e-companion to "Who Is Next: Patient Prioritization under Emergency Department Blocking", by W. Li, Z. Sun, L.J. Hong

Appendix A. Further Summary Statistics, Robustness Checks, and Discussions

Table 5 The five-number summary of the five measures of *BlockLevel* and *WaitRoomCensus*.

Variables	Min	First quartile	Median (Mean)	Third quartile	Max
BlockLevel					
Measure 1	0	0.152	0.228 (0.244)	0.321	1
Measure 2	0	0.167	0.267 (0.288)	0.367	1
Measure 3	0	0.147	0.211 (0.236)	0.310	1
Measure 4	0	0.162	0.241 (0.261)	0.342	1
Measure 5	0	0.260	0.370 (0.387)	0.500	1
WaitRoomCensus	0	9	13 (12.9)	17	37

Table 6 Accuracy of the two disposition prediction models used in the discrete-choice model.

	AUC (95% CI)	Sensitivity (Recall)	Specificity	PPV (Precision)	NPV
Logit model	0.783 (0.780–0.787)	0.718	0.706	0.458	0.878
Probit model	0.783 (0.780–0.787)	0.718	0.710	0.460	0.877

Note. PPV = positive predictive value; NPV = negative predictive value.

Table 7 Bed capacity estimated using the 90th and 95th percentiles of the total number of patients in ED beds.

Hour	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
90th	42	41	40	38	37	36	34	35	37	39	41	46	48	50	49	50	48	49	48	49	47	46	45	41
95th	44	43	41	40	39	38	37	37	39	41	43	47	50	52	53	52	51	50	49	51	49	48	46	44

Table 8 Cramer's V correlation between categorical variables in the disposition prediction model and discrete-choice model.

Variables	*Triage Level*	*Age Group*	*Gender*	*Arrival Mode*	*Address*	*Chief Complaint*
Triage Level	1.000					
Age Group	0.078	1.000				
Gender	0.069	0.080	1.000			
Arrival Mode	0.168	0.313	0.061	1.000		
Address	0.053	0.044	0.087	0.127	1.000	
Chief Complaint	0.420	0.242	0.252	0.421	0.106	1.000

Robustness Checks. We test the robustness of our findings by deviating from the baseline model (Model 1). Specifically, we use four other different proxies of ED blocking level in Models 2, 6,7, and 8 (see details in Section 4.4). We use a probit model to predict patient dispositions in Model 3 and control the quadratic term of *WaitRoomCensus* in Model 4. We remove visit records of patients with triage orders in Model 5. Model 9 (Model 10) is the same as Model 1 except that the observations when the ED blocking level by measure 1 is below (above) the median are used to train the disposition prediction model. Complete estimation results are provided in Tables 11 and 12. The quadratic term of *WaitRoomCensus* in Model 4 is not significant for all triage levels thus not included in the estimation results.

Table 9 The upper section of the table shows the point-biserial correlation between categorical variables and numerical variables. The last two rows shows the Pearson correlation between the numerical variables. The pairwise correlation coefficients between different measures of *BlockLevel* are close to 1 and thus not included in the table.

	Measures of *BlockLevel*					*WaitRoom*	*Wait*
Variables	Measure 1	Measure 2	Measure 3	Measure 4	Measure 5	*Census*	*Time*
Disposition (Admit=1)	0.009	0.008	0.009	0.008	0.003	0.028	−0.016
Triage Level							
Level 2	0.014	0.018	0.014	0.015	0.009	0.033	−0.109
Level 3	0.018	0.017	0.017	0.017	0.011	−0.015	0.101
Level 4,5	−0.040	−0.045	−0.040	−0.041	−0.025	−0.023	0.011
Age Group							
0-18 years	−0.013	−0.010	−0.013	−0.012	−0.009	0.003	−0.012
18-40 years	−0.010	−0.006	−0.010	−0.008	−0.006	−0.021	−0.052
40-55 years	−0.002	−0.002	−0.002	−0.002	0.001	−0.001	−0.016
55-70 years	0.009	0.008	0.008	0.008	0.008	0.016	0.007
>70 years	0.009	0.004	0.009	0.007	0.001	0.009	0.074
Gender (Male=1)	−0.001	−0.002	−0.002	−0.001	−0.001	0.007	−0.032
Arrival Mode (Ambulance=1)	−0.028	−0.027	−0.028	−0.028	−0.033	0.012	−0.010
Address							
Region A	−0.005	−0.004	−0.005	−0.005	−0.002	0.006	−0.002
Region B	0.008	0.008	0.008	0.008	0.004	−0.014	0.027
Region C	0.004	0.004	0.003	0.004	0.006	0.013	−0.017
Region D	0.006	0.003	0.006	0.004	0.005	0.002	−0.013
Others	−0.020	−0.019	−0.020	−0.019	−0.019	0.003	−0.012
Chief Complaint †							
Shortness of Breath	0.015	0.015	0.015	0.015	0.009	0.009	0.018
Chest Pain (Cardiac Feature)	0.020	0.020	0.020	0.020	0.017	0.009	−0.013
Headache	0.009	0.009	0.009	0.008	0.009	−0.001	0.005
Depression / Suicidal	−0.002	0.001	−0.002	−0.001	0.000	0.001	0.038
Vomiting And / Or Nausea	0.000	−0.001	0.000	0.000	0.000	−0.007	0.019
WaitRoomCensus	−0.079	−0.059	−0.081	−0.082	−0.050	1.000	0.291
WaitTime	−0.033	−0.008	−0.034	−0.025	−0.020	0.291	1.000

