Online Companion for “Actor-Critic Like Stochastic Adaptive Search for Continuous Simulation Optimization”

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A. Connections between Model-based Algorithms and Policy Gradient Methods

We show that certain model-based algorithms, including MARS, closely resemble in form and structure to the policy gradient (PG) methods in RL.

Let $\mathcal{S} : \Re \to \Re^+ \cup \{0\}$ be a non-negative function. Clearly, if $\mathcal{S}$ is increasing, then the original problem (1) is equivalent to

$$\max_{\mathbf{x}\in\mathcal{X}} \mathcal{S}(H(\mathbf{x})). \tag{A-1}$$

We consider how to use the principle of PG methods to solve (A-1). To this end, we interpret the problem as a single-period MDP with a given (dummy) initial state s and an action space $\mathcal{X}$. The underlying process terminates after one time step with reward $R(s, \mathbf{x}) := \mathcal{S}(H(\mathbf{x}))$ when action $\mathbf{x} \in \mathcal{X}$ is taken. Since the action space $\mathcal{X}$ is uncountable, a regular PG algorithm for solving the MDP begins with a parameterized policy (action-selection probability) $\pi_\eta(\mathbf{x}|s)$ for all $\mathbf{x} \in \mathcal{X}$ and aims to find an optimal policy within the parameterized class that (locally) maximizes the expected return $E_{\pi_\eta}[\mathcal{S}(H(X))]$. In the vanilla PG algorithm, this is done by iteratively updating the policy parameter along the gradient of the performance measure $J(\eta) := E_{\pi_\eta}[\mathcal{S}(H(X))]$. Specifically, let η_k denote the parameter found at the kth iteration. A new parameter η_{k+1} can then be obtained from the gradient ascent method

$$\eta_{k+1} = \eta_k + \alpha_k \nabla_\eta J(\eta)\big|_{\eta=\eta_k} = \eta_k + \alpha_k E_{\pi_\eta}\left[\mathcal{S}(H(X))\nabla_\eta \ln \pi_\eta(X|s)\right]\Big|_{\eta=\eta_k}, \tag{A-2}$$

where the gradient can be estimated using the unbiased estimator

$$\frac{1}{M}\sum_{i=1}^{M} \mathcal{S}(H(\mathbf{x}_i))\nabla_\eta \ln \pi_\eta(\mathbf{x}_i|s)\big|_{\eta=\eta_k}, \tag{A-3}$$

with $\mathbf{x}_1, \ldots, \mathbf{x}_M$ being i.i.d. randomly generated from $\pi_{\eta_k}(\mathbf{x}|s)$.

Note that since the logarithm function is increasing, maximizing the reward $E_{\pi_\eta}[\mathcal{S}(H(X))]$ is equivalent to maximizing $\ln E_{\pi_\eta}[\mathcal{S}(H(X))]$. So (A-2) could instead be implemented using the gradient of $\ln E_{\pi_\eta}[\mathcal{S}(H(X))]$, leading to

$$\eta_{k+1} = \eta_k + \alpha_k \nabla_\eta \ln J(\eta)\big|_{\eta=\eta_k} = \eta_k + \alpha_k \frac{E_{\pi_\eta}[\mathcal{S}(H(X))\nabla_\eta \ln \pi_\eta(X|s)]\big|_{\eta=\eta_k}}{E_{\pi_{\eta_k}}[\mathcal{S}(H(X))]}. \tag{A-4}$$

The benefit of (A-4) is that the gradient in (A-2) is now normalized by the performance measure $E_{\pi_{\eta_k}}[\mathcal{S}(H(X))]$, so that the resulting estimator $\frac{\frac{1}{M}\sum_{i=1}^{M}\mathcal{S}(H(\mathbf{x}_i))\nabla_\eta \ln \pi_\eta(\mathbf{x}_i|s)|_{\eta=\eta_k}}{\frac{1}{M}\sum_{i=1}^{M}\mathcal{S}(H(\mathbf{x}_i))}$ typically has a lower variance than (A-3), albeit at the expense of inducing a ratio bias.

When π_η belongs to the NEFs with mean parameter function η, it can be viewed as a reparameterization of f_θ in terms of the mean parameter function $\eta = m(\theta)$, i.e., $\pi_\eta = f_{m^{-1}(\eta)}$. In other words, π_η is just f_θ but is written as function of η. This implies that the gradient in (A-4) can be expressed as a gradient with respect to θ. In particular, it can be seen that $\nabla_\eta \ln \pi_\eta(\mathbf{x}|s) = (\mathbf{J}_m^T)^{-1}\nabla_\theta \ln f_\theta(\mathbf{x})$, where $\mathbf{J}_m$ is the Jacobian of the transformation $m(\cdot)$. Substituting this into (A-4) then yields the following recursion:

$$\begin{aligned}\eta_{k+1} &= \eta_k + \alpha_k (\mathbf{J}_m^T)^{-1} \frac{E_{f_\theta}[\nabla_\theta \ln f_\theta(X)\mathcal{S}(H(X))]}{E_{f_{\theta_k}}[\mathcal{S}(H(X))]}\Bigg|_{\theta=\theta_k} \\ &= \eta_k - \alpha_k (\mathbf{J}_m^T)^{-1}\nabla_\theta \mathcal{D}(\tilde{g}_k, f_\theta)\big|_{\theta=\theta_k},\end{aligned} \tag{A-5}$$

where we have defined $\tilde{g}_k(\mathbf{x}) = \frac{\mathcal{S}(H(\mathbf{x}))f_{\theta_k}(\mathbf{x})}{E_{f_{\theta_k}}[\mathcal{S}(H(X))]}$.

Notice that (A-5) is derived from a pure RL perspective by viewing optimization problem (A-1) as a single-period MDP. It only differs in form from (4) by a scaling factor $(\mathbf{J}_m^T)^{-1}$. Without such a scale term, it can be shown that (A-5) coincides with the cross-entropy method; see, e.g., Hu (2015). If in addition, the function $\mathcal{S}$ is set to $\mathcal{S}(H(\mathbf{x})) = e^{\frac{H(\mathbf{x})}{T_k}}/f_{\theta_k}(\mathbf{x})$, then $\tilde{g}_k$ is identical to the Boltzmann density (2), in which case (A-5) becomes the same as MARS.

B. Some Ancillary Results

We state and prove some ancillary results, which will be used throughout the analysis.

