E-companion to Finding Minimum Volume Circumscribing Ellipsoids Using Generalized Copositive Programming

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1 Reachability in Linear Dynamical Systems

1.1 Ellipsoidal Approximation

We consider a discrete-time linear dynamical system

$$\boldsymbol{x}(0) = \boldsymbol{0}, \quad \boldsymbol{x}(t+1) = \boldsymbol{W}_1 \boldsymbol{x}(t) + \boldsymbol{W}_2 \boldsymbol{u}(t) \ \forall t \in [T-1].$$

Here T is the number of time steps, $\boldsymbol{x}(t) \in \mathbb{R}^K$ represents the state at time t, and $\boldsymbol{u}(t) \in \mathbb{R}^J$ is a vector of controls selected from a convex and compact *control set* $\mathcal{U}$. At any time, the state $\boldsymbol{x}(t)$ depends on the previous state and the control exercised in the previous time step. Finally, the matrices $\boldsymbol{W}_1 \in \mathbb{R}^{K \times K}$ and $\boldsymbol{W}_2 \in \mathbb{R}^{K \times J}$ determine the dynamics of the system.

The *reachable set* $\mathcal{W}_T$—the set of all states reachable at time T—is defined as follows:

$$\mathcal{W}_T = \{\boldsymbol{x}(T) : \boldsymbol{x}(t+1) = \boldsymbol{W}_1 \boldsymbol{x}(t) + \boldsymbol{W}_2 \boldsymbol{u}(t) \ \forall t \in [T-1], \ \boldsymbol{u}(t) \in \mathcal{U} \ \forall t \in [T-1], \ \boldsymbol{x}(0) = \boldsymbol{0}\}.$$

Note that the reachable set can be defined recursively as

$$\mathcal{W}_0 = \{\boldsymbol{0}\} \ \text{ and } \ \mathcal{W}_{t+1} = \boldsymbol{W}_1 \mathcal{W}_t + \boldsymbol{W}_2 \mathcal{U} = \{\boldsymbol{W}_1 \boldsymbol{x} + \boldsymbol{W}_2 \boldsymbol{u} : \boldsymbol{x} \in \mathcal{W}_t, \boldsymbol{u} \in \mathcal{U}\} \ \forall t \in [T]. \tag{EC.1}$$

A fundamental problem in the study of linear dynamical systems involves verifying whether a state is reachable at time T [2, Chapter 3]. For a single state, reachability can be determined by solving a convex feasibility problem whose size is quadratic in the number of time steps T (since both the number of variables and constraints are proportional to T).

When T grows large, determining the reachability of multiple states becomes computationally expensive (since the feasibility problem must be solved separately for every state). For this reason, several authors have proposed outer ellipsoidal approximations for $\mathcal{W}_T$ [1, 3]. A state is then deemed reachable if it falls within the approximating ellipsoid. Checking this condition is much faster than solving the convex feasibility problem. The reduction in time, however, can lead to false positives (states which lie inside the ellipsoid but are not reachable). Thus, generating a tight ellipsoidal approximation is desirable.

The previous studies describe ellipsoidal approximations for $\mathcal{W}_T$ only when the control set $\mathcal{U}$ is an ellipsoid. In contrast, in this section, we consider the case when the control set $\mathcal{U} = \{\boldsymbol{u} \in \mathbb{R}^J : \boldsymbol{S}\boldsymbol{u} \leq \boldsymbol{t}\}$ is a polytope, where $\boldsymbol{S} \in \mathbb{R}^{M\times J}$ and $\boldsymbol{t} \in \mathbb{R}^M$.

We find the ellipsoidal approximation for $\mathcal{W}_T$ as follows. Let $\mathcal{E}(\boldsymbol{A}_t, \boldsymbol{b}_t)$ be the ellipsoid found at time t that contains $\mathcal{W}_t$. First, note that $\mathcal{W}_1 = \boldsymbol{W}_2\mathcal{U} = \{\boldsymbol{W}_2\boldsymbol{u} : \boldsymbol{S}\boldsymbol{u} \leq \boldsymbol{t}\}$, which is a polytope. Therefore, we can use Theorem 2 to find an ellipsoid $\mathcal{E}(\boldsymbol{A}_1, \boldsymbol{b}_1)$ that contains $\mathcal{W}_1$. Next, at each step, we use the ellipsoid $\mathcal{E}(\boldsymbol{A}_t, \boldsymbol{b}_t)$ that contains $\mathcal{W}_t$ to generate the ellipsoid $\mathcal{E}(\boldsymbol{A}_{t+1}, \boldsymbol{b}_{t+1})$ that contains $\mathcal{W}_{t+1}$. To find $\mathcal{E}(\boldsymbol{A}_{t+1}, \boldsymbol{b}_{t+1})$ from $\mathcal{E}(\boldsymbol{A}_t, \boldsymbol{b}_t)$, we solve the following optimization problem:

$$
\begin{array}{ll}
\text{minimize} & -\log\det(\boldsymbol{A}) \\
\text{subject to} & \boldsymbol{A} \in \mathbb{S}^K,\ \boldsymbol{b} \in \mathbb{R}^K,\ \boldsymbol{N} \in \mathbb{R}_+^{J\times J},\ \boldsymbol{F} \in \mathbb{S}^K,\ \boldsymbol{g} \in \mathbb{R}^K,\ h \in \mathbb{R}, \\
& \begin{bmatrix} \boldsymbol{F} & \boldsymbol{g} \\ \boldsymbol{g}^\top & h-1 \end{bmatrix} \preceq -\boldsymbol{\mathcal{S}}^\top \boldsymbol{N} \boldsymbol{\mathcal{S}} + \lambda \boldsymbol{J}_t, \\
& \begin{bmatrix} \boldsymbol{F} & \boldsymbol{g} & (\boldsymbol{A}\overline{\boldsymbol{W}})^\top \\ \boldsymbol{g}^\top & h & \boldsymbol{b}^\top \\ \boldsymbol{A}\overline{\boldsymbol{W}} & \boldsymbol{b} & \mathbb{I} \end{bmatrix} \succeq \boldsymbol{0},
\end{array} \tag{EC.2}
$$

where

$$
\overline{\boldsymbol{W}} = \begin{bmatrix} \boldsymbol{W}_1 & \boldsymbol{W}_2 \end{bmatrix},\ \ \boldsymbol{\mathcal{S}} = \begin{bmatrix} \boldsymbol{0} & -\boldsymbol{S} & \boldsymbol{t} \end{bmatrix} \in \mathbb{R}^{M\times(K+J+1)},\ \ \text{and}\ \ \boldsymbol{J}_t = \begin{bmatrix} \boldsymbol{A}_t^2 & \boldsymbol{0} & \boldsymbol{A}_t^\top \boldsymbol{b}_t \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{b}_t^\top \boldsymbol{A}_t & \boldsymbol{0} & \boldsymbol{b}_t^\top \boldsymbol{b}_t - 1 \end{bmatrix} \in \mathbb{S}^{K+J+1}.
$$