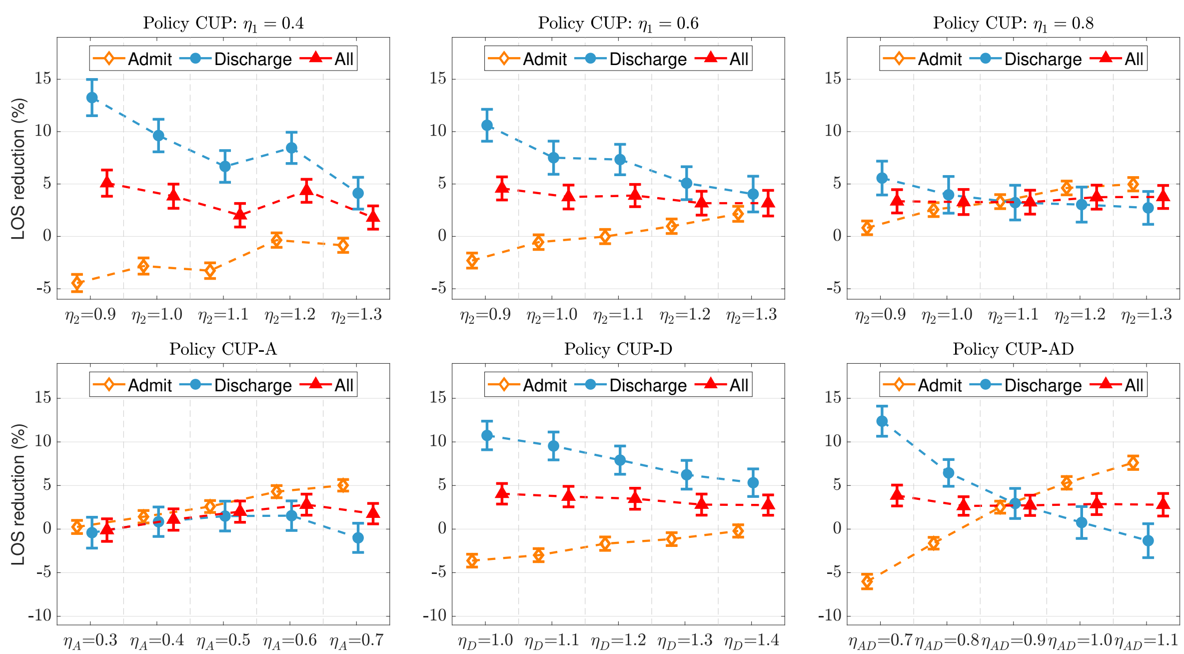
† The remaining 165 chief complaint codes are not shown for the sake of space.

Further Discussion on Confounding Factors. In Section 4.2 of the paper, we exploited several strategies to rule out endogeneity in the predicted disposition *Disposition*. However, there are other possible confounding variables and it is difficult to exhaust them all. Take patient age and gender as examples. One may argue that senior patients (or female patients) are discriminated against. Hence, their priorities of being seen is low, which leads to longer waiting time; as a result, they are more likely to be admitted (Sun et al. 2013, Richardson and Bryant 2004). Our estimation results show that both age and gender are significantly associated with *Disposition* and the choice probability, which supports this discrimination-driven mechanism. We argue that a confounding variable is problematic only if (1) it affects prioritization decisions; (2) it negatively associates with *Disposition*; and (3) the effect varies with *BlockLevel*. The three requirements are hardly met at the same time; especially, it is hard to believe that discrimination happens only when *BlockLevel* is high if it were to exist. Hence, we believe that such confounding variables do not undermine our results and conclusions.

Table 10 Specifications for the 10 models estimated in this paper. The McFadden pseudo-R^2 and its equivalent R^2 are 0.069 and 0.151, respectively, for all models except Model 5 (which are 0.067 and 0.147 respectively).

Model ID	With $WaitRoomCensus^2$	With Triage Orders	Blocking Level	Prediction Method	Data Used for Prediction	Number of Observations
1	no	yes	Measure 1	logit	all	83,189
2	no	yes	Measure 2	logit	all	83,189
3	no	yes	Measure 1	probit	all	83,189
4	yes	yes	Measure 1	logit	all	83,189
5	no	no	Measure 1	logit	all	50,885
6	no	yes	Measure 3	logit	all	83,189
7	no	yes	Measure 4	logit	all	83,189
8	no	yes	Measure 5	logit	all	83,189
9	no	yes	Measure 1	logit	lower half	83,189
10	no	yes	Measure 1	logit	upper half	83,189

Figure 6 The 95% confidence interval for the percentage reduction in LOS by using Policy CUP over Policy UP (the three figures in the top row) and by using Policy CUP-A, CUP-D, CUP-AD over Policy UP (the three figures in the bottom row), respectively, when $\eta_1 \in \{0.4, 0.6, 0.8\}$, $\eta_2 \in \{0.9, 1.0, 1.1, 1.2, 1.3\}$, $\eta_A \in \{0.3, 0.4, 0.5, 0.6, 0.7\}$, $\eta_D \in \{1.0, 1.1, 1.2, 1.3, 1.4\}$, and $\eta_{AD} \in \{0.7, 0.8, 0.9, 1.0, 1.1\}$.



Appendix B. Heterogeneity in Physician Behavior.

Prioritizing discharge patients when the ED blocking level is high is not an explicit policy in the study ED. We believe that this is a spontaneous reaction of practitioners who face ED blocking on a daily basis and understand the causes and consequences of ED blocking profoundly. Such decisions may depend on the training, past experience, and risk attitude of decision makers. Hence, it is plausible that different decision makers will behave differently. To gain more insights, we perform a similar analysis using the baseline model

Table 11 Complete estimation results for Models 1-5. See Table 10 for detailed model specifications.