Proposition B.1 *Let $u_1, u_2, \ldots$ be a sequence of real numbers, $u_i \in [0,1]$ for all i. Then*

$$\sum_{i=1}^{m}\Big[\prod_{j=i+1}^{m}(1-u_j)\Big]u_i \leq 1, \ \forall m \geq 1.$$

Proof. We prove the result by induction. When $m=1$, we have $\sum_{i=1}^{m}\Big[\prod_{j=i+1}^{m}(1-u_j)\Big]u_i = u_1 \leq 1$. Suppose the result holds for $m=k$, i.e., $\sum_{i=1}^{k}\Big[\prod_{j=i+1}^{k}(1-u_j)\Big]u_i \leq 1$. Then by our induction hypothesis, for $m=k+1$ we have

$$\begin{aligned}
\sum_{i=1}^{k+1}\Big[\prod_{j=i+1}^{k+1}(1-u_j)\Big]u_i &= \sum_{i=1}^{k}\Big[\prod_{j=i+1}^{k+1}(1-u_j)\Big]u_i + u_{k+1} \\
&= (1-u_{k+1})\sum_{i=1}^{k}\Big[\prod_{j=i+1}^{k}(1-u_j)\Big]u_i + u_{k+1} \\
&\leq (1-u_{k+1}) + u_{k+1} \\
&= 1.
\end{aligned}$$

□

Proposition B.2 *Let $u_1, u_2, \ldots$ be a sequence of real numbers, $u_i \in [0,1]$ for all i. Then for any two positive integers $m < k$,*

$$\sum_{i=1}^{m}\Big[\prod_{j=i+1}^{k}(1-u_j)\Big]u_i \leq \prod_{j=m+1}^{k}(1-u_j).$$

Proof. Since $m < k$, we can write

$$\sum_{i=1}^{m}\Big[\prod_{j=i+1}^{k}(1-u_j)\Big]u_i = \sum_{i=1}^{m}\Big[\prod_{j=i+1}^{m}(1-u_j)\Big]\Big[\prod_{j=m+1}^{k}(1-u_j)\Big]u_i$$

$$
\begin{aligned}
&= \prod_{j=m+1}^{k} (1-u_j) \sum_{i=1}^{m} \Big[\prod_{j=i+1}^{m} (1-u_j) \Big] u_i \\
&\leq \prod_{j=m+1}^{k} (1-u_j),
\end{aligned}
$$

where the inequality follows from Proposition B.1. □

Proposition B.3 *If Assumption A5 holds, then we have*

$$\sum_{i=1}^{m} \Big[\prod_{j=i+1}^{m} \big(1-f(j)\big)^2 \Big] f^2(i) \leq f(m), \ \forall m \geq 1.$$

Proof. We prove the result by mathematical induction. When $m=1$, we have $\sum_{i=1}^{m} \Big[\prod_{j=i+1}^{m} \big(1-f(j)\big)^2 \Big] f^2(i) = f^2(1) \leq f(1)$. Now, suppose the result is true for $m=k$, i.e., $\sum_{i=1}^{k} \Big[\prod_{j=i+1}^{k} \big(1-f(j)\big)^2 \Big] f^2(i) \leq f(k)$. Then, for $m=k+1$ we have

$$
\begin{aligned}
\sum_{i=1}^{k+1} \Big[\prod_{j=i+1}^{k+1} \big(1-f(j)\big)^2 \Big] f^2(i) &= \sum_{i=1}^{k} \Big[\prod_{j=i+1}^{k+1} \big(1-f(j)\big)^2 \Big] f^2(i) + f^2(k+1) \\
&= \big(1-f(k+1)\big)^2 \sum_{i=1}^{k} \Big[\prod_{j=i+1}^{k} \big(1-f(j)\big)^2 \Big] f^2(i) + f^2(k+1) \\
&\leq \big(1-f(k+1)\big)^2 f(k) + f^2(k+1) \\
&\leq \big(1-f(k+1)\big) f(k+1) + f^2(k+1) \\
&= f(k+1),
\end{aligned}
$$

where the last inequality is due to the fact that

$$\big(1-f(k+1)\big) f(k) = \frac{(k+2)^{\gamma_1}-1}{(k+2)^{\gamma_1}} \frac{1}{(k+1)^{\gamma_1}} = \frac{\big(1+\frac{1}{k+1}\big)^{\gamma_1} - \big(\frac{1}{k+1}\big)^{\gamma_1}}{(k+2)^{\gamma_1}} \leq \frac{1}{(k+2)^{\gamma_1}} = f(k+1).$$

□

C. Proof of Lemma 2

For the class of Lipschitz continuous function $H(\cdot)$, the ideal Boltzmann distribution constructed based on $H(\cdot)$ will converge (in a weak sense) to the Dirac measure at the optimal solution $\mathbf{x}^*$ as $T_k \to 0$ (A6). The result was proved in Hu and Hu (2011) under more general conditions on $H(\cdot)$ than Assumption A1. Thus, for any given $\epsilon > 0$, we know that there exists an integer $N' > 0$ such that $||E_{g_i}[\Gamma(X)] - \Gamma(\mathbf{x}^*)|| < \frac{\epsilon}{2}$ for all $i \geq N'$. Now, consider a $k > N'$, we have

$$
\begin{aligned}
&\sum_{i=1}^{k} \Big[\prod_{j=i+1}^{k} (1-\alpha_j) \Big] \alpha_i ||E_{g_i}[\Gamma(X)] - \Gamma(\mathbf{x}^*)|| \\
=& \sum_{i=1}^{N'} \Big[\prod_{j=i+1}^{k} (1-\alpha_j) \Big] \alpha_i ||E_{g_i}[\Gamma(X)] - \Gamma(\mathbf{x}^*)|| + \sum_{i=N'+1}^{k} \Big[\prod_{j=i+1}^{k} (1-\alpha_j) \Big] \alpha_i ||E_{g_i}[\Gamma(X)] - \Gamma(\mathbf{x}^*)|| \\
<& 2B \sum_{i=1}^{N'} \Big[\prod_{j=i+1}^{k} (1-\alpha_j) \Big] + \frac{\epsilon}{2} \sum_{i=N'+1}^{k} \Big[\prod_{j=i+1}^{k} (1-\alpha_j) \Big] \alpha_i \\
\leq& 2BN' \prod_{j=N'+1}^{k} (1-\alpha_j) + \frac{\epsilon}{2} \qquad \text{(by Proposition B.1)} \\
\leq& 2BN' e^{-\sum_{j=N'+1}^{k} \alpha_j} + \frac{\epsilon}{2},
\end{aligned}
$$

where recall that $B = \max_{\mathbf{x} \in \mathcal{X}} \|\Gamma(\mathbf{x})\|$. Finally, since ϵ is arbitrary, the proof is completed by noticing that $\sum_{j=N'+1}^{\infty} \alpha_j = \infty$ by A4. $\square$

D. Derivation of the Upper Bound (10)

According to Pinsker's inequality (e.g., Topsoe 2001), we have

$$
||E_{\hat{g}_i}[\Gamma(X)] - E_{g_i}[\Gamma(X)]|| \leq B \int_{\mathcal{X}} |g_i(\mathbf{x}) - \hat{g}_i(\mathbf{x})| d\mathbf{x} \leq B \sqrt{2\mathcal{D}(g_i, \hat{g}_i)}, \tag{A-6}
$$

where recall that $B = \max_{\mathbf{x} \in \mathcal{X}} \|\Gamma(\mathbf{x})\|$.

