Appendices

A. Proofs

Proof of Lemma 1

(a) We first prove $\phi_t(u, v_{[t-1]}; s)$ is increasing convex in u and decreasing in s by induction. At $t = T$, we have

$$\phi_T(u, v_{[T-1]}; s) = \mathbb{E}_{v_T}\{u \vee \omega_T(v_{[T]})\} - s,$$

which is increasing convex in u and decreasing in s. Suppose $\phi_{t+1}(u, v_{[t]}; s)$ is increasing convex in u and decreasing in s, then

$$\phi_t(u, v_{[t-1]}; s) = \mathbb{E}_{v_t}\{u \vee \omega_t(v_{[t]}) \vee \phi_{t+1}(u, v_{[t]}; s)\} - s,$$

which is again increasing in u and decreasing in s by the induction hypothesis.

Next, we prove $\phi_t(u, v_{[t-1]}) - u$ is decreasing in u by induction. At $t = T$, we have

$$\phi_T(u, v_{[T-1]}) - u = \mathbb{E}_{v_T}\{u \vee \omega_T(v_{[T]})\} - u - s = \mathbb{E}_{v_T}\{(\omega_T(v_{[T]}) - u)^+\} - s,$$

which is decreasing in u. Suppose $\phi_{t+1}(u, v_{[t]}) - u$ is decreasing in u, then

$$\phi_t(u, v_{[t-1]}) - u = \mathbb{E}_{v_t}\{u \vee \omega_t(v_{[t]}) \vee \phi_{t+1}(u, v_{[t]})\} - u - s = \mathbb{E}_{v_t}\{0 \vee (\omega_t(v_{[t]}) - u) \vee (\phi_{t+1}(u, v_{[t]}) - u)\} - s,$$

which is also decreasing in u by the induction hypothesis.

Lastly, we show $\phi_t(u, v_{[t-1]}; T)$ is increasing in T by induction. At $t = T$, we have

$$\begin{aligned}\phi_T(u, v_{[T-1]}; T+1) &= \mathbb{E}_{v_T}\{u \vee \omega_T(v_{[T]}) \vee \phi_{T+1}(u, v_{[T]}; T+1)\} - s \\ &\geq \mathbb{E}_{v_T}\{u \vee \omega_T(v_{[T]})\} - s = \phi_T(u, v_{[T-1]}; T),\end{aligned}$$

where the last equality holds by the definition of $\phi_T(u, v_{[T-1]}; T)$. Suppose $\phi_{t+1}(u, v_{[t]}; T+1) \geq \phi_{t+1}(u, v_{[t]}; T)$, then

$$\begin{aligned}\phi_t(u, v_{[t-1]}; T+1) &= \mathbb{E}_{v_t}\{u \vee \omega_t(v_{[t]}) \vee \phi_{t+1}(u, v_{[t]}; T+1)\} - s \\ &\geq \mathbb{E}_{v_t}\{u \vee \omega_t(v_{[t]}) \vee \phi_{t+1}(u, v_{[t]}; T)\} - s = \phi_t(u, v_{[t-1]}; T),\end{aligned}$$

where the inequality follows from the induction hypothesis.

(b) We prove via induction. At period $t = T$, because $\phi_T(u, v_{[T-1]}) = \mathbb{E}_{v_T}\{u \vee \omega_T(v_{[T]})\} - s$, then

$$\frac{\partial \phi_T(u, v_{[T-1]})}{\partial u} = \mathbb{E}_{v_T}\{\mathbb{1}_{\{\omega_T(v_{[T]}) < u\}}\} = \xi_T(u, v_{[T-1]}) \leq 1.$$

Suppose $\frac{\partial \phi_{t+1}(u, v_{[t]})}{\partial u} = \xi_{t+1}(u, v_{[t]}) \leq 1$, then

$$\begin{aligned}\frac{\partial \phi_t(u, v_{[t-1]})}{\partial u} &= \mathbb{E}_{v_t}\{\mathbb{1}_{\{u > \omega_t(v_{[t]}) \vee \phi_{t+1}(u, v_{[t]})\}} + \mathbb{1}_{\{\phi_{t+1}(u, v_{[t]}) \geq \omega_t(v_{[t]}) \vee u\}} \tfrac{\partial \phi_{t+1}(u, v_{[t]})}{\partial u}\} \\ &= \mathbb{E}_{v_t}\{\mathbb{1}_{\{u > \omega_t(v_{[t]}) \vee \phi_{t+1}(u, v_{[t]})\}} + \mathbb{1}_{\{\phi_{t+1}(u, v_{[t]}) \geq \omega_t(v_{[t]}) \vee u\}} \xi_{t+1}(u, v_{[t]})\} = \xi_t(u, v_{[t-1]}) \leq 1,\end{aligned}$$

where the second equality and the inequality hold by the induction hypothesis. □

Proof of Lemma 2

(a) We first prove $\phi_t(u, v_{[t-1]}; k)$ is increasing in k by induction. At $t = T$, we have

$$\phi_T(u, v_{[T-1]}; k) = \mathbb{E}_{v_T}\{u \vee \omega_T(v_{[T]}; k)\} - s,$$

which is increasing in k by the definition of $\omega_T(v_{[T]}; k)$. Suppose $\phi_{t+1}(u, v_{[t]}; k)$ is increasing in k, then

$$\phi_t(u, v_{[t-1]}; k) = \mathbb{E}_{v_t}\big\{u \vee \omega_t(v_{[t]}; k) \vee \phi_{t+1}(u, v_{[t]}; k)\big\} - s,$$

is also increasing in k by the definition of $\omega_t(v_{[t]}; k)$ and the induction hypothesis.

(b) We prove $\phi_t(u, v_{[t-1]}) \leq u \vee \omega_{t-1}(v_{[t-1]})$ when $u \geq \check{u}$ by induction. At period $t = T$,

$$\begin{aligned}\phi_T(u, v_{[T-1]}) = \mathbb{E}_{v_T}\{u \vee \omega_T(v_{[T]})\} - s &\leq \mathbb{E}_{v_T}\{u \vee \omega_{T-1}(v_{[T-1]}) \vee v_T\} - s \\ &\leq u \vee \omega_{T-1}(v_{[T-1]}),\end{aligned}$$

where the first inequality comes from the fact that $w_t(v_{[t]}) \leq w_{t-1}(v_{[t-1]}) \vee v_t$ because of recall with finite memory, and the last inequality follows from $\mathbb{E}_v\{(v-u)^+\} \leq s$ for $u \geq \check{u}$. Now suppose $\phi_{t+1}(u, v_{[t]}) \leq u \vee \omega_t(v_{[t]})$ for $u \geq \check{u}$. Then, when $u \geq \check{u}$, we have

$$\begin{aligned}\phi_t(u, v_{[t-1]}) &= \mathbb{E}_{v_t}\{u \vee \omega_t(v_{[t]}) \vee \phi_{t+1}(u, v_{[t]})\} - s \\ &\leq \mathbb{E}_{v_t}\{u \vee \omega_t(v_{[t]}) \vee u \vee \omega_t(v_{[t]})\} - s \\ &= \mathbb{E}_{v_t}\{u \vee \omega_t(v_{[t]})\} - s \\ &\leq \mathbb{E}_{v_t}\{u \vee \omega_{t-1}(v_{[t-1]}) \vee v_t\} - s \leq u \vee \omega_{t-1}(v_{[t-1]}),\end{aligned}$$

where the first inequality holds by the induction hypothesis, the second inequality follows from the fact that $\omega_t(v_{[t]}) \leq \omega_{t-1}(v_{[t-1]}) \vee v_t$ due to recall with finite memory, and the last inequality again follows from $\mathbb{E}_v\{(v-u)^+\} \leq s$ for $u \geq \check{u}$. Note that with $\omega_1(v_{[1]}) = v_1, \phi_2(\check{u}, v_{[1]}) \leq \check{u} \vee v_1$, we have $\phi_1(\check{u}) = \mathbb{E}_{v_1}\{\check{u} \vee v_1 \vee \phi_2(\check{u}, v_{[1]})\} - s = \mathbb{E}_{v_1}\{\check{u} \vee v_1\} - s = \check{u}$.

Next we show $\phi_t(u, v_{[t-1]}) > u$ if $u < \check{u}$. For $t = 1, 2, ..., T$,

$$\phi_t(u, v_{[t-1]}) = \mathbb{E}_{v_t}\{u \vee \omega_t(v_{[t]}) \vee \phi_{t+1}(u, v_{[t]})\} - s \geq \mathbb{E}_{v_t}\{u \vee v_t\} - s > u,$$

where the first inequality holds because $\omega_t(v_{[t]}) \geq v_t$ and the last inequality follows from $\mathbb{E}_v\{(v-u)^+\} > s$ for $u < \check{u}$.

(c) Given any u, let $\check{s}(u) = \mathbb{E}_v[(v-u)^+]$. To prove $\phi_1(z; s)$ is submodular in $(z, s) \in [0, u] \times [0, \check{s}(u)]$, it suffices to show that $\xi_1(z; s)$ is decreasing in s. We use $\check{u}(s)$ to make explicit the relationship between $\check{u}$ and s, that is, $\check{u}(s) = \{u : \mathbb{E}_v[(v-u)^+] - s = 0\}$. For any $[z, s] \in [0, u] \times [0, \check{s}(u)]$, it is clear that

$$\mathbb{E}_v[(v-z)^+] \geq \mathbb{E}_v[(v-u)^+] = \check{s}(u) \geq s,$$

implying that $z \le \check{u}(s)$. By part (b), when $z \le \check{u}(s)$, we have $\phi_t(z, v_{[t-1]}; s) \ge z$ for $t = 1, ..., T$. It follows that

$$\xi_t(z, v_{[t-1]}; s) = \mathbb{E}_{v_t}\left[\mathbb{1}_{\left\{\phi_{t+1}(z, v_{[t]}; s) - \omega_t(v_{[t]}) > 0\right\}} \xi_{t+1}(z, v_{[t]}; s)\right],$$

with $\xi_{T+1}(z, v_{[T]}; s) = 1$. We prove $\xi_t(z, v_{[t-1]}; s)$ is decreasing in s by induction. At period $t = T+1$, $\xi_{T+1}(z, v_{[T]}; s) = 1$ is decreasing in s. Suppose $\xi_{t+1}(z, v_{[t]}; s)$ is decreasing in s. By Lemma 1(a) that $\phi_{t+1}(z, v_{[t]}; s)$ is decreasing in s, we know $\mathbb{1}_{\left\{\phi_{t+1}(z, v_{[t]}; s) - \omega_t(v_{[t]}) > 0\right\}}$ is decreasing in s, and hence $\xi_t(z, v_{[t-1]}; s)$ is also decreasing in s. Therefore, $\xi_1(z; s)$ is decreasing in s, implying that $\phi_1(z; s)$ is submodular over $(z, s) \in [0, u] \times [0, \check{s}(u)]$.

