^{\mathbb{P}^{\lambda_a^\star,\lambda_b^\star}}[N^b_h(N^b_h - N^a_h)] \end{cases}, \tag{15}$$

where $\lambda_b^\star$ and $\lambda_b^\star$ are simultaneous optimizers of (13) and (14) respectively (depending on the action of market takers having the opposite behavior).

We now investigate theoretically the existence of a Nash equilibrium associated with (15). First we introduce some notations.

- Let Ω be the set of piece-wise constant functions with jumps of size 1. Consider[7] $X = (N^a, N^b)^\top$ be the canonical processes in Ω^2 and $\mathbb{F} = (\mathcal{F}_s)_{0 \le s \le h}$ the smallest filtration for which X is adapted.
- Let $\mathbb{P}$ be a probability measure on $(\Omega^2, \mathcal{F}_h)$ such that

  $$M_s = X_s - s\mathcal{L}_0, \text{ with } \mathcal{L}_0 := (\lambda_0, \lambda_0)^\top,\ 0 < \lambda_0 < \lambda_+,\ s \in [0, h],$$

  is a local martingale. A proof of the existence of such measure $\mathbb{P}$ is given in Jacod (1975). We set $M^a_r := M_{1,r}$ (resp. $M^b_r := M_{2,r}$) the first (resp. the second) component of M. Moreover to any pair $(\lambda^a, \lambda^b) \in \mathcal{U}^2$ of admissible controls we associate $\mathbb{P}^{\lambda^a,\lambda^b}$ the measure defined by

  $$\frac{\mathrm{d}\mathbb{P}^{\lambda^a,\lambda^b}}{\mathrm{d}\mathbb{P}} = \exp\Big(\int_0^h \log\big(\frac{\lambda^a_s}{\lambda_0}\big)\mathrm{d}N^a_s - \big(\lambda^a_s - \lambda_0\big)\mathrm{d}s + \log\big(\frac{\lambda^b_s}{\lambda_0}\big)\mathrm{d}N^b_s - \big(\lambda^b_s - \lambda_0\big)\mathrm{d}s\Big).$$

  Hence, under the measure $\mathbb{P}^{\lambda^a,\lambda^b}$,

  $$\Big(X_s - \int_0^s (\lambda^a_u, \lambda^b_u)^\top \mathrm{d}u\Big)_{0 \le s \le h}$$

[6] Rigorously speaking we should write $V_0^{i,\alpha,\beta}(\lambda_a, \lambda_b, h)$ instead of $V_h^{i,\alpha,\beta}(\lambda_a, \lambda_b)$ with $i \in \{a, b\}$, since we define here the value function of each market taker at time 0 and h is a time horizon. Since we consider only value functions of market takers at time 0, we make this slight abuse of notation.

[7] Here for the notation $\top$ denotes the transposition of a vector to identify as usual any element of $\mathbb{N}^2$ with a column vector.

is a martingale.

- For $(E, \|\cdot\|)$ a normed space, any $0 \leq s \leq t \leq h$ and $p > 1$, we define

$$\mathcal{H}^p_{s,t}(E) = \{Y,\ E-\text{valued and } \mathbb{F}-\text{adapted process s.t.}, \mathbb{E}[(\int_s^t \|Y_r\|^2 \mathrm{d}r)^{\frac{p}{2}}] < +\infty\}$$

$$\mathcal{S}^p_{s,t}(E) = \{Y,\ E-\text{valued and } \mathbb{F}-\text{adapted process s.t.}, \mathbb{E}[\sup_{s\leq t}\|Y_r\|^p \mathrm{d}r] < +\infty\}$$

$$\mathbb{L}^p(E) = \{\xi,\ E-\text{valued } \mathcal{F}_h-\text{measurable random variable, s.t. } \mathbb{E}[\|\xi\|^p] < +\infty\}.$$

When $s = 0$ we omit the index s in the previous definitions. If $E = \mathbb{R}^2$, we set $\|\cdot\|_2$ and $\|\cdot\|_1$ the classical Manhattan norm and Euclidean norm on $\mathbb{R}^2$ respectively. For any $\mathbb{R}^2-$valued process $Y := (Y_r)_{0\leq r\leq h}$, we denote by $Y_{r,1}$ and $Y_{r,2}$ its first and second coordinates respectively for any time $r \in [0, h]$.

- For any $z \in \mathbb{R}^2$ and $\varepsilon^a, \varepsilon^b \in [\lambda_-, \lambda_+]$, we set

$$(\mathbf{L}) \begin{cases} \lambda^\star_a(z, \varepsilon^a) &= \mathbf{1}_{z_1>0}\lambda_- + \mathbf{1}_{z_1<0}\lambda_+ + \varepsilon^a \mathbf{1}_{z_1=0} \\ \lambda^\star_b(z, \varepsilon^b) &= \mathbf{1}_{z_2>0}\lambda_- + \mathbf{1}_{z_2<0}\lambda_+ + \varepsilon^b \mathbf{1}_{z_2=0}. \end{cases}$$

Note that both $z_1\lambda^\star_a(z, \varepsilon^a)$ and $z_2\lambda^\star_b(z, \varepsilon^b)$ do not depend on ε^a and ε^b. To alleviate notations, when one of these products appears, we will denote it simply by $z_1\lambda^\star_a(z)$ and $z_2\lambda^\star_b(z)$ respectively.

- For any $z, \tilde{z} \in \mathbb{R}^2$ and any $\varepsilon \in [\lambda_-, \lambda_+]$, we set $H^{a,\star}(z, \tilde{z}, \varepsilon) = z_1\lambda^\star_a(z) + z_2\lambda^\star_b(\tilde{z}, \varepsilon)$ and $H^{b,\star}(z, \tilde{z}, \varepsilon) = z_2\lambda^\star_b(z) + z_1\lambda^\star_a(\tilde{z}, \varepsilon)$.

- for $x \in \mathbb{N}^2$ we define $g^a(x) = x_1(x_1 - x_2)$ and $g^b(x) = x_2(x_2 - x_1)$.

- Let U be a map from $[0, h] \times \mathbb{N}^2$ into $\mathbb{R}$. For any $(s, \alpha, \beta) \in [0, h] \times \mathbb{N}^2$ we set

$$(\mathbf{D}) \begin{cases} D_aU(s, \alpha, \beta) = & U(s, \alpha + 1, \beta) - U(s, \alpha, \beta) \\ D_bU(s, \alpha, \beta) = & U(s, \alpha, \beta + 1) - U(s, \alpha, \beta)) \\ DU(s, \alpha, \beta) = & (D_aU(s, \alpha, \beta), D_bU(s, \alpha, \beta))^\top. \end{cases}$$

We first provide a very general result by associated to the existence of a Nash equilibrium for (15) a system of coupled ODE on $\mathbb{N}^2$, as a direct extension of (Dockner et al., 2000, Theorem 8.5).

Proposition C.1. *Assume that there exist two maps $\varepsilon^a, \varepsilon^b$ from $[0, h] \times \mathbb{N}^2$ into $[\lambda_-, \lambda_+]$ such that the following coupled system*

$$(\mathbf{S}) \begin{cases} \partial_s V^a + H^{a,\star}(DV^a, DV^b, \varepsilon^b) = 0, \quad s \in [0, h),\ (\alpha, \beta) \in \mathbb{N}^2 \\ V^a(h, \alpha, \beta) = g^a(\alpha, \beta), \\ \partial_s V^b + H^{b,\star}(DV^b, DV^a, \varepsilon^a) = 0, \quad s \in [0, h),\ (\alpha, \beta) \in \mathbb{N}^2 \\ V^b(h, \alpha, \beta) = g^b(\alpha, \beta), \end{cases}$$

has a continuously differentiable (in time) solution denoted by (V^a, V^b) *on* $[0,h] \times \mathbb{N}^2$ *and assume moreover that*

$$DV^i(\cdot, N^a_\cdot, N^b_\cdot) \in \mathcal{H}^2_h(\mathbb{R}^2),\ i = a, b.$$

Then, $(\lambda^\star_a(DV^a, \varepsilon^a), \lambda^\star_b(DV^b, \varepsilon^b))$ *is a Nash equilibrium for* (15).

Proof. The proof follows a standard verification argument. Notice however that we need feedback control for the thresholds $(\varepsilon^a, \varepsilon^b)$ in order to have classical HJB equations. See for instance (Dockner et al., 2000, Theorem 8.5). □

Although the previous result provides sufficient conditions to get a Nash equilibrium for the stochastic differential game (15), it is quite hard to justify such existence in practice. Note indeed that the optimizers $\lambda^\star_a$ and $\lambda^\star_b$ are singular in view of their definition **(L)**. Thus, the main difficulty encountered in this proposition is to solve the bang-bang type system **(S)** of ODEs on $\mathbb{N}^2$ for relevant thresholds $\varepsilon^a, \varepsilon^b$. As far as we now, we have no PDE results ensuring the existence of a solution to **(S)**.