If $(\boldsymbol{A}^*, \boldsymbol{b}^*, \boldsymbol{N}^*, \boldsymbol{F}^*, \boldsymbol{g}^*, h^*)$ is an optimal solution to (EC.2), then we use $\mathcal{E}(\boldsymbol{A}_{t+1}, \boldsymbol{b}_{t+1}) = \mathcal{E}(\boldsymbol{A}^*, \boldsymbol{b}^*)$ as the ellipsoid for next time step. In Proposition EC.1, presented below, we prove that $\mathcal{E}(\boldsymbol{A}^*, \boldsymbol{b}^*)$ indeed contains $\mathcal{W}_{t+1}$. Thus, repeating this procedure, we obtain the ellipsoidal approximation $\mathcal{E}(\boldsymbol{A}_T, \boldsymbol{b}_T)$ that contains $\mathcal{W}_T$. In Figure 1, we depict the evolution of $\mathcal{E}_T$ and $\mathcal{W}_T$ as T is increased for the case when $J = K = 2$,

$$
\boldsymbol{W}_1 = \begin{bmatrix} 0.9202 & -0.0396 \\ 0.0777 & 0.9800 \end{bmatrix} \quad \boldsymbol{W}_1 = \mathbb{I}, \quad \text{and} \quad \mathcal{U} = \{\boldsymbol{u} \in \mathbb{R}^2 : -\mathbf{e} \leq \boldsymbol{u} \leq \mathbf{e}, \|\boldsymbol{u}\|_1 \leq 1.4\}.
$$

Proposition EC.1. *Let $\mathcal{E}(\boldsymbol{A}_t, \boldsymbol{b}_t) = \{\boldsymbol{x} \in \mathbb{R}^K : \|\boldsymbol{A}_t\boldsymbol{x} + \boldsymbol{b}_t\| \leq 1\}$ be the ellipsoid containing $\mathcal{W}_t$. If the matrix $\boldsymbol{A} \in \mathbb{S}_{++}^K$ and the vector $\boldsymbol{b} \in \mathbb{R}^K$ satisfy the constraints of* (EC.2)*, then the ellipsoid $\mathcal{E}(\boldsymbol{A}, \boldsymbol{b})$ contains $\mathcal{W}_{t+1}$.*

Proof. See Section 2.1 of the electronic companion. □

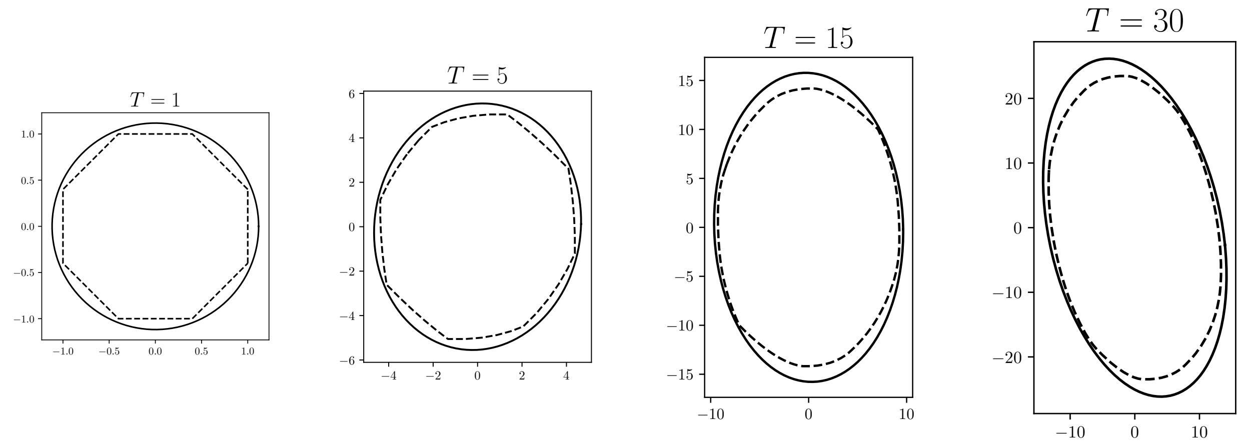

Figure 1. Reachability in Linear Dynamical Systems. For different values of T, the dotted curves depict the boundary of the actual reachable set $\mathcal{W}_T$, and the solid curves depict the ellipsoidal approximation $\mathcal{E}_T$ generated by our method.

Note that the time needed to generate ellipsoidal approximation to $\mathcal{W}_T$ is linear in T, since we solve T semidefinite programs whose size does not depend on T. This is an improvement from checking the reachability exactly, where the size of the problems is quadratic in T. We now provide numerical experiments to demonstrate the effectiveness of our approach on randomized instances of linear dynamical problems.