Similarly, to prove $\phi_1(u; T)$ is submodular in (u, T), it suffices to show that $\xi_1(u; T)$ is decreasing in T. For any vector $v_{[t-1]}$, we let $v^0_{[t]} = (0, v_1, ..., v_{t-1})$ (note that $v^0_{[1]} = 0$). That is, $v^0_{[t]}$ is a vector of valuations from t outside alternatives with the first component being zero: $v^0_1 = 0$, and all subsequent components being $v_{[t-1]}$: $v^0_\tau = v_{\tau-1}, \tau = 2, ..., t$. By the definition of $\omega_t(\cdot)$, we have $\omega_t(v^0_{[t]}) = \omega_{t-1}(v_{[t-1]})$ for any $t = 1, ..., T+1$ (we interpret $\omega_0(\cdot) = 0$). Now by part (b), when $u < \check{u}$, we have $\phi_t(u, v_{[t-1]}) > u$, then

$$\xi_1(u; T+1) = \mathbb{P}(\phi_2(u, v_{[1]}; T+1) > \omega_1(v_{[1]}), ..., \phi_{T+2}(u, v_{[T+1]}; T+1) > \omega_{T+1}(v_{[T+1]})).$$

We first show that $\phi_t(u, v_{[t-1]}; T+1) - \omega_{t-1}(v_{[t-1]})$ is decreasing in v_1 by induction. When $t = T+2$,

$$\phi_{T+2}(u, v_{[T+1]}; T+1) - \omega_{T+1}(v_{[T+1]}) = u - \omega_{T+1}(v_{[T+1]}),$$

which is decreasing in v_1 since $\omega_{T+1}(v_{[T+1]})$ is increasing in v_1. Suppose $\phi_{t+1}(u, v_{[t]}; T+1) - \omega_t(v_{[t]})$ is decreasing in v_1, then

$$\begin{aligned}
&\phi_t(u, v_{[t-1]}; T+1) - \omega_{t-1}(v_{[t-1]}) \\
=&\mathbb{E}_{v_t}\{u \vee \omega_t(v_{[t]}) \vee \phi_{t+1}(u, v_{[t]}; T+1)\} - \omega_{t-1}(v_{[t-1]}) - s \\
=&\mathbb{E}_{v_t}\{(u - \omega_t(v_{[t]})) \vee 0 \vee (\phi_{t+1}(u, v_{[t]}; T+1) - \omega_t(v_{[t]}))\} + \omega_t(v_{[t]}) - \omega_{t-1}(v_{[t-1]}) - s,
\end{aligned}$$

which is decreasing in v_1 because $\omega_t(v_{[t]})$ is increasing in v_1 and

$$\omega_t(v_{[t]}; k) - \omega_{t-1}(v_{[t-1]}; k) = \begin{cases} (v_t - \max_{1 \le \tau \le t-1} v_\tau)^+, & \text{if } k \ge t\ ; \\ \max_{2 \le \tau \le t} v_\tau - \max_{1 \le \tau \le t-1} v_\tau, & \text{if } k = t-1\ ; \\ \max_{t-k+1 \le \tau \le t} v_\tau - \max_{t-k \le \tau \le t-1} v_\tau, & \text{if } 1 \le k \le t-2\ ; \end{cases}$$

is decreasing in v_1 regardless of the value of k. It follows that

$$\begin{aligned}\xi_1(u;T+1) &= \mathbb{P}(\phi_2(u,v_{[1]};T+1) > \omega_1(v_{[1]}),...,\phi_{T+2}(u,v_{[T+1]};T+1) > \omega_{T+1}(v_{[T+1]}))\\ &\le \mathbb{P}(\phi_2(u,0;T+1) > 0,...,\phi_{T+2}(u,(0,v_2,...,v_{T+1});T+1) > \omega_{T+1}((0,v_2,...,v_{T+1})))\\ &\le \mathbb{P}(\phi_3(u,(0,v_2);T+1) > \omega_2((0,v_2))...,\phi_{T+2}(u,(0,v_2,...,v_{T+1});T+1) > \omega_{T+1}((0,v_2,...,v_{T+1})))\\ &= \mathbb{P}(\phi_3(u,(0,v_1);T+1) > \omega_2((0,v_1))...,\phi_{T+2}(u,(0,v_1,...,v_T);T+1) > \omega_{T+1}((0,v_1,...,v_T)))\\ &= \mathbb{P}(\phi_3(u,v^0_{[2]};T+1) > \omega_2(v^0_{[2]}),...,\phi_{T+2}(u,v^0_{[T+1]};T+1) > \omega_{T+1}(v^0_{[T+1]})),\end{aligned}$$

where the first inequality holds because $\phi_t(u,v_{[t-1]}) - \omega_{t-1}(v_{t-1})$ is decreasing in v_1, the second equality follows from the stationarity of v_t, and the third equality comes from the definition of $v^0_{[t]}$. To show $\xi_1(u;T+1) \le \xi_1(u;T)$, it suffices to show

$$\mathbb{P}(\phi_3(u,v^0_{[2]};T+1) > \omega_2(v^0_{[2]}),...,\phi_{T+2}(u,v^0_{[T+1]};T+1) > \omega_{T+1}(v^0_{[T+1]})) = \xi_1(u;T).$$

To this end, we prove $\phi_{t+1}(u,v^0_{[t]};T+1) = \phi_t(u,v_{[t-1]};T)$ by induction. When $t=T$,

$$\phi_{T+1}(u,v^0_{[T]};T+1) = \mathbb{E}_{v_T}\{u \vee \omega_{T+1}((v^0_{[T]},v_T))\} - s = \mathbb{E}_{v_T}\{u \vee \omega_T(v_{[T]})\} - s = \phi_T(u,v_{[T-1]};T),$$

where in the second equality we used $(v^0_{[T]},v_T) = v^0_{[T+1]}$ and $\omega_{T+1}(v^0_{[T+1]}) = \omega_T(v_{[T]})$. Suppose $\phi_{t+2}(u,v^0_{[t+1]};T+1) = \phi_{t+1}(u,v_{[t]};T)$, then

$$\begin{aligned}\phi_{t+1}(u,v^0_{[t]};T+1) &= \mathbb{E}_{v_t}\{u \vee \omega_{t+1}((v^0_{[t]},v_t)) \vee \phi_{t+2}(u,(v^0_{[t]},v_t);T+1)\} - s\\ &= \mathbb{E}_{v_t}\{u \vee \omega_{t+1}(v^0_{[t+1]}) \vee \phi_{t+2}(u,v^0_{[t+1]};T+1)\} - s\\ &= \mathbb{E}_{v_t}\{u \vee \omega_t(v_{[t]}) \vee \phi_{t+1}(u,v_{[t]};T)\} - s = \phi_t(u,v_{[t-1]};T),\end{aligned}$$

where the second equality holds because $(v^0_{[t]},v_t) = v^0_{[t+1]}$, and the third equality comes from the induction hypothesis and the fact that $\omega_{t+1}(v^0_{[t+1]}) = \omega_t(v_{[t]})$. Therefore,

$$\begin{aligned}&\mathbb{P}(\phi_3(u,v^0_{[2]};T+1) > \omega_2(v^0_{[2]}),...,\phi_{T+2}(u,v^0_{[T+1]};T+1) > \omega_{T+1}(v^0_{[T+1]}))\\ =&\mathbb{P}(\phi_2(u,v_{[1]};T) > \omega_1(v_{[1]}),...,\phi_{T+1}(u,v_{[T]};T) > \omega_T(v_{[T]})) = \xi_1(u;T),\end{aligned}$$

and hence $\xi_1(u;T+1) \le \xi_1(u;T)$, implying that $\phi_1(u;T)$ is submodular in (u,T) when $u < \check{u}$. □

Proof of Proposition 1

(a) When $s < \check{s}(u)$, $\mathbb{E}_v[(v-u)^+] = \check{s}(u) > s$, implying that $u < \check{u}(s)$. By Lemma 2(b), $\phi_1(u) > u$, and the seller's revenue is then $\bar{\Phi}^{\text{FB}}(u;s) = \phi_1(u;s) - \phi_1(0;s)$ which is decreasing in s because $\phi_1(z;s)$ is submodular in $(z,s) \in [0,u] \times [0,\check{s}(u)]$ by Lemma 2(c); when $s \ge \check{s}(u)$, we have $u \ge \check{u}(s)$, by Lemma 2(b), $\phi_1(u) \le u$, and the seller's revenue is then $\bar{\Phi}^{\text{FB}}(u;s) = u - \phi_1(0;s)$ which is increasing in s by Lemma 1(a).

(b) When $u \geq \check{u}$, by Lemma 2(b), $\phi_1(u) \leq u$, and the seller's revenue is then $\bar{\Phi}^{\text{FB}}(u;T) = u - \phi_1(0;T)$ which is decreasing in T by Lemma 1(a). When $u < \check{u}$, by Lemma 2(b), $\phi_1(u) > u$, and the seller's revenue becomes $\bar{\Phi}^{\text{FB}}(u;T) = \phi_1(u;T) - \phi_1(0;T)$ which is decreasing in T because $\phi_1(u;T)$ is submodular in (u,T) by Lemma 2(c). □

Proof of Proposition 2

Under the dynamic pricing mechanism where the seller commits to a price path $p_0, p_1, ..., p_T$ at period 0, let the customer's utility-to-go at period t be $l_t^d(u)$, then we have

$$l_0^d(u) = (u - p_0) \vee l_1^d(u),$$

and for $t = 1, ..., T$,

$$l_t^d(u, v_{[t-1]}) = \mathbb{E}_{v_t}\{(u - p_t) \vee \omega_t(v_{[t]}) \vee l_{t+1}^d(u, v_{[t]})\},$$

with $l_{T+1}^d(u, v_{[T]}) = 0$.

If there exists $t \leq T-1$ such that $p_t > p_{t+1}$, then because $l_{t+1}^d(u, v_{[t]}) \geq u - p_{t+1}$, it follows that $u - p_t < u - p_{t+1} \leq l_{t+1}^d(u, v_{[t]}), \forall u \in [0,1]$. This implies that no customer will purchase the product at period t, and thus no profit is generated at period t. The seller can achieve at least the same profit by decreasing p_t to p_{t+1}, and hence the optimal prices must satisfy $p_t^* \leq p_{t+1}^*$. □

Proof of Proposition 3

Here, we focus on deriving a further revenue upper bound to the relaxed problem (4). We first show in the following auxiliary proposition that by using the local IC constraints, one can eliminate the transfers $z_t(u, v_{[t]})$ and reformulate the seller's revenue function.

Proposition A1. *For any mechanism $\mathcal{M}$ that satisfies the truthtelling constraints in (4), the revenue $\Phi(\mathcal{M})$ can be computed as:*

$$\Phi(\mathcal{M}) = \mathbb{E}_u\big\{q_0(u)l_0(u;1) + (1 - q_0(u)l_0(u;0)\big\} - \check{\Psi}_0(0,0),$$

where for $t = 1, ..., T$,

$$l_t(u, v_{[t]}; 1) = \alpha_t^0(u, v_{[t]})\lambda(u) + \alpha_t^1(u, v_{[t]})\omega_t(v_{[t]}) + \alpha_t^2(u, v_{[t]})\mathbb{E}_{v_{t+1}} l_{t+1}(u, v_{[t+1]}; 1) - s\mathbb{1}_{\{t>0\}},$$

and

$$l_t(u, v_{[t]}; 0) = \beta_t^1(u, v_{[t]})\omega_t(v_{[t]}) + \beta_t^2(u, v_{[t]})\mathbb{E}_{v_{t+1}}[q_{t+1}(u, v_{[t+1]})l_{t+1}(u, v_{[t+1]}; 1) + (1 - q_{t+1}(u, v_{[t+1]}))l_{t+1}(u, v_{[t+1]}; 0)] - s\mathbb{1}_{\{t>0\}},$$

with $l_{T+1}(\cdot) = 0$.

Proof: See Appendix D.1. □

Here, $l_t(u,v_{[t]};1)$ and $l_t(u,v_{[t]};0)$ can be interpreted as the dynamic virtual surplus of the seller for type $(u,v_{[t]})$ customer conditional on the seller's product being allocated or not, respectively (see Pavan et al. 2014 for a similar interpretation). We can then characterize a revenue upper bound to problem (4) by using the revenue function derived in Proposition A1 and dropping the incentive compatibility constraint in problem (4) to arrive at the problem:

$$\begin{aligned}
\max_{\mathcal{M}} \quad & \mathbb{E}[\Phi_0(u)] \\
\text{s.t.} \quad & \check{\Psi}_0(u,u) \geq \phi_1(0), && \forall u \in [0,1]; \\
& q_t(u,v_{[t]}), \alpha_t^i(u,v_{[t]}), \beta_t^j(u,v_{[t]}) \in [0,1], && \forall t \in \{1,...,T\}, i \in \{0,1,2\}, j \in \{1,2\}; \\
& \sum_{i=0}^{2} \alpha_t^i(u,v_{[t]}) = 1, \sum_{i=1}^{2} \beta_t^i(u,v_{[t]}) = 1, && \forall t \in \{1,...,T\}.
\end{aligned} \tag{A.1}$$

Without the incentive compatibility constraint, problem (A.1) can then be solved via dynamic programming. We let $l_t^*(u,v_{[t]};x_t)$ denote $l_t(u,v_{[t]};x_t)$ evaluated at the optimal mechanism. Then from the recursive equations in Proposition A1, $l_t^*(u,v_{[t]};1)$ satisfies the following Bellman equation:

$$\begin{aligned}
l_t^*(u,v_{[t]};1) = \max & \left\{ \alpha_t^0(u,v_{[t]})\lambda(u) + \alpha_t^1(u,v_{[t]})\omega_t(v_{[t]}) + \alpha_t^2(u,v_{[t]})\mathbb{E}_{v_{t+1}} l_{t+1}^*(u,v_{[t+1]};1) - s\mathbb{1}_{\{t>0\}} \right\} \\
\text{s.t.} \quad & \alpha_t^i(u,v_{[t]}), \beta_t^j(u,v_{[t]}) \in [0,1], \quad \forall t \in \{1,...,T\}, i \in \{1,2,3\}, j \in \{1,2\}; \\
& \sum_{i=0}^{2} \alpha_t^i(u,v_{[t]}) = 1, \sum_{i=1}^{2} \beta_t^i(u,v_{[t]}) = 1, \quad \forall t \in \{1,...,T\}.
\end{aligned}$$

with $l_{T+1}^*(\cdot) = 0$. Since the objective is linear, we arrive at the optimal solution:

$$\begin{aligned}
\alpha_t^{0*}(u,v_{[t]}) &= \mathbb{1}\{\lambda(u) \geq \omega_t(v_{[t]}) \vee \mathbb{E}l_{t+1}^*(u,v_{[t+1]};1)\}, \\
\alpha_t^{1*}(u,v_{[t]}) &= \mathbb{1}\{\omega_t(v_{[t]}) \geq \lambda(u) \vee \mathbb{E}l_{t+1}^*(u,v_{[t+1]};1)\}, \\
\alpha_t^{2*}(u,v_{[t]}) &= 1 - \alpha_t^{0*}(u,v_{[t]}) - \alpha_t^{1*}(u,v_{[t]}).
\end{aligned}$$