Inspired by Hamadène and Mu (2014), we thus propose to study a smooth approximation of **(S)** and then to build a sequence of processes converging (up to a subsequence) to a Nash equilibrium for the game (15).

Let $n \in \mathbb{N}$. We consider the smoothed control functions for any $z \in \mathbb{R}$

$$\lambda^n(z) = \begin{cases} \lambda_+ & \text{if } z \leq -\frac{1}{n} \\ \lambda_- & \text{if } z \geq \frac{1}{n} \\ n\frac{\lambda_- - \lambda_+}{2} z + \frac{\lambda_+ + \lambda_-}{2} & \text{if } z \in (-\frac{1}{n}, \frac{1}{n}). \end{cases}$$

The functions λ^n and $z \longmapsto z\lambda^n(z)$ are Lipschitz continuous. Also consider Φ_n, the truncation function defined for any $x \in \mathbb{R}$ by

$$\Phi_n(x) = (x \wedge n) \vee (-n).$$

Hence, we introduction the smoother of $H^\star$ denoted by $H^{\star,n}$ and defined by for any $(z_1, z_2, \tilde{z}) \in \mathbb{R}^3$ by

$$H^{\star,n}(z_1, z_2, \tilde{z}) = \Phi_n(z_1 \lambda^n(z_1)) + \Phi_n(z_2)\lambda^n(\tilde{z}).$$

Theorem C.1. *For any* $n \in \mathbb{N}$*, there exists a unique (viscosity) solution denoted by* $V^{a,n}$ *to the following system of integro-PDEs*

$$(\mathbf{S^n})\begin{cases} \partial_s V^{a,n} + H^{\star,n}(D_a V^{a,n}, D_b V^{a,n}, D_b V^{b,n}) = 0, \ s \in [0,h), \ (\alpha,\beta) \in \mathbb{N}^2, \\ V^{a,n}(h,\alpha,\beta) = g^a(\alpha,\beta), \\ \partial_s V^{b,n} + H^{\star,n}(D_b V^{b,n}, D_a V^{b,n}, D_a V^{a,n}) = 0, \ s \in [0,h), \ (\alpha,\beta) \in \mathbb{N}^2, \\ V^{b,n}(h,\alpha,\beta) = g^b(\alpha,\beta). \end{cases}$$

Moreover,

- *The system* $(\mathbf{S^n})$ *admits a unique viscosity solution.*
- *There exists a subsequence* $(n_k)_{k\geq 0}$ *and two measurable applications* V^a, V^b *from* $[0,h]\times\mathbb{N}^2$ *into* $\mathbb{R}$ *such that for any* $(s,\alpha,\beta)\in[0,h]\times\mathbb{N}^2$

$$\lim_{k\to+\infty} V^{i,n_k}(s,\alpha,\beta) = V^i(s,\alpha,\beta), \ i\in\{a,b\}$$

and

$$\lim_{n\to+\infty} DV^{i,n}(s,\alpha,\beta) = DV^i(s,\alpha,\beta), \ i\in\{a,b\}.$$

- *Moreover* $\lambda^{n_k}(D_a V^{a,n_k}(\cdot,N^a,N^b))\mathbf{1}_{D_aV^a(\cdot,N^a,N^b)=0}$ *and* $\lambda^{n_k}(D_b V^{b,n_k}(\cdot,N^a,N^b))\mathbf{1}_{D_bV^b(\cdot,N^a,N^b)=0}$ *converges weakly in* $\mathcal{H}^2_h(\mathbb{R}^2)$ *to some progressively measurable and* $[\lambda_-,\lambda^+]$*-valued processes denoted respectively by* θ *and* ϑ*.*

Thus, $(\lambda_a^\star,\lambda_b^\star) = (\lambda_a^\star(DV^a(s,N_s^a,N_s^b),\theta_s),\lambda_b^\star(DV^b(s,N_s^a,N_s^b),\vartheta_s))_{0\leq s\leq t}$ *is a Nash equilibrium for the game* (15) *and* $V_h^{i,\alpha,\beta}(\lambda_a^\star,\lambda_b^\star) = V^i(0,\alpha,\beta), \ i\in\{a,b\}$*.*

We give here the sketch of the proof of this result. The details are postponed to Appendix C.2.

Sketch of the proof of Theorem C.1 The proof will be divided in three steps. The main tool used is the theory of BSDE with jumps (see Tang and Li (1994); Buckdahn and Pardoux (1994); Barles et al. (1997)) and their representations through integro-partial differential equations.

Step 1. We associated to the system $(\mathbf{S^n})$ a two dimensional BSDE for which it is well-known that there exists a unique solutions in appropriate spaces.

Step 2. By mimicking the proof of Theorem 2.5 in Hamadène and Mu (2014) extended to the case of counting processes, we prove that the solution of the BSDE associated to $(\mathbf{S^n})$ converges up to a subsequence to a solution of a two-dimensional BSDE associated with the system $(\mathbf{S})$.

Step 3. We prove that this approximation provides a Nash equilibrium for the game (15) with well-chosen thresholds obtained in Step 2 as limits of functions of the components of the solution to the approached BSDE considered, see Proposition C.2 below.

We conclude thanks to semi-linear Feynman-Kac formula for BSDEs and the system $(\mathbf{S^n})$ established in Step 1, together with convergence results.

C.2 Proof of Theorem C.1

For the proof we follow the methodology of Hamadène and Mu (2014). First we introduce a series of smoothed BSDE with Lipschitz generator by smoothing the controls $\lambda_a^\star$, $\lambda_b^\star$. Then we show that the solution of the smoothed BSDE converges (up to a subsequence) almost surely towards a solution of Equation (28).

We have the following a priori estimates results which is a consequence of the BDG inequalities and of the Gronwall Lemma.

Lemma C.1. *For $(s,x) \in [0,h] \times \mathbb{N}^2$ let $X^{s,x}$ be the process in Ω defined onto $[s,h]$ by*

$$X_u^{s,x} = x + X_u - X_s.$$

We have for any $s \in [0,h]$ and $\rho > 0$

$$\mathbb{E}[\sup_{s\leq u\leq h} \|X_u^{s,x}\|_1^\rho] \leq C_\rho(1 + |x_1|^\rho + |x_2|^\rho)$$

and for any $(\lambda_a, \lambda_b) \in \mathcal{U}^2$

$$\mathbb{E}^{\mathbb{P}^{\lambda_a,\lambda_b}}[\sup_{s\leq u\leq h} \|X_u^{s,x}\|_1^\rho] \leq C_\rho(1 + |x_1|^\rho + |x_2|^\rho)$$

We now turn to the proof of Theorem C.1.

C.2.1 Step 1: Approximation, existence and uniqueness

From now, $s \in [0,h)$. We recall the definition of smoothed control functions

$$\lambda^n(z) = \begin{cases} \lambda_+ & \text{if } z \leq -\frac{1}{n} \\ \lambda_- & \text{if } z \geq \frac{1}{n} \\ n\frac{\lambda_- - \lambda_+}{2} z + \frac{\lambda_+ + \lambda_-}{2} & \text{if } z \in (-\frac{1}{n}, \frac{1}{n}) \end{cases}$$

Consider Φ_n, the truncation function

$$\Phi_n(x) = (x \wedge n) \vee (-n).$$

Now we define the system of smoothed BSDEs for any $u \in [s,h]$:

$$(\mathbf{J^n}) \begin{cases} -\mathrm{d}Y_u^{a,n;s,x} & = (H^{\star,n}(Z_{1,u}^{a,n;s,x}, Z_{2,u}^{a,n;s,x}, Z_{2,u}^{b,n;s,x}) - \mathcal{L}_0 \cdot Z_u^{a,n;s,x})\mathrm{d}u - Z_u^{a,n;s,x} \cdot \mathrm{d}M_u, \\ Y_h^{a,n;s,x} & = g^a(X_h^{s,x}) \\ -\mathrm{d}Y_u^{b,n;s,x} & = (H^{\star,n}(Z_{2,u}^{b,n;s,x}, Z_{1,u}^{b,n;s,x}, Z_{1,u}^{a,n;s,x}) - \mathcal{L}_0 \cdot Z_u^{b,n;s,x})\mathrm{d}u - Z_u^{b,n;s,x} \cdot \mathrm{d}M_u, \\ Y_h^{b,n;s,x} & = g^b(X_h^{s,x}), \end{cases}$$

with $Z_u^{i,n;s,x} = (Z_{1,u}^{i,n;s,x}, Z_{2,u}^{i,n;s,x})^\top$ for any $i \in \{a,b\}$.