	Model 1	Model 2	Model 3	Model 4	Model 5
Triage Level = 2					
CTAS×WaitTime	0.661***	0.663***	0.661***	0.66***	0.372***
	(0.009)	(0.009)	(0.009)	(0.009)	(0.013)
CTAS×WaitTime^2	−0.049***	−0.049***	−0.049***	−0.048***	−0.023***
	(0.001)	(0.001)	(0.001)	(0.001)	(0.002)
CTAS×Disposition	0.435***	0.507***	0.43***	0.436***	0.762***
	(0.098)	(0.100)	(0.124)	(0.098)	(0.128)
CTAS×Disposition×BlockLevel	−1.583***	−1.601***	−1.63***	−1.584***	−1.497***
	(0.224)	(0.200)	(0.236)	(0.224)	(0.306)
Triage Level = 3					
CTAS	−0.302***	−0.278***	−0.299***	−0.251***	−0.443***
	(0.040)	(0.041)	(0.041)	(0.053)	(0.053)
CTAS×WaitTime	1.035***	1.036***	1.036***	1.036***	0.942***
	(0.010)	(0.010)	(0.010)	(0.010)	(0.016)
CTAS×WaitTime^2	−0.077***	−0.077***	−0.077***	−0.077***	−0.079***
	(0.001)	(0.001)	(0.001)	(0.001)	(0.003)
CTAS×BlockLevel	0.288*	0.17	0.283*	0.289*	0.458**
	(0.118)	(0.106)	(0.122)	(0.118)	(0.147)
CTAS×WaitRoomCensus	−1.034***	−1.041***	−1.034***	−1.354***	−1.002***
	(0.060)	(0.060)	(0.060)	(0.229)	(0.083)
CTAS×Disposition	−0.141	−0.018	−0.163	−0.14	−0.236
	(0.115)	(0.116)	(0.143)	(0.115)	(0.153)
CTAS×Disposition×BlockLevel	−2.791***	−2.802***	−2.827***	−2.792***	−2.235***
	(0.266)	(0.237)	(0.267)	(0.266)	(0.363)
Triage Level = 4, 5					
CTAS	−0.603***	−0.579***	−0.6***	−0.543***	−0.786***
	(0.047)	(0.048)	(0.049)	(0.064)	(0.059)
CTAS×WaitTime	1.088***	1.087***	1.089***	1.09***	1.042***
	(0.015)	(0.015)	(0.015)	(0.015)	(0.021)
CTAS×WaitTime^2	−0.085***	−0.084***	−0.085***	−0.085***	−0.089***
	(0.002)	(0.002)	(0.002)	(0.002)	(0.003)
CTAS×BlockLevel	0.427**	0.304*	0.389**	0.427**	0.606***
	(0.137)	(0.123)	(0.138)	(0.137)	(0.164)
CTAS×WaitRoomCensus	−0.577***	−0.585***	−0.577***	−0.952***	−0.65***
	(0.077)	(0.077)	(0.077)	(0.285)	(0.098)
CTAS×Disposition	−0.235	−0.012	−0.27	−0.236	−0.585*
	(0.214)	(0.218)	(0.234)	(0.214)	(0.287)
CTAS×Disposition×BlockLevel	−3.419***	−3.725***	−3.24***	−3.414***	−2.014*
	(0.653)	(0.583)	(0.622)	(0.653)	(0.831)
Control Variables					
Age Group (Base=18-40 years)					
0-18 years	0.058	0.057	0.059	0.058	0.053
40-55 years	−0.057***	−0.057***	−0.055***	−0.057***	−0.043**
55-70 years	−0.086***	−0.086***	−0.082***	−0.086***	−0.079***
>70 years	−0.264***	−0.264***	−0.257***	−0.264***	−0.244***
Gender (Male=1)	0.067***	0.068***	0.069***	0.067***	0.065***
Arrival Mode (Ambulance=1)	0.066***	0.067***	0.07**	0.066***	0.078***
Chief Complaint (Base=Abdominal Pain)†					
Shortness of Breath	0.101***	0.101***	0.103***	0.1***	0.003
Chest Pain (Cardiac Features)	−0.041	−0.042	−0.045	−0.04	−0.034
Headache	−0.029	−0.03	−0.033	−0.029	−0.141***
Depression / Suicidal	−0.359***	−0.36***	−0.358***	−0.359***	−0.583***
Vomiting And / Or Nausea	0.045	0.045	0.045	0.045	0.042

† The other 164 chief complaint codes are not shown for the sake of space. ***p<0.001; **p<0.01; *p<0.05
Note. The quadratic terms of *WaitRoomCensus* in Model 4 (not significant across all triage levels).

Table 12 Complete estimation results for Models 6-10. See Table 10 for detailed model specifications.

	Model 6	Model 7	Model 8	Model 9	Model 10
Triage Level = 2					
CTAS×WaitTime	0.661***	0.661***	0.661***	0.661***	0.661***
	(0.009)	(0.009)	(0.009)	(0.009)	(0.009)
CTAS×WaitTime^2	−0.049***	−0.049***	−0.049***	−0.049***	−0.049***
	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)
CTAS×Disposition	0.423***	0.439***	0.391***	0.441***	0.422***
	(0.098)	(0.099)	(0.104)	(0.098)	(0.097)
CTAS×Disposition×BlockLevel	−1.585***	−1.503***	−0.904***	−1.618***	−1.532***
	(0.230)	(0.213)	(0.165)	(0.221)	(0.224)
Triage Level = 3					
CTAS	−0.295***	−0.289***	−0.291***	−0.302***	−0.302***
	(0.040)	(0.040)	(0.043)	(0.040)	(0.040)
CTAS×WaitTime	1.035***	1.035***	1.034***	1.036***	1.035***
	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)
CTAS×WaitTime^2	−0.077***	−0.077***	−0.077***	−0.077***	−0.077***
	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)
CTAS×BlockLevel	0.269*	0.225*	0.157	0.279*	0.296*
	(0.121)	(0.112)	(0.087)	(0.116)	(0.119)
CTAS×WaitRoomCensus	−1.034***	−1.035***	−1.038***	−1.034***	−1.034***
	(0.060)	(0.060)	(0.060)	(0.060)	(0.060)
CTAS×Disposition	−0.173	−0.139	−0.134	−0.14	−0.152
	(0.115)	(0.115)	(0.121)	(0.114)	(0.113)
CTAS×Disposition×BlockLevel	−2.748***	−2.625***	−1.798***	−2.788***	−2.76***
	(0.274)	(0.253)	(0.196)	(0.266)	(0.265)
Triage Level = 4, 5					
CTAS	−0.598***	−0.594***	−0.546***	−0.604***	−0.603***
	(0.047)	(0.048)	(0.051)	(0.047)	(0.048)
CTAS×WaitTime	1.088***	1.088***	1.085***	1.088***	1.089***
	(0.015)	(0.015)	(0.015)	(0.015)	(0.015)
CTAS×WaitTime^2	−0.085***	−0.085***	−0.084***	−0.085***	−0.085***
	(0.002)	(0.002)	(0.002)	(0.002)	(0.002)
CTAS×BlockLevel	0.417**	0.369**	0.131	0.416**	0.44**
	(0.141)	(0.130)	(0.099)	(0.135)	(0.138)
CTAS×WaitRoomCensus	−0.576***	−0.576***	−0.595***	−0.576***	−0.578***
	(0.077)	(0.077)	(0.077)	(0.077)	(0.077)
CTAS×Disposition	−0.268	−0.208	−0.238	−0.23	−0.252
	(0.213)	(0.215)	(0.230)	(0.211)	(0.215)
CTAS×Disposition×BlockLevel	−3.384***	−3.313***	−2.11***	−3.391***	−3.45***
	(0.671)	(0.621)	(0.468)	(0.645)	(0.661)
Control Variables					
Age Group (Base=18-40 years)					
0-18 years	0.058	0.058	0.057	0.05	0.067*
40-55 years	−0.057***	−0.057***	−0.056***	−0.056***	−0.057***
55-70 years	−0.086***	−0.086***	−0.085***	−0.085***	−0.086***
>70 years	−0.264***	−0.264***	−0.263***	−0.264***	−0.264***
Gender (Male=1)	0.068***	0.068***	0.068***	0.067***	0.068***
Arrival Mode (Ambulance=1)	0.066***	0.066***	0.067***	0.067***	0.066***
Chief Complaint (Base=Abdominal Pain)†					
Shortness of Breath	0.1***	0.101***	0.098***	0.11***	0.093***
Chest Pain (Cardiac Features)	−0.041	−0.041	−0.043	−0.037	−0.043
Headache	−0.029	−0.03	−0.032	−0.028	−0.03
Depression / Suicidal	−0.359***	−0.359***	−0.359***	−0.356***	−0.362***
Vomiting And / Or Nausea	0.045	0.045	0.045	0.05	0.042