It follows that

$$l_t^*(u,v_{[t]};1) = \lambda(u) \vee \omega_t(v_{[t]}) \vee \mathbb{E}_{v_{t+1}} l_{t+1}^*(u,v_{[t+1]};1) - s\mathbb{1}_{\{t>0\}}, \tag{A.2}$$

with $l_{T+1}^*(\cdot) = 0$. Similarly, $l_t^*(u,v_{[t]};0)$ satisfies the following Bellman equation:

$$\begin{aligned}
l_t^*(u,v_{[t]};0) = \max & \left\{ \beta_t^1(u,v_{[t]})\omega_t(v_{[t]}) \right. \\
& \left. + \beta_t^2(u,v_{[t]})\mathbb{E}_{v_{t+1}}[q_{t+1}(u,v_{[t+1]}) l_{t+1}^*(u,v_{[t+1]};1) + (1 - q_{t+1}(u,v_{[t+1]})) l_{t+1}^*(u,v_{[t+1]};0)] - s\mathbb{1}_{\{t>0\}} \right\} \\
\text{s.t.} \quad & q_t(u,v_{[t]}), \alpha_t^i(u,v_{[t]}), \beta_t^j(u,v_{[t]}) \in [0,1], \quad \forall t \in \{1,...,T\}, i \in \{0,1,2\}, j \in \{1,2\}; \\
& \sum_{i=0}^{2} \alpha_t^i(u,v_{[t]}) = 1, \sum_{i=1}^{2} \beta_t^i(u,v_{[t]}) = 1, \quad \forall t \in \{1,...,T\},
\end{aligned}$$

with $l^*_{T+1}(\cdot)=0$. Since the objective is linear, we arrive at the optimal solution:

$$q^*_{t+1}(u,v_{[t+1]})=\mathbb{1}\{l^*_{t+1}(u,v_{[t+1]};1)\geq l^*_{t+1}(u,v_{[t+1]};0)\},$$
$$\beta^{1*}_t(u,v_{[t]})=\mathbb{1}\{\omega_t(v_{[t]})\geq \mathbb{E}_{v_{t+1}}[l^*_{t+1}(u,v_{[t+1]};1)\vee l^*_{t+1}(u,v_{[t+1]};0)]\},$$
$$\beta^{2*}_t(u,v_{[t]})=1-\beta^{1*}_t(u,v_{[t]}).$$

It is immediate that

$$l^*_t(u,v_{[t]};0)=\omega_t(v_{[t]})\vee \mathbb{E}_{v_{t+1}}[l^*_{t+1}(u,v_{[t+1]};1)\vee l^*_{t+1}(u,v_{[t+1]};0)]-s\mathbb{1}_{\{t>0\}}, \tag{A.3}$$

with $l^*_{T+1}(\cdot)=0$.

Since $\check{\Psi}_0(u,u)$ is increasing in u by Lemma D4, the IR constraint in problem (A.1) is satisfied if and only if $\check{\Psi}_0(0,0)\geq\phi_1(0)$. At the optimal solution, the IR constraint for the lowest-type customer should be binding, i.e., $\check{\Psi}_0(0,0)=\phi_1(0)$. It follows that

$$\bar{\Phi}=\mathbb{E}_u\max[q_0(u)l^*_0(u;1)+(1-q_0(u))l^*_0(u;0)]-\phi_1(0).$$

Again by linearity, the optimal solution is $q^*_0(u)=\mathbb{1}[l^*_0(u;1)\geq l^*_0(u;0)]$, and correspondingly we have

$$\bar{\Phi}=\mathbb{E}_u\{l^*_0(u;1)\vee l^*_0(u;0)\}-\phi_1(0).$$

By the equation of $l^*_t(u,v_{[t]};1)$ in (A.2) and $l^*_t(u,v_{[t]};0)$ in (A.3), we have

$$\begin{aligned}
&l^*_t(u,v_{[t]};1)\vee l^*_t(u,v_{[t]};0)\\
=&\lambda(u)\vee\omega_t(v_{[t]})\vee\mathbb{E}_{v_{t+1}}[l^*_{t+1}(u,v_{[t+1]};1)]\vee\omega_t(v_{[t]})\vee\mathbb{E}_{v_{t+1}}[l^*_{t+1}(u,v_{[t+1]};1)\vee l^*_{t+1}(u,v_{[t+1]};0)]-s\mathbb{1}_{\{t>0\}}\\
=&\lambda(u)\vee\omega_t(v_{[t]})\vee\mathbb{E}_{v_{t+1}}[l^*_{t+1}(u,v_{[t+1]};1)\vee l^*_{t+1}(u,v_{[t+1]};0)]-s\mathbb{1}_{\{t>0\}},
\end{aligned}$$

with $l^*_{T+1}(u,v_{[T+1]};1)\vee l^*_{T+1}(u,v_{[T+1]};0)=0$. Note that for $t=1,2,...,T,\mathbb{E}_{v_t}[l^*_t(u,v_{[t]};1)\vee l^*_t(u,v_{[t]};0)]$ is defined by the Bellman equation of the same form as $\phi_t(u,v_{[t-1]})$, except that the fallback option value for the seller's product becomes $\lambda(u)$ instead of u. In other words, we have $\mathbb{E}_{v_t}[l^*_t(u,v_{[t]};1)\vee l^*_t(u,v_{[t]};0)]=\phi_t(\lambda(u),v_{[t-1]})$ for $t=1,2,...,T$. Then, $\bar{\Phi}$ can be further simplified as follows:

$$\begin{aligned}
\bar{\Phi}&=\mathbb{E}_u[l^*_0(u;1)\vee l^*_0(u;0)]-\phi_1(0)\\
&=\mathbb{E}_u[\lambda(u)\vee\mathbb{E}_{v_1}[l^*_1(u,v_{[1]};1)\vee l^*_1(u,v_{[1]};0)]]-\phi_1(0)=\mathbb{E}_u\{\lambda(u)\vee\phi_1(\lambda(u))\}-\phi_1(0).
\end{aligned}$$

By the definition of $\underline{u}$ and $\hat{u}$, and by noting that $\phi_1(\lambda(u))=\phi_1(0)$ for $u<\underline{u}$, we then have

$$\begin{aligned}
\bar{\Phi}&=\int_0^{\underline{u}}\phi_1(0)dF(u)+\int_{\underline{u}}^{\hat{u}}\phi_1(\lambda(u))dF(u)+\int_{\hat{u}}^1\lambda(u)dF(u)-\phi_1(0)\\
&=\int_{\underline{u}}^{\hat{u}}\phi_1(\lambda(u))dF(u)+\int_{\hat{u}}^1\lambda(u)dF(u)-\bar{F}(\underline{u})\phi_1(0)\\
&=\int_{\underline{u}}^{\hat{u}}[\phi_1(\lambda(u))-\phi_1(0)]dF(u)+\bar{F}(\hat{u})(\hat{u}-\phi_1(0)),
\end{aligned}$$

where the last equality holds because $\int_{\hat{u}}^1\lambda(u)dF(u)=\int_{\hat{u}}^1 udF(u)-\int_{\hat{u}}^1\bar{F}(u)du=\bar{F}(\hat{u})\hat{u}$. □

Proof of Theorem 1

We first show that under the optimal mechanism $\mathcal{M}^*$, customer with valuation u indeed behaves as stated in Theorem 1. We then show the revenue collected under such behavior achieves the revenue upper bound $\bar{\Phi}$.

We let $w_t(u,\tilde{u})$ be the utility-to-go at period $t, t=0,1,...,T$ of a customer who has valuation $u \in [0,1]$ but chooses the option contract $(D(\tilde{u}), p(\tilde{u})), \tilde{u} \in [\underline{u}, 1]$. Then, by definition,

$$w_0(u,\tilde{u}) = (u - p(\tilde{u})) \vee w_1(u,\tilde{u}) - D(\tilde{u}),$$

and for $t = 1, ..., T$,

$$w_t(u,\tilde{u}; v_{[t-1]}) = \mathbb{E}_{v_t}\{(u - p(\tilde{u})) \vee \omega_t(v_{[t]}) \vee w_{t+1}(u,\tilde{u}; v_{[t]})\} - s,$$

where we let $w_{T+1}(u,\tilde{u}) = u - p(\tilde{u})$. One can see that upon paying the deposit $D(\tilde{u})$, the customer is essentially engaged in a search process with a fallback value $u - p(\tilde{u})$ as the one we analyzed in Section 3. Hence, for $t = 1, ..., T$, we have $w_t(u,\tilde{u}; v_{[t-1]}) = \phi_t(u - p(\tilde{u}), v_{[t-1]})$, and consequently

$$w_0(u,\tilde{u}) = (u - p(\tilde{u})) \vee \phi_1(u - p(\tilde{u})) - D(\tilde{u}).$$

Lemma A1 below summarizes the properties of $w_0(u,\tilde{u})$.

Lemma A1. *For any $u \in [0,1]$ and $\tilde{u} \in [\underline{u}, 1]$, $w_0(u,\tilde{u})$ is decreasing in $\tilde{u}$ for $\tilde{u} \geq u$; and is increasing in $\tilde{u}$ for $\tilde{u} < u$.*

Proof: See Appendix D.2. □

Next, we show that the customer will self-select the contract that is consistent with his valuation for the seller's product and the contract is individual rational by discussing the customer's choices among three cases.

- When $u \in [\hat{u}, 1]$, the customer's surplus from choosing the contract $(D(u), p(u))$ is the same as choosing $(D(\hat{u}), p(\hat{u}))$, i.e., $w_0(u,u) = w_0(u,\hat{u})$. It is hence sufficient for us to argue that $w_0(u,\hat{u}) \geq w_0(u,\tilde{u})$ for $\tilde{u} \in [\underline{u}, \hat{u}]$, and $w_0(u,\hat{u}) \geq \phi_1(0)$. By Lemma A1, we know that $w_0(u,\tilde{u})$ is increasing in $\tilde{u}$ for $\tilde{u} \leq u$. Hence,

$$w_0(u,\tilde{u}) \leq w_0(u,\hat{u}), \quad \forall \tilde{u} \in [\underline{u}, \hat{u}].$$

In addition, because $u - p(\hat{u}) \geq \hat{u} - p(\hat{u}) = \lambda(\hat{u})$, by Lemma 1(a) and the definition of $\hat{u}$, we have

$$u - p(\hat{u}) \geq \phi_1(u - p(\hat{u})).$$

This implies that customer with valuation $u \in [\hat{u}, 1]$ will exercise the option immediately without any search. Then,

$$\begin{aligned} w_0(u,\hat{u}) &= (u - p(\hat{u})) \vee \phi_1(u - p(\hat{u})) - D(\hat{u}) \\ &\geq \hat{u} - p(\hat{u}) - D(\hat{u}) \\ &= \lambda(\hat{u}) - D(\hat{u}) = \lambda(\hat{u}) - \phi_1(\lambda(\hat{u})) + \phi_1(0) + \int_{\underline{u}}^{\hat{u}} \xi_1(\lambda(\nu))d\nu \geq \phi_1(0), \end{aligned}$$

where the last inequality holds because $\phi_1(\lambda(\hat{u})) \leq \lambda(\hat{u})$. Therefore, the customer will participate in the seller's mechanism.

- When $u \in [\underline{u}, \hat{u})$, by Lemma A1, $w_0(u,u) \geq w_0(u,\tilde{u}), \forall \tilde{u} \in [\underline{u}, 1]$, which implies it is never optimal for the customer to choose the contract designed for customer with valuation $\tilde{u} \neq u$. In addition,

$$\begin{aligned} w_0(u,u) = \lambda(u) \vee \phi_1(u - p(u)) - D(u) &= \phi_1(\lambda(u)) - \phi_1(\lambda(u)) + \phi_1(0) + \int_{\underline{u}}^{u} \xi_1(\lambda(\nu))d\nu \\ &= \phi_1(0) + \int_{\underline{u}}^{u} \xi_1(\lambda(\nu))d\nu \geq \phi_1(0), \end{aligned}$$

where the second equality is due to $\phi_1(\lambda(u)) \geq \lambda(u)$ for $u \leq \hat{u}$.