From Proposition 2.1. in Buckdahn and Pardoux (1994) since Φ_n is Lipschitz continuous there exists a unique solution to $(\mathbf{J^n})$ such that

$$\big((Y^{a,n;s,x}, Z^{a,n;s,x}), (Y^{b,n;s,x}, Z^{b,n;s,x})\big) \in \big(\mathcal{S}^2_{s,h}(\mathbb{R}) \times \mathcal{H}^2_{s,h}(\mathbb{R}^2)\big)^2.$$

Moreover (Proposition 3.8. in Buckdahn and Pardoux (1994)) there exist measurable deterministic functions $V^{a,n}$, $V^{b,n}$ defined on $[s,h] \times \mathbb{N}^2$ with values in $\mathbb{R}$ such that:

$$\forall u \in [s,h],\ Y_u^{i,n;s,x} = V^{i,n}(s, X_u^{s,x}) \text{ and } Z_u^{i,n;s,x} = DV^{i,n}(u, X_{u^-}^{s,x}),\ \text{for } i = a, b. \tag{16}$$

From Theorem 3.4. in Barles et al. (1997), we know that the unique solution of $(\mathbf{J^n})$ provides a unique viscosity solution denoted by $(V^{a,n}, V^{b,n})$ to $(\mathbf{S^n})$ and given by (16).

Before going to the convergence of $Y^{i,n}$ and $Z^{i,n}$, notice that by considering the generator functions

$$\begin{cases} H^{a,n}(u,x) &= \big(\Phi_n(D_aV^{a,n}(u,x)\lambda^n(D_aV^{a,n}(u,x))) + \Phi_n(D_bV^{a,n}(u,x))\lambda^n(D_bV^{b,n}(u,x))\big) \\ H^{b,n}(u,x) &= \big(\Phi_n(D_bV^{b,n}(u,x)\lambda^n(D_bV^{b,n}(u,x))) + \Phi_n(D_aV^{b,n}(u,x))\lambda^n(D_aV^{a,n}(u,x))\big), \end{cases}$$

we deduce from (16) that

$$H^{a,n}(u, X_{u^-}^{s,x}) = H^{\star,n}(Z_{1,u}^{a,n;s,x}, Z_{2,u}^{a,n;s,x}, Z_{2,u}^{b,n;s,x}),$$

and

$$H^{b,n}(u, X_{u^-}^{s,x}) = H^{\star,n}(Z_{2,u}^{b,n;s,x}, Z_{1,u}^{b,n;s,x}, Z_{1,u}^{a,n;s,x}),$$

so that $(\mathbf{J^n})$ becomes

$$(\widetilde{\mathbf{J^n}}) \begin{cases} -\mathrm{d}Y_u^{a,n;s,x} &= (H^{a,n}(s, X_{u^-}^{s,x}) - \mathcal{L}_0 \cdot Z_u^{a,n;s,x})\mathrm{d}u - Z_u^{a,n;s,x} \cdot \mathrm{d}M_u,\ Y_h^{a,n;s,x} = g^a(X_h^{s,x}) \\ -\mathrm{d}Y_u^{b,n;s,x} &= (H^{b,n}(u, X_{u^-}^{s,x}) - \mathcal{L}_0 \cdot Z_u^{b,n;s,x})\mathrm{d}u - Z_u^{b,n;s,x} \cdot \mathrm{d}M_u,\ Y_h^{b,n;s,x} = g^b(X_h^{s,x}). \end{cases}$$

C.2.2 Step 2: Convergence to the solution of a bang-bang system of BSDEs

From now, we consider any index i equals to a or b, we set $x \in \mathbb{N}^2$ and $s \in [0,h]$.

Step 2a. Uniform estimates.

In order to use dominated convergence we give some uniform a priori estimates for processes $(Y^{i,n;s,x}, Z^{i,n;s,x})$.

We first aim at using a comparison principle to control the upper bound of $Y^{i,n}$ and introduce the following BSDE

$$\overline{Y}_u^{i,n;s,x} = g^i(X_h^{s,x}) + \int_u^h 4\lambda^+ \|\overline{Z}_r^{i,n;s,x}\|_1 \mathrm{d}r - \int_u^h \overline{Z}_r^{i,n;s,x} \cdot \mathrm{d}M_r, \ s \le u \le h. \tag{17}$$

Once again according to Buckdahn and Pardoux (1994) there exists a unique solution $(\overline{Y}^{i,n;s,x}, \overline{Z}^{i,n;s,x})$ of the above BSDE in the space $\mathcal{S}^2_{s,h}(\mathbb{R}) \times \mathcal{H}^2_{s,h}(\mathbb{R}^2)$ and there exists deterministic measurable functions $\overline{V}^{i,n}$ such that for any $u \in [s,h]$:

$$\overline{Y}_u^{i,n;s,x} = \overline{V}^{i,n}(u, X_u^{s,x}).$$

By comparison theorem for BSDE (see for instance[8] Theorem 2.5 in Royer (2006)), for any time $s \le u \le h$ we get

$$Y_u^{i,n;s,x} \le \overline{Y}_u^{i,n;s,x}, \mathbb{P} - a.s. \tag{18}$$

We now give a uniform estimates of $\overline{Y}^{i,n;s,x}$ to get a uniform estimates for $Y^{i,n;s,x}$ in view of the previous relation. Consider the bi-dimensional process:

$$M_u^{i,n} = M_u - 4\lambda_+ \mathrm{sign}(\overline{Z}_u^{i,n;s,x}),$$

where the sign is taken coordinate by coordinate. The process $M^{i,n} = (M_1^{i,n}, M_2^{i,n})$ is a bi-dimensional martingale under the probability $\mathbb{P}^{i,n}$ equivalent to $\mathbb{P}$ with density given by

$$\mathcal{E}_h^{i,n} = \exp\Big(\int_0^h \log(\frac{\gamma_{t,1}^{i,n}}{\lambda_0}) \mathrm{d}N_t^a - (\gamma_{t,1}^{i,n} - \lambda_0)\mathrm{d}t + \log(\frac{\gamma_{t,2}^{i,n}}{\lambda_0}) \mathrm{d}N_t^b - (\gamma_{t,2}^{i,n} - \lambda_0)\mathrm{d}t \Big)$$

with

$$\gamma_{t,j}^{i,n} = \lambda_0 + 4\lambda_+ \mathrm{sign}(\overline{Z}_{j,t}^{i,n;t,x}).$$

Consequently we get

$$\overline{V}^{i,n}(s,x) = \mathbb{E}^{\mathbb{P}^{i,n}}[g_i(X_h^{s,x})].$$

By polynomial growth of g_i we deduce that there exists a positive constant $\tilde{C}$ such that

$$|\overline{V}^{i,n}(s,x)| \le \tilde{C} \mathbb{E}^{\mathbb{P}^{i,n}}[\|X_h^{s,x}\|_2^2].$$

Note that there exists a positive constant $\tilde{\kappa}$ such that

$$\mathbb{E}^{\mathbb{P}^{i,n}}[\|X_h^{s,x}\|_2^2] \le \tilde{\kappa}(\|x\|_2^2 + 1).$$

The previous equation implies the following polynomial growth bound

$$|\overline{V}^{i,n}(s,x)| \le C(1 + \|x\|_2^2),$$

[8] To be more accurate, we identify our pair of processes as a compound Poisson process with jumps in $\{-1, 1\}$, so that we are in the framework of Royer (2006) for a compensator $\lambda(dx) = \lambda_0(\delta_1(dx) + \delta_{-1}(dx))$.

where $C := \tilde{C}\tilde{\kappa} > 0$.
According to the comparison result (18) together with (16), we deduce that there exists some positive constant C, which does not depend on n, such that

$$V^{i,n}(s,x) \leq C(1 + |x_1|^2 + |x_2|^2).$$

Similarly, by considering a BSDE similar to (17) but with a minus sign in the generator, we get

$$V^{i,n}(s,x) \geq -C(1 + |x_1|^2 + |x_2|^2).$$

We thus deduce that for any $(s,x) \in [0,h] \times \mathbb{N}^2$ and $p \geq 1$ the following estimate holds for some positive constant C_p

$$\mathbb{E}[\sup_{s \leq u \leq h} |Y_u^{i,n;s,x}|^p] \leq C_p(1 + |x_1|^{2p} + |x_2|^{2p}). \tag{19}$$

Moreover, the characterization (16) allows to transfer the prior estimates of $Y^{i,n;s,x}$ to $Z^{i,n;s,x}$. In particular we get that for any $p \geq 1$

$$\mathbb{E}[\sup_{s \leq u \leq h} |Z_u^{i,n;s,x}|^p] \leq C_p(1 + |x_1|^{2p} + |x_2|^{2p}). \tag{20}$$

Note that the constant C_p does not depend on n, so that Estimates (19) and (20) are uniform with respect to n.