1.2 Numerical Experiments

For our experiments, we consider the control set $\mathcal{U} = \{\boldsymbol{u} \in \mathbb{R}^K : -\mathbf{e} \leq \boldsymbol{u} \leq \mathbf{e}, \|\boldsymbol{u}\|_1 \leq \sqrt{K}\}$, $T = 30$ time steps, $J = K$ and $\boldsymbol{W}_2 = \mathbb{I}$. We randomly generate $\boldsymbol{W}_1$ such that the maximum eigenvalue is less than 1 and the condition number is at most 2, which ensures that the system is stable. For given lengths of the state vector K, this is achieved with the following MATLAB code:

```
function W1 = generate_W1(K)
    max_eigenvalue = 1;
    condition_number = 1 + rand(); % choose randomly between 1 and 2
    X = randn(K);
    [U, S, V] = svd(X);
    singular_values = linspace(condition_number, 1, K);
    W1 = U * diag(singular_values) * V'; % update the singular values
    W1 = max_eigenvalue * W1 / max(abs(eig(W1)));
end
```

Next, we explain the metric that we use to compare the volumes of the sets $\mathcal{E}_T$ and $\mathcal{W}_T$. For a convex

K	γ	MVE Time (s)	Exact Time (s)
2	1.18	0.13	52.3
5	1.24	0.48	87.1
10	1.30	3.2	205
20	1.39	55.5	325

Table 1. Reachability in Linear Dynamical Systems: The quality of the approximation of $\mathcal{E}_T$ (γ), the time required to generate $\mathcal{E}_T$ and check the membership of $10,000$ points in $\mathcal{E}_T$ ("MVE Time"), and the time required to check the membership of the same points in $\mathcal{W}_T$ ("Exact Time") for different lengths of the state vector.

and compact set $S \subseteq \mathbb{R}^K$, let $\gamma(S) = \left(\prod_{k=1}^K \rho_k(S)\right)^{1/K}$, where $\rho_k(S) = \max\{x_k : \boldsymbol{x} \in S\} - \min\{x_k : \boldsymbol{x} \in S\},\ k \in [K]$. Here, $\rho_k(S)$ represents the length of the projection of S onto the k^{th} coordinate, and $\gamma(S)$ represents the geometric mean of these lengths. We use the parameter $\gamma = \gamma(\mathcal{E}_T)/\gamma(\mathcal{W}_T)$ to measure the closeness of the sizes of $\mathcal{E}_T$ and $\mathcal{W}_T$. We can see that $\gamma \geq 1$ since $\mathcal{W}_T \subseteq \mathcal{E}_T$, and the closer γ is to one, the better $\mathcal{E}_T$ approximates $\mathcal{W}_T$. As an example, in the case depicted in Figure 1, $\gamma = 1.06$ at time step $T = 30$.

For different values of K, we perform the experiment on 100 randomly generated instances. For each instance, we record the time that it takes to check the membership within $\mathcal{E}_T$ and $\mathcal{W}_T$ of $10,000$ points sampled randomly from $[-10, 10]^K$. In Table 1, we present 1) average solution quality (γ), 2) the total computational time required for generating $\mathcal{E}_T$ and then checking the membership of the randomly sampled points inside $\mathcal{E}_T$, and 3) the computational time required to establish the membership of these points in $\mathcal{W}_T$. We can observe that, for $K = 20$, $\mathcal{E}_T$ is about 39% bigger in size than $\mathcal{W}_T$. However, the computational time needed to find $\mathcal{E}_T$ and then checking the membership inside it is much less than checking the feasibility of these points exactly.

2 Proofs

2.1 Propositions

Proof of Proposition 1: For the set $\{\boldsymbol{x} \in \mathbb{R}^K : \boldsymbol{S}\boldsymbol{x} \leq \boldsymbol{t}, \|\boldsymbol{Q}\boldsymbol{x} + \boldsymbol{q}\|^2 \leq 1\}$, then $\mathcal{E}_{\text{sproc}} = \mathcal{E}(\boldsymbol{A}, \boldsymbol{b})$, where $\boldsymbol{A}$ and $\boldsymbol{b}$ are optimal in the following optimization problem (see Appendix **??**):

$$\begin{array}{ll} \text{minimize} & -\log\det(\boldsymbol{A}) \\ \text{subject to} & \boldsymbol{A} \in \mathbb{S}^K, \boldsymbol{b} \in \mathbb{R}^K,\ \lambda \in \mathbb{R}_+,\ \boldsymbol{\mu} \in \mathbb{R}^J_+, \\ & \begin{bmatrix} \mathbf{0} & \frac{1}{2}\boldsymbol{S}^\top\boldsymbol{\mu} & \boldsymbol{A} \\ \frac{1}{2}\boldsymbol{\mu}^\top\boldsymbol{S} & 1 - \boldsymbol{\mu}^\top\boldsymbol{t} & \boldsymbol{b}^\top \\ \boldsymbol{A} & \boldsymbol{b} & \mathbb{I} \end{bmatrix} + \lambda \begin{bmatrix} \boldsymbol{Q}^2 & \boldsymbol{Q}\boldsymbol{q} & \mathbf{0} \\ \boldsymbol{q}^\top\boldsymbol{Q} & \boldsymbol{q}^\top\boldsymbol{q} - 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \succeq \mathbf{0}. \end{array} \tag{EC.3}$$

The dual of (EC.3) can be written as

$$\begin{array}{ll} \text{maximize} & K + \log\det(-2\boldsymbol{F}) - \kappa - \operatorname{tr}(\boldsymbol{\Gamma}) \\ \text{subject to} & \boldsymbol{\Omega} \in \mathbb{S}^K,\ \boldsymbol{\xi} \in \mathbb{R}^K,\ \boldsymbol{F} \in \mathbb{S}^K,\ \boldsymbol{\Gamma} \in \mathbb{S}^K, \kappa \in \mathbb{R}, \\ & \boldsymbol{S}\boldsymbol{\xi} \leq \kappa\boldsymbol{t}, \\ & \operatorname{tr}\left(\begin{bmatrix} \boldsymbol{\Omega} & \boldsymbol{\xi} \\ \boldsymbol{\xi}^\top & \kappa \end{bmatrix} \begin{bmatrix} \boldsymbol{Q}^2 & \boldsymbol{Q}\boldsymbol{q} \\ \boldsymbol{q}^\top\boldsymbol{Q} & \boldsymbol{q}^\top\boldsymbol{q} - 1 \end{bmatrix} \right) \leq 0, \\ & \begin{bmatrix} \boldsymbol{\Omega} & \boldsymbol{\xi} & \boldsymbol{F} \\ \boldsymbol{\xi}^\top & \kappa & \mathbf{0} \\ \boldsymbol{F} & \mathbf{0} & \boldsymbol{\Gamma} \end{bmatrix} \succeq \mathbf{0}. \end{array} \tag{EC.4}$$