For customer with valuation $u \in [\underline{u}, \hat{u})$, since $\phi_1(\lambda(u)) \geq \lambda(u)$, the customer will continue searching at the initial period. Then the customer's utility to go at each period is:

$$w_t(u,u;v_{[t-1]}) = \mathbb{E}_{v_t}\{(\lambda(u)) \vee \omega_t(v_{[t]}) \vee w_{t+1}(u,u;v_{[t]})\} - s = \phi_t(\lambda(u), v_{[t-1]}) - s \geq \phi_t(0, v_{[t-1]}),$$

implying that the mechanism satisfies the interim phases individual rationality constraint.

- When $0 \leq u < \underline{u}$, by Lemma A1, $w_0(u,\tilde{u})$ is decreasing in $\tilde{u}$ for $\tilde{u} \geq u$ with $\tilde{u} \in [\underline{u}, 1]$, then we have

$$w_0(u,\tilde{u}) \leq w_0(u,\underline{u}) = \phi_1(u - p(\underline{u})) - D(\underline{u}) = \phi_1(u - p(\underline{u})) - \phi_1(\lambda(\underline{u})) + \phi_1(0) \leq \phi_1(0).$$

Here the first equality holds by $u - p(\underline{u}) \leq \lambda(\underline{u}) \leq \lambda(\hat{u})$ and Lemma A1, and the last inequality holds because $\phi_1(u)$ is increasing in u. Therefore, when $u < \underline{u}$, the customer searches without purchasing from the seller.

Given that customer indeed behaves as stated in Theorem 1, we know from the definition of $\hat{u}$ that for customer with valuation $u \in [\underline{u}, \hat{u}]$, he will return to purchase the product with probability $\xi_1(\lambda(u))$ and for customer with valuation $u \in (\hat{u}, 1]$, he will purchase seller's product without any search at period 0. Therefore, the revenue that the seller can achieve is

$$\int_{\underline{u}}^{\hat{u}} [D(u) + \xi_1(\lambda(u))p(u)]dF(u) + \int_{\hat{u}}^{1} [D(\hat{u}) + p(\hat{u})]dF(u)$$

$$= \int_{\underline{u}}^{\hat{u}} [\phi_1(\lambda(u)) - \phi_1(0) + \xi_1(\lambda(u))p(u)]dF(u) - \int_{\underline{u}}^{\hat{u}} \int_{\underline{u}}^{u} \xi_1(\lambda(\nu))d\nu dF(u) + \int_{\hat{u}}^{1} [D(\hat{u}) + p(\hat{u})]dF(u)$$
$$= \int_{\underline{u}}^{\hat{u}} [\phi_1(\lambda(u)) - \phi_1(0) + \xi_1(\lambda(u))p(u)]dF(u) + \bar{F}(\hat{u}) \int_{\underline{u}}^{\hat{u}} \xi_1(\lambda(u))du - \int_{\underline{u}}^{\hat{u}} \xi_1(\lambda(u))p(u)dF(u)$$
$$+ \int_{\hat{u}}^{1} [D(\hat{u}) + p(\hat{u})]dF(u)$$
$$= \int_{\underline{u}}^{\hat{u}} [\phi_1(\lambda(u)) - \phi_1(0)]dF(u) + \bar{F}(\hat{u}) \left(\int_{\underline{u}}^{\hat{u}} \xi_1(\lambda(u))du + D(\hat{u}) + p(\hat{u}) \right)$$
$$= \int_{\underline{u}}^{\hat{u}} [\phi_1(\lambda(u)) - \phi_1(0)]dF(u) + \bar{F}(\hat{u}) \left(\int_{\underline{u}}^{\hat{u}} \xi_1(\lambda(u))du + \phi_1(\lambda(\hat{u})) - \phi_1(0) - \int_{\underline{u}}^{\hat{u}} \xi_1(\lambda(u))du + \hat{u} - \lambda(\hat{u}) \right),$$
$$= \int_{\underline{u}}^{\hat{u}} [\phi_1(\lambda(u)) - \phi_1(0)]dF(u) + \bar{F}(\hat{u})(\hat{u} - \phi_1(0)) = \bar{\Phi},$$

where the second equality holds by integration by parts and our construction that $p(u) = \frac{\bar{F}(u)}{f(u)}$ for $\underline{u} \le u < \hat{u}$, and the second to the last equality comes from the fact that $\phi_1(\lambda(\hat{u})) = \lambda(\hat{u})$. This establishes that $\mathcal{M}^*$ achieves the revenue upper bound $\bar{\Phi}$ and concludes our proof. □

Proof of Proposition 4

The proof of part (a) directly follows from the proof of Theorem 1. Here we focus on part (b). By the definition of $\hat{u}$ and $\check{u}$, we have $\lambda(\hat{u}) = \check{u}$. For $u < \hat{u}$, we then have $\lambda(u) < \check{u}$, and by Lemma 2(b), we know $\lambda(u) < \phi_t(\lambda(u), v_{[t-1]})$ for $t = 1, 2, ..., T$. That is, the customer never exercises option before searching all the alternatives. It follows that customer's utility is

$$\lambda(u) \vee \phi_1(\lambda(u)) - D(u) = \phi_1(\lambda(u)) - D(u). \quad \square$$

Proof of Proposition 5

Note that $\phi_1(\lambda(u); T)$ being increasing in T directly comes from Lemma 1(a). We show next that the seller's optimal revenue $\bar{\Phi}(T)$ is decreasing in T. Recall that the seller's optimal revenue is:

$$\bar{\Phi}(T) = \int_{\underline{u}}^{\hat{u}} [\phi_1(\lambda(u); T) - \phi_1(0; T)]dF(u) + \bar{F}(\hat{u})(\hat{u} - \phi_1(0; T)).$$

By Lemma 2(c), $\phi_1(u; T)$ is submodular over (u, T) when $u < \check{u}$ and because $\lambda(u) \in [0, \check{u})$ when $u \in [\underline{u}, \hat{u})$, we have $\phi_1(\lambda(u); T) - \phi_1(0; T)$ is decreasing in T when $u \in [\underline{u}, \hat{u})$. In addition, because $\phi_1(0; T)$ is increasing in T by Lemma 1, $\bar{\Phi}(T)$ is decreasing in T.

We next investigate the seller's revenue when the length of the time horizon goes to infinity. Let $\phi_1(u; T, k)$ be the customer's utility-to-go from searching when his valuation for the seller's product is u, the length of search horizon is T and the memory length is k. We first show that for any $u \in [0, \check{u}]$ and any $k \ge 1$, $\phi_1(u; T, k)$ converges to $\check{u}$ as $T \to \infty$. To this end, note first that $\phi_1(u; T, k) \le \phi_1(\check{u}; T, k) = \check{u}$ for any $u \in [0, \check{u}]$ since $\phi_1(u; T, k)$ is increasing in u by Lemma 1(a). In addition, because $\phi_1(u; T, k)$ is increasing in T by Lemma 1(a), then by monotone convergence theorem, we know $\phi_1(u;T,k)$ has a limit, which we denote by $\phi(u;k) := \lim_{T\to+\infty}\phi_1(u;T,k) \le \check{u}$ for any $k \ge 1$. By Lemma 2(a), $\phi_1(u;T,k)$ is increasing in k, and hence we must have $\phi(u;1) \le \phi(u;k) \le \check{u}$ for any $k \ge 1$. Therefore, to show that $\phi(u;k) \equiv \check{u}$ for any $u \in [0,\check{u}]$, it is sufficient to show that $\phi(u;1) \equiv \check{u}$ for any $u \in [0,\check{u}]$, i.e., it is sufficient to consider the case with no recall. With no recall, we can simplify the value function $\phi_t(u, v_{[t-1]}; T, 1)$ as $\phi_t(u;T,1)$. In particular, for $u \in [0,\check{u}]$ we have

$$\phi_1(u;T,1) = \mathbb{E}_v\{u \vee v \vee \phi_2(u;T,1)\} - s = \mathbb{E}_v\{v \vee \phi_2(u;T,1)\} - s = \mathbb{E}_v\{v \vee \phi_1(u;T-1,1)\} - s,$$

where the second equality is due to $u \le \phi_2(u;T,1)$ for $u \le \check{u}$ by Lemma 2(b). Taking limits on both sides, we arrive at

$$\begin{aligned}\phi(u;1) &= \lim_{T\to+\infty}\phi_1(u;T,1) = \lim_{T\to+\infty}\mathbb{E}_v\{v \vee \phi_1(u;T-1,1)\} - s\\ &= \mathbb{E}_v\{v \vee \lim_{T\to+\infty}\phi_1(u;T-1,1)\} - s = \mathbb{E}_v\{v \vee \phi(u;1)\} - s.\end{aligned}$$

By the definition of $\check{u}$, $\phi(u;1) \equiv \check{u}$, and hence $\phi(u;k) \equiv \check{u}$ for any $u \in [0,\check{u}]$ and $k \ge 1$.

Therefore, when the length of time horizon goes to infinity, the seller's optimal revenue becomes:

$$\begin{aligned}\lim_{T\to+\infty}\bar{\Phi}(T) &= \lim_{T\to+\infty}\int_{\underline{u}}^{\hat{u}}[\phi_1(\lambda(u);T) - \phi_1(0;T)]dF(u) + (\hat{u} - \phi_1(0;T))\bar{F}(\hat{u})\\ &= (\hat{u} - \check{u})\bar{F}(\hat{u}) = p(\hat{u})\bar{F}(\hat{u}),\end{aligned}$$

where the last equality holds because $\lambda(\hat{u}) = \hat{u} - p(\hat{u}) = \check{u}$. It can be easily verified that the revenue can be achieved by giving an exploding offer at time zero with a posted price $p(\hat{u})$.

Finally, we establish that the rate of convergence is exponential. It suffices to show that there exist a constant M such that

$$0 \le \bar{\Phi}(T) - p(\hat{u})\bar{F}(\hat{u}) \le MG(\check{u})^T.$$

We first show $\xi_t(u, v_{[t-1]}) \le G(\check{u})^{T+1-t}$ for $u < \check{u}$ by induction. At period $t = T$,

$$\xi_T(u, v_{[T-1]}) = \mathbb{E}_{v_T}\{\mathbb{1}_{\{\omega_T(v_{[T]}) \le u\}}\} \le \mathbb{E}_{v_T}\{\mathbb{1}_{\{v_T \le \check{u}\}}\} = G(\check{u}),$$

where the inequality is due to $\omega_T(v_{[T]}) \ge v_T$ and $u < \check{u}$. Suppose $\xi_{t+1}(u, v_{[t]}) \le G(\check{u})^{T-t}$. By Lemma 2(b), we have $\phi_t(u, v_{[t-1]}) = \mathbb{E}_{v_t}\{\omega_t(v_{[t]}) \vee \phi_{t+1}(u, v_{[t]})\} - s$ when $u < \check{u}$, then

$$\xi_t(u, v_{[t-1]}) = \mathbb{E}_{v_t}\{\mathbb{1}_{\{\phi_{t+1}(u, v_{[t]}) - \omega_t(v_{[t]}) > 0\}}\xi_{t+1}(u, v_{[t]})\} \le \mathbb{P}(\phi_{t+1}(u, v_{[t]}) - \omega_t(v_{[t]}) > 0)G(\check{u})^{T-t}.$$

where the inequality is from the induction hypothesis. Note that by Lemma 2(b), we have $\phi_{t+1}(\check{u}, v_{[t]}) \le \check{u} \vee \omega_t(v_{[t]})$, and then

$$\begin{aligned}\mathbb{P}(\omega_t(v_{[t]}) < \phi_{t+1}(u, v_{[t]})) &\le \mathbb{P}(\omega_t(v_{[t]}) < \phi_{t+1}(\check{u}, v_{[t]}))\\ &\le \mathbb{P}(\omega_t(v_{[t]}) < \check{u} \vee \omega_t(v_{[t]})) = \mathbb{P}(\omega_t(v_{[t]}) < \check{u}) \le \mathbb{P}(v_t \le \check{u}) = G(\check{u}).\end{aligned}$$

It follows that when $u < \check{u}$, $\xi_t(u, v_{[t-1]}) \le G(\check{u})^{T-t+1}$.