Step 2b. Convergence of the solutions of the smoothed BSDE.

We now turn to the convergence of $(Y^{i,n;s,x}, Z^{i,n;s,x})$, in $\mathcal{S}^2_{s,h}(\mathbb{R}) \times \mathcal{H}^2_{s,h}(\mathbb{R}^2)$. For any $q \leq 2$, there exists a positive constant $\tilde{C}$ which does not depend on n such that

$$\mathbb{E}[\int_0^h |H^{i,n}(r, X^{0,0}_{r^-})|^q \mathrm{d}r \leq \mathbb{E}[\int_0^h 2\lambda_+ \|Z_r^{i,n;0,0}\|_1^q \mathrm{d}r] \leq \tilde{C}.$$

The sequence $(H^{i,n})_{n\geq 0}$ is bounded in $\mathbb{L}^2\big([0,h] \times \mathbb{N}^2, \mathrm{d}r \times \mu(0,0;r,\mathrm{d}x)\big)$ where $\mu(0,0;r,\mathrm{d}x)$ is the law of $X^{0,0}_{r^-}$ under $\mathbb{P}$. Thus there exists a subsequence $(n_k)_{k\geq 0}$ such that $(H^{i,n_k})_{k\geq 0}$ converges weakly in $\mathbb{L}^2([0,h] \times \mathbb{R}, \mu(0,0;r,\mathrm{d}x)\mathrm{d}r)$. We omit the index k and still write n instead of n_k to reduce the notations.

We now prove that for any $(s,x) \in [0,h] \times \mathbb{N}^2$, $(V^{i,n}(s,x))_{n\geq 0}$ is a Cauchy sequence. We set the function $\Delta^{i,n,m}(t,x,z_n,z_m) := H^{i,n}(t,x) - H^{i,m}(t,x) - \mathcal{L}_0 \cdot (z_n - z_m)$ with $(n,m) \in \mathbb{N}$ and $(t,x,z_n,z_m) \in [0,T] \times \mathbb{N}^2 \times \mathbb{R}^2 \times \mathbb{R}^2$. Let $\delta \in [0,h-s]$ and $k \in \mathbb{N}$, we have

$$\begin{aligned} |V^{i,n}(s,x) - V^{i,m}(s,x)| &= |\mathbb{E}[\int_s^h \Delta^{i,n,m}(r, X^{s,x}_{r^-}, Z_r^{i,n;s,x}, Z_r^{i,m;s,x})\mathrm{d}r]| \\ &\leq E_-^{s+\delta,h} + E_+^{s+\delta,h} + E^{s,s+\delta}, \end{aligned} \tag{21}$$

with

$$E_-^{s+\delta,h} := |\mathbb{E}[\int_{s+\delta}^h \mathbf{1}_{\|X_{r^-}^{s,x}\|_\infty \le k} \Delta^{i,n,m}(r, X_{r^-}^{s,x}, Z_r^{i,n;s,x}, Z_r^{i,m;s,x}) \mathrm{d}r]|,$$

$$E_+^{s+\delta,h} := |\mathbb{E}[\int_{s+\delta}^h \mathbf{1}_{\|X_{r^-}^{s,x}\|_\infty > k} \Delta^{i,n,m}(r, X_{r^-}^{s,x}, Z_r^{i,n;s,x}, Z_r^{i,m;s,x}) \mathrm{d}r]|,$$

and

$$E^{s,s+\delta} := |\mathbb{E}[\int_s^{s+\delta} \Delta^{i,n,m}(r, X_{r^-}^{s,x}, Z_r^{i,n;s,x}, Z_r^{i,m;s,x}) \mathrm{d}r]|.$$

We obtain from (20) that there exists some constant C independent of n and m such that

$$E^{s,s+\delta} \le C\delta.$$

We now turn to $E_+^{s+\delta,h}$. By using Cauchy Schwarz and Markov inequalities together with the prior inequalities (19) and (20), there exists a positive constant $\hat{C}$ again independent of n and m such that for any positive integer k

$$\begin{aligned} E_+^{s+\delta,h} &\le |\mathbb{E}[\int_{s+\delta}^h \mathbf{1}_{\|X_{r^-}^{s,x}\|_\infty > k} \mathrm{d}r]|^{\frac{1}{2}} |\mathbb{E}[\int_{s+\delta}^h \Delta^{i,n,m}(r, X_{r^-}^{s,x}, Z_r^{i,n;s,x}, Z_r^{i,m;s,x})^2 \mathrm{d}r]|^{\frac{1}{2}} \\ &\le \frac{\hat{C}}{\sqrt{k}}. \end{aligned}$$

Finally, we note that

$$E_-^{s+\delta,h} = \Big| \sum_{(p,q)\in\mathbb{N}^2} \int_s^h \Delta^{i,n,m}(r, p, q, DV^{i,n}(t,p,q), DV^{i,m}(t,p,q)) \mathbb{P}\big(X_r^{t,(0,0)} = (p,q)\big) \phi_{s,x}(r,p,q) \mathrm{d}r \Big|$$

with

$$\phi_{s,x}(r,p,q) = \mathbf{1}_{p\le k} \mathbf{1}_{q \le k} \mathbf{1}_{r \ge s+\delta} \frac{\mathbb{P}\big(X_r^{t,x} = (p,q)\big)}{\mathbb{P}\big(X_r^{t,(0,0)} = (p,q)\big)}.$$

Since

$$\mathbb{P}\big(X_r^{t,(0,0)} = (p,q)\big)^{-1} = e^{2\lambda_0 r} \frac{p!q!}{(\lambda_0 r)^{p+q}}$$

is bounded for p and q lower than k and r lower than h. The function $\phi_{s,x}$ is bounded and thus in $\mathbb{L}^2([0,h] \times \mathbb{N}^2, \mu(0,0;s,\mathrm{d}x) \times \mathrm{d}s))$ consequently by weak convergence of $H^{i,n}$, we have that $E_-^{s+\delta,h}$ goes to 0 when m,n go to infinity. Hence, taking the limit when δ goes to 0 and k,n,m go to infinity, we deduce from (21) that $(V^{i,n}(s,x))_{n\ge 0}$ is a Cauchy sequence. We thus denote by $V^i(s,x)$ the limit of $(V^{i,n}(s,x))_{n\ge0}$. We recall that V^i depends on the subsequence $(n_k)_{k\ge0}$

We have the $\mathbb{P}$-almost sure convergence (up to the subsequence) of $Y_u^{i,n;s,x}$ since $Y_u^{i,n;s,x} = V^{i,n}(u, X_u^{s,x})$. We denote by $Y^{i;s,x}$ the almost sure limit of $Y^{i,n;s,x}$. Notice moreover that in view of (**D**), we have

$$\lim_{n\to+\infty} DV^{i,n}(s,x) = DV^i(s,x),\ (s,x) \in [0,h] \times \mathbb{N}^2. \tag{22}$$

By Equation (19) and Lebesgue dominated convergence theorem we have for any $\rho \geq 1$

$$\mathbb{E}[\int_s^h |Y_r^{i,n;s,x} - Y_r^{i;s,x}|^\rho \mathrm{d}r] \underset{n\to+\infty}{\to} 0. \tag{23}$$

Let now n, m be two positive integers. From Ito's formula applied to $(Y^{i,n;s,x} - Y^{i,m;s,x})^2$ we get for any $s \leq u \leq h$

$$\begin{aligned}
&|Y_u^{i,n;s,x} - Y_u^{i,m;s,x}|^2 \\
&= -\int_u^h |Z_{1,r}^{i,n;s,x} - Z_{1,r}^{i,m;s,x}|^2 \mathrm{d}(M_r^a + \lambda_0 r) - \int_u^h |Z_{2,r}^{i,n;s,x} - Z_{2,r}^{i,m;s,x}|^2 \mathrm{d}(M_r^b + \lambda_0 r) \\
&\quad + 2\int_u^h (Y_r^{i,n;s,x} - Y_r^{i,m;s,x})\big((H^{i,n} - H^{i,m})(r, X_{r^-}^{t,x}) - \mathcal{L}_0 \cdot (Z_r^{i,n;s,x} - Z_r^{i,m;s,x})\big)\mathrm{d}r \\
&\quad - 2\int_u^h (Y_r^{i,n;s,x} - Y_r^{i,m;s,x})(Z_r^{i,n;s,x} - Z_r^{i,m;s,x}) \cdot \mathrm{d}M_r.
\end{aligned} \tag{24}$$