† The remaining 164 chief complaint codes are not shown for the sake of space. ***p<0.001; **p<0.01; *p<0.05

(Model 1) for individual physicians. Our data include 130 physicians with 83,139 patient visit records, giving an average of 639.5 visits per physician. The most productive physician in our dataset treated 2,685 patients. However, even for this physician, the sample size is not sufficient due to the large number of control variables in our model. Hence, we remove the chief complaint control variables (170 categories) from the model specification. We make the following observation from our estimation results in Table 13.

Observation. *ED physicians exhibit heterogeneous patient prioritization behavior.*

It appears that all five physicians consider patients' estimated dispositions in their prioritization decisions to a certain extent. However, their behavior are different from each other and vary across triage levels, both qualitatively and quantitatively. The estimation results for the terms corresponding to the effect of ED blocking ($CTAS \times Disposition \times BlockLevel$) are more consistent across the five physicians for triage levels 2 and 3. However, these terms for levels 4 and 5 patients are mostly insignificant. A closer look at the data reveals that most patients of levels 4 and 5 are discharge patients: the total number of admitted patients varies between 18 to 46 patients for the five physicians over the 2-year study period. This might explain why, for all five physicians, the interaction $CTAS \times Disposition \times BlockLevel$ of triage levels 4 and 5 has larger standard error than that of triage levels 2 and 3, thus producing insignificant results. We note that the coefficients for *WaitTime* and its quadratic term are mostly significant across all five physicians. This is not unexpected, as the service levels specified by the triage protocol CTAS are explicitly dependent on patient waiting time (e.g., 90% of CTAS 3 patients should be seen within 30 minutes upon arrival). Hence, waiting time is a consensus factor for patient prioritization.

Appendix C. Proofs of Propositions 1 and 2.

Proof of Proposition 1: Denote the number of servers and the number of patients at the assess and test queues by n_i and x_i, respectively, where $i = 1$ corresponds to the assess queue, and $i = 2$ corresponds to the test queue. Denote the service rate of each server at the assess and test queues by λ^1 and λ^2, respectively. Since the assess and test queues form a cyclic network with K patients, we have $x_1 + x_2 = K$, $x_i \geq 0$, $i = 1, 2$. Thus, the system state can be fully described by x_2. When $x_2 = i$, the total service rates for assessment and for testing are $\lambda_i^1 \equiv \min\{n_1, K - i\}\lambda^1$ and $\lambda_i^2 \equiv \min\{n_2, i\}\lambda^2$, respectively. Hence, both λ_i^1 and λ_i^2 are increasing and concave in the number of patients at the assess and test queues, respectively. Note that this network structure holds for any positive integer number of servers at each queue, i.e., $1 \leq n_i \leq \infty$, $i = 1, 2$. We can use a continuous-time Markov chain to model the dynamic of the cyclic network and the balance equations can be written as $\hat{p}\lambda_i^1 \pi_i = \lambda_{i+1}^2 \pi_{i+1}$, $i = 0, 1, \cdots, K - 1$, where $\hat{p}$ is the probability of patients joining the test queue after physician assessment and $\{\pi_i,\ i = 0, 1, \cdots, K\}$ is the stationary distribution. Together with $\sum_{i=0}^{K} \pi_i = 1$, we can solve the system of linear equations and get

$$\pi_0 = \frac{1}{1 + \sum_{k=1}^{K} \prod_{j=0}^{k-1} (\hat{p}\lambda_j^1 / \lambda_{j+1}^2)}, \quad \pi_i = \frac{\prod_{j=0}^{i-1} (\hat{p}\lambda_j^1 / \lambda_{j+1}^2)}{1 + \sum_{k=1}^{K} \prod_{j=0}^{k-1} (\hat{p}\lambda_j^1 / \lambda_{j+1}^2)}, \quad i = 1, 2, \cdots, K.$$

Table 13 Key determinants of patient prioritization decisions for five selected physicians.