Recall that the seller's optimal revenue is:

$$\begin{aligned}\bar{\Phi}(T) &= \int_{\underline{u}}^{\hat{u}} [\phi_1(\lambda(u)) - \phi_1(0)]dF(u) + \bar{F}(\hat{u})(\hat{u} - \phi_1(0)) \\ &= \int_{\underline{u}}^{\hat{u}} [\phi_1(\lambda(u)) - \phi_1(0)]dF(u) + \bar{F}(\hat{u})(\lambda(\hat{u}) + p(\hat{u}) - \phi_1(0)) \\ &= \int_{\underline{u}}^{\hat{u}} [\phi_1(\lambda(u)) - \phi_1(0)]dF(u) + \bar{F}(\hat{u})(\phi_1(\lambda(\hat{u})) + p(\hat{u}) - \phi_1(0)) \\ &\le \int_{\underline{u}}^{\hat{u}} [\phi_1(\lambda(\hat{u})) - \phi_1(0)]dF(u) + \bar{F}(\hat{u})(\phi_1(\lambda(\hat{u})) - \phi_1(0)) + \bar{F}(\hat{u})p(\hat{u}) \\ &= (\phi_1(\check{u}) - \phi_1(0))\bar{F}(\underline{u}) + \bar{F}(\hat{u})p(\hat{u}) \le \check{u}\xi_1(\check{u})\bar{F}(\underline{u}) + \bar{F}(\hat{u})p(\hat{u}) \le \check{u}\bar{F}(\underline{u})G(\check{u})^T + p(\hat{u})\bar{F}(\hat{u}),\end{aligned}$$

where in the second inequality we used the convexity of $\phi_1(\cdot)$ and the last inequality uses the bound $\xi_1(\check{u}) \le G(\check{u})^T$. Since we have already shown that $\bar{\Phi}(T)$ is decreasing in T with the limit $p(\hat{u})\bar{F}(\hat{u})$, we must have $\bar{\Phi}(T) \ge p(\hat{u})\bar{F}(\hat{u})$. Therefore, $\bar{\Phi}(T) - p(\hat{u})\bar{F}(\hat{u}) = O(G(\check{u})^T)$. □

Proof of Proposition 6

To emphasize the dependence on s, we denote $\check{u}(s)$ to be the unique solution to equation $\mathbb{E}_v(v - u)^+ - s = 0$ and similarly let $\hat{u}(s)$ be the solution to $\phi_1(\lambda(u); s) = \lambda(u)$ (or equivalently $\lambda(u) = \check{u}(s)$). The seller's optimal revenue as a function of search cost s can be written as:

$$\bar{\Phi}(s) = \int_{\underline{u}}^{\hat{u}(s)} [\phi_1(\lambda(u); s) - \phi_1(0; s)]dF(u) + \int_{\hat{u}(s)}^{1} [\lambda(u) - \phi_1(0; s)]dF(u).$$

By taking first order derivative of $\bar{\Phi}(s)$ over s, we have

$$\begin{aligned}\frac{d\bar{\Phi}(s)}{ds} &= \int_{\underline{u}}^{\hat{u}(s)} [\frac{\partial\phi_1(\lambda(u); s)}{\partial s} - \frac{\partial\phi_1(0; s)}{\partial s}]dF(u) + [\phi_1(\lambda(\hat{u}(s)); s) - \phi_1(0; s)]f(\hat{u}(s)))\frac{d\hat{u}(s)}{ds} \\ &\quad + \int_{\hat{u}(s)}^{1} -\frac{\partial\phi_1(0; s)}{\partial s}dF(u) - [\lambda(\hat{u}(s)) - \phi_1(0; s)]f(\hat{u}(s))\frac{d\hat{u}(s)}{ds} \\ &= \int_{\underline{u}}^{\hat{u}(s)} [\frac{\partial\phi_1(\lambda(u); s)}{\partial s} - \frac{\partial\phi_1(0; s)}{\partial s}]dF(u) - \int_{\hat{u}(s)}^{1} \frac{\partial\phi_1(0; s)}{\partial s}dF(u).\end{aligned}$$

Note that $\check{u}(0) = 1$ and $\check{u}(\bar{v}) = 0$. By the definition of $\hat{u}(s)$, we have $\hat{u}(0) = 1$ and $\hat{u}(\bar{v}) = \underline{u}$. It then follows that

$$\frac{d\bar{\Phi}(s)}{ds}\Big|_{s=0} = \int_{\underline{u}}^{1} [\frac{\partial\phi_1(\lambda(u); s)}{\partial s} - \frac{\partial\phi_1(0; s)}{\partial s}]\Big|_{s=0} dF(u) \le 0,$$

where the inequality holds because $\phi_1(z; s)$ is submodular over $(z, s) \in [0, u] \times [0, \check{s}(u)]$ by Lemma 2(c). In addition,

$$\frac{d\bar{\Phi}(s)}{ds}\Big|_{s=\bar{v}} = -\int_{\underline{u}}^{1} \frac{\partial\phi_1(0; s)}{\partial s}\Big|_{s=\bar{v}} dF(u) \ge 0,$$

where the inequality holds because $\phi_1(u; s)$ is decreasing in s by Lemma 1(a). □

Proof of Corollary 1

With $s=\bar{v}$, we have $\check{u}=0$, and correspondingly $\hat{u}=\underline{u}$. It follows from Theorem 1 that the optimal mechanism $\mathcal{M}^*$ satisfies: if $u<\underline{u}$, the customer leaves without any purchase; if $u\geq\underline{u}$, the customer pays a deposit $D(\underline{u})=\phi_1(\lambda(\underline{u}))=\lambda(\underline{u})=0$ and purchases the seller's product at a strike price $p(\underline{u})=\frac{\bar{F}(\underline{u})}{f(\underline{u})}$. The corresponding optimal revenue is $\bar{\Phi}=\underline{u}\bar{F}(\underline{u})=p(\underline{u})\bar{F}(\underline{u})$. Effectively, $\mathcal{M}^*$ is equivalent to the seller making an exploding offer by selling the product at price $p(\underline{u})$, and the customer with valuation above $\underline{u}$ would purchase the product at period 0. □

B. Revelation Principle

B.1. General Mechanism

Following Myerson (1986) and Sugaya and Wolitzky (2021), we consider a multistage game with general communication between the seller and the customer. In each period $t=0,1,...,T$, we let $S_t=[0,1]$ denote the set of possible private signals received by the customer which could be his valuation for the seller's or for the outside alternative. We use $x_t\in X_t=\{0,1\}$ to denote the state of ownership with $x_t=1$ indicating that the customer possesses the seller's product at the *end* of period t and $x_t=0$ otherwise. The customer also has a set of all possible reports R_t to send to the seller and a set of possible actions A_t which depends on the state of ownership at period t.[9] In our problem, we let $A_t=\{0,1,2\}$ if $x_t=1$ and $A_t=\{1,2\}$ if $x_t=0$, where we use "0" to denote the option of stopping the search and purchasing the seller's product; "1" to denote stopping the search and turning to outside alternatives; and "2" to denote continuing the search. The seller has a set of all possible messages M_t to send to the customer and decides on the probability of allocation $\hat{q}_t$ and payment $\hat{z}_t$ with $(\hat{q}_t,\hat{z}_t)\in B_t$, where $B_t=[0,1]\times\mathbb{R}$ if $x_{t-1}=0$ and $B_t=\{0\}\times\mathbb{R}$ if $x_{t-1}=1$. For each $t=0,1,...,T$, let $A^t=\prod_{\tau=0}^t A_\tau$, and we analogously define $X^t=\prod_{\tau=0}^t X_\tau$, $S^t=\prod_{\tau=0}^t S_\tau$, $R^t=\prod_{\tau=0}^t R_\tau$, $M^t=\prod_{\tau=0}^t M_\tau$, and $B^t=\prod_{\tau=0}^t B_\tau$.[10]

The sequence of events within each period $t=0,1,...,T$ is as follows:

1. The customer's valuation for either the seller's product ($u\in S_0$) or the outside alternative ($v_t\in S_t, t=1,2,...,T$) is realized and privately observed by the customer.
2. The customer chooses a report $r_t\in R_t$ to send to the seller.
3. Given the customer's reports $r^t:=(r_0,...,r_t)\in R^t$ so far, the seller decides both the allocation probability $\hat{q}_t$ and payment $\hat{z}_t$ with $(\hat{q}_t,\hat{z}_t)\in B_t$, and the seller chooses a message $m_t\in M_t$ to send back to the customer.

[9] To simplify the notation, we follow the notation in Bergemann and Välimäki (2019) in making such dependence implicit.

[10] We use superscript t to denote indices from 0 to t while the notation $v_{[t]}$ stands for the vectors with indices from 1 to t.

4. Based on the allocation outcome and the message from the seller, the customer takes an action $a_t \in A_t$. In particular, if the product has been allocated to the customer, i.e., $x_t = 1$, the customer then chooses whether to take the seller's product $a_t = 0$ or the outside alternative $a_t = 1$, or to continue searching $a_t = 2$; if the product has not been allocated yet, i.e., $x_t = 0$, the customer chooses whether to stop and take the outside alternative $a_t = 1$ or to continue searching $a_t = 2$.

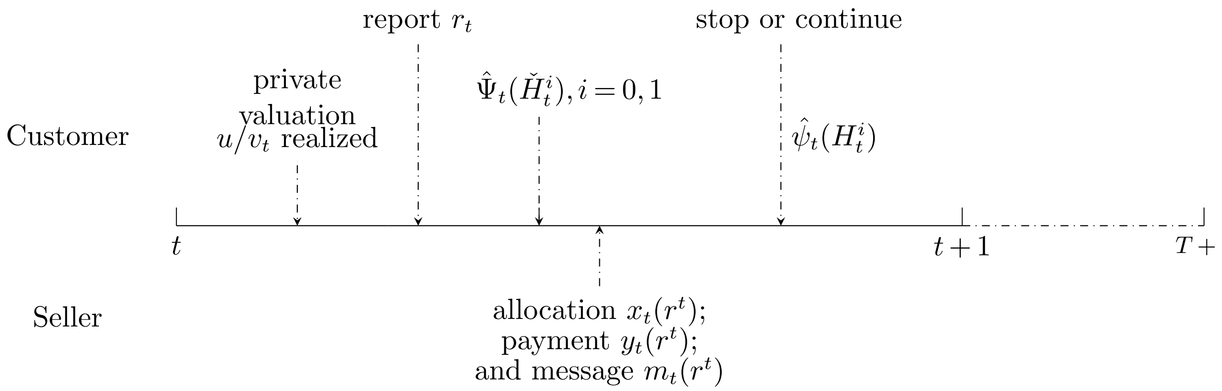

Figure 6 Sequence of Events under the General Communication Game

Figure 6 depicts the sequence of events under the general communication game. Note that Sugaya and Wolitzky (2021) consider the sequence of events where all the players (the seller and the customer in our problem) take actions simultaneously after the seller sends messages to the agent. However, in our model, the seller's allocation and transfer decisions should be made and observed before the customer takes the stopping decisions. As a result, the revelation principal shown in Sugaya and Wolitzky (2021) cannot be directly applied.

We let $\bar{\mathcal{H}}_t = R^{t-1} \times M^{t-1} \times X^{t-1} \times A^{t-1}$ be the set of public history which records the variables observable to both the seller and the customer at the beginning of period t. The set of public history after the customer's report in period t is then $R_t \times \bar{\mathcal{H}}_t$. The seller's mechanism consists of the allocation and payment functions: $\hat{q}_t : R_t \times \bar{\mathcal{H}}_t \to [0, 1]$ and $\hat{z}_t : R_t \times \bar{\mathcal{H}}_t \to \mathbb{R}$, and the message generating functions (also referred to as the mediation plan in Sugaya and Wolitzky 2021): $\hat{\mu}_t : R_t \times \bar{\mathcal{H}}_t \to \Delta(M_t)$ that determines the message to be sent (possibly in a random fashion) as a function of the public history. Here, $\Delta(X)$ denotes the set of probability distributions on a set X. The tuple $\{(R_t, M_t)_{t=0}^T, (\hat{\mu}_t, \hat{q}_t, \hat{z}_t)_{t=0}^T\}$ then completely describes a general mechanism of the seller.