To prove the theorem, we construct a pair of primal and dual feasible solutions which generate the same objective function values to their respective problems. To this end, consider the following solution to the primal problem (EC.3):

$$\boldsymbol{A} = \boldsymbol{Q},\ \boldsymbol{b} = \boldsymbol{q},\ \boldsymbol{\mu} = \mathbf{0},\ \lambda = 1.$$

This solution is feasible since $\boldsymbol{\mu} \geq \mathbf{0}$, $\lambda \geq 0$ and

$$\begin{bmatrix} \mathbf{0} & \frac{1}{2}\boldsymbol{S}^\top\boldsymbol{\mu} & \boldsymbol{A} \\ \frac{1}{2}\boldsymbol{\mu}^\top\boldsymbol{S} & 1 - \boldsymbol{\mu}^\top\boldsymbol{t} & \boldsymbol{b}^\top \\ \boldsymbol{A} & \boldsymbol{b} & \mathbb{I} \end{bmatrix} + \lambda \begin{bmatrix} \boldsymbol{Q}^2 & \boldsymbol{Q}\boldsymbol{q} & \mathbf{0} \\ \boldsymbol{q}^\top\boldsymbol{Q} & \boldsymbol{q}^\top\boldsymbol{q} - 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \boldsymbol{Q}^2 & \boldsymbol{Q}\boldsymbol{q} & \boldsymbol{Q} \\ \boldsymbol{q}^\top\boldsymbol{Q} & \boldsymbol{q}^\top\boldsymbol{q} & \boldsymbol{q}^\top \\ \boldsymbol{Q} & \boldsymbol{q} & \mathbb{I} \end{bmatrix} = \begin{bmatrix} \boldsymbol{Q} \\ \boldsymbol{q}^\top \\ \mathbb{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{Q} \\ \boldsymbol{q}^\top \\ \mathbb{I} \end{bmatrix}^\top \succeq \mathbf{0}.$$

Next, consider the following solution to the dual problem (EC.4):

$$\boldsymbol{F} = -\frac{1}{2}\boldsymbol{Q}^{-1},\ \boldsymbol{\Gamma} = \frac{\mathbb{I}}{2},\ \kappa = \frac{K}{2},\ \boldsymbol{\xi} = \kappa\boldsymbol{x}_c,\ \boldsymbol{\Omega} = \kappa\boldsymbol{x}_c\boldsymbol{x}_c^\top + 2\boldsymbol{F}^2,$$

where $\boldsymbol{x}_c = -\boldsymbol{Q}^{-1}\boldsymbol{q}$ is the center of the ellipsoid $\mathcal{E}$. We claim that this solution is feasible to (EC.4). Under the assumption that the center of the ellipsoid lies inside the polytope, we get that $\boldsymbol{S}\boldsymbol{x}_c \leq \boldsymbol{t}$, which implies that $\boldsymbol{S}\boldsymbol{\xi} \leq \kappa \boldsymbol{t}$. Next, we have that

$$\begin{bmatrix} \boldsymbol{\Omega} & \boldsymbol{\xi} & \boldsymbol{F} \\ \boldsymbol{\xi}^\top & \kappa & \boldsymbol{0} \\ \boldsymbol{F} & \boldsymbol{0} & \boldsymbol{\Gamma} \end{bmatrix} = \begin{bmatrix} \kappa \boldsymbol{x}_c \boldsymbol{x}_c^\top + 2\boldsymbol{F}^2 & \kappa \boldsymbol{x}_c & \boldsymbol{F} \\ \kappa \boldsymbol{x}_c^\top & \kappa & \boldsymbol{0} \\ \boldsymbol{F} & \boldsymbol{0} & \frac{\mathbb{I}}{2} \end{bmatrix} = \kappa \begin{bmatrix} \boldsymbol{x}_c \\ 1 \\ \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_c \\ 1 \\ \boldsymbol{0} \end{bmatrix}^\top + \frac{1}{2} \begin{bmatrix} 2\boldsymbol{F} \\ \boldsymbol{0} \\ \mathbb{I} \end{bmatrix} \begin{bmatrix} 2\boldsymbol{F} \\ \boldsymbol{0} \\ \mathbb{I} \end{bmatrix}^\top \succeq \boldsymbol{0},$$

where the last inequality uses the fact that $\kappa = K/2 \geq 0$. Also,

$$\begin{aligned} \operatorname{tr}\left(\begin{bmatrix} \boldsymbol{\Omega} & \boldsymbol{\xi} \\ \boldsymbol{\xi}^\top & \kappa \end{bmatrix} \begin{bmatrix} \boldsymbol{Q}^2 & \boldsymbol{Q}\boldsymbol{q} \\ \boldsymbol{q}^\top \boldsymbol{Q} & \boldsymbol{q}^\top \boldsymbol{q} - 1 \end{bmatrix} \right) \quad &= \operatorname{tr}(\boldsymbol{\Omega}\boldsymbol{Q}^2) + 2\boldsymbol{\xi}^\top \boldsymbol{Q}\boldsymbol{q} + \kappa(\boldsymbol{q}^\top \boldsymbol{q} - 1) \\ &= \operatorname{tr}((\kappa \boldsymbol{x}_c \boldsymbol{x}_c^\top + 2\boldsymbol{F}^2)\boldsymbol{Q}^2) + 2\kappa \boldsymbol{x}_c^\top \boldsymbol{Q}\boldsymbol{q} + \kappa(\boldsymbol{q}^\top \boldsymbol{q} - 1) \\ &= \kappa \boldsymbol{x}_c^\top \boldsymbol{Q}^2 \boldsymbol{x}_c + \frac{1}{2}\operatorname{tr}(\mathbb{I}) + 2\kappa \boldsymbol{x}_c^\top \boldsymbol{Q}\boldsymbol{q} + \kappa(\boldsymbol{q}^\top \boldsymbol{q} - 1) \\ &= \frac{1}{2}\operatorname{tr}(\mathbb{I}) - \kappa = \frac{K}{2} - \frac{K}{2} = 0. \end{aligned}$$