In addition, since

$$
\frac{g_i(\mathbf{x})}{\hat{g}_i(\mathbf{x})} = \frac{e^{H(\mathbf{x})/T_i}}{\int_{\mathcal{X}} e^{H(\mathbf{x})/T_i} d\mathbf{x}} \frac{\int_{\mathcal{X}} e^{S_i(\mathbf{x})/T_i} d\mathbf{x}}{e^{S_i(\mathbf{x})/T_i}} = \exp\left(\frac{H(\mathbf{x}) - S_i(\mathbf{x})}{T_i} \right) \Big/ E_{\hat{g}_i}\left[\exp\left(\frac{H(X) - S_i(X)}{T_i} \right) \right],
$$

we have

$$
\begin{aligned}
\mathcal{D}(g_i, \hat{g}_i) = E_{g_i}\left[\ln \frac{g_i(X)}{\hat{g}_i(X)} \right] &= E_{g_i}\left[\frac{H(X) - S_i(X)}{T_i} \right] - \ln E_{\hat{g}_i}\left[\exp\left(\frac{H(X) - S_i(X)}{T_i} \right) \right] \\
&\leq E_{g_i}\left[\frac{H(X) - S_i(X)}{T_i} \right] - E_{\hat{g}_i}\left[\frac{H(X) - S_i(X)}{T_i} \right] \quad \text{(by Jensen's inequality)} \\
&\leq \frac{\max_{\mathbf{x} \in \mathcal{X}} |H(\mathbf{x}) - S_i(\mathbf{x})|}{T_i} \int_{\mathcal{X}} |g_i(\mathbf{x}) - \hat{g}_i(\mathbf{x})| d\mathbf{x} \\
&\leq \frac{\max_{\mathbf{x} \in \mathcal{X}} |H(\mathbf{x}) - S_i(\mathbf{x})|}{T_i} \sqrt{2\mathcal{D}(g_i, \hat{g}_i)},
\end{aligned} \tag{A-7}
$$

where the last step again follows from Pinsker's inequality. Equation (A-7) shows that $\sqrt{2\mathcal{D}(g_i,\hat{g}_i)} \le \frac{2}{T_i}\max_{\mathbf{x}\in\mathcal{X}}|H(\mathbf{x})-S_i(\mathbf{x})|$. Substitute this into (A-6), we get

$$||E_{\hat{g}_i}[\Gamma(X)] - E_{g_i}[\Gamma(X)]|| \le \frac{2B}{T_i}\max_{\mathbf{x}\in\mathcal{X}}|H(\mathbf{x})-S_i(\mathbf{x})| \tag{A-8}$$

For any $\mathbf{x}\in\mathcal{X}$, let $\mathbf{x}' = \arg\min_{\mathbf{x}_l\in\Lambda_{\ell_i}} d(\mathbf{x},\mathbf{x}_l)$. Note that

$$\begin{aligned}
|H(\mathbf{x})-S_i(\mathbf{x})| &\le |H(\mathbf{x})-H(\mathbf{x}')| + |H(\mathbf{x}')-S_i(\mathbf{x}')| + |S_i(\mathbf{x}')-S_i(\mathbf{x})| \\
&\le (\mathcal{L}_1+\mathcal{L}_2)d(\mathbf{x},\mathbf{x}') + |H(\mathbf{x}')-S_i(\mathbf{x}')| \\
&= (\mathcal{L}_1+\mathcal{L}_2)d(\mathbf{x},\Lambda_{\ell_i}) + |H(\mathbf{x}')-H_i(\mathbf{x}')|.
\end{aligned}$$

This implies that

$$\max_{\mathbf{x}\in\mathcal{X}}|H(\mathbf{x})-S_i(\mathbf{x})| \le (\mathcal{L}_1+\mathcal{L}_2)\max_{\mathbf{x}\in\mathcal{X}} d(\mathbf{x},\Lambda_{\ell_i}) + \max_{\mathbf{x}_l\in\Lambda_{\ell_i}}|H_i(\mathbf{x}_l)-H(\mathbf{x}_l)|. \tag{A-9}$$

Finally, combining (A-8) and (A-9) yields the desired upper bound in (10).

E. Proof of Lemma 3

For simplicity, we consider the case where $\mathcal{X}$ is characterized by box constraints, i.e., $\mathcal{X}=[a_1,b_1]\times\ldots\times[a_d,b_d]$. Let $\epsilon_1>0$ be given, define $L=\max_{i=1,\ldots,d}(b_i-a_i)$ and $n=\sqrt{d}L(\mathcal{L}_1+\mathcal{L}_2)/\epsilon_1$. Then, partition $\mathcal{X}$ into n^d hyper-rectangles having sides of lengths $(b_i-a_i)/n$ for $i=1,\ldots,d$. For more complex solution spaces, one may instead use a triangulation of the domain (e.g., Cooper 2006). It is easy to see that the diameter of each rectangle is no greater than $\sqrt{d}L/n$. Next, consider the event $\mathcal{A}_i :=$ {each of the n^d hyper-rectangles contains at least one point from Λ_{ℓ_i}}. It is clear that $\mathcal{A}_i\subseteq\mathcal{A}_{i+1}$, $\forall i$. Thus, if $\mathcal{A}_m$ occurs for some $1\le m<k$, then we must have $\max_{\mathbf{x}\in\mathcal{X}} d(\mathbf{x},\Lambda_{\ell_i}) < \sqrt{d}L/n = \epsilon_1/(\mathcal{L}_1+\mathcal{L}_2)$, $\forall i\ge m$. This, together with Proposition B.1, implies

$$2B\sum_{i=m+1}^{k}\Big[\prod_{j=i+1}^{k}(1-\alpha_j)\Big]\alpha_i\frac{\mathcal{L}_1+\mathcal{L}_2}{T_i}\max_{\mathbf{x}\in\mathcal{X}} d(\mathbf{x},\Lambda_{\ell_i}) < \frac{2B\epsilon_1}{T_k}\sum_{i=m+1}^{k}\Big[\prod_{j=i+1}^{k}(1-\alpha_j)\Big]\alpha_i \le \frac{2B\epsilon_1}{T_k}.$$

Now for any given $\epsilon > 0$, take $\epsilon_1 = T_k\epsilon/(2B)$. It follows that

$$\begin{aligned}
P\bigg(2B\sum_{i=m+1}^{k}\Big[\prod_{j=i+1}^{k}(1-\alpha_j)\Big]\alpha_i\frac{\mathcal{L}_1+\mathcal{L}_2}{T_i}\max_{\mathbf{x}\in\mathcal{X}} d(\mathbf{x},\Lambda_{\ell_i})\geq\epsilon\bigg) &\leq P\big(\mathcal{A}_m^c\big) \\
&\leq n^d\big(1-\lambda/n^d\big)^{\ell_m} \\
&\leq n^d\exp\big(-\lambda\ell_m/n^d\big) \\
&\leq \bigg(\frac{2\sqrt{d}L(\mathcal{L}_1+\mathcal{L}_2)B}{\epsilon T_k}\bigg)^d \exp\bigg(-\frac{\lambda\epsilon^d T_k^d\lfloor m^{\gamma_3}\rfloor}{(2\sqrt{d}L(\mathcal{L}_1+\mathcal{L}_2)B)^d}\bigg),
\end{aligned}$$

where the last step follows from $\ell_m = \lfloor m^{\gamma_3}\rfloor$ (A8). Hence, the proof is completed by setting $c_1 = \big(2\sqrt{d}L(\mathcal{L}_1+\mathcal{L}_2)B\big)^d$ and $c_2 = \lambda/c_1$. $\square$