	MD 1	MD 2	MD 3	MD 4	MD 5
Triage Level = 2					
CTAS×WaitTime	0.562***	0.732***	0.687***	1.045***	0.614***
	(0.046)	(0.071)	(0.080)	(0.092)	(0.105)
CTAS×WaitTime^2	−0.036***	−0.041***	−0.049***	−0.086***	−0.006
	(0.005)	(0.008)	(0.007)	(0.013)	(0.017)
CTAS×Disposition	1.213	2.85*	1.557	1.888*	1.104
	(0.689)	(1.340)	(2.901)	(0.950)	(0.821)
CTAS×Disposition×BlockLevel	−4.724**	−5.525	−8.313*	−6.584*	−4.856*
	(1.743)	(2.836)	(3.570)	(3.044)	(2.466)
Triage Level = 3					
CTAS	−0.565*	0.163	0.579	0.096	−0.122
	(0.232)	(0.378)	(0.489)	(0.394)	(0.344)
CTAS×WaitTime	1.211***	0.371***	0.924***	1.246***	1.181***
	(0.073)	(0.104)	(0.087)	(0.109)	(0.101)
CTAS×WaitTime^2	−0.117***	0.087***	−0.066***	−0.103***	−0.069***
	(0.013)	(0.018)	(0.012)	(0.020)	(0.014)
CTAS×WaitRoomCensus	−1.225***	−1.273**	−1.033	−0.972	−1.441**
	(0.349)	(0.450)	(0.540)	(0.587)	(0.532)
CTAS×Disposition	0.561	2.904*	−1.349	0.689	−0.447
	(0.809)	(1.366)	(3.916)	(1.048)	(0.854)
CTAS×Disposition×BlockLevel	−6.346***	−11.9***	−11.54*	−7.374*	−2.59
	(1.836)	(3.099)	(4.980)	(3.013)	(2.274)
Triage Level = 4, 5					
CTAS	−0.424	−0.904	−0.071	0.263	0.177
	(0.268)	(0.481)	(0.575)	(0.456)	(0.418)
CTAS×WaitTime	0.844***	1.378***	1.09***	1.187***	1.088***
	(0.065)	(0.176)	(0.175)	(0.114)	(0.112)
CTAS×WaitTime^2	−0.05***	−0.079**	−0.1**	−0.078***	−0.06***
	(0.006)	(0.029)	(0.037)	(0.010)	(0.013)
CTAS×WaitRoomCensus	−0.659	−1.529*	−0.455	−0.458	−0.654
	(0.459)	(0.647)	(0.651)	(0.771)	(0.696)
CTAS×Disposition	−0.964	2.007	0.085	−1.046	−2.799
	(1.578)	(2.207)	(8.541)	(1.901)	(1.794)
CTAS×Disposition×BlockLevel	−4.538	−8.568	−27.234	−2.185	1.863
	(4.938)	(7.227)	(16.612)	(6.264)	(6.372)
Control Variables					
Age Group (Base=18-40 years)					
0-18 years	−0.254	−0.077	−0.263	0.364	−0.059
40-55 years	0.043	−0.047	−0.09	0.126	−0.1
55-70 years	−0.073	−0.149	−0.113	0.09	0.07
>70 years	−0.235	−0.543	−0.437	0.048	−0.181
Gender (Male=1)	0.074	−0.003	0.026	0.059	−0.056
Arrival Mode (Ambulance=1)	0.132	0.047	0.014	0.112	0.135
Observations	2685	1337	1208	1204	1145
McFadden pseudo R^2	0.062	0.093	0.071	0.094	0.104
(Equivalent linear model R^2)	(0.136)	(0.204)	(0.156)	(0.206)	(0.228)

Notes. Robust standard errors are shown in the parentheses. ***p<0.001; **p<0.01; *p<0.05

We are interested in the rate that patients complete their treatment and exit ED (either discharged home or admitted to hospital wards), $\mu_1(K)$, which can be written as

$$\mu_1(K) = \sum_{i=0}^{K}(1-\hat{p})\lambda_i^1\pi_i = \frac{(1-\hat{p})\left[\lambda_0^1 + \sum_{i=1}^{K}\lambda_i^1\prod_{j=0}^{i-1}(\hat{p}\lambda_j^1/\lambda_{j+1}^2)\right]}{1+\sum_{k=1}^{K}\prod_{j=0}^{k-1}(\hat{p}\lambda_j^1/\lambda_{j+1}^2)}.$$

Let $D(K,t)$ be the cumulative number of patients whose assessments are completed between 0 and t. Then,

$$\lim_{t\to\infty}\frac{D(K,t)}{t} = \sum_{i=0}^{K}\hat{p}\lambda_i^1\pi_i = \frac{\hat{p}}{1-\hat{p}}\mu_1(K).$$

From Theorem 14.D.1 of Shanthikumar and Yao (1994), $D(K,t)$ is increasing and concave in K. Therefore, $\mu_1(K)$ is increasing and concave in K. It is obvious that $\mu_1(0)=0$, which completes the proof. □

Next, we prove Proposition 2 by the value iteration algorithm. We first study the discounted net social benefit version of the MDP. The uniformization constant is $\Lambda = \mu_1(0)+\mu_2+\alpha$, where α is the continuous discount factor. Without loss of generality, we can redefine the time unit so that $\Lambda = 1$. Let $v(x)$ be the total discounted net social benefits starting from state x. Then, the optimality equations become $v = Tv$, where the operator T is defined as:

$$\begin{aligned} Tv(x) &= \mu_1(x)\max\left\{R_1+v(x)-c_1, R_2+v(x+1)-c_2, R_0+(1-p)v(x)+pv(x+1)\right\} \\ &\quad + \mu_2 v(x-1) + [\mu_1(0)-\mu_1(x)]\, v(x), \text{ if } x\in\mathcal{S}\setminus\{0\}, \end{aligned} \tag{3}$$

$$Tv(0) = \mu_1(0)\max\left\{R_1+v(0)-c_1, R_2+v(1)-c_2, R_0+(1-p)v(0)+pv(1)\right\}+\mu_2 v(0), \tag{4}$$

where $v(\cdot):\mathcal{S}\to\mathbb{R}$ and we let $v(x)=-\infty, \forall x\notin\mathcal{S}$ for notational convenience. Each term in the maximization is the cost-to-go if the corresponding action in $\mathcal{A}$ is taken. The fact that the state and action spaces are finite, one-period utility and costs are stationary and bounded, and that the discounting factor for the uniformized system $(\mu_1(0)+\mu_2)/(\alpha+\mu_1(0)+\mu_2)<1$ implies that the maximum in the optimality equations $v(x)=Tv(x)$ is achieved and that there exists an optimal policy that is stationary and deterministic (see Chapter 6 in Puterman 2005). This also implies that we may restrict our attention to this class of policies. We next define $Dv(x)\equiv v(x)-v(x+1),\ 0\le x<B$, and then $Tv(x)$ simplifies into

$$Tv(B) = \mu_2 v(B-1)+\mu_1(0)v(B),$$

$$Tv(x) = \mu_1(x)\max\{R_0-pDv(x), R_1-c_1, R_2-c_2-Dv(x)\}+\mu_2 v(x-1)+\mu_1(0)v(x),\ 0<x<B, \tag{5}$$

$$Tv(0) = \mu_1(0)\max\{R_0-pDv(0), R_1-c_1, R_2-c_2-Dv(0)\}+\mu_2 v(0)+\mu_1(0)v(0).$$

Let $\mathcal{F}$ be the set of functions defined on S such that if $f(x)\in\mathcal{F}$, then (i) $Df(x)\ge 0$ for $0\le x\le B-1$; (ii) $Df(x)\le Df(x+1)$ for $0\le x\le B-2$. Then, we have the following result.