Therefore, all constraints in the dual problem are satisfied. Finally, both of these solutions give an objective function value of $-\log\det(\boldsymbol{Q})$. Thus, $\boldsymbol{A} = \boldsymbol{Q}$ and $\boldsymbol{b} = \boldsymbol{q}$ is an optimal solution to the primal problem, which implies that $\mathcal{E}_{\text{sproc}} = \mathcal{E}(\boldsymbol{A}, \boldsymbol{b}) = \mathcal{E}(\boldsymbol{Q}, \boldsymbol{q})$. Furthermore, the solution is unique because the feasible region is convex and the objective function $-\log\det(\boldsymbol{A})$ is strictly convex in the space of positive definite matrices. □

Proof of Proposition 2: To prove this result, we demonstrate that any feasible solution in (18) can be used to construct a feasible solution to (14) with the same or lower objective value. To this end, consider a feasible solution $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{V}, \boldsymbol{v})$ of (18). Now consider the following solution to (14):

$$\boldsymbol{\Lambda} = -\boldsymbol{V}, \ \boldsymbol{\rho} = \boldsymbol{v}.$$

The last two constraints of (18) imply that the constraints of (15) are satisfied. Also note that

$$\begin{aligned} -\log\det(\boldsymbol{A}) \quad &= \log\det\left(\frac{1}{2}(\boldsymbol{V}^\top \boldsymbol{S} + \boldsymbol{S}^\top \boldsymbol{V})\right) \\ &\geq \log\det\left(\frac{1}{2}(\boldsymbol{V}^\top \boldsymbol{S} + \boldsymbol{S}^\top \boldsymbol{V})\right) + K(\boldsymbol{t}^\top \boldsymbol{v} - 1) \\ &= \log\det\left(-\frac{1}{2}(\boldsymbol{\Lambda}^\top \boldsymbol{S} + \boldsymbol{S}^\top \boldsymbol{\Lambda})\right) + K(\boldsymbol{t}^\top \boldsymbol{\rho} - 1). \end{aligned} \tag{EC.5}$$

Here, the first equality follows since $\boldsymbol{A}$ is symmetric, which implies that $\boldsymbol{A} = (\boldsymbol{A} + \boldsymbol{A}^\top)/2 = (\boldsymbol{V}^\top \boldsymbol{S} + \boldsymbol{S}^\top \boldsymbol{V})/2$, and the inequality follows from the first constraint of (18) since $\left\|\boldsymbol{V}^\top \boldsymbol{t} + \boldsymbol{b}\right\| \geq 0$. Therefore, the result holds. □

Proof of Proposition 3: The PLD restriction of (24) can be written as follows:

$$\begin{aligned} \inf_{\boldsymbol{x}, \boldsymbol{y}(\cdot)} \quad & \boldsymbol{c}^\top \boldsymbol{x} + \sup_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[(\boldsymbol{D}\tilde{\boldsymbol{\xi}} + \boldsymbol{d})^\top \boldsymbol{y}(\tilde{\boldsymbol{\xi}})], \\ \text{s.t.} \quad & \boldsymbol{x} \in \mathcal{X}, \\ & \boldsymbol{T}_\ell(\boldsymbol{x})^\top \boldsymbol{\xi} + h_\ell(\boldsymbol{x}) \leq (\boldsymbol{W}_\ell \boldsymbol{\xi} + \boldsymbol{w}_\ell)^\top (\boldsymbol{Y}_j \boldsymbol{\xi} + \boldsymbol{y}_j) \quad \forall \boldsymbol{\xi} \in \Xi_j \ \forall j \in [J] \ \forall \ell \in [L]. \end{aligned} \tag{EC.6}$$

First, for the objective function, observe that

$$
\begin{aligned}
\sup_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[(\boldsymbol{D}\tilde{\boldsymbol{\xi}}+\boldsymbol{d})^\top \boldsymbol{y}(\tilde{\boldsymbol{\xi}})] = \quad & \sup_{\nu(\cdot)\geq 0} \quad \sum_{j\in[J]} \int_{\boldsymbol{\xi}\in\Xi_j} (\boldsymbol{D}\boldsymbol{\xi}+\boldsymbol{d})^\top (\boldsymbol{Y}_j\boldsymbol{\xi}+\boldsymbol{y}_j)\nu(d\boldsymbol{\xi}) \\
& \text{s.t.} \quad \sum_{j\in[J]} \int_{\boldsymbol{\xi}\in\Xi_j} \nu(d\boldsymbol{\xi}) = 1, \\
& \qquad\ \ \sum_{j\in[J]} \int_{\boldsymbol{\xi}\in\Xi_j} \boldsymbol{\xi}\nu(d\boldsymbol{\xi}) = \boldsymbol{\mu}, \\
& \qquad\ \ \sum_{j\in[J]} \int_{\boldsymbol{\xi}\in\Xi_j} \boldsymbol{\xi}\boldsymbol{\xi}^\top \nu(d\boldsymbol{\xi}) \preceq \boldsymbol{\Sigma}.
\end{aligned}
$$

By weak duality, we get that

$$
\begin{aligned}
\sup_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[(\boldsymbol{D}\tilde{\boldsymbol{\xi}}+\boldsymbol{d})^\top \boldsymbol{y}(\tilde{\boldsymbol{\xi}})] \leq \inf \quad & \alpha + \boldsymbol{\beta}^\top\boldsymbol{\mu} + \mathrm{tr}(\boldsymbol{\Gamma}\boldsymbol{\Sigma}) \\
\text{s.t.} \quad & \alpha\in\mathbb{R}, \boldsymbol{\beta}\in\mathbb{R}^K, \boldsymbol{\Gamma}\in\mathbb{S}^K_+, \\
& \alpha + \boldsymbol{\beta}^\top\boldsymbol{\xi} + \boldsymbol{\xi}^\top\boldsymbol{\Gamma}\boldsymbol{\xi} \geq (\boldsymbol{D}\boldsymbol{\xi}+\boldsymbol{d})^\top(\boldsymbol{Y}_j\boldsymbol{\xi}+\boldsymbol{y}_j) \quad \forall\boldsymbol{\xi}\in\Xi_j\ \forall j\in[J].
\end{aligned}
$$