F. Proofs of Lemmas in Section 4.1

Proof of Lemma 4 For a given sampled solution $\mathbf{x}_l \in \Lambda_{\ell_k}$, define an indicator function $J_i(\mathbf{x}_l) = 1 - I_i(\mathbf{x}_l)$. It follows that

$$\mu_i(\mathbf{x}_l) := E[J_i(\mathbf{x}_l)|\mathcal{F}_{i-1}] = P\big(\mathbf{x}_i \notin B(\mathbf{x}_l, r_i)|\mathcal{F}_{i-1}\big) \leq 1 - \lambda\frac{\mathcal{V}(B(\mathbf{x}_l, r_i))}{\mathcal{V}(\mathcal{X})} = 1 - 2\kappa r_i^d \leq 1 - 2\kappa r_k^d$$

for some constant $\kappa > 0$, where $\mathcal{V}(A)$ is the volume of a set $A \subseteq \Re^d$, and the last step follows from A7. Next, let $Y_k(\mathbf{x}_l) = \sum_{i=l+1}^k \big(J_i(\mathbf{x}_l) - \mu_i(\mathbf{x}_l)\big)$. Clearly, the sequence $\{Y_k(\mathbf{x}_l),\ k \geq l+1\}$ forms a martingale. Thus, by noting that $N_k(\mathbf{x}_l) = k - l - \sum_{i=l+1}^k J_i(\mathbf{x}_l)$, we have

$$\begin{aligned}
P\big(N_k(\mathbf{x}_l) \leq \rho_k\big) &= P\Big(\sum_{i=l+1}^{k} J_i(\mathbf{x}_l) \geq k - l - \rho_k\Big) \\
&= P\Big(Y_k(\mathbf{x}_l) \geq k - l - \rho_k - \sum_{i=l+1}^{k}\mu_i(\mathbf{x}_l)\Big) \\
&\leq P\big(Y_k(\mathbf{x}_l) \geq k - l - \rho_k - (k-l)(1-2\kappa r_k^d)\big) \\
&\leq P\big(Y_k(\mathbf{x}_l) \geq 2\kappa(k-l)r_k^d - \rho_k\big) \\
&\leq P\big(Y_k(\mathbf{x}_l) \geq \rho_k\big) \\
&\leq e^{-\frac{\rho_k^2}{2k}},
\end{aligned}$$

where we have applied Azuma's inequality (Azuma 1967) in the last inequality. $\square$

Proof of Lemma 5 For a sampled solution $\mathbf{x}_l \in \Lambda_{\ell_k}$, it is easy to observe that both $H_l(\mathbf{x}_l)$ and $H(\mathbf{x}_l)$ are bounded. Without loss of generality, we assume that $H(\mathbf{x}_l)$ has the same bounding constant as $H_l(\mathbf{x}_l)$. Then, due to the projection at Step 2, the initial estimation error at $\mathbf{x}_l$ satisfies $|\varepsilon_l(\mathbf{x}_l)| = |H_l(\mathbf{x}_l) - H(\mathbf{x}_l)| \le 2\mathcal{H}$. Let ρ_k be defined as in Lemma 4. Therefore, for any given $\epsilon > 0$, we have

$$
\begin{aligned}
& P\Big(\Big|\prod_{i=l+1}^{k}\big(1-\beta_i(\mathbf{x}_l)I_i(\mathbf{x}_l)\big)\cdot\varepsilon_l(\mathbf{x}_l)\Big| \ge \epsilon T_k\Big) \le P\Big(\prod_{i=l+1}^{k}\big(1-\beta_i(\mathbf{x}_l)I_i(\mathbf{x}_l)\big) \ge \frac{\epsilon T_k}{2\mathcal{H}}\Big)\\
\le & \sum_{m=1}^{k-l} P\Big(\prod_{i=l+1}^{k}\big(1-\beta_i(\mathbf{x}_l)I_i(\mathbf{x}_l)\big) \ge \frac{\epsilon T_k}{2\mathcal{H}},\, N_k(\mathbf{x}_l)=m\Big) + P\big(N_k(\mathbf{x}_l)=0\big)\\
\le & \sum_{m=1}^{k-l} P\Big(\prod_{i=1}^{N_k(\mathbf{x}_l)}\big(1-f(i)\big) \ge \frac{\epsilon T_k}{2\mathcal{H}},\, N_k(\mathbf{x}_l)=m\Big) + P\big(N_k(\mathbf{x}_l)=0\big)\\
\le & \sum_{m=\lfloor\rho_k\rfloor+1}^{k-l} P\Big(\prod_{i=1}^{m}\big(1-f(i)\big) \ge \frac{\epsilon T_k}{2\mathcal{H}}\Big) + P\big(N_k(\mathbf{x}_l)\le\lfloor\rho_k\rfloor\big)\\
\le & \sum_{m=\lfloor\rho_k\rfloor+1}^{k-l} P\Big(\prod_{i=1}^{m}\Big(1-\frac{1}{i+1}\Big) \ge \frac{\epsilon T_k}{2\mathcal{H}}\Big) + P\big(N_k(\mathbf{x}_l)\le\rho_k\big) \qquad \text{(by Assumption A6)}\\
\le & \sum_{m=\lfloor\rho_k\rfloor+1}^{k-l} P\Big(\frac{1}{m+1} \ge \frac{\epsilon T_k}{2\mathcal{H}}\Big) + e^{-\frac{\rho_k^2}{2k}} \qquad \text{(by Lemma 4)}\\
\le & \sum_{m=\lfloor\rho_k\rfloor+1}^{k-l} P\Big(\frac{1}{m} \ge \frac{\epsilon T_k}{2\mathcal{H}}\Big) + e^{-\frac{\rho_k^2}{2k}}\\
\le & \big(k-l-\lfloor\rho_k\rfloor\big)P\Big(\frac{1}{\rho_k T_k} \ge \frac{\epsilon}{2\mathcal{H}}\Big) + e^{-\frac{\rho_k^2}{2k}}.
\end{aligned}
$$

It can be easily seen from A7 and A8 that $\rho_k = \Omega\big(k^{\frac{1}{2}}\big)$. Therefore, we have $\rho_k T_k = \Omega\big(k^{\frac{1}{4}}\big)$ from A6, which implies $\rho_k T_k \to \infty$, and thus there exists an integer $N_1 > 0$ such that $\frac{1}{\rho_k T_k} < \frac{\epsilon}{2\mathcal{H}}$ for all $k \ge N_1$. Thus, the desired result follows by the fact that the probability term in the last step becomes identically zero for all $k \ge N_1$. $\square$