Lemma 1. *If $v\in\mathcal{F}$, then $Tv\in\mathcal{F}$.*

Proof: We first prove that $DTv(x)=Tv(x)-Tv(x+1)\ge 0$ for $0\le x\le B-1$. Plug in Equation (5) into $DTv(x)$, we have

$$DTv(x) = \mu_1(x)\max\{R_0-pDv(x), R_1-c_1, R_2-c_2-Dv(x)\}-\mu_1(x+1)\max\{R_0-pDv(x+1),$$

$$R_1 - c_1, R_2 - c_2 - Dv(x+1)\} + \mu_2 Dv(x-1) + \mu_1(0)Dv(x), \text{ if } 0 < x \le B-1,$$

$$DTv(0) = \mu_1(0)\max\{R_0 - pDv(0), R_1 - c_1, R_2 - c_2 - Dv(0)\} - \mu_1(1)\max\{R_0 - pDv(1), R_1 - c_1,$$
$$R_2 - c_2 - Dv(1)\} + \mu_1(0)Dv(0).$$

Since $Dv(x) \ge 0$ for any $0 \le x \le B-1$, we have

$$\begin{aligned} DTv(x) \ge\ & \mu_1(x)\max\{R_0 - pDv(x), R_1 - c_1, R_2 - c_2 - Dv(x)\} + \mu_1(0)Dv(x) \\ & - \mu_1(x+1)\max\{R_0 - pDv(x+1), R_1 - c_1, R_2 - c_2 - Dv(x+1)\}. \end{aligned} \tag{6}$$

We next consider six separate cases to show the right-hand side of (6) is non-negative for any $0 \le x \le B-1$.
Case 1. When $Dv(x) \le Dv(x+1) \le R_2 - R_1 - (1-p)^{-1}c_2$, we have

$$\begin{aligned} DTv(x) &\ge \mu_1(x)[R_2 - c_2 - Dv(x)] - \mu_1(x+1)[R_2 - c_2 - Dv(x+1)] + \mu_1(0)Dv(x) \\ &= D\mu_1(x)(R_2 - c_2) + \mu_1(x+1)Dv(x+1) + [\mu_1(0) - \mu_1(x)]\,Dv(x) \ge 0. \end{aligned}$$

The last inequality follows since $\mu_1(x)$ is concave and non-increasing in x and $Dv(x) \ge 0$ for any $0 \le x \le B-1$.
Case 2. When $Dv(x) \le R_2 - R_1 - (1-p)^{-1}c_2 \le Dv(x+1) \le R_2 - R_1 + p^{-1}c_1$, we have

$$\begin{aligned} DTv(x) &\ge \mu_1(x)[R_2 - c_2 - Dv(x)] - \mu_1(x+1)[R_0 - pDv(x+1)] + \mu_1(0)Dv(x) \\ &\ge (1-p)\mu_1(x)(R_2 - R_1 - (1-p)^{-1}c_2) + p\mu_1(x+1)(R_2 - R_1 - (1-p)^{-1}c_2) \ge \mu_1(x+1)Dv(x) \ge 0. \end{aligned}$$

The second and third inequalities follow from $\mu_1(x) \ge \mu_1(x+1)$, $Dv(x) \ge 0$ and $Dv(x+1) \ge R_2 - R_1 - (1-p)^{-1}c_2 \ge 0$; the last inequality follows from $\mu_1(x) \ge 0$ and $Dv(x) \ge 0$.
Case 3. When $Dv(x) \le R_2 - R_1 - (1-p)^{-1}c_2 \le R_2 - R_1 + p^{-1}c_1 \le Dv(x+1)$, we have

$$\begin{aligned} DTv(x) &\ge \mu_1(x)[R_2 - c_2 - Dv(x)] - \mu_1(x+1)(R_1 - c_1) + \mu_1(0)Dv(x) \\ &\ge \mu_1(x+1)(R_2 - R_1 - c_2 + c_1) \ge \mu_1(x+1)[R_2 - R_1 - (1-p)^{-1}c_2 + c_1] \ge 0. \end{aligned}$$

The second inequality follows from the non-increasing property of $\mu_1(x)$ and the last inequality follows from $R_2 - R_1 - (1-p)^{-1}c_2 \ge Dv(x) \ge 0$.
Case 4. When $R_2 - R_1 - (1-p)^{-1}c_2 \le Dv(x) \le Dv(x+1) \le R_2 - R_1 + p^{-1}c_1$, we have

$$\begin{aligned} DTv(x) &\ge \mu_1(x)[R_0 - pDv(x)] - \mu_1(x+1)[R_0 - pDv(x+1)] + \mu_1(0)Dv(x) \\ &= (\mu_1(x) - \mu_1(x+1))R_0 + [\mu_1(0) - p\mu_1(x)]\,Dv(x) + \mu_1(x+1)pDv(x+1) \ge 0. \end{aligned}$$

The last inequality follows from that $\mu_1(x)$ is non-increasing and $Dv(x) \ge 0$.
Case 5. When $R_2 - R_1 - (1-p)^{-1}c_2 \le Dv(x) \le R_2 - R_1 + p^{-1}c_1 \le Dv(x+1)$, we have

$$DTv(x) \ge \mu_1(x)[R_0 - pDv(x)] - \mu_1(x+1)(R_1 - c_1) + \mu_1(0)Dv(x)$$

$$\geq \mu_1(x)R_0 - \mu_1(x+1)(R_1 - c_1) \geq \mu_1(x+1)p(R_2 - R_1 + p^{-1}c_1) \geq 0.$$

The second and third inequalities follow from that $\mu_1(x)$ is non-increasing and $R_2 - R_1 + p^{-1}c_1 \geq 0$.
Case 6. When $R_2 - R_1 + p^{-1}c_1 \leq Dv(x) \leq Dv(x+1)$, we have

$$DTv(x) \geq \mu_1(x)\,(R_1 - c_1) - \mu_1(x+1)\,(R_1 - c_1) + \mu_1(0)Dv(x) \geq 0.$$

The last equality follows from that $\mu_1(x)$ is non-increasing. Hence, we conclude that $DTv(x) \geq 0$.