The constraint of the optimization problem above holds if and only if, for all $j\in[J]$, the optimal value of the problem

$$
\begin{aligned}
\inf_{\boldsymbol{\xi}\in\mathbb{R}^K} \quad & \alpha + \boldsymbol{\beta}^\top\boldsymbol{\xi} + \boldsymbol{\xi}^\top\boldsymbol{\Gamma}\boldsymbol{\xi} - (\boldsymbol{D}\boldsymbol{\xi}+\boldsymbol{d})^\top(\boldsymbol{Y}_j\boldsymbol{\xi}+\boldsymbol{y}_j) \\
\text{s.t.} \quad & \boldsymbol{S}_j\boldsymbol{\xi}\leq\boldsymbol{t}_j, \quad \|\boldsymbol{A}_j\boldsymbol{\xi}+\boldsymbol{b}_j\|^2\leq 1,
\end{aligned}
$$

is ≥ 0. Next, using the $\mathcal{S}$-procedure (Lemma 9), we get that this constraint holds if the first semidefinite constraint of (25) holds. Therefore, replacing the former by the latter, we get an upper bound on the optimal decision rules problem. Similarly, the final constraint of (EC.6) is equivalent to the constraint that, for all $j\in[J]$ and $\ell\in[L]$, the optimal value of the following optimization problem is greater than or equal to 0:

$$
\begin{aligned}
\inf_{\boldsymbol{\xi}\in\mathbb{R}^K} \quad & (\boldsymbol{W}_\ell\boldsymbol{\xi}+\boldsymbol{w}_\ell)^\top(\boldsymbol{Y}_j\boldsymbol{\xi}+\boldsymbol{y}_j) - \boldsymbol{T}_\ell(\boldsymbol{x})^\top\boldsymbol{\xi} + h_\ell(\boldsymbol{x}) \\
\text{s.t.} \quad & \boldsymbol{S}_j\boldsymbol{\xi}\leq\boldsymbol{t}_j, \quad \|\boldsymbol{A}_j\boldsymbol{\xi}+\boldsymbol{b}_j\|^2\leq 1.
\end{aligned}
$$

Using Lemma 9, we get that the above constraint holds if the second semidefinite constraint of (25) holds. Therefore, the SDP (25) provides a feasible decision rule approximation, and the optimal value of (25) provides an upper bound to the optimal value of the DRO model (24). ☐

Proof of Proposition EC.1: Consider the set $\overline{\mathcal{W}} = \boldsymbol{W}_1\mathcal{E}(\boldsymbol{A}_t,\boldsymbol{b}_t) + \boldsymbol{W}_2\mathcal{U} = \{\boldsymbol{W}_1\boldsymbol{x}+\boldsymbol{W}_2\boldsymbol{u} : \boldsymbol{S}\boldsymbol{u}\leq\boldsymbol{t},\ \|\boldsymbol{A}_t\boldsymbol{x}+\boldsymbol{b}_t\|^2\leq 1\}$. Since $\mathcal{W}_t\subseteq\mathcal{E}(\boldsymbol{A}_t,\boldsymbol{b}_t)$, we get that $\mathcal{W}_{t+1} = \boldsymbol{W}_1\mathcal{W}_t+\boldsymbol{W}_2\mathcal{U}\subseteq\boldsymbol{W}_1\mathcal{E}(\boldsymbol{A}_t,\boldsymbol{b}_t)+\boldsymbol{W}_2\mathcal{U} = \overline{\mathcal{W}}$. Therefore, $\mathcal{W}_{t+1}\subseteq\overline{\mathcal{W}}$. Next, we find sufficient conditions that imply that an ellipsoid contains $\overline{\mathcal{W}}$, which, in turn, implies that the ellipsoid contains $\mathcal{W}_{t+1}$. To this end, note that $\overline{\mathcal{W}}$ is an special case of the setup in Remark 2 with $\mathcal{P} = \{(\boldsymbol{x},\boldsymbol{u})\in\mathbb{R}^{K+J} : \boldsymbol{S}\boldsymbol{u}\leq\boldsymbol{t},\ \|\boldsymbol{A}_t\boldsymbol{x}+\boldsymbol{b}_t\|^2\leq 1\}$, $\boldsymbol{C} = \overline{\boldsymbol{W}}$ and $\boldsymbol{d} = \boldsymbol{0}$. Therefore, the ellipsoid $\mathcal{E}(\boldsymbol{A}_{t+1}, \boldsymbol{b}_{t+1})$ contains $\overline{\mathcal{W}}$ if and only if there exist $\boldsymbol{F} \in \mathbb{S}^{K+J},\ \boldsymbol{g} \in \mathbb{R}^{K+J},\ h \in \mathbb{R}$ such that

$$\begin{bmatrix} \boldsymbol{F} & \boldsymbol{g} \\ \boldsymbol{g}^\top & h-1 \end{bmatrix} \preceq_{\mathcal{C}(\mathcal{K})} \mathbf{0} \quad \text{and} \quad \begin{bmatrix} \boldsymbol{F} & \boldsymbol{g} & (\boldsymbol{A}_{t+1}\overline{\boldsymbol{W}})^\top \\ \boldsymbol{g}^\top & h & \boldsymbol{b}_{t+1}^\top \\ \boldsymbol{A}_{t+1}\overline{\boldsymbol{W}} & \boldsymbol{b}_{t+1} & \mathbb{I} \end{bmatrix} \succeq \mathbf{0},$$

where $\mathcal{K} = \{(\boldsymbol{x}, \boldsymbol{u}, \tau) \in \mathbb{R}^{K+J+1} : \boldsymbol{S}\boldsymbol{u} \leq \tau \boldsymbol{t},\ \|\boldsymbol{A}_t\boldsymbol{x} + \tau\boldsymbol{b}_t\|^2 \leq \tau^2\}$. Since the set $\mathcal{P}$ is defined by linear and quadratic inequalities, using Theorem 4, we get that the constraints of (EC.2) imply that $\mathcal{E}(\boldsymbol{A}_{t+1}, \boldsymbol{b}_{t+1})$ contains $\overline{\mathcal{W}}$. Hence, $\mathcal{E}(\boldsymbol{A}_{t+1}, \boldsymbol{b}_{t+1})$ contains $\mathcal{W}_{t+1}$. $\square$