Given the seller's mechanism, the customer maximizes his own total payoff by finding the optimal reporting and stopping strategies. Formally, for each t, we let $\mathcal{H}'_t = S^t \times \bar{\mathcal{H}}_t = S^t \times R^{t-1} \times M^{t-1} \times X^{t-1} \times A^{t-1}$ be the set of private history right before the customer makes a report, and $\mathcal{H}_t =$

$M_t \times X_t \times R_t \times \mathcal{H}'_t = S^t \times R^t \times M^t \times X^t \times A^{t-1}$ be the one after the seller's action and message are revealed. The customer's reporting and stopping strategies are then defined to be functions $\delta_t : \mathcal{H}'_t \to R_t$ and $\eta_t : \mathcal{H}_t \to A_t$ respectively. For convenience, we partition $\mathcal{H}_t$ into two sets $\mathcal{H}^1_t$ and $\mathcal{H}^0_t$ which respectively represent the set of history in which the customer either possesses or does not possess the seller's product post allocation:

$$\mathcal{H}^1_t = \left\{(s^t, r^t, m^t, x^t, a^{t-1}) \in \mathcal{H}_t : \exists \tau \le t \text{ s.t. } x_\tau = 1\right\},$$
$$\mathcal{H}^0_t = \left\{(s^t, r^t, m^t, x^t, a^{t-1}) \in \mathcal{H}_t : x_\tau = 0, \forall \tau \le t\right\}.$$

For any $\bar{h}_t \in \bar{\mathcal{H}}_t$, $h'_t = (s^t, \bar{h}_t) \in \mathcal{H}'_t$ and $h_t = (m_t, x_t, r_t, h'_t) \in \mathcal{H}_t$, we let $\hat{\psi}_t(h_t)$ be the customer's utility-to-go at period t right after the seller's allocation and message decision, and $\hat{\Psi}_t((r_t, h'_t)) = \mathbb{E}[\hat{\psi}_t(h_t)|r_t, h'_t]$ be the corresponding expected utility-to-go right before the seller's allocation and message decision. Note that the expectation here is taken with respect to the randomized allocation and message specified by $\hat{q}_t$ and $\hat{\mu}_t$. It then follows that $\hat{\psi}_t(h_t)$ must satisfy the following Bellman equation:

$$\hat{\psi}_t(h_t) = \begin{cases} u \vee \omega_t(v_{[t]}) \vee \mathbb{E}_{v_{t+1}}\left[\max_{r_{t+1}} \hat{\Psi}_{t+1}((r_{t+1}, h'_{t+1}))\right] - \hat{z}_t((r_t, \bar{h}_t)) - s\mathbb{1}_{\{t>0\}}, & \text{if } h_t \in \mathcal{H}^1_t \ ; \\ \omega_t(v_{[t]}) \vee \mathbb{E}_{v_{t+1}}\left[\max_{r+1} \hat{\Psi}_{t+1}((r_{t+1}, h'_{t+1}))\right] - \hat{z}_t((r_t, \bar{h}_t)) - s\mathbb{1}_{\{t>0\}}, & \text{if } h_t \in \mathcal{H}^0_t \ . \end{cases}$$

with $\hat{\psi}_{T+1}(\cdot) = 0$. The optimal stopping strategy at period t denoted as $\hat{\eta}^*_t(h_t)$, is then defined by the optimal solution to the above equation while the optimal reporting strategy denoted as $\hat{\delta}^*_t(h'_t)$, is specified by $\hat{\delta}^*_t(h'_t) \in \arg\max_{r_t} \hat{\Psi}_t((r_t, h'_t))$.

Before we establish the revelation principle, note first that there is no loss of optimality for the seller to focus on mechanisms that are independent of the historical messages. Intuitively, it is sufficient for the seller to use determinsitc message generating policies since her payoff is linear in the probability of a message history. As a result, a message history is completely determined by the historical reports, allocations and the customer's actions, and can be eliminated from the state variables. This observation is summarized in the lemma below.

Lemma B2. *It is without loss of optimality for the seller to use mechanisms* $(\hat{\mu}_t, \hat{q}_t, \hat{z}_t)_{t=0}^T$ *that are independent of* $m^t \in M^t$.

Proof: See Appendix D.3. □

With Lemma B2 and the observation that the only possible action history $a^{t-1} \in A^{t-1}$ upon entering period t is $a^{t-1} = \mathbf{2}^{t-1}$, we can write the allocation, payment, and message generating functions as functions of historical reports $r^t \in R^t$ only, i.e., $\hat{q}_t(r^t), \hat{z}_t(r^t), \hat{\mu}_t(r^t)$, where the dependence on x^t is made implicit. Similarly, for the customer, the public history can be summarized using only historical reports $r^t \in R^t$, which itself is uniquely determined by the signal history $s^t \in S^t$ via the recursion $r_\tau = \hat{\delta}_\tau(s^\tau, r^{\tau-1})$ for $\tau = 0, 1, ..., t$. Hence, we will use the shorthand notation $\hat{\delta}_t(s^t)$ for the customer's reporting strategy and $\hat{\eta}_t(s^t)$ for the stopping strategy.

B.2. Proof of Revelation Principle

As we have defined in Section 4, a mechanism is *direct* if $R_t = S_t$ and $M_t = A_t$. That is, in a direct mechanism, at each period, the agent is asked to report his private type $(\tilde{u}, \tilde{v}_{[t]})$ in S_t to the seller, and in return, the seller will send the customer a recommended action in A_t. We use $q_t(\tilde{u}, \tilde{v}_{[t]}), z_t(\tilde{u}, \tilde{v}_{[t]})$ to denote the corresponding allocation and transfer function in a direct mechanism, and let $\alpha_t^i(\tilde{u}, \tilde{v}_{[t]})$ and $\beta_t^i(\tilde{u}, \tilde{v}_{[t]})$ denote the probability of recommending action $i \in A_t$ when $x_t = 1$ and $x_t = 0$ respectively. Recall that a direct mechanism is said to be incentive compatible if it satisfies both obedience constraints (OB_t) and truth-telling constraints (TR_t). The revelation principle states that it is without loss of optimality for the seller to consider only incentive-compatible direct mechanisms since given any general mechanism one can construct an incentive-compatible direct mechanism such that the seller obtains the same revenue. The construction follows the same intuition as in the static setting whereupon obtaining a report $(\tilde{u}, \tilde{v}_{[t]})$, one then simply simulates what a customer with type $(\tilde{u}, \tilde{v}_{[t]})$ would get under a general mechanism. We formally state and prove it in the proposition below.

Proposition B2. *Given any mechanism* $\{(R_t, M_t)_{t=0}^T, (\hat{\mu}_t, \hat{q}_t, \hat{z}_t)_{t=0}^T\}$, *there exists an incentive-compatible direct mechanism* $\left((q_t(u, v_{[t]}), z_t(u, v_{[t]}), (\alpha_t^i(u, v_{[t]}))_{i=0}^2, (\beta_t^i(u, v_{[t]}))_{i=1}^2\right)_{t=0}^T$ *such that the seller obtains the same total expected revenue.*

Proof: Given any mechanism $\{(R_t, M_t)_{t=0}^T, (\hat{\mu}_t, \hat{q}_t, \hat{z}_t)_{t=0}^T\}$, we let $\hat{\delta}_t^*((u, v_{[t]}))$ and $\hat{\eta}_t^*((u, v_{[t]}))$ be the customer's optimal reporting and stopping strategy under the mechanism. The direct mechanism is then constructed as follows:

$$\begin{aligned}
&q_t(u, v_{[t]}) = \hat{q}_t((\hat{\delta}_\tau^*(u, v_{[\tau]}))_{\tau=0}^t), \quad z_t(u, v_{[t]}) = \hat{z}_t((\hat{\delta}_\tau^*(u, v_{[\tau]}))_{\tau=0}^t), \\
&\alpha_t^i(u, v_{[t]}) = \mathbb{1}_{\{\hat{\eta}_t^*(u, v_{[t]}) = i\}}, i = 0, 1, 2, \text{ when } x_t = 1, \\
&\beta_t^i(u, v_{[t]}) = \mathbb{1}_{\{\hat{\eta}_t^*(u, v_{[t]}) = i\}}, i = 1, 2, \text{ when } x_t = 0.
\end{aligned} \tag{B.1}$$

That is, given any reported type $(u, v_{[t]})$, the constructed direct mechanism follows exactly the same allocation and transfer as what the general mechanism would do for the customer whose true type is $(u, v_{[t]})$ and reports $(\hat{\delta}_\tau^*(u, v_{[\tau]}))_{\tau=0}^t$. In addition, the constructed direct mechanism recommends the optimal action taken by type-$(u, v_{[t]})$ customer in the general mechanism with probability one. Clearly, as long as our constructed mechanism in (B.1) is incentive compatible, then it will result in the same revenue collected from every type-$(u, v_{[t]})$ customer since by truthful reporting, the allocations and transfers are the same, and by obedience, type-$(u, v_{[t]})$ customer takes the same actions under both mechanisms.

To show incentive compatibility, recall first that $\psi_t(u, \tilde{u}; v_{[t]}, \tilde{v}_{[t]}; x_t)$ and $\tilde{\psi}_t(u, \tilde{u}; v_{[t]}, \tilde{v}_{[t]}; x_t)$ denote the customer's ex-post utility-to-go under the direct mechanism when he follows and ignores the seller's recommendation respectively. We similarly use the more explicit notation $\hat{\psi}_t(u, v_{[t]}; r^t; x_t)$ to denote the customer's ex-post utility-to-go under the general mechanism (note that we have used Lemma B2 here in eliminating the dependence on m^t). Next, we show that

$$\hat{\psi}_t(u, v_{[t]}; (\hat{\delta}^*_\tau(u, v_{[\tau]}))_{\tau=0}^t; x_t) = \psi_t(u, u; v_{[t]}, v_{[t]}; x_t), \tag{B.2}$$

$$\hat{\psi}_t(u, v_{[t]}; (\hat{\delta}^*_\tau(\tilde{u}, \tilde{v}_{[\tau]}))_{\tau=0}^t; x_t) \geq \tilde{\psi}_t(u, \tilde{u}; v_{[t]}, \tilde{v}_{[t]}; x_t) \geq \psi_t(u, \tilde{u}; v_{[t]}, \tilde{v}_{[t]}; x_t), \tag{B.3}$$

via induction. Equation (B.2) claims that if the customer is always obedient and truthtelling in the direct mechanism then he obtains the same payoff as in the general mechanism. Inequalities in (B.3) show that, given the history that the type-$(u, v_{[t]})$ customer misreports by mimicking a type-$(\tilde{u}, \tilde{v}_{[t]})$ customer, his utility-to-go under the general mechanism is no less than his best possible payoff under the direct mechanism which in turn dominates his payoff by following the seller's recommendation. The boundary case when $t = T+1$ clearly holds. Now suppose (B.2) and (B.3) hold for $t+1$. Then by $\hat{q}_{t+1}((\hat{\delta}^*_\tau(\tilde{u}, \tilde{v}_{[\tau]}))_{\tau=0}^{t+1}) = q_{t+1}(\tilde{u}, \tilde{v}_{[t+1]})$, we have

$$\hat{\Psi}_{t+1}(u, v_{[t+1]}; (\hat{\delta}^*_\tau(u, v_{[\tau]}))_{\tau=0}^{t+1}; x_t) = \Psi_{t+1}(u, u; v_{[t+1]}, v_{[t+1]}; x_t), \tag{B.4}$$

$$\hat{\Psi}_{t+1}(u, v_{[t+1]}; (\hat{\delta}^*_\tau(\tilde{u}, \tilde{v}_{[\tau]}))_{\tau=0}^{t+1}; x_t) \geq \tilde{\Psi}_{t+1}(u, \tilde{u}; v_{[t+1]}, \tilde{v}_{[t+1]}; x_t) \geq \Psi_{t+1}(u, \tilde{u}; v_{[t+1]}, \tilde{v}_{[t+1]}; x_t). \tag{B.5}$$