Next, we show that $DTv(x+1) \geq DTv(x)$ for $0 \leq x \leq B-2$. When $0 < x \leq B-2$, we have

$$\begin{aligned} &DTv(x+1) - DTv(x) \\ =& 2\mu_1(x+1)\max\{R_0 - pDv(x+1), R_1 - c_1, R_2 - c_2 - Dv(x+1)\} \\ &- \mu_1(x)\max\{R_0 - pDv(x), R_1 - c_1, R_2 - c_2 - Dv(x)\} + \mu_1(0)[Dv(x+1) - Dv(x)] \\ &- \mu_1(x+2)\max\{R_0 - pDv(x+2), R_1 - c_1, R_2 - c_2 - Dv(x+2)\} + \mu_2[Dv(x) - Dv(x-1)]. \end{aligned} \tag{7}$$

When $x = 0$, we have

$$\begin{aligned} DTv(x+1) - DTv(x) =& 2\mu_1(1)\max\{R_0 - pDv(1), R_1 - c_1, R_2 - c_2 - Dv(1)\} \\ &- \mu_1(0)\max\{R_0 - pDv(0), R_1 - c_1, R_2 - c_2 - Dv(0)\} + \mu_1(0)[Dv(1) - Dv(0)] \\ &- \mu_1(2)\max\{R_0 - pDv(2), R_1 - c_1, R_2 - c_2 - Dv(2)\} + \mu_2 Dv(0). \end{aligned} \tag{8}$$

Hence, combining Equations (7) and (8) and for any $0 \leq x \leq B-2$, we have

$$\begin{aligned} &DTv(x+1) - DTv(x) \\ \geq& 2\mu_1(x+1)\max\{R_0 - pDv(x+1), R_1 - c_1, R_2 - c_2 - Dv(x+1)\} - \mu_1(x)\max\{R_0 - pDv(x), R_1 - c_1, R_2 - c_2 \\ &- Dv(x)\} - \mu_1(x+2)\max\{R_0 - pDv(x+2), R_1 - c_1, R_2 - c_2 - Dv(x+2)\} + \mu_1(0)\,[Dv(x+1) - Dv(x)] \\ \geq& [2\mu_1(x+1) - \mu_1(x+2)]\max\{R_0 - pDv(x+1), R_1 - c_1, R_2 - c_2 - Dv(x+1)\} \\ &- \mu_1(x)\max\{R_0 - pDv(x), R_1 - c_1, R_2 - c_2 - Dv(x)\} + \mu_1(0)[Dv(x+1) - Dv(x)] \equiv G(x), \end{aligned}$$

where the first inequality holds since $Dv(0) \geq 0$ and $Dv(x) - Dv(x-1) \geq 0$ for $0 < x \leq B-2$, and the second inequality holds since $Dv(x+2) \geq Dv(x+1)$. We next show $G(x) \geq 0$ for any $0 \leq x \leq B-2$ by considering six separate cases.
Case 1. When $Dv(x) \leq Dv(x+1) \leq R_2 - R_1 - (1-p)^{-1}c_2$, we have

$$\begin{aligned} G(x) &= [2\mu_1(x+1) - \mu_1(x+2)][R_2 - c_2 - Dv(x+1)] - \mu_1(x)[R_2 - c_2 - Dv(x)] + \mu_1(0)[Dv(x+1) - Dv(x)] \\ &= [2\mu_1(x+1) - \mu_1(x+2) - \mu_1(x)][R_2 - c_2 - Dv(x+1)] + [\mu_1(0) - \mu_1(x)][Dv(x+1) - Dv(x)] \geq 0, \end{aligned}$$

where the inequality follows from that $\mu_1(x)$ is concave and non-increasing, $Dv(x) \leq Dv(x+1)$, and $Dv(x+1) \leq R_2 - R_1 - (1-p)^{-1}c_2 < R_2 - c_2$.

Case 2. When $Dv(x) \le R_2 - R_1 - (1-p)^{-1}c_2 \le Dv(x+1) \le R_2 - R_1 + p^{-1}c_1$, we have

$$\begin{aligned} G(x) &= [2\mu_1(x+1) - \mu_1(x+2)][R_0 - pDv(x+1)] - \mu_1(x)[R_2 - c_2 - Dv(x)] + \mu_1(0)[Dv(x+1) - Dv(x)] \\ &\ge [2\mu_1(x+1) - \mu_1(x+2)]R_0 - \mu_1(x)(R_2 - c_2) + [\mu_1(0) - 2p\mu_1(x+1) + p\mu_1(x+2)]Dv(x+1) \\ &\quad + [\mu_1(x) - \mu_1(0)][R_2 - R_1 - (1-p)^{-1}c_2], \end{aligned}$$

where the inequality follows since $\mu_1(x) \le \mu_1(0)$ and $Dv(x) \le R_2 - R_1 - (1-p)^{-1}c_2$. If $\mu_1(0) - 2p\mu_1(x+1) + p\mu_1(x+2) \ge 0$, we have

$$\begin{aligned} G(x) \ge& [2\mu_1(x+1) - \mu_1(x+2)]R_0 - \mu_1(x)(R_2 - c_2) + [\mu_1(0) - 2p\mu_1(x+1) + p\mu_1(x+2)][R_2 - R_1 - c_2/(1-p)] \\ &+ [\mu_1(x) - \mu_1(0)][R_2 - R_1 - (1-p)^{-1}c_2] = [D\mu_1(x+1) - D\mu_1(x)][R_1 + p(1-p)^{-1}c_2] \ge 0, \end{aligned}$$

where the last inequality follows from that $\mu_1(x)$ is concave and using $Dv(x+1) \ge R_2 - R_1 - (1-p)^{-1}c_2$. If $\mu_1(0) - 2p\mu_1(x+1) + p\mu_1(x+2) \le 0$, using $Dv(x+1) \le R_2 - R_1 + p^{-1}c_1$, we have

$$\begin{aligned} G(x) &\ge [D\mu_1(x+1) - D\mu_1(x)]R_1 + [\mu_1(0) - p\mu_1(x)](1-p)^{-1}c_2 + [\mu_1(0) - 2p\mu_1(x+1) + p\mu_1(x+2)]p^{-1}c_1 \\ &= [D\mu_1(x+1) - D\mu_1(x)](R_1 - c_1) + [\mu_1(0) - p\mu_1(x)][p^{-1}c_1 + (1-p)^{-1}c_2] \ge 0, \end{aligned}$$

where the last inequality follows from the concavity of $\mu_1(x)$ and $R_1 \ge c_1$.