2.2 Claims in Example 1

In this section, we prove that in Example 1, $R_{\text{sdp}} = O(K^{1/4})$, and $R_{\text{smvie}} = \Theta(\sqrt{K})$. Consider the solution

$$\boldsymbol{A} = k_1\mathbb{I} + k_2\mathbf{e}\mathbf{e}^\top,\ \boldsymbol{b} = k_3\mathbf{e},\ \boldsymbol{N} = \begin{bmatrix} \mathbf{0} & k_4\mathbb{I} & \mathbf{0} \\ k_4\mathbb{I} & \mathbf{0} & k_5\mathbf{e} \\ \mathbf{0} & k_5\mathbf{e}^\top & 0 \end{bmatrix},$$

$$\text{where,} \quad k_1 = \sqrt{\frac{K^2-1}{(\sqrt{K}-1)K^2}},\ k_2 = \frac{1}{K}\left(1 + \frac{1}{K} - k_1\right),\ k_3 = -\frac{1}{\sqrt{K}},\ k_4 = \frac{k_1^2}{2},\ k_5 = k_2\left(k_1 + \frac{K}{2}k_2\right).$$

It can be checked that this solution is feasible to (13). Therefore, $R_{\text{sdp}} \leq (1/\det(\boldsymbol{A}))^{1/K}$. The eigenvalues of $\boldsymbol{A}$ are $k_1 + Kk_2$, and k_1 with a multiplicity of $K-1$. Therefore,

$$R_{\text{sdp}} \leq (1/\det(\boldsymbol{A}))^{1/K} = \left((k_1 + Kk_2)\, k_1^{K-1}\right)^{-\frac{1}{K}} = \left(\left(1 + \frac{1}{K}\right) k_1^{K-1}\right)^{-\frac{1}{K}} = O\left(\frac{1}{k_1}\right) = O\left(K^{\frac{1}{4}}\right).$$

Next, for $\mathcal{E}_{\text{smvie}}$, consider the following solution to the primal problem (15): $\boldsymbol{B} = m_1\mathbb{I} + m_2\mathbf{e}\mathbf{e}^\top,\ \boldsymbol{d} = m_3\mathbf{e}$, and the following solution to the dual problem (14): $\boldsymbol{\rho} = 1/\sqrt{K}\left[\mathbf{0};\ \mathbf{e};\ 1\right],\ \boldsymbol{\Lambda} = \left[\mathbf{0};\ m_4\mathbb{I} + m_5\mathbf{e}\mathbf{e}^\top;\ m_6\mathbf{e}^\top\right]$, where

$$m_1 = \frac{K}{\sqrt{K+1}},\ m_2 = \frac{1}{K+1} - \frac{1}{\sqrt{K+1}},\ m_3 = \frac{\sqrt{K}}{K+1},\ m_4 = \frac{\sqrt{K+1}}{K},\ m_5 = \frac{1-\sqrt{K+1}}{K^2},\ m_6 = -\frac{1}{K}.$$

It can be verified that these solutions have the same objective function value, and are feasible—and therefore optimal—to their respective problems. The eigenvalues of $\boldsymbol{B}$ are $m_1 + Km_2$, and m_1 with a multiplicity of $K-1$. Therefore

$$\det(\boldsymbol{B}) = (m_1 + Km_2)m_1^{K-1} = \frac{K}{K+1}\left(\frac{K}{\sqrt{K+1}}\right)^{K-1} = \frac{K^K}{(\sqrt{K+1})^{K+1}},$$

which implies that

$$\lim_{K\to\infty} \frac{R_{\text{smvie}}}{\sqrt{K}} = \lim_{K\to\infty} \frac{\det(\boldsymbol{B})^{1/K}}{\sqrt{K}} = 1.$$

Therefore, $R_{\text{smvie}} = \Theta(\sqrt{K})$.

2.3 Claims in Example 2

Using $\mathcal{S}$-procedure (Lemma 9), we substitute the semidefinite programming approximation for the inequalities of (26), yielding the following approximation to (26):

$$\begin{aligned}
\inf \quad & x \\
\text{s.t.} \quad & x \in \mathbb{R},\ \boldsymbol{Y} \in \mathbb{R}^{K\times K},\ \boldsymbol{y} \in \mathbb{R}^K,\ \boldsymbol{\rho}_1 \geq \boldsymbol{0},\ \boldsymbol{\rho}_2 \geq \boldsymbol{0},\ \lambda_1 \geq 0,\ \lambda_2 \geq 0, \\
& \begin{bmatrix} \frac{1}{2}(\boldsymbol{Y}+\boldsymbol{Y}^\top) & \frac{1}{2}(\boldsymbol{S}^\top\boldsymbol{\rho}_1 + \boldsymbol{Y}^\top\mathbf{e} + \boldsymbol{y}) \\ \frac{1}{2}(\boldsymbol{S}^\top\boldsymbol{\rho}_1 + \boldsymbol{Y}^\top\mathbf{e} + \boldsymbol{y})^\top & -1 + \mathbf{e}^\top\boldsymbol{y} - \boldsymbol{t}^\top\boldsymbol{\rho}_1 \end{bmatrix} + \lambda_1 \boldsymbol{J}(s) \succeq \boldsymbol{0}, \\
& \begin{bmatrix} -\frac{1}{2}(\boldsymbol{Y}+\boldsymbol{Y}^\top) & \frac{1}{2}(\boldsymbol{S}^\top\boldsymbol{\rho}_2 + \boldsymbol{Y}^\top\mathbf{e} - \boldsymbol{y}) \\ \frac{1}{2}(\boldsymbol{S}^\top\boldsymbol{\rho}_2 - \boldsymbol{Y}^\top\mathbf{e} - \boldsymbol{y})^\top & x - \mathbf{e}^\top\boldsymbol{y} - \boldsymbol{t}^\top\boldsymbol{\rho}_2 \end{bmatrix} + \lambda_2 \boldsymbol{J}(s) \succeq \boldsymbol{0},
\end{aligned} \tag{EC.7}$$

where

$$J(s) = \frac{1}{K}\begin{bmatrix} 4\mathbb{I} & -2\mathbf{e} \\ -2\mathbf{e}^\top & -Ks \end{bmatrix}.$$