Using Young's inequality and the definitions of H^n and H^m we deduce that there exists a positive constant $\tilde{c}$ (independent of n and m) such that for any $\varepsilon > 0$

$$\begin{aligned}
&|Y_u^{i,n;s,x} - Y_u^{i,m;s,x}|^2 + \int_u^h \lambda_0 \|Z_r^{i,n;s,x} - Z_r^{i,m;s,x}\|_2^2 \mathrm{d}r \\
&\leq \tilde{c}\varepsilon|\lambda_+|^2 \int_u^h \big(\|Z_r^{i,n;s,x}\|_2^2 + \|Z_r^{i,m;s,x}\|_2^2\big)\mathrm{d}r + \frac{1}{\varepsilon}\int_u^h |Y_r^{i,n;s,x} - Y_r^{i,m;s,x}|^2 \mathrm{d}r \\
&\quad - 2\int_u^h (Y_r^{i,n;s,x} - Y_r^{i,m;s,x})(Z_r^{i,n;s,x} - Z_r^{i,m;s,x}) \cdot \mathrm{d}M_r \\
&\quad - \int_u^h |Z_{1,r}^{i,n;s,x} - Z_{1,r}^{i,m;s,x}|^2 \mathrm{d}M_r^a - \int_u^h |Z_{2,r}^{i,n;s,x} - Z_{2,r}^{i,m;s,x}|^2 \mathrm{d}M_r^b.
\end{aligned}$$

For $u = s$, by taking the expectation and by choosing n, m large enough, we obtain from (20) and (22), (23) and the fact that ε is arbitrary small that the following convergence holds

$$\limsup_{n,m\to+\infty} \mathbb{E}[\int_s^h \|Z_r^{i,n;s,x} - Z_r^{i,m;s,x}\|_2^2 \mathrm{d}r] = 0. \tag{25}$$

Hence, $(Z^{i,n;s,x})_{n\in\mathbb{N}}$ is a Cauchy sequence (along the subsequence) and thus converges in $\mathcal{H}_{s,h}^2(\mathbb{R}^2)$ to some process $(Z_u^{i;s,x})_{s\leq u\leq h}$.

Similarly, by using (24) and by noting that $-\int_u^h |Z_{1,r}^{i,n;s,x} - Z_{1,r}^{i,m;s,x}|^2 \mathrm{d}(M_r^a + \lambda_0 r) \leq 0$ and $-\int_u^h |Z_{2,r}^{i,n;s,x} -$

$Z_{2,r}^{i,m;s,x}|^2 \mathrm{d}(M_r^b + \lambda_0 r) \leq 0$ since $M_\cdot^\alpha + \lambda_0 \cdot = X_\cdot$ is a non decreasing process for $\alpha \in \{a,b\}$, we have

$$
\begin{aligned}
&\mathbb{E}[\sup_{u\in[s,h]} |Y_u^{i,n;s,x} - Y_u^{i,m;s,x}|^2] \\
&\leq \tilde{c}\varepsilon|\lambda_+|^2 \mathbb{E}[\int_s^h \big(\|Z_r^{i,n;s,x}\|_2^2 + \|Z_r^{i,m;s,x}\|_2^2\big)\mathrm{d}r] + \frac{1}{\varepsilon}\mathbb{E}[\int_s^h |Y_r^{i,n;s,x} - Y_r^{i,m;s,x}|^2 \mathrm{d}r] \\
&+ 2\mathbb{E}[\int_0^h |Y_r^{i,n;s,x} - Y_r^{i,m;s,x}||Z_{1,r}^{i,n;s,x} - Z_{1,r}^{i,m;s,x}|(dN_r^a + \lambda_0 dr)] \\
&+ 2\mathbb{E}[\int_0^h |Y_r^{i,n;s,x} - Y_r^{i,m;s,x}||Z_{2,r}^{i,n;s,x} - Z_{2,r}^{i,m;s,x}|(dN_r^b + \lambda_0 dr)] \\
&\leq \tilde{c}\varepsilon|\lambda_+|^2 \mathbb{E}[\int_s^h \big(\|Z_r^{i,n;s,x}\|_2^2 + \|Z_r^{i,m;s,x}\|_2^2\big)\mathrm{d}r] + \frac{1}{\varepsilon}\mathbb{E}[\int_s^h |Y_r^{i,n;s,x} - Y_r^{i,m;s,x}|^2 \mathrm{d}r] \\
&+ 2\mathbb{E}[\int_0^h |Y_r^{i,n;s,x} - Y_r^{i,m;s,x}||Z_{1,r}^{i,n;s,x} - Z_{1,r}^{i,m;s,x}|(dM_r^a + 2\lambda_0 dr)] \\
&+ 2\mathbb{E}[\int_0^h |Y_r^{i,n;s,x} - Y_r^{i,m;s,x}||Z_{2,r}^{i,n;s,x} - Z_{2,r}^{i,m;s,x}|(dM_r^b + 2\lambda_0 dr)] \\
&\leq \tilde{c}\varepsilon|\lambda_+|^2 \mathbb{E}[\int_s^h \big(\|Z_r^{i,n;s,x}\|_2^2 + \|Z_r^{i,m;s,x}\|_2^2\big)\mathrm{d}r] + \frac{1}{\varepsilon}\mathbb{E}[\int_s^h |Y_r^{i,n;s,x} - Y_r^{i,m;s,x}|^2 \mathrm{d}r] \\
&+ 4\lambda_0 \mathbb{E}[\int_0^h |Y_r^{i,n;s,x} - Y_r^{i,m;s,x}|\|Z_r^{i,n;s,x} - Z_r^{i,m;s,x}\|_1 dr].
\end{aligned}
$$

By using again Young inequality for the last term in the previous inequality with the same ε, we deduce that there exists a positive constant $c > 0$ independent of n, m and ε such that

$$
\begin{aligned}
&\mathbb{E}[\sup_{u\in[s,h]} |Y_u^{i,n;s,x} - Y_u^{i,m;s,x}|^2] \\
&\leq c\big(\varepsilon|\lambda_+|^2 \mathbb{E}[\int_s^h \big(\|Z_r^{i,n;s,x}\|_2^2 + \|Z_r^{i,m;s,x}\|_2^2\big)\mathrm{d}r] + \frac{1}{\varepsilon}\mathbb{E}[\int_s^h |Y_r^{i,n;s,x} - Y_r^{i,m;s,x}|^2 \mathrm{d}r]\big).
\end{aligned}
$$

Since ε is arbitrary and because of Equations (19), (20) and (23) we deduce

$$
\limsup_{n,m\to+\infty} \mathbb{E}[\sup_{u\in[s,t]} |Y_u^{i,n;s,x} - Y_u^{i,m;s,x}|^2] = 0.
$$

So we have the convergence of $(Y^{i,n;s,x})_{n\geq 0}$ in $\mathcal{S}^2_{s,h}(\mathbb{R})$ towards a process $(Y_u^{i;s,x})_{s\leq u\leq h}$ up to a subsequence.

Step 2c. Convergence of the generator

We study the convergence of $(H^{i,n})_{n\geq 0}$, for $i \in \{a,b\}$ (still along the subsequence introduced in Step 2b.). We focus on $(H^{a,n})_{n\geq 0}$, the proof is identical for $(H^{b,n})_{n\geq 0}$. Recall that

$$
H^{a,n}(u, X_{u^-}^{s,x}) = \Phi_n(Z_{1,u}^{a,n;s,x}\lambda_a^\star(Z_{1,u}^{a,n;s,x})) + \Phi_n(Z_{2,u}^{a,n;s,x})\lambda^n(Z_{2,u}^{b,n;s,x}).
$$

First note that

$$\Phi_n(Z_{1,u}^{a,n;s,x}\lambda_a^\star(Z_{1,u}^{a,n;s,x})) \underset{n\to+\infty}{\to} Z_{1,u}^{a;s,x}\lambda_a^\star(Z_{1,u}^{a;s,x})$$

with convergence taking place $\mathbb{P}$-a.s. and in $\mathcal{H}_{s,u}^2(\mathbb{R}^2)$ by dominated convergence and uniform integrability of $(\|Z^{a,n;s,x}\|_2^2)_{n\geq 0}$. We split the remaining part in a continuous and a non continuous parts

$$\Phi_n(Z_{2,u}^{a,n;s,x})\lambda^n(Z_{2,u}^{b,n;s,x}) = \Phi_n(Z_{2,u}^{a,n;s,x})\lambda^n(Z_{2,u}^{b,n;s,x})\mathbf{1}_{Z_u^{b;s,x}\neq 0} + \Phi_n(Z_{2,u}^{a,n;s,x})\lambda^n(Z_{2,u}^{b,n;s,x})\mathbf{1}_{Z_{2,u}^{b;s,x}=0}.$$

We have the convergence of $\Phi_n(Z_{2,u}^{a,n;s,x})\lambda^n(Z_{2,u}^{b,n;s,x})\mathbf{1}_{Z_{2,u}^{b;s,x}\neq 0}$, $\mathrm{d}s\times\mathrm{d}\mathbb{P}$ $a.e$ and the convergence also holds in $\mathcal{H}_{s,h}^2(\mathbb{R}^2)$. Moreover, $(\lambda^n(Z_{2,u}^{b,n;s,x})\mathbf{1}_{Z_u^{b;s,x}=0})_{n\geq 0}$ being bounded we denote by ϑ a weak limit in $\mathcal{H}_{s,h}^2(\mathbb{R}^2)$.