Proof of Lemma 6 For a sampled solution $\mathbf{x}_l \in \Lambda_{\ell_k}$, let $l+1 \le \tau_1 < \tau_2 < \cdots < \tau_{N_k(\mathbf{x}_l)} \le k$ be an increasing sequence of random times at which the events $\{I_j(\mathbf{x}_l)=1\}$, $j = l+1,\ldots,k$, occur. Then, for any given $\epsilon > 0$, we have

$$
P\Big(\sum_{i=l+1}^{k}\Big[\prod_{j=i+1}^{k}\big(1-\beta_j(\mathbf{x}_l)I_j(\mathbf{x}_l)\big)\Big]\beta_i(\mathbf{x}_l)I_i(\mathbf{x}_l)z(\mathbf{x}_i) \ge \epsilon T_k\Big)
$$

$$
\begin{aligned}
&= \sum_{m=0}^{k-l} P\Big(\sum_{i=l+1}^{k} \Big[\prod_{j=i+1}^{k} \Big(1-\beta_j(\mathbf{x}_l) I_j(\mathbf{x}_l)\Big)\Big] \beta_i(\mathbf{x}_l) I_i(\mathbf{x}_l) z(\mathbf{x}_i) \geq \epsilon T_k,\ N_k(\mathbf{x}_l) = m \Big) \\
&= \sum_{m=1}^{k-l} P\Big(\sum_{i=1}^{N_k(\mathbf{x}_l)} \Big[\prod_{j=i+1}^{N_k(\mathbf{x}_l)} \big(1-f(j)\big)\Big] f(i) z(\mathbf{x}_{\tau_i}) \geq \epsilon T_k,\ N_k(\mathbf{x}_l) = m \Big) \\
&= \sum_{m=1}^{k-l} P\Big(\prod_{j=1}^{N_k(\mathbf{x}_l)} \big(1-f(j)\big) \cdot \sum_{i=1}^{N_k(\mathbf{x}_l)} \frac{f(i) z(\mathbf{x}_{\tau_i})}{\prod_{j=1}^{i} \big(1-f(j)\big)} \geq \epsilon T_k,\ N_k(\mathbf{x}_l) = m \Big) \\
&\leq \sum_{m=\lfloor \rho_k \rfloor + 1}^{k-l} P\Big(\prod_{j=1}^{m} \big(1-f(j)\big) \cdot \sum_{i=1}^{m} \frac{f(i) z(\mathbf{x}_{\tau_i})}{\prod_{j=1}^{i} \big(1-f(j)\big)} \geq \epsilon T_k \Big) + P\big(N_k(\mathbf{x}_l) \leq \lfloor \rho_k \rfloor\big) \\
&\leq \sum_{m=\lfloor \rho_k \rfloor + 1}^{k-l} P\Big(\sum_{i=1}^{m} \frac{f(i) z(\mathbf{x}_{\tau_i})}{\prod_{j=1}^{i} \big(1-f(j)\big)} \geq \frac{\epsilon T_k}{\prod_{j=1}^{m} \big(1-f(j)\big)} \Big) + e^{-\frac{\rho_k^2}{2k}} \qquad \text{(by Lemma 4)} \\
&\leq \sum_{m=\lfloor \rho_k \rfloor + 1}^{k-l} P\left(\exp\left(\sum_{i=1}^{m} \frac{f(i) z(\mathbf{x}_{\tau_i})}{\prod_{j=1}^{i} \big(1-f(j)\big)} \cdot t \right) \geq \exp\left(\frac{\epsilon T_k}{\prod_{j=1}^{m} \big(1-f(j)\big)} \cdot t \right) \right) + e^{-\frac{\rho_k^2}{2k}} \\
&\leq \sum_{m=\lfloor \rho_k \rfloor + 1}^{k-l} E\left[\exp\left(\sum_{i=1}^{m} \frac{f(i) z(\mathbf{x}_{\tau_i})}{\prod_{j=1}^{i} \big(1-f(j)\big)} \cdot t \right) \right] \cdot \exp\left(- \frac{\epsilon T_k}{\prod_{j=1}^{m} \big(1-f(j)\big)} \cdot t \right) + e^{-\frac{\rho_k^2}{2k}},
\end{aligned}
\tag{A-10}
$$

where t is any positive real number, and we have applied Markov's inequality for the last step.

Next, we use the sub-Gaussian(σ^2) property of $z(\mathbf{x}_k)$ (A2) to bound the expectation term in (A-10). Let τ be an integer such that $l+m \leq \tau \leq k$. Since $\{\tau_m = \tau\} = \big\{ \sum_{j=l+1}^{\tau-1} I_j(\mathbf{x}_l) = m-1 \big\} \cap \big\{ \mathbf{x}_\tau \in B(\mathbf{x}_l, r_\tau) \big\}$, we have $\{\tau_m = \tau\} \in \mathcal{F}_\tau$. Therefore,

$$
\begin{aligned}
& E\left[\exp\left(\sum_{i=1}^{m} \frac{f(i) z(\mathbf{x}_{\tau_i})}{\prod_{j=1}^{i} \big(1-f(j)\big)} \cdot t \right) \right] = \sum_{\tau=l+m}^{k} E\left[\exp\left(\sum_{i=1}^{m} \frac{f(i) z(\mathbf{x}_{\tau_i})}{\prod_{j=1}^{i} \big(1-f(j)\big)} \cdot t \right) I\{\tau_m = \tau\} \right] \\
&= \sum_{\tau=l+m}^{k} E\left[\exp\left(\sum_{i=1}^{m-1} \frac{f(i) z(\mathbf{x}_{\tau_i})}{\prod_{j=1}^{i} \big(1-f(j)\big)} \cdot t \right) I\{\tau_m = \tau\} E\left[\exp\left(\frac{f(m) z(\mathbf{x}_\tau)}{\prod_{j=1}^{m} \big(1-f(j)\big)} \cdot t \right) \middle| \mathcal{F}_\tau \right] \right] \\
&\leq \exp\left(\frac{1}{2} \sigma^2 t^2 \frac{f^2(m)}{\prod_{j=1}^{m} \big(1-f(j)\big)^2} \right) \cdot \sum_{\tau=l+m}^{k} E\left[\exp\left(\sum_{i=1}^{m-1} \frac{f(i) z(\mathbf{x}_{\tau_i})}{\prod_{j=1}^{i} \big(1-f(j)\big)} \cdot t \right) I\{\tau_m = \tau\} \right] \\
&= \exp\left(\frac{1}{2} \sigma^2 t^2 \frac{f^2(m)}{\prod_{j=1}^{m} \big(1-f(j)\big)^2} \right) \cdot E\left[\exp\left(\sum_{i=1}^{m-1} \frac{f(i) z(\mathbf{x}_{\tau_i})}{\prod_{j=1}^{i} \big(1-f(j)\big)} \cdot t \right) \right] \\
&\quad \cdots
\end{aligned}
$$

$$\le \exp\left(\frac{1}{2}\sigma^2 t^2 \cdot \sum_{i=1}^{m} \frac{f^2(i)}{\prod_{j=1}^{i}\big(1-f(j)\big)^2}\right).$$