It follows from (B.4) and (B.5) that for any $\tilde{v}_{t+1}$, we have

$$\begin{aligned}\Psi_{t+1}(u, u; v_{[t+1]}, v_{[t+1]}; x_t) &= \hat{\Psi}_{t+1}(u, v_{[t+1]}; (\hat{\delta}^*_\tau(u, v_{[\tau]}))_{\tau=0}^{t+1}; x_t) \\ &\geq \hat{\Psi}_{t+1}(u, v_{[t+1]}; (\hat{\delta}^*_\tau(u, v_{[\tau]}))_{\tau=0}^{t}, \hat{\delta}^*_{t+1}(u, v_{[t]}, \tilde{v}_{t+1})); x_t) \\ &\geq \Psi_{t+1}(u, u; v_{[t+1]}, (v_{[t]}, \tilde{v}_{t+1}); x_t),\end{aligned} \tag{B.6}$$

where the first inequality uses the fact that $\hat{\delta}^*_{t+1}(u, v_{[t+1]})$ is an optimal report given the true type being $(u, v_{[t+1]})$. By the recursive definition of $\hat{\psi}_t$, when $x_t = 1$, we then have (B.2) also holds for t:

$$\begin{aligned}&\hat{\psi}_t(u, v_{[t]}; (\hat{\delta}^*_\tau(u, v_{[\tau]}))_{\tau=0}^t; x_t = 1) \\ =&u \vee \omega_t(v_{[t]}) \vee \mathbb{E}_{v_{t+1}} \hat{\Psi}_{t+1}(u, v_{[t+1]}; (\hat{\delta}^*_\tau(u, v_{[\tau]}))_{\tau=0}^{t+1}; 1) - \hat{z}_t((\hat{\delta}^*_\tau(u, v_{[\tau]}))_{\tau=0}^t) - s\mathbb{1}_{\{t>0\}} \\ =&\alpha_t^0(u, v_{[t]})u + \alpha_t^1(u, v_{[t]})\omega_t(v_{[t]}) + \alpha_t^2(u, v_{[t]})\mathbb{E}_{v_{t+1}} \Psi_{t+1}(u, u; v_{[t+1]}, v_{[t+1]}; 1) - z_t(u, v_{[t]}) - s\mathbb{1}_{\{t>0\}} \\ =&\alpha_t^0(u, v_{[t]})u + \alpha_t^1(u, v_{[t]})\omega_t(v_{[t]}) + \alpha_t^2(u, v_{[t]})\mathbb{E}_{v_{t+1}} \max_{\tilde{v}_{t+1}} \Psi_{t+1}(u, u; v_{[t+1]}, (v_{[t]}, \tilde{v}_{t+1}); 1) - z_t(u, v_{[t]}) - s\mathbb{1}_{\{t>0\}} \\ =&\psi_t(u, u; v_{[t]}, v_{[t]}; x_t = 1),\end{aligned}$$

where we used (B.1) and (B.4) in the second equality and the third equality is due to (B.6). In addition, given any report history $(\hat{\delta}^*_\tau(\tilde{u}, \tilde{v}_{[\tau]}))_{\tau=0}^t$

$$\hat{\psi}_t(u, v_{[t]}; (\hat{\delta}^*_\tau(\tilde{u}, \tilde{v}_{[\tau]}))_{\tau=0}^t; x_t = 1)$$

$$
\begin{aligned}
&\geq u \vee \omega_t(v_{[t]}) \vee \mathbb{E}_{v_{t+1}} \max_{\tilde{v}_{t+1}} \hat{\Psi}_{t+1}(u, v_{[t+1]}; (\hat{\delta}^*_\tau(\tilde{u}, \tilde{v}_{[\tau]}))_{\tau=0}^{t+1}; 1) - \hat{z}_t((\hat{\delta}^*_\tau(\tilde{u}, \tilde{v}_{[\tau]}))_{\tau=0}^{t}) - s\mathbb{1}_{\{t>0\}} \\
&\geq u \vee \omega_t(v_{[t]}) \vee \mathbb{E}_{v_{t+1}} \max_{\tilde{v}_{t+1}} \tilde{\Psi}_{t+1}(u, \tilde{u}; v_{[t+1]}, \tilde{v}_{[t+1]}; 1) - z_t(\tilde{u}, \tilde{v}_{[t]}) - s\mathbb{1}_{\{t>0\}} \\
&= \tilde{\psi}_t(u, \tilde{u}; v_{[t]}, \tilde{v}_{[t]}; x_t = 1) \\
&\geq \alpha_t^0(\tilde{u}, \tilde{v}_{[t]})u + \alpha_t^1(\tilde{u}, \tilde{v}_{[t]})\omega_t(v_{[t]}) + \alpha_t^2(\tilde{u}, \tilde{v}_{[t]})\mathbb{E}_{v_{t+1}} \max_{\tilde{v}_{t+1}} \Psi_{t+1}(u, \tilde{u}; v_{[t+1]}, \tilde{v}_{[t+1]}; 1) - z_t(\tilde{u}, \tilde{v}_{[t]}) - s\mathbb{1}_{\{t>0\}} \\
&= \psi_t(u, \tilde{u}; v_{[t]}, \tilde{v}_{[t]}; x_t = 1),
\end{aligned}
$$

where the first inequality is due to the fact that for any $\tilde{v}_{t+1}$, $\hat{\delta}^*_{t+1}(\tilde{u}, \tilde{v}_{[t+1]})$ is always a feasible report and the second and third inequalities used (B.5). One can similarly show that (B.2) and (B.3) also hold for $x_t = 0$.

Finally, note that (B.2) and (B.3) would imply (OB_t) and (TR_t) since

$$
\psi_t(u, u; v_{[t]}, v_{[t]}; x_t) = \hat{\psi}_t(u, v_{[t]}; (\hat{\delta}^*_\tau(u, v_{[\tau]}))_{\tau=0}^{t}; x_t) \geq \tilde{\psi}_t(u, u; v_{[t]}, v_{[t]}; x_t)
$$

and

$$
\begin{aligned}
\Psi_t(u, u; v_{[t]}, v_{[t]}; x_{t-1}) &= \hat{\Psi}_t(u, v_{[t]}; (\hat{\delta}^*_\tau(u, v_{[\tau]}))_{\tau=0}^{t}; x_{t-1}) \geq \tilde{\Psi}_t(u, u; v_{[t]}, (v_{[t-1]}, \tilde{v}_t); x_{t-1}), \\
\Psi_0(u, u) &= \hat{\Psi}_0(u; \hat{\delta}^*_0(u)) \geq \tilde{\Psi}_0(u, \tilde{u}).
\end{aligned}
$$

□

C. Correlated Case

In this section, we examine an extension where the customer's valuation for the outside alternative is positively correlated with that for the seller's product. Specifically, at each period, by paying a search cost s, the customer gets an outside alternative with net utility $v_t = \gamma u + \varepsilon_t$ where $0 \leq \gamma \leq 1$ captures the strength of correlation, and $\varepsilon_t \geq 0$ are independent with u and are i.i.d across time. We focus here on recall with finite memory so that

$$
\omega_t(v_{[t]}) = \max_{(t-k)^+ + 1 \leq \tau \leq t} v_\tau = \gamma u + \max_{(t-k)^+ + 1 \leq \tau \leq t} \varepsilon_\tau = \gamma u + \omega_t(\varepsilon_{[t]}).
$$

Customer's ex-ante utility-to-go at period t in the first-best benchmark case is similarly given by

$$
\phi_t(u, \varepsilon_{[t-1]}) = \mathbb{E}_{\varepsilon_t}[u \vee (\gamma u + \omega_t(\varepsilon_{[t]})) \vee \phi_{t+1}(u, \varepsilon_{[t]})] - s,
$$

with $\phi_{T+1}(u, \varepsilon_{[T]}) = 0$. Different from the independent case considered in (1), here $\phi_1(0)$ is no longer the customer's reservation utility at the first period in the absence of the seller. Without the seller's product, u still affects the customer's valuation for the outside alternatives and the customer's ex-ante utility-to-go at period t, denoted as $\hat{\phi}_t(u, \varepsilon_{[t-1]})$, now satisfies the following Bellman equation instead:

$$
\hat{\phi}_t(u, \varepsilon_{[t-1]}) = \mathbb{E}_{\varepsilon_t}[(\gamma u + \omega_t(\varepsilon_{[t]})) \vee \hat{\phi}_{t+1}(u, \varepsilon_{[t]})] - s,
$$

with $\hat{\phi}_{T+1}(u,\varepsilon_{[T]})=0$. Since $\gamma u+\omega_T(\varepsilon_{[T]})\geq 0$, one can easily show via induction that

$$\hat{\phi}_t(u,\varepsilon_{[t-1]})=\gamma u+\nu_t(\varepsilon_{[t-1]}),$$

where

$$\nu_t(\varepsilon_{[t-1]})=\mathbb{E}_{\varepsilon_t}[\omega_t(\varepsilon_{[t]})\vee\nu_{t+1}(\varepsilon_{[t]})]-s,$$

with $\nu_{T+1}(\varepsilon_{[T]})=0$. Then the customer's reservation utility at the first period is $\hat{\phi}_1(u)=\gamma u+\nu_1$.

Given that the customer continues searching to period t, let $\xi_t(u,\varepsilon_{[t-1]})$ be the customer's purchasing probability of the seller's product thereafter. By definition, $\xi_t(u,\varepsilon_{[t-1]})$ satisfies the following recursive equation:

$$\xi_t(u,\varepsilon_{[t-1]})=\mathbb{E}\left[\mathbb{1}_{\{u\geq(\gamma u+\omega_t(\varepsilon_{[t]}))\vee\phi_{t+1}(u,\varepsilon_{[t]})\}}+\mathbb{1}_{\{\phi_{t+1}(u,\varepsilon_{[t]})>u\vee(\gamma u+\omega_t(\varepsilon_{[t]}))\}}\xi_{t+1}(u,\varepsilon_{[t]})\right],$$

with $\xi_{T+1}(u,\varepsilon_{[T]})=0$. We derive some properties of $\phi_t(u,\varepsilon_{[t-1]})$ and $\xi_t(u,\varepsilon_{[t-1]})$ in the following lemma.

Lemma C3. *(a) $\phi_t(u,\varepsilon_{[t-1]})$ is increasing convex in u, $\phi_t(u,\varepsilon_{[t-1]})-u$ is decreasing in u and $\phi_t(u,\varepsilon_{[t-1]})-\gamma u$ is increasing in u;*

(b) Let $\eta_t(u,\varepsilon_{[t-1]})=\frac{\partial\phi_t(u,\varepsilon_{[t-1]})}{\partial u}$. Then $\eta_t(u,\varepsilon_{[t-1]})=\gamma+(1-\gamma)\xi_t(u,\varepsilon_{[t-1]})$.

Proof: See Appendix D.4. □

We now turn to the problem with information asymmetry. One can define the customer's and the seller's utility-to-go functions ψ_t and φ_t similarly as in Section 4, and define the truth-telling and obedience constraints correspondingly. We note that the individual rationality constraints now become

$$\mathbb{E}_{\varepsilon_t}[\Psi_t(u,u;\varepsilon_{[t]},\varepsilon_{[t]};x_{t-1})]\geq\hat{\phi}_t(u,\varepsilon_{[t-1]}),\quad\forall u\in[0,1],t=1,2,...,T,\quad\text{and }\Psi_0(u,u)\geq\hat{\phi}_1(u).$$

Similar to Section 4, we focus on a relaxed problem of the same form as problem (4) to derive a revenue upper bound, and then propose a feasible mechanism that achieves the revenue upper bound. The following proposition characterizes the revenue upper bound.

Proposition C3. *A revenue upper bound is $\bar{\Phi}=\mathbb{E}_u\{\lambda(u)\vee\phi_1(\lambda(u))\}-\phi_1(0)$.*

The proof follows the same steps as that of Proposition 3 and hence omitted.

When $0\leq\gamma<1$, let $\hat{u}=\inf\{u\in[0,1],\lambda(u)-\phi_1(\lambda(u))\geq 0\}$ and $\underline{u}$ be the solution to $\lambda(u)=0$. The following theorem constructs a feasible mechanism that achieves the revenue upper bound.

Theorem 2. *When $\gamma=1$, it is optimal to post a constant price $(0-\phi_1(0))^+$. When $0\leq\gamma<1$, the optimal mechanism $\mathcal{M}^*$ consists of a menu of American option contracts $\{(D(u),T(u)),u\in[\underline{u},1]\}$ and it satisfies:*

(a) if $u<\underline{u}$, then the customer with valuation u will search without purchasing from the seller;

(b) if $u \geq \underline{u}$*, then the customer with valuation* u *is charged a non-refundable deposit*

$$D(u)=\begin{cases}\phi_1(\lambda(u))+\gamma p(u)-\phi_1(0)-\int_0^u \eta_1(\lambda(\nu))d\nu, & \text{if } u\in[\underline{u},\hat{u});\\ \phi_1(\lambda(\hat{u}))+\gamma p(\hat{u})-\phi_1(0)-\int_0^{\hat{u}} \eta_1(\lambda(\nu))d\nu, & \text{if } u\in[\hat{u},1],\end{cases}$$

at period 0 so that he can purchase the seller's product at a strike price

$$T(u)=\begin{cases}(1-\gamma)p(u), & \text{if } u\in[\underline{u},\hat{u});\\ (1-\gamma)p(\hat{u}), & \text{if } u\in[\hat{u},1],\end{cases}$$

at any time during the horizon. Here $p(u)=\frac{\bar{F}(u)}{f(u)}$.

Proof: See Appendix D.5. □

Compared to the independence case with $\gamma=0$, Theorem 2 shows that when the customer has a baseline utility γu for all outside alternatives, it is optimal for the seller to respond by giving a $1-\gamma$ percent discount to the strike price and charge the amount reduced to the deposit instead.

We remark that our assumption $\varepsilon_t \geq 0$ is critical for our upper bound in Proposition C3 to be tight. Lu and Wang (2021) analyze a two-period version of our model here by restricting $\varepsilon_t \leq 0$ and show that the revenue upper bound obtained by assuming future private signals are observable is no longer tight. They further establish that using a constant pricing strategy in this case is optimal.

D. Proof of Auxiliary Results

Before we prove Proposition A1, we first show in the following lemma that the customer's utility function $\check{\psi}_t(u,\tilde{u};v_{[t]};x_t)$ in the relaxed problem (4) has a similar property as its first-best counterpart (see Lemma 1).