Now we show that for any stopping time $\tau\in[s,h]$ we have in the sense of weak convergence in $\mathbb{L}^2(\mathbb{R})$:

$$\int_s^\tau \Phi_n(Z_{2,u}^{a,n;s,x})\lambda^n(Z_{2,u}^{b,n;s,x})\mathbf{1}_{Z_{2,u}^{b;s,x}=0}\mathrm{d}u \underset{n\to+\infty}{\to} \int_s^\tau Z_{2,u}^{a;s,x}\vartheta_u\mathbf{1}_{Z_{2,u}^{b;s,x}}\mathrm{d}u. \tag{26}$$

We have

$$\begin{aligned}\int_s^\tau \Phi_n(Z_{2,u}^{a,n;s,x})\lambda^n(Z_{2,u}^{b,n;s,x})\mathbf{1}_{Z_{2,u}^{b;s,x}=0}\mathrm{d}u &= \int_s^\tau(\Phi_n(Z_{2,u}^{a,n;s,x}) - Z_{2,u}^{a;s,x})\lambda^n(Z_{2,u}^{b,n;s,x}))\mathbf{1}_{Z_{2,u}^{b,s,x}=0}\mathrm{d}u\\ &+\int_s^\tau Z_{2,u}^{a;s,x}\lambda^n(Z_{2,u}^{b,n;s,x})\mathbf{1}_{Z_{2,u}^{b;s,x}=0}\mathrm{d}s.\end{aligned}$$

The first term in the previous equality converges to 0 in $\mathbb{L}^2(\mathbb{R})$ by dominated convergence therefore it converges weakly. Now we show that the second term converges weakly. We prove that for any random variable $\xi\in\mathbb{L}^2(\mathbb{R})$ and $\mathcal{F}_h$-measurable the following convergence holds

$$\mathbb{E}[\xi\int_s^\tau Z_{2,u}^{a;s,x}\lambda^n(Z_{2,u}^{b,n;s,x})\mathbf{1}_{Z_{2,u}^{b;s,x}=0}\mathrm{d}u] \underset{n\to+\infty}{\to} \mathbb{E}[\xi\int_s^\tau Z_{2,u}^{a;s,x}\vartheta_u\mathbf{1}_{Z_{2,u}^{b;s,x}=0}\mathrm{d}u]. \tag{27}$$

Using a martingale decomposition result for martingales associated to jump processes, see Davis (1976), to the conditional expectation of ξ with respect to the filtration $\mathcal{F}$ we have

$$\mathbb{E}[\xi|\mathcal{F}_\tau] = \mathbb{E}[\xi] + \int_s^\tau \Lambda_u\cdot\mathrm{d}M_u$$

for some $\Lambda\in\mathcal{H}_{s,h}^2(\mathbb{R}^2)$. Consequently

$$\begin{aligned}\mathbb{E}[\xi\int_s^\tau Z_{2,u}^{a;s,x}\lambda^n(Z_{2,u}^{b,n;s,x})\mathbf{1}_{Z_{2,u}^{b;s,x}=0}\mathrm{d}u] &= \mathbb{E}[\int_s^\tau\Lambda_u\cdot\mathrm{d}M_u\int_s^\tau Z_{2,u}^{a;s,x}\lambda^n(Z_{2,u}^{b,n;s,x})\mathbf{1}_{Z_{2,u}^{b;s,x}=0}\mathrm{d}u]\\ &+\mathbb{E}[\xi]\mathbb{E}[\int_s^\tau Z_{2,u}^{a;s,x}\lambda^n(Z_{2,u}^{b,n;s,x})\mathbf{1}_{Z_{2,u}^{b;s,x}=0}\mathrm{d}u].\end{aligned}$$

Notice moreover that

$$\mathbb{E}[\xi]\mathbb{E}[\int_s^\tau Z_{2,u}^{a;s,x}\lambda^n(Z_{2,u}^{b,n;s,x})\mathbf{1}_{Z_{2,u}^{b;s,x}=0}\mathrm{d}u] \underset{n\to+\infty}{\to} \mathbb{E}[\xi]\mathbb{E}[\int_s^\tau Z_{2,u}^{a;s,x}\vartheta_u\mathbf{1}_{Z_{2,u}^{b;s,x}=0}\mathrm{d}u]$$

since $\lambda^n(Z_{2,u}^{b,n;s,x})\mathbf{1}_{Z_{2,u}^{b;s,x}=0}$ converges to $\vartheta_u\mathbf{1}_{Z_{2,u}^{b;s,x}=0}$ and since $Z^{a;s,x}\in\mathcal{H}_{s,h}^2(\mathbb{R}^2)$. Using Ito's formula, we get

$$\begin{aligned}
&\mathbb{E}[\int_s^\tau \Lambda_u\cdot\mathrm{d}M_u\int_s^\tau Z_{2,u}^{a;s,x}\lambda^n(Z_{2,u}^{b,n;s,x})\mathbf{1}_{Z_{2,u}^{b;s,x}=0}\mathrm{d}u]\\
&=\mathbb{E}\Big[\int_s^\tau\big(\int_s^u Z_{r,2}^{a;s,x}\lambda^n(Z_{r,2}^{b,n;s,x})\mathbf{1}_{Z_{r,2}^{b;s,x}=0}\mathrm{d}r\big)\Lambda_u\cdot\mathrm{d}M_u\Big]\\
&+\mathbb{E}\Big[\int_s^\tau\int_s^u \Lambda_r\cdot\mathrm{d}M_r Z_{2,u}^{a;s,x}\lambda^n(Z_{2,u}^{b,n;s,x})\mathbf{1}_{Z_{2,u}^{b;s,x}=0}\mathrm{d}u\Big].
\end{aligned}$$

The first term is equal to zero. Concerning the second term, we set $\psi_r=\int_s^r\Lambda_u\cdot\mathrm{d}M_u$. Hence, for any $\kappa\geq 0$

$$\begin{aligned}
&\mathbb{E}[\int_s^\tau \psi_u Z_{2,u}^{a;s,x}(\lambda^n(Z_{2,u}^{b,n;s,x})\mathbf{1}_{Z_{2,u}^{b;s,x}=0}-\vartheta_u)\mathrm{d}u]\\
&=\mathbb{E}[\int_s^\tau \psi_u Z_{2,u}^{a;s,x}\mathbf{1}_{|\psi_u Z_{2,u}^{a;s,x}|<\kappa}(\lambda^n(Z_{2,u}^{b,n;s,x})-\vartheta_u)\mathbf{1}_{Z_{2,u}^{b;s,x}=0}\mathrm{d}u]\\
&+\mathbb{E}[\int_s^\tau \psi_u Z_{2,u}^{a;s,x}\mathbf{1}_{|\psi_u Z_{2,u}^{a;s,x}|\geq\kappa}(\lambda^n(Z_{2,u}^{b,n;s,x})\ -\vartheta_u)\mathbf{1}_{Z_{2,u}^{b;s,x}=0}\mathrm{d}u].
\end{aligned}$$

The first term in the previous expression converges to 0 since $\lambda^n(Z_{2,\cdot}^{b,n;s,x})\mathbf{1}_{Z_{2,\cdot}^{b;s,x}=0}$ converges weakly towards ϑ. The second one goes to zero when κ goes to infinity as $\psi\|Z^{a;s,x}\|_2$ is in $\mathcal{H}_{s,h}^2(\mathbb{R})$. We have proved the convergence (27). Hence, the convergence (26) holds weakly in $\mathbb{L}^2(\mathbb{R})$.