Substituting the above into (A-10) and optimizing the resulting bounds with respect to t, we have

$$\begin{aligned}
(A-10) &\le \sum_{m=\lfloor\rho_k\rfloor+1}^{k-l} \exp\left(\frac{1}{2}\sigma^2 t^2 \cdot \sum_{i=1}^{m} \frac{f^2(i)}{\prod_{j=1}^{i}\big(1-f(j)\big)^2} - \frac{\epsilon T_k}{\prod_{j=1}^{m}(1-f(j))} \cdot t\right) + e^{-\frac{\rho_k^2}{2k}} \\
&\le \sum_{m=\lfloor\rho_k\rfloor+1}^{k-l} \exp\left(-\frac{\epsilon^2 T_k^2}{2\sigma^2 \sum_{i=1}^{m}\Big[\prod_{j=i+1}^{m}\big(1-f(j)\big)^2\Big] f^2(i)}\right) + e^{-\frac{\rho_k^2}{2k}} \\
&\le \sum_{m=\lfloor\rho_k\rfloor+1}^{k-l} \exp\left(-\frac{\epsilon^2 T_k^2}{2\sigma^2 f(m)}\right) + e^{-\frac{\rho_k^2}{2k}} \qquad \text{(by Proposition B.3)} \\
&\le k e^{-\frac{\epsilon^2 T_k^2}{2\sigma^2 f(\rho_k)}} + e^{-\frac{\rho_k^2}{2k}}.
\end{aligned}$$

A similar argument can be used to show that

$$P\left(\sum_{i=l+1}^{k}\Big[\prod_{j=i+1}^{k}\big(1-\beta_j(\mathbf{x}_l)I_j(\mathbf{x}_l)\big)\Big]\beta_i(\mathbf{x}_l)I_i(\mathbf{x}_l)z(\mathbf{x}_i) \le -\epsilon T_k\right) \le k e^{-\frac{\epsilon^2 T_k^2}{2\sigma^2 f(\rho_k)}} + e^{-\frac{\rho_k^2}{2k}}.$$

Finally, combining the above two cases yields the claimed result. □

Proof of Lemma 7 Similar to the proof of Lemma 6, for a sampled solution $\mathbf{x}_l \in \Lambda_{\ell_k}$, let $l+1 \le \tau_1 < \tau_2 < \cdots < \tau_{N_k(\mathbf{x}_l)} \le k$ be the sequence of random times at which the events $\{I_j(\mathbf{x}_l)=1\},\ j=l+1,\ldots,k$, occur. Because $H(\cdot)$ is Lipschitz continuous and r_k is non-increasing, we have $|B_{\tau_i}(\mathbf{x}_l)| \le \mathcal{L}_1 r(\tau_i) \le \mathcal{L}_1 r(l+i) \le \mathcal{L}_1 r(i)$. Therefore, for any given $\epsilon > 0$, we have

$$\begin{aligned}
&P\left(\Big|\sum_{i=l+1}^{k}\Big[\prod_{j=i+1}^{k}\big(1-\beta_j(\mathbf{x}_l)I_j(\mathbf{x}_l)\big)\Big]\beta_i(\mathbf{x}_l)I_i(\mathbf{x}_l)B_i(\mathbf{x}_l)\Big| \ge \epsilon T_k\right) \\
&\le \sum_{m=1}^{k-l} P\left(\sum_{i=1}^{N_k(\mathbf{x}_l)}\Big[\prod_{j=i+1}^{N_k(\mathbf{x}_l)}\big(1-f(j)\big)\Big] f(i)|B_{\tau_i}(\mathbf{x}_l)| \ge \epsilon T_k,\ N_k(\mathbf{x}_l)=m\right) \\
&\le \sum_{m=\lfloor\rho_k\rfloor+1}^{k-l} P\left(\sum_{i=1}^{m}\Big[\prod_{j=i+1}^{m}\big(1-f(j)\big)\Big] f(i) r(i) \ge \frac{\epsilon T_k}{\mathcal{L}_1}\right) + P\big(N_k(\mathbf{x}_l) \le \lfloor\rho_k\rfloor\big) \\
&\le \sum_{m=\lfloor\rho_k\rfloor+1}^{k-l} P\left(\sum_{i=1}^{m}\Big[\prod_{j=i+1}^{m}\big(1-f(j)\big)\Big] f(i) r(i) \ge \frac{\epsilon T_k}{\mathcal{L}_1}\right) + e^{-\frac{\rho_k^2}{2k}}. \qquad \text{(by Lemma 4)} \qquad \text{(A-11)}
\end{aligned}$$

Next, we investigate the probability term in (A-11). For any integer m, we have

$$
\begin{aligned}
&P\Big(\sum_{i=1}^{m}\Big[\prod_{j=i+1}^{m}\big(1-f(j)\big)\Big]f(i)r(i)\geq\frac{\epsilon T_k}{\mathcal{L}_1}\Big)\\
&\leq P\Big(\sum_{i=1}^{\lfloor\sqrt{m}\rfloor}\Big[\prod_{j=i+1}^{m}\big(1-f(j)\big)\Big]f(i)r(i)\geq\frac{\epsilon T_k}{2\mathcal{L}_1}\Big)+P\Big(\sum_{i=\lfloor\sqrt{m}\rfloor+1}^{m}\Big[\prod_{j=i+1}^{m}\big(1-f(j)\big)\Big]f(i)r(i)\geq\frac{\epsilon T_k}{2\mathcal{L}_1}\Big)\\
&\leq P\Big(r_1\sum_{i=1}^{\lfloor\sqrt{m}\rfloor}\Big[\prod_{j=i+1}^{m}\big(1-f(j)\big)\Big]f(i)\geq\frac{\epsilon T_k}{2\mathcal{L}_1}\Big)+P\Big(r\big(\sqrt{m}\big)\sum_{i=\lfloor\sqrt{m}\rfloor+1}^{m}\Big[\prod_{j=i+1}^{m}\big(1-f(j)\big)\Big]f(i)\geq\frac{\epsilon T_k}{2\mathcal{L}_1}\Big)\\
&\leq P\Big(\prod_{i=\lfloor\sqrt{m}\rfloor+1}^{m}\big(1-f(i)\big)\geq\frac{\epsilon T_k}{2\mathcal{L}_1 r_1}\Big)+P\Big(r\big(\sqrt{m}\big)\geq\frac{\epsilon T_k}{2\mathcal{L}_1}\Big)\\
&\leq P\Big(\prod_{i=\lfloor\sqrt{m}\rfloor+1}^{m}\Big(1-\frac{1}{i+1}\Big)\geq\frac{\epsilon T_k}{2\mathcal{L}_1 r_1}\Big)+P\Big(r\big(\sqrt{m}\big)\geq\frac{\epsilon T_k}{2\mathcal{L}_1}\Big)\\
&\leq P\Big(\frac{\sqrt{m}+1}{m+1}\geq\frac{\epsilon T_k}{2\mathcal{L}_1 r_1}\Big)+P\Big(r\big(\sqrt{m}\big)\geq\frac{\epsilon T_k}{2\mathcal{L}_1}\Big),
\end{aligned}
$$

where we have applied Propositions B.2 and B.1 in the third inequality. Substituting the above bound into (A-11) and noticing that both $\frac{\sqrt{m}+1}{m+1}$ and $r(\sqrt{m})$ are non-increasing in m and $m\geq\lfloor\rho_k\rfloor+1\geq\rho_k$. we obtain

$$
(A-11)\leq\big(k-l-\lfloor\rho_k\rfloor\big)\left(P\Big(\frac{\sqrt{\rho_k}+1}{(\rho_k+1)T_k}\geq\frac{\epsilon}{2\mathcal{L}_1 r_1}\Big)+P\Big(\frac{r\big(\sqrt{\rho_k}\big)}{T_k}\geq\frac{\epsilon}{2\mathcal{L}_1}\Big)\right)+e^{-\frac{\rho_k^2}{2k}}. \quad \text{(A-12)}
$$