Case 3. When $Dv(x) \le R_2 - R_1 - (1-p)^{-1}c_2 \le R_2 - R_1 + p^{-1}c_1 \le Dv(x+1)$, we have

$$\begin{aligned} G(x) &= [2\mu_1(x+1) - \mu_1(x+2)](R_1 - c_1) - \mu_1(x)[R_2 - c_2 - Dv(x)] + \mu_1(0)[Dv(x+1) - Dv(x)] \\ &\ge [2\mu_1(x+1) - \mu_1(x+2)](R_1 - c_1) - \mu_1(x)(R_2 - c_2) + \mu_1(0)R_2 - R_1 + c_1/p \\ &\quad + [\mu_1(x) - \mu_1(0)]R_2 - R_1 - c_2/(1-p) \\ &= [D\mu_1(x+1) - D\mu_1(x)](R_1 - c_1) + [\mu_1(0)/p - \mu_1(x)]c_1/(1-p) \ge 0, \end{aligned}$$

where the last inequality follows from the non-increasing and concave property of $\mu_1(x)$ and $R_1 \ge c_1$.

Case 4. When $R_2 - R_1 - (1-p)^{-1}c_2 \le Dv(x) \le Dv(x+1) \le R_2 - R_1 + p^{-1}c_1$, we have

$$\begin{aligned} G(x) &= [2\mu_1(x+1) - \mu_1(x+2)][R_0 - pDv(x+1)] - \mu_1(x)[R_0 - pDv(x)] + \mu_1(0)[Dv(x+1) - Dv(x)] \\ &= [D\mu_1(x+1) - D\mu_1(x)][R_0 - pDv(x+1)] + [\mu_1(0) - p\mu_1(x)][Dv(x+1) - Dv(x)] \\ &\ge [D\mu_1(x+1) - D\mu_1(x)][R_0 - p(R_2 - R_1 + p^{-1}c_1)] + [\mu_1(0) - p\mu_1(x)][Dv(x+1) - Dv(x)] \ge 0, \end{aligned}$$

where the last inequality follows from the non-increasing and concave property of $\mu_1(x)$, $Dv(x+1) \ge Dv(x)$ and $R_1 \ge c_1$.

Case 5. When $R_2 - R_1 - (1-p)^{-1}c_2 \le Dv(x) \le R_2 - R_1 + p^{-1}c_1 \le Dv(x+1)$, we have

$$G(x) = [2\mu_1(x+1) - \mu_1(x+2)](R_1 - c_1) - \mu_1(x)[R_0 - pDv(x)] + \mu_1(0)[Dv(x+1) - Dv(x)]$$

$$\geq [2\mu_1(x+1) - \mu_1(x+2)](R_1 - c_1) - \mu_1(x)R_0 + [p\mu_1(x) - \mu_1(0)]Dv(x+1) + \mu_1(0)Dv(x+1)$$
$$\geq [2\mu_1(x+1) - \mu_1(x+2)](R_1 - c_1) - \mu_1(x)R_0 + p\mu_1(x)(R_2 - R_1 + p^{-1}c_1)$$
$$= [D\mu_1(x+1) - D\mu_1(x)](R_1 - c_1) \geq 0,$$

where the first inequality follows from that $p\mu_1(x) \leq \mu_1(0)$ and $Dv(x+1) \geq Dv(x)$; the second inequality follows from $Dv(x+1) \geq R_2 - R_1 + p^{-1}c_1$; and the last inequality follows from the concavity of $\mu_1(x)$ and $R_1 \geq c_1$.

Case 6. When $R_2 - R_1 + p^{-1}c_1 \leq Dv(x) \leq Dv(x+1)$, we have

$$G(x) = [D\mu_1(x+1) - D\mu_1(x)](R_1 - c_1) + \mu_1(0)[Dv(x+1) - Dv(x)] \geq 0.$$

where the inequality follows from the concavity of $\mu_1(x)$, $Dv(x+1) \geq Dv(x)$, and $R_1 \geq c_1$. Combining the results of the six cases, we conclude that $DTv(x+1) \geq DTv(x)$ for all $0 \leq x \leq B-2$. □

Proof of Proposition 2: We prove our results by verifying the conditions in Theorem 6.11.3 of Puterman (2005). It is obvious that the state space $\mathcal{S}$ is countable and Assumptions 6.10.1 and 6.10.2 of Puterman (2005) hold. Hence, we only need to show that conditions (a), (b), and (c) in Theorem 6.11.3 of Puterman (2005) hold. Condition (a) holds by Lemma 1. Next, consider a stationary policy π that if decision makers choose an admit patient at $1 \leq x \leq B$, then they will choose admit patients at x' where $x' \leq x$; if decision makers choose a discharge patient at $0 \leq x \leq B-1$, then they will choose discharge patients at x' where $x' \geq x$; otherwise, choose the first patient in line. After that, π follows the optimal policy. From the optimality equations, we find that $v \in \mathcal{F}$ implies that policy π is an optimal policy. Hence, condition (b) holds. Finally, condition (c) holds, i.e., $\mathcal{F}$ is closed, because the limit of any convergent sequence of functions that satisfy the two conditions of $\mathcal{F}$ will satisfy them as well. This concludes the proof that there exists an optimal stationary policy whose value function belongs to $\mathcal{F}$ for the discount net social benefits version of the problem, and the same structural results can be extended to the average net social benefit version by verifying three SEN conditions given in Section 7.2 of Sennott (1999) in a fairly straightforward manner. □

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