We deduce that $\int_s^\tau H^{a,n}(u,X_u^{s,x})\mathrm{d}u$ converges weakly to $\int_0^\tau H^{a,\star}(Z_u^{a;s,x},Z_u^{b;s,x},\vartheta_u)\mathrm{d}u$ in $\mathbb{L}^2(\mathbb{R})$ along the subsequence $(n_k)_{k\geq0}$.

Step 2d. Convergence to the solution of a bang-bang BSDE

If we write the first BSDE in the system ($\mathbf{J^n}$) in a forward way, we get

$$Y_\tau^{a,n;s,x}=Y_s^{a,n;s,x}-\int_s^\tau H^{a,n}(u,X_{u^-}^{s,x})\mathrm{d}u+\int_s^\tau Z_u^{a,n;s,x}\mathrm{d}M_u.$$

We recall that we write n instead of n_k so that all the convergence that we obtain has to be understood up to a subsequence. Thus, from the almost sure and $\mathcal{S}_{s,h}^2(\mathbb{R})$ convergence of $(Y^{a,n;s,x})_{n\geq0}$ to $Y^{a;s,x}$ together with

$$\int_s^\tau Z_u^{a,n;s,x}\cdot\mathrm{d}M_u \underset{n\to+\infty}{\to}\int_s^\tau Z_u^{a;s,x}\cdot\mathrm{d}M_u,\ \text{in }\mathbb{L}^2(\mathbb{R}),$$

and the convergence of the generator $H^{a,n}$ proved in Step 2c, we deduce that

$$Y_\tau^{a;s,x} = Y_s^{a;s,x} - \int_s^\tau H^{a,\star}(Z_u^{a;s,x}, Z_u^{b;s,x}, \vartheta_u)\mathrm{d}u + \int_s^\tau Z_u^{a;s,x}\mathrm{d}M_u,\ \mathbb{P}-a.s.$$

This result being true for any stopping time $\tau \in [s,h]$, the processes on both sides are indistinguishable and we have

$$\mathbb{P}-a.s.\ Y_u^{a;s,x} = Y_s^{a;s,x} - \int_s^u H^{a,\star}(Z_r^{a;s,x}, Z_r^{b;s,x}, \vartheta_r)\mathrm{d}r + \int_s^u Z_r^{a;s,x}\mathrm{d}M_r,\ \forall u \in [s,h].$$

Finally we have

$$\mathbb{P}-a.s.\ Y_u^{a;s,x} = g^a(X_h^{s,x}) + \int_u^h H^{a,\star}(Z_r^{a;s,x}, Z_r^{b;s,x}, \vartheta_r)\mathrm{d}r - \int_u^h Z_r^{a;s,x}\mathrm{d}M_r,\ \forall u \in [s,h].$$

with $Y^{a;s,x} \in \mathcal{S}^2_{s,h}(\mathbb{R})$ and $Z^{a;s,x} \in \mathcal{H}^2_{s,h}(\mathbb{R}^2)$. We have the same result by considering the index b and by denoting θ_u the almost sure limit of $(\lambda^n(Z_u^{a,n;s,x})\mathbf{1}_{Z_u^{a,n;s,x}=0})_{n\geq 0}$ which holds also in $\mathcal{H}^2_{s,h}$ by the dominated convergence theorem.

Step 3: Nash equilibrium and conclusion.

We have seen in the previous step that we can build ϑ and θ, which are functions of (u, N_u^a, N_u^b) ensuring the existence of a solution a solution $(Y^a, Y^b, Z^a, Z^b) \in (\mathcal{S}^2_{s,h}(\mathbb{R}))^2 \times (\mathcal{H}^2_{s,h}(\mathbb{R}^2))^2$ to the following coupled BSDE (by taking $s=0$),

$$\begin{cases} -\mathrm{d}Y_u^a = & H^{a,\star}(Z_u^a, Z_u^b, \vartheta_u) - Z_u^a \cdot \mathrm{d}M_u,\ Y_h^a = g^a(X_t^{0,0}) \\ -\mathrm{d}Y_u^b = & H^{b,\star}(Z_u^a, Z_u^b, \theta_u) - Z_u^b \cdot \mathrm{d}M_u,\ Y_h^b = g^b(X_t^{0,0}). \end{cases} \tag{28}$$

We could rely this BSDE to the system (**S**) and use Proposition C.1. However, we are not able to prove the continuous differentiability of the functions V^i with respect to the time variable. It is why we use the theory of BSDEs similarly to Hamadène and Mu (2014) with the proposition below to conclude.

Proposition C.2 (Extension of Theorems 2.5 and 2.6 in Hamadène and Mu (2014))**.** *There exist a pair of deterministic functions V^a, V^b and some adapted processes ϑ and θ with values in $[\lambda_-, \lambda_+]$ such that*

- *BSDE* (28) *admits a solution $(Y^a, Y^b, Z^a, Z^b) \in (\mathcal{S}^2_h(\mathbb{R}))^2 \times (\mathcal{H}^2_h(\mathbb{R}^2))^2$,*
- *V^a and V^b are two deterministic measurable functions with polynomial growth from $[0,h]\times\mathbb{R}^2$ to $\mathbb{R}$ such that $\mathbb{P}-as$, $\forall u \leq h,\ Y_u^a = V^a(u, X_u)$ and $Y_u^b = V^b(u, X_u)$.*
- *The pair of controls $(\lambda_a^\star(Z_u^a, \theta_u), \lambda_b^\star(Z_u^b, \vartheta_u))_{u\leq t}$ defined by* (**L**) *where ϑ and θ are obtained as an almost sure (up to a subsequence) and $\mathcal{H}^2_h(\mathbb{R}^2)$ limits of $\lambda^n(Z_u^{b,n})\mathbf{1}_{Z_u^b=0}$ and $\lambda^n(Z_u^{a,n})\mathbf{1}_{Z_u^a=0}$*

respectively is a bang-bang type Nash equilibrium point of the non zero-sum stochastic differential game (15)*.*

Proof. Properties 1. and 2. are direct consequences of the proof made in Step 2. Property 3. is obtained by adapting the proof of Proposition 2.4 in Hamadène and Mu (2014) to the jump case, with minimizations instead of maximizations. □

Hence, Step 1 provides that the system ($\mathbf{S^n}$) admits a unique viscosity solution given by the unique solution of ($\widetilde{\mathbf{J^n}}$) which approaches the solution of (28) so that $\lambda^n(Z_u^{b,n})\mathbf{1}_{Z_u^b=0}$ and $\lambda^n(Z_u^{a,n})\mathbf{1}_{Z_u^a=0}$ converge almost surely up to a subsequence (and in fact in $\mathcal{H}_h^2(\mathbb{R}^2)$) to a Nash equilibrium for the game (15) by using Proposition C.2. This concludes the proof of Theorem C.1.

C.3 Proof of Corollary 3.1 and numerical method

In Theorem C.1 we only get convergence results up to a subsequence. However numerically we observe that the sequence $(V^{i,n})_{n\geq 0}$ converges for $i = a$ or b. Therefore to approach the solution of the system ($\mathbf{S}$) we solve the approached system ($\mathbf{S^n}$) for n large. To implement the numerical method we need to bound the domain. In practice this means that there is only a limited number of orders in auctions. Thus we consider the new system

$$(\mathbf{S^n_Q})\begin{cases} \partial_s V^{a,n} + H^{a,n}(D_a^Q V^{a,n}, D_b^Q V^{a,n}, D_b^Q V^{b,n}) = 0,\ s\in[0,h),\ (\alpha,\beta)\in\{0,\dots,Q\}^2, \\ V^{a,n}(h,\alpha,\beta) = g^a(\alpha,\beta), \\ \partial_s V^{b,n} + H^{b,n}(D_b^Q V^{b,n}, D_a^Q V^{b,n}, D_a^Q V^{a,n})) = 0,\ s\in[0,h),\ (\alpha,\beta)\in\{0,\dots,Q\}^2, \\ V^{b,n}(h,\alpha,\beta) = g^b(\alpha,\beta), \end{cases}$$

on the domain $[0,h]\times\{0,\dots,Q\}^2$. The operators (D_a^Q, D_b^Q) are defined similarly to (D_a, D_b) with the following boundary conditions

$$D_a^Q V(s,Q,m) = 0 \text{ and } D_b^Q V(s,n,Q) = 0 \text{ for any } (s,n,m)\in[0,h]\times\{0,\dots,Q\}^2.$$

Interpreting ($\mathbf{S^n_b}$) as an ordinary differential equation in $\mathbb{R}^{(Q+1)^2}$ according to Cauchy-Lipschitz Theorem we have existence of a solution $(V_Q^{a,n}, V_Q^{b,n})$ for the system ($\mathbf{S^n_Q}$) which is unique.