The dual of the SDP (EC.7) is given by

$$\begin{aligned}
\sup \quad & h_1 \\
\text{s.t.} \quad & \boldsymbol{F}_1, \boldsymbol{F}_2 \in \mathbb{S}^K,\ \ \boldsymbol{g}_1, \boldsymbol{g}_2 \in \mathbb{R}^K,\ \ h_1, h_2 \in \mathbb{R} \\
& \begin{bmatrix} \boldsymbol{F}_1 & \boldsymbol{g}_1 \\ \boldsymbol{g}_1^\top & h_1 \end{bmatrix} \succeq \boldsymbol{0}, \quad \begin{bmatrix} \boldsymbol{F}_2 & \boldsymbol{g}_2 \\ \boldsymbol{g}_2^\top & h_2 \end{bmatrix} \succeq \boldsymbol{0}, \\
& \operatorname{tr}\left(\boldsymbol{J}(s)\begin{bmatrix} \boldsymbol{F}_1 & \boldsymbol{g}_1 \\ \boldsymbol{g}_1^\top & h_1 \end{bmatrix}\right) \leq 0, \quad \operatorname{tr}\left(\boldsymbol{J}(s)\begin{bmatrix} \boldsymbol{F}_2 & \boldsymbol{g}_2 \\ \boldsymbol{g}_2^\top & h_2 \end{bmatrix}\right) \leq 0, \\
& \boldsymbol{g}_1 - \boldsymbol{g}_2 = (h_2 - h_1)\mathbf{e}, \\
& \boldsymbol{F}_1 - \boldsymbol{F}_2 = \mathbf{e}(\boldsymbol{g}_2 - \boldsymbol{g}_1)^\top, \\
& \boldsymbol{S}\boldsymbol{g}_1 \leq h_1\boldsymbol{t},\ \boldsymbol{S}\boldsymbol{g}_2 \leq h_2\boldsymbol{t}, \\
& h_2 = 1.
\end{aligned}$$

Consider the following cases:

- Case 1: $0 \leq s \leq 2$. Consider the following solution to the dual problem:

  $$\boldsymbol{F}_2 = \frac{1}{4}\mathbf{e}\mathbf{e}^\top,\ \boldsymbol{g}_2 = \frac{1}{2}\mathbf{e},\ h_2 = 1,\ h_1 = \frac{9}{8-s},\ \boldsymbol{F}_1 = \boldsymbol{F}_2 + \frac{1+s}{8-s}\mathbf{e}\mathbf{e}^\top,\ \boldsymbol{g}_1 = \boldsymbol{g}_2 - \frac{1+s}{8-s}\mathbf{e},$$

  and the following solution to the primal problem:

  $$\lambda_2 = 0,\ \lambda_1 = \frac{1}{8-s},\ \boldsymbol{\rho}_1 = \boldsymbol{0},\ \boldsymbol{\rho}_2 = \boldsymbol{0},\ \boldsymbol{y} = \frac{8}{K(8-s)}\mathbf{e},\ \boldsymbol{Y} = -\frac{2}{K(8-s)}\left(\mathbb{I} + \frac{1}{K}\mathbf{e}\mathbf{e}^\top\right)$$

- Case 2: $2 \leq s \leq 4$. Consider the following solution to the dual problem:

  $$\boldsymbol{F}_2 = \left(\frac{s}{4}\right)^2\mathbf{e}\mathbf{e}^\top,\ \boldsymbol{g}_2 = \frac{s}{4}\mathbf{e},\ h_2 = 1,\ h_1 = 1 + \frac{s}{4},\ \boldsymbol{F}_1 = \boldsymbol{F}_2 + \frac{s}{4}\mathbf{e}\mathbf{e}^\top,\ \boldsymbol{g}_1 = \boldsymbol{g}_2 - \frac{s}{4}\mathbf{e} = \boldsymbol{0},$$

and the following solution to the primal problem:

$$\lambda_2 = 0,\ \lambda_1 = \frac{1}{4+s},\ \boldsymbol{\rho}_1 = \frac{2(s-2)}{K(4+s)}\begin{bmatrix}\mathbf{0}\\ \mathbf{e}\end{bmatrix},\ \boldsymbol{\rho}_2 = \mathbf{0},\ \boldsymbol{y} = \frac{4+2s}{K(4+s)}\mathbf{e},\ \boldsymbol{Y} = -\frac{2}{K(4+s)}\left(\mathbb{I} + \frac{1}{K}\mathbf{e}\mathbf{e}^\top\right)$$

- Case 3: $s \geq 4$. Consider the following solution to the dual problem:

$$\boldsymbol{F}_2 = \mathbf{e}\mathbf{e}^\top,\ \boldsymbol{g}_2 = \mathbf{e},\ h_2 = 1,\ h_1 = 2,\ \boldsymbol{F}_1 = 2\mathbf{e}\mathbf{e}^\top,\ \boldsymbol{g}_1 = \mathbf{0},$$

and the following solution to the primal problem:

$$\lambda_2 = 0,\ \lambda_1 = 0,\ \boldsymbol{\rho}_1 = \frac{1}{K}\begin{bmatrix}\mathbf{0}\\ \mathbf{e}\end{bmatrix},\ \boldsymbol{\rho}_2 = \frac{1}{K}\begin{bmatrix}\mathbf{e}\\ \mathbf{0}\end{bmatrix},\ \boldsymbol{y} = \frac{1}{K}\mathbf{e},\ \boldsymbol{Y} = \mathbf{0}.$$

In all the three cases, the primal and the dual solutions are feasible to their respective problems, and provide the same objective value which corresponds to the one presented in Example 2.

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