Recall that $\rho_k=\kappa(k-\ell_k)r_k^d$ for some constant $\kappa>0$. It thus follows from A6, A7, and A8 that there exists a constant $c>0$ such that

$$
\sqrt{\rho_k}T_k=(\kappa(k-\ell_k)r_k^d)^{1/2}T_k\geq ck^{\frac{1}{2}}k^{-\frac{\gamma_2 d}{2}}k^{-\frac{\gamma_1}{4}}=ck^{\frac{2-\gamma_1-2\gamma_2 d}{4}}
$$

for k sufficiently large. Because $\gamma_1\in(0,1]$ and $\gamma_2\in(0,\frac{1}{2d})$, this implies that $\frac{\sqrt{\rho_k}+1}{(\rho_k+1)T_k}\leq\frac{1+1/\sqrt{\rho_k}}{\sqrt{\rho_k}T_k}\to 0$ as $k\to\infty$. On the other hand, since $\sqrt{\rho_k}=(\kappa(k-\ell_k)r_k^d)^{1/2}\geq c'k^{\frac{1-\gamma_2 d}{2}}>k^{\frac{1}{4}}$ for some constant $c'>0$ and k sufficiently large, A7 implies that $\frac{r(\sqrt{\rho_k})}{T_k}\leq\frac{r(k^{1/4})}{T_k}\to 0$ as $k\to\infty$. Consequently, we conclude that there exists an integer $N_2>0$ such that $\frac{\sqrt{\rho_k}+1}{(\rho_k+1)T_k}<\frac{\epsilon}{2\mathcal{L}_1 r_1}$ and $\frac{r(\sqrt{\rho_k})}{T_k}<\frac{\epsilon}{2\mathcal{L}_1}$ for all $k\geq N_2$, i.e., both probability terms in (A-12) become identically zero for all $k\geq N_2$. This completes the proof of the lemma. $\square$

Proof of Lemma 8 For any given $\epsilon>0$, Lemmas 5 and 7 imply that there exist integers $N_1'>0$ and $N_2'>0$ such that

$$
P\Big(\Big|\prod_{i=l+1}^{k}\big(1-\beta_i(\mathbf{x}_l)I_i(\mathbf{x}_l)\big)\cdot\epsilon_l(\mathbf{x}_l)\Big|\geq\frac{\epsilon T_k}{3}\Big)\leq e^{-\frac{\rho_k^2}{2k}},\ \forall k\geq N_1',
$$

and

$$P\Big(\Big|\sum_{i=l+1}^{k}\Big[\prod_{j=i+1}^{k}\big(1-\beta_j(\mathbf{x}_l)I_j(\mathbf{x}_l)\big)\Big]\beta_i(\mathbf{x}_l)I_i(\mathbf{x}_l)B_i(\mathbf{x}_l)\Big|\geq\frac{\epsilon T_k}{3}\Big)\leq e^{-\frac{\rho_k^2}{2k}},\ \forall k\geq N_2'.$$

Therefore, for all $k\geq N_3=\max\{N_1',N_2'\}$, we have

$$\begin{aligned}
P\Big(\max_{\mathbf{x}_l\in\Lambda_{\ell_k}}|H_k(\mathbf{x}_l)-H(\mathbf{x}_l)|\geq\epsilon T_k\Big)&=P\Big(\cup_{\mathbf{x}_l\in\Lambda_{\ell_k}}\big\{|H_k(\mathbf{x}_l)-H(\mathbf{x}_l)|\geq\epsilon T_k\big\}\Big)\\
&\leq|\Lambda_{\ell_k}|\max_{\mathbf{x}_l\in\Lambda_{\ell_k}}P\Big(|H_k(\mathbf{x}_l)-H(\mathbf{x}_l)|\geq\epsilon T_k\Big)\\
&\leq\lfloor k^{\gamma_3}\rfloor\max_{\mathbf{x}_l\in\Lambda_{\ell_k}}\Bigg(P\Big(\Big|\prod_{i=l+1}^{k}\big(1-\beta_i(\mathbf{x}_l)I_i(\mathbf{x}_l)\big)\cdot\epsilon_l(\mathbf{x}_l)\Big|\geq\frac{\epsilon T_k}{3}\Big)\\
&\qquad+P\Big(\Big|\sum_{i=l+1}^{k}\Big[\prod_{j=i+1}^{k}\big(1-\beta_j(\mathbf{x}_l)I_j(\mathbf{x}_l)\big)\Big]\beta_i(\mathbf{x}_l)I_i(\mathbf{x}_l)z(\mathbf{x}_i)\Big|\geq\frac{\epsilon T_k}{3}\Big)\\
&\qquad+P\Big(\Big|\sum_{i=l+1}^{k}\Big[\prod_{j=i+1}^{k}\big(1-\beta_j(\mathbf{x}_l)I_j(\mathbf{x}_l)\big)\Big]\beta_i(\mathbf{x}_l)I_i(\mathbf{x}_l)B_i(\mathbf{x}_l)\Big|\geq\frac{\epsilon T_k}{3}\Big)\Bigg)\\
&\leq 2k^{\gamma_3+1}e^{-\frac{\epsilon^2T_k^2}{18\sigma^2f(\rho_k)}}+4k^{\gamma_3}e^{-\frac{\rho_k^2}{2k}},
\end{aligned}$$

where the first term in the last step is obtained by replacing ϵ in Lemma 6 with $\epsilon/3$. $\square$

Proof of Corollary 1 Fix an $\epsilon>0$, let $\epsilon_1=\frac{\epsilon}{2B}$, and define $\mathcal{B}_i=\big\{\max_{\mathbf{x}_l\in\Lambda_{\ell_i}}|H_i(\mathbf{x}_l)-H(\mathbf{x}_l)|<\epsilon_1T_i\big\}$. Clearly, if $\mathcal{B}_{m+1}\cap\ldots\cap\mathcal{B}_k$ occurs, then by Proposition B.1,

$$\sum_{i=m+1}^{k}\Big[\prod_{j=i+1}^{k}(1-\alpha_j)\Big]\alpha_i\frac{2B}{T_i}\max_{\mathbf{x}_l\in\Lambda_{\ell_i}}|H_i(\mathbf{x}_l)-H(\mathbf{x}_l)|<2B\epsilon_1=\epsilon.$$