Remember that in our model the auction starts at time $\tau = \inf\{s > 0 \text{ s.t. } N_s^a + N_s^b > 0\}$. Consequently market takers optimize their behavior by controlling the processes $(N^a_{\tau+\cdot}, N^b_{\tau+\cdot})$. Now remark that

$$I^2_{\tau+h} = N^a_{\tau+h}(N^a_{\tau+h} - N^b_{\tau+h}) + N^b_{\tau+h}(N^b_{\tau+h} - N^a_{\tau+h}).$$

Consequently, the symmetry of the problem with respect to a and b leads to

$$\mathbb{E}[I^2_{\tau+h}] = \mathbb{P}(N^a_\tau = 1)\big(V^a(0,1,0) + V^b(0,1,0)\big) + \mathbb{P}(N^b_\tau = 1)\big(V^a(0,0,1) + V^b(0,0,1)\big).$$

Now we assume that market takers controls their intensities using a pair of Nash Equilibrium controls $(\lambda_a^\star, \lambda_b^\star)$ obtained in Theorem C.1 as limit of the smoothed problem. According to the first point of Theorem C.1 and since $V^a(0,0,1) = V^b(0,1,0)$ and $V^b(0,0,1) = V^a(0,1,0)$, we get Corollary 3.1 so that

$$\mathbb{E}[I_h^2] = \lim_{n\to+\infty} V^{a,n}(0,1,0) + V^{b,n}(0,1,0) = V^a(0,1,0) + V^b(0,1,0).$$

Let $\bar{V}^{a,n}$ (resp. $\bar{V}^{b,n}$) be defined as the backward form of the solutions $V^{a,n}$ (resp. $V^{b,n}$) of $(\mathbf{S^n})$, more precisely

$$\bar{V}^{i,n}(s,\cdot,\cdot) = V^{i,n}(h-s,\cdot,\cdot),\ s \in [0,h],\ \text{for } i \in \{a,b\}.$$

In the same way, we denote by $(\bar{V}_Q^{a,n}, \bar{V}_Q^{b,n})$ the backward versions of the solution $(V_Q^{a,n}, V_Q^{b,n})$ of $(\mathbf{S_Q^n})$. The functions $(\bar{V}_Q^{a,n}, \bar{V}_Q^{b,n})$ are computed by solving the backward system $(\mathbf{S_Q^n})$.

Finally note that

$$\mathbb{E}[I_h^2] = \lim_{n\to+\infty} \bar{V}^{a,n}(h,1,0) + \bar{V}^{b,n}(h,1,0) \approx \bar{V}_Q^{a,n}(h,1,0) + \bar{V}_Q^{b,n}(h,1,0).$$

Hence we use the quantity $\bar{V}_Q^{a,n}(h,1,0) + \bar{V}_Q^{b,n}(h,1,0)$ for $n = 1000$ and $Q = 100$ to approach more accurately $\mathbb{E}[I_h^2]$.

D Model extension: Market makers can cancel their limit orders

We can extend our model and allow market makers to revise their position before the auction clearing by cancelling their limit orders. Formally a market maker arrived at time $\tau \le \tau_i^{cl}$ will maintain its position until the auction clearing at time t with a probability $\theta(t - \tau_i^{cl})$, where θ is a $[0,1]$-valued decreasing function such that $\theta(0) = 1$. Hence, the number of market makers present at the $i-th$ auction clearing is

$$\tilde{N}_{\tau_{i-1}^{cl}} - \tilde{N}_{\tau_i^{cl}}\ ,\quad \text{with}\ \ \tilde{N}_s = \sum_{j=N^{mm}_{\tau_{i-1}^{cl}}+1}^{N^{mm}_{\tau_i^{cl}}} \mathbf{1}_{X_k \le \theta(\tau_k - \tau_i^{cl})},$$

where $(X_j)_{j\ge0}$ is a sequence of i.i.d. random variables with uniform law on $[0,1]$. We can show that during auction time $(\overline{N}_s)_{0\le s\le h} = (\tilde{N}_{\tau_i^{op}+s})_{s\ge0}$ has the same law than an inhomogeneous Poisson process with intensity

$$\lambda(s) = \mu\theta(t-s).$$

Moreover we still have an explicit formula for E.

$$E^{mid}(h) = (1 - e^{-m_h}\frac{\nu}{\nu+\mu})^{-1} e^{\nu h} \int_h^{+\infty} \nu e^{-\nu t}\Big((\sigma_f^2 \frac{t}{6} + \sigma^2) e^{-m_t} \int_0^{m_t} \frac{e^s - 1}{s} \mathrm{d}s + \sigma_f^2 \frac{t}{3}(1 - e^{-m_t})\Big)\mathrm{d}t$$

and

$$E(h) = E^{mid}(h) + \frac{\mathbb{E}[I^2_{\tau_1^{op}+h}]}{K^2}(1 - e^{-m_h}\frac{\nu}{\nu+\mu})^{-1}e^{\nu h}\int_h^{+\infty}\nu e^{-m_t}e^{-m_t}\int_0^{m_t}\frac{1}{s}\int_0^s\frac{e^u-1}{u}\mathrm{d}u\mathrm{d}s\mathrm{d}t$$

with

$$m_t = \int_0^t \mu\theta(s)\mathrm{d}s.$$

E Proof of Lemma 2.1

Consider for any $s > \tau_1^{cl}$, $X_s = (\overline{P}_s^{cl} - \overline{P}_s)^2$. We show that $(X_s)_{s>\tau_1^{cl}}$ is a regenerative process with renewal times given by $(\tau_i^{cl})_{i\geq 1}$.

Consider $\tau_i^{cl} \leq s < \tau_{i+1}^{cl}$ we have

$$\overline{P}_s^{cl} - \overline{P}_s = \frac{1}{N_{\Delta_i}^{i,mm}}\sum_{k=1}^{N_{\Delta_i}^{i,mm}}(P_{\tau_i^{cl}} - P_{\tau_{i-1}^{cl}+\tau_k^{i,mm}}) + \frac{1}{N_{\Delta_i}^{i,mm}}\sum_{k=1}^{N_{\Delta_i}^{i,mm}} g_k + \frac{I_{\Delta_i}^i}{KN_{\Delta_i}^{i,mm}}. \tag{29}$$

According to Assumption 1 the process $(N_t^{i,mm}, I_t^i)_{t\geq 0}$ is independent from $\mathcal{F}_{\tau_{i-1}^{cl}}$ with same law as $(N_t^{mm}, I_t)_{t\geq 0}$. Same results holds for $(P_{\tau_{i-1}^{cl}+t} - P_{\tau_{i-1}^{cl}})_{t\geq 0}$ and $(P_t - P_0)_{t\geq 0}$ since P is a Brownian motion. Consequently $N_{\Delta_i}^{i,mm}$, $I_{\Delta_i}^i$ and $(P_{\tau_{i-1}^{cl}+t} - P_{\tau_{i-1}^{cl}})_{t\geq 0}$ are independent from $(X_s)_{s<\tau_i^{cl}}$ with same law as $N_{\tau_1^{cl}}^{mm}$, $I_{\tau_1^{cl}}$ and $(P_t - P_0)_{t\geq 0}$.

Thus according to (29) and since X is piecewise continuous with jump at times $(\tau_i^{cl})_{i\geq 1}$, for any $\tau_i^{cl} \leq s < \tau_{i+1}^{cl}$, X_s is independent of $(X_s)_{s<\tau_i^{cl}}$ and has the same distribution than $X_{\tau_1^{cl}}$. Thus X is regenerative with renewal times equal to $(\tau_i^{cl})_{i\geq>1}$

Thus according to Theorem 3.1 Chap VI in Asmussen (2008) we have the almost sure convergence

$$\begin{aligned}
\frac{\int_0^t X_s\mathrm{d}s}{t} \underset{t\to+\infty}{\to} & \frac{\mathbb{E}[\int_{\tau_1^{cl}}^{\tau_2^{cl}} X_s\mathrm{d}s]}{\mathbb{E}[\tau_2^{cl} - \tau_1^{cl}]} \\
= & \frac{\mathbb{E}[\tau_2^{cl} - \tau_1^{cl}]\mathbb{E}[X_{\tau_1^{cl}}]}{\mathbb{E}[\tau_2^{cl} - \tau_1^{cl}]} \\
= & \mathbb{E}[X_{\tau_1^{cl}}] = \mathbb{E}[(P_{\tau_1^{cl}} - P_{\tau_1^{cl}}^{cl})^2].
\end{aligned}$$

Thus we get the stated result.