Thus, we know from Lemma 8 that there exists an $N_3'>0$ such that for all $m\geq N_3'$,

$$\begin{aligned}
P\Big(\sum_{i=m+1}^{k}\Big[\prod_{j=i+1}^{k}(1-\alpha_j)\Big]\alpha_i\frac{2B}{T_i}\max_{\mathbf{x}_l\in\Lambda_{\ell_i}}|H_i(\mathbf{x}_l)-H(\mathbf{x}_l)|\geq\epsilon\Big)&\leq P\Big(\big(\mathcal{B}_{m+1}\cap\ldots\cap\mathcal{B}_k\big)^c\Big)\\
&\leq\sum_{i=m+1}^{k}P\big(\mathcal{B}_i^c\big)\\
&\leq(k-m)\Big(2k^{\gamma_3+1}e^{-\frac{\epsilon_1^2T_k^2}{18\sigma^2f(\rho_m)}}+4k^{\gamma_3}e^{-\frac{\rho_m^2}{2k}}\Big)\\
&\leq 2k^{\gamma_3+2}e^{-\frac{c_3\epsilon^2T_k^2}{f(\rho_m)}}+4k^{\gamma_3+1}e^{-\frac{\rho_m^2}{2k}},
\end{aligned}$$

where $c_3:=\frac{1}{72\sigma^2B^2}$. $\square$

G. Proof of Theorem 1

For any given $\epsilon > 0$, let $\epsilon' = \epsilon/5$ and consider the event $\{\Delta_{k+1} \geq \epsilon\}$. From (9a)-(9b), we have that

$$P\big(||\Delta_{k+1}|| \geq \epsilon\big) \leq P\bigg(\prod_{j=1}^{k}(1-\alpha_j)\cdot||\Delta_1|| \geq \epsilon'\bigg) + P\bigg(\sum_{i=1}^{k}\Big[\prod_{j=i+1}^{k}(1-\alpha_j)\Big]\alpha_i||E_{g_i}[\Gamma(X)] - \Gamma(\mathbf{x}^*)|| \geq \epsilon'\bigg)$$
$$+ P\bigg(\sum_{i=1}^{k}\Big[\prod_{j=i+1}^{k}(1-\alpha_j)\Big]\alpha_i||E_{\hat{g}_i}[\Gamma(X)] - E_{g_i}[\Gamma(X)]|| \geq 3\epsilon'\bigg). \tag{A-13}$$

By Lemmas 1 and 2, there must be an ϵ-dependent integer $\mathcal{N}_1 > 0$ so that the first two probability terms on the right-hand side of (A-13) can be made identically zero for all $k \geq \mathcal{N}_1$. Consequently, let $\zeta \in (0,1)$ be given as in A4, we obtain that for all $k \geq \mathcal{N}_1$

$$\begin{aligned} P\big(||\Delta_{k+1}|| \geq \epsilon\big) &\leq P\bigg(\sum_{i=1}^{k}\Big[\prod_{j=i+1}^{k}(1-\alpha_j)\Big]\alpha_i||E_{\hat{g}_i}[\Gamma(X)] - E_{g_i}[\Gamma(X)]|| \geq 3\epsilon'\bigg) \\ &\leq P\bigg(\sum_{i=1}^{\lfloor\zeta k\rfloor}\Big[\prod_{j=i+1}^{k}(1-\alpha_j)\Big]\alpha_i||E_{\hat{g}_i}[\Gamma(X)] - E_{g_i}[\Gamma(X)]|| \geq \epsilon'\bigg) \\ &\quad + P\bigg(\sum_{i=\lfloor\zeta k\rfloor+1}^{k}\Big[\prod_{j=i+1}^{k}(1-\alpha_j)\Big]\alpha_i||E_{\hat{g}_i}[\Gamma(X)] - E_{g_i}[\Gamma(X)]|| \geq 2\epsilon'\bigg). \end{aligned} \tag{A-14}$$

Note that since $\|\Gamma(\mathbf{x})\|$ is bounded by B on $\mathcal{X}$, we have from Proposition B.2 that

$$\begin{aligned} \sum_{i=1}^{\lfloor\zeta k\rfloor}\Big[\prod_{j=i+1}^{k}(1-\alpha_j)\Big]\alpha_i||E_{\hat{g}_i}[\Gamma(X)] - E_{g_i}[\Gamma(X)]|| &\leq 2B\sum_{i=1}^{\lfloor\zeta k\rfloor}\Big[\prod_{j=i+1}^{k}(1-\alpha_j)\Big]\alpha_i \\ &\leq 2B\prod_{i=\lfloor\zeta k\rfloor+1}^{k}(1-\alpha_i) \\ &\leq 2Be^{-\sum_{i=\lfloor\zeta k\rfloor+1}^{k}\alpha_i}, \end{aligned}$$

which tends to zero as $k \to \infty$ by A4. This indicates the existence of another ϵ-dependent integer $\mathcal{N}_2 > 0$ so that for all $k \geq \max\{\mathcal{N}_1, \mathcal{N}_2\}$ (A-14) can be reduced to

$$\begin{aligned} P\big(||\Delta_{k+1}|| \geq \epsilon\big) &\leq P\bigg(\sum_{i=\lfloor\zeta k\rfloor+1}^{k}\Big[\prod_{j=i+1}^{k}(1-\alpha_j)\Big]\alpha_i||E_{\hat{g}_i}[\Gamma(X)] - E_{g_i}[\Gamma(X)]|| \geq 2\epsilon'\bigg) \\ &\leq P\bigg(\sum_{i=\lfloor\zeta k\rfloor+1}^{k}\Big[\prod_{j=i+1}^{k}(1-\alpha_j)\Big]\alpha_i\frac{2B(\mathcal{L}_1+\mathcal{L}_2)}{T_i}\max_{\mathbf{x}\in\mathcal{X}} d(\mathbf{x},\Lambda_{\ell_i}) \geq \epsilon'\bigg) \end{aligned}$$

$$+ P\bigg(\sum_{i=\lfloor \zeta k \rfloor +1}^{k} \Big[\prod_{j=i+1}^{k} (1-\alpha_j) \Big] \alpha_i \frac{2B}{T_i} \max_{\mathbf{x}_l \in \Lambda_{\ell_i}} |H_i(\mathbf{x}_l) - H(\mathbf{x}_l)| \geq \epsilon' \bigg),$$

where the last step is a result of inequality (10). Hence, a straightforward application of Lemma 3 and Corollary 1 (with $m = \lfloor \zeta k \rfloor$) yields the following bound on the probability:

$$P\big(||\Delta_{k+1}|| \geq \epsilon\big) \leq \frac{c_1}{\epsilon^d T_k^d} e^{-c_2 \epsilon^d T_k^d \lfloor m^{\gamma_3} \rfloor} + 2k^{\gamma_3+2} e^{-\frac{c_3 \epsilon^2 T_k^2}{f(\rho_m)}} + 4k^{\gamma_3+1} e^{-\frac{\rho_m^2}{2k}},$$

for all $k \geq \max\{\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3\}$, where $\mathcal{N}_3 = N_3'/\zeta$ and N_3' is given in Corollary 1.

Finally, by our choices of $f(k)$, r_k, and ℓ_k (Assumptions A5, A7, and A8), it is a simple exercise to verify that when $m = \lfloor \zeta k \rfloor$, $\rho_m = \kappa(m - \ell_m) r_m^d = \Omega(k^{1-\gamma_2 d})$ and $f(\rho_m) = 1/(1+\rho_m)^{\gamma_1} = O(k^{-\gamma_1(1-\gamma_2 d)})$. Therefore, through a suitable choice of the integer $\mathcal{N} > 0$ and constants $C_1, C_2, C_3, C_4 > 0$, we obtain the bound shown in Theorem 1. $\square$

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