Lemma D4. *For* $t=1,2,...,T$*,* $\check{\psi}_t(u,\tilde{u};v_{[t]};x_t)$ *is increasing convex in* u*, and* $\check{\Psi}_t(u,\tilde{u};v_{[t]};x_{t-1})$ *is increasing convex in* u*.*

Proof: We prove $\check{\psi}_t(u,\tilde{u};v_{[t]};x_t)$ and $\check{\Psi}_t(u,\tilde{u};v_{[t]};x_{t-1})$ are increasing convex in u by induction. At period $t=T+1$,

$$\check{\psi}_{T+1}(u,\tilde{u};v_{[T+1]};x_{T+1})=\check{\Psi}_{T+1}(u,\tilde{u};v_{[T+1]};x_T)=0.$$

Suppose $\check{\Psi}_{t+1}(u,\tilde{u};v_{[t+1]};x_t)$ is increasing convex in u, then

$$\check{\psi}_t(u,\tilde{u};v_{[t]};1)=\alpha_t^1(\tilde{u},v_{[t]})u+\alpha_t^2(\tilde{u},v_{[t]})\omega_t(v_{[t]})+\alpha_t^3(\tilde{u},v_{[t]})\mathbb{E}_{v_{t+1}}\check{\Psi}_{t+1}(u,\tilde{u};v_{[t+1]};1)-z_t(\tilde{u},v_{[t]})-s\mathbb{1}_{\{t>0\}}.$$

By our induction hypothesis, $\check{\psi}_t(u,\tilde{u};v_{[t]};1)$ is then a weighted average of an increasing linear function and increasing convex functions, and hence is also increasing convex in u. Similarly, we can prove that $\check{\psi}_t(u,\tilde{u};v_{[t]};0)$ is increasing convex in u. Then it is clear that

$$\Psi_t(u,\tilde{u};v_{[t]};x_{t-1})=\mathbb{E}[\psi_t(u,\tilde{u};v_{[t]};x_t)|x_{t-1}]$$

is increasing convex in u. □

D.1. Proof of Proposition A1

For convenience, we denote for $t=0,1,...,T$, $h_t(u,v_{[t]};x_t)=\check{\psi}_t(u,u;v_{[t]};x_t)+\varphi_t(u,v_{[t]};x_t)$, $H_t(u,v_{[t]};x_{t-1})=\mathbb{E}[h_t(u,v_{[t]};x_t)|x_{t-1}]$, and $l_t(u,v_{[t]};x_t)=h_t(u,v_{[t]};x_t)-\frac{\partial\psi_t(u,\tilde{u};v_{[t]};x_t)}{\partial u}\Big|_{\tilde{u}=u}\frac{\bar{F}(u)}{f(u)}$.

We first derive a recursive formulation for $h_t(u,v_{[t]};1), t=0,1,2,...,T$.

$$\begin{aligned}h_t(u,v_{[t]};1)=&\check{\psi}_t(u,u;v_{[t]};1)+\varphi_t(u,v_{[t]};1)\\=&\alpha_t^0(u,v_{[t]})u+\alpha_t^1(u,v_{[t]})\omega_t(v_{[t]})+\alpha_t^2(u,v_{[t]})\mathbb{E}_{v_{t+1}}[\check{\Psi}_{t+1}(u,u;v_{[t+1]};1)]-z_t(u,v_{[t]})-s\mathbb{1}_{\{t>0\}}\\&+z_t(u,v_{[t]})+\alpha_t^2(u,v_{[t]})\mathbb{E}_{v_{t+1}}[\Phi_{t+1}(u,v_{[t+1]};1)]\\=&\alpha_t^0(u,v_{[t]})u+\alpha_t^1(u,v_{[t]})\omega_t(v_{[t]})+\alpha_t^2(u,v_{[t]})\mathbb{E}_{v_{t+1}}[H_{t+1}(u,v_{[t+1]};1)]-s\mathbb{1}_{\{t>0\}}.\end{aligned}\tag{D.1}$$

Similarly,

$$\begin{aligned}h_t(u,v_{[t]};0)=&\check{\psi}_t(u,u;v_{[t]};0)+\varphi_t(u,v_{[t]};0)\\=&\beta_t^1(u,v_{[t]})\omega_t(v^t)+\beta_t^2(u,v_{[t]})\mathbb{E}_{v_{t+1}}[\check{\Psi}_{t+1}(u,u;v_{[t+1]};0)]-s\mathbb{1}_{\{t>0\}}-z_t(u,v_{[t]})\\&+z_t(u,v_{[t]})+\beta_t^2(u,v_{[t]})\mathbb{E}_{v_{t+1}}[\Phi_{t+1}(u,v_{[t+1]};0)]\\=&\beta_t^1(u,v_{[t]})\omega_t(v_{[t]})+\beta_t^2(u,v_{[t]})\mathbb{E}_{v_{t+1}}[H_{t+1}(u,v_{[t+1]};0)]-s\mathbb{1}_{\{t>0\}}.\end{aligned}$$

Next, we derive a recursive formulation for $\frac{\partial\check{\psi}_t(u,\tilde{u};v_{[t]};x_t)}{\partial u}\Big|_{\tilde{u}=u}$ for $t=1,2,...,T$. By definition,

$$\begin{aligned}\check{\psi}_t(u,\tilde{u};v_{[t]};1)=&\alpha_t^0(\tilde{u},v_{[t]})u+\alpha_t^1(\tilde{u},v_{[t]})\omega_t(v_{[t]})+\alpha_t^2(\tilde{u},v_{[t]})\mathbb{E}_{v_{t+1}}\left[\check{\Psi}_{t+1}(u,\tilde{u};v_{[t+1]};1)\right]-z_t(\tilde{u},v_{[t]})-s\mathbb{1}_{\{t>0\}},\\\check{\psi}_t(u,\tilde{u};v_{[t]};0)=&\beta_t^1(\tilde{u},v_{[t]})\omega_t(v_{[t]})+\beta_t^2(\tilde{u},v_{[t]})\mathbb{E}_{v_{t+1}}\left[\check{\Psi}_{t+1}(u,\tilde{u};v_{[t+1]};0)\right]-z_t(\tilde{u},v_{[t]})-s\mathbb{1}_{\{t>0\}}.\end{aligned}$$

Because $\check{\psi}_t(u,\tilde{u};v_{[t]};x_t)$ is convex in u by Lemma D4, implying that $\check{\psi}_t(u,\tilde{u};v_{[t]};x_t)$ is almost everywhere differentiable. Therefore, whenever $\check{\psi}_t(u,\tilde{u};v_{[t]};x_t)$ is differentiable over u, we have for $t=1,...,T$

$$\begin{aligned}\frac{\partial\check{\psi}_t(u,\tilde{u};v_{[t]};1)}{\partial u}\Big|_{\tilde{u}=u}=&\alpha_t^0(u,v_{[t]})+\alpha_t^2(u,v_{[t]})\mathbb{E}_{v_{t+1}}\left[\frac{\partial\check{\Psi}_{t+1}(u,\tilde{u};v_{[t+1]};1)}{\partial u}\Big|_{\tilde{u}=u}\right]\\=&\alpha_t^0(u,v_{[t]})+\alpha_t^2(u,v_{[t]})\mathbb{E}_{v_{t+1}}\left[\frac{\partial\check{\psi}_{t+1}(u,\tilde{u};v_{[t+1]};1)}{\partial u}\Big|_{\tilde{u}=u}\right],\end{aligned}\tag{D.2}$$

and

$$\begin{aligned}\frac{\partial\check{\psi}_t(u,\tilde{u};v_{[t]};0)}{\partial u}\Big|_{\tilde{u}=u}=&\beta_t^2(u,v_{[t]})\mathbb{E}_{v_{t+1}}\left[\frac{\partial\check{\Psi}_{t+1}(u,\tilde{u};v_{[t+1]};0)}{\partial u}\Big|_{\tilde{u}=u}\right]\\=&\beta_t^2(u,v_{[t]})\mathbb{E}_{v_{t+1}}\Big[q_{t+1}(u,v_{[t+1]})\frac{\partial\check{\psi}_{t+1}(u,\tilde{u};v_{[t+1]};1)}{\partial u}\Big|_{\tilde{u}=u}\\&+(1-q_{t+1}(u,v_{[t+1]}))\frac{\partial\check{\psi}_{t+1}(u,\tilde{u};v_{[t+1]};0)}{\partial u}\Big|_{\tilde{u}=u}\Big],\end{aligned}\tag{D.3}$$

and

$$\frac{\partial \check{\Psi}_0(u,\tilde{u})}{\partial u}\Big|_{\tilde{u}=u} = q_0(u)\frac{\partial \check{\psi}_0(u,\tilde{u};1)}{\partial u}\Big|_{\tilde{u}=u} + (1-q_0(u))\frac{\partial \check{\psi}_0(u,\tilde{u};0)}{\partial u}\Big|_{\tilde{u}=u}. \tag{D.4}$$

By using the recursive equations for $h_t(u,v_{[t]};1)$ in (D.1) and $\frac{\partial \check{\psi}_t(u,\tilde{u};v_{[t]};1)}{\partial u}\Big|_{\tilde{u}=u}$ in (D.2), we derive the recursive equations for $l_t(u,v_{[t]};1), t=0,1,2,...,T$ as follows:

$$\begin{aligned}
l_t(u,v_{[t]};1) =& h_t(u,v_{[t]};1) - \frac{\partial \check{\psi}_t(u,\tilde{u};v_{[t]};1)}{\partial u}\Big|_{\tilde{u}=u}\frac{\bar{F}(u)}{f(u)} \\
=& \alpha_t^0(u,v_{[t]})u + \alpha_t^1(u,v_{[t]})\omega_t(v_{[t]}) + \alpha_t^2(u,v_{[t]})\mathbb{E}_{v_{t+1}}H_{t+1}(u,v_{[t+1]};1) - s\mathbb{1}_{\{t>0\}} \\
& - \left(\alpha_t^0(u,v^t) + \alpha_t^2(u,v^t)\mathbb{E}_{v_{t+1}}\left[\frac{\partial \check{\Psi}_{t+1}(u,\tilde{u};v_{[t+1]};1)}{\partial u}\Big|_{\tilde{u}=u}\right]\right)\frac{\bar{F}(u)}{f(u)} \\
=& \alpha_t^0(u,v_{[t]})u + \alpha_t^1(u,v_{[t]})\omega_t(v_{[t]}) + \alpha_t^2(u,v_{[t]})\mathbb{E}_{v_{t+1}}h_{t+1}(u,v_{[t+1]};1) - s\mathbb{1}_{\{t>0\}} \\
& - \left(\alpha_t^0(u,v^t) + \alpha_t^2(u,v^t)\mathbb{E}_{v_{t+1}}\left[\frac{\partial \check{\psi}_{t+1}(u,\tilde{u};v_{[t+1]};1)}{\partial u}\Big|_{\tilde{u}=u}\right]\right)\frac{\bar{F}(u)}{f(u)} \\
=& \alpha_t^0(u,v_{[t]})\lambda(u) + \alpha_t^1(u,v_{[t]})\omega_t(v_{[t]}) + \alpha_t^2(u,v_{[t]})\mathbb{E}_{v_{t+1}}l_{t+1}(u,v_{[t+1]};1) - s\mathbb{1}_{\{t>0\}},
\end{aligned}$$

Similarly,

$$\begin{aligned}
& l_t(u,v_{[t]};0) \\
=& h_t(u,v_{[t]};0) - \frac{\partial \check{\psi}_t(u,\tilde{u};v_{[t]};0)}{\partial u}\Big|_{\tilde{u}=u}\frac{\bar{F}(u)}{f(u)} \\
=& \beta_t^1(u,v_{[t]})\omega_t(v_{[t]}) + \beta_t^2(u,v_{[t]})\mathbb{E}_{v_{t+1}}[H_{t+1}(u,v_{[t]};0)] - s\mathbb{1}_{\{t>0\}} - \beta_t^2(u,v_{[t]})\mathbb{E}_{v_{t+1}}\frac{\partial \check{\Psi}_{t+1}(u,\tilde{u};v_{[t+1]};0)}{\partial u}\Big|_{\tilde{u}=u} \\
=& \beta_t^1(u,v_{[t]})\omega_t(v_{[t]}) + \beta_t^2(u,v_{[t]})\mathbb{E}_{v_{t+1}}[q_{t+1}(u,v_{[t+1]})h_{t+1}(u,v_{[t+1]};1) + (1-q_{t+1}(u,v_{[t+1]}))h_{t+1}(u,v_{[t+1]};0)] \\
& - s\mathbb{1}_{\{t>0\}} - \beta_t^2(u,v_{[t]})\mathbb{E}_{v_{t+1}}\Big[q_{t+1}(u,v_{[t+1]})\frac{\partial \check{\psi}_{t+1}(u,\tilde{u};v_{[t+1]};1)}{\partial u}\Big|_{\tilde{u}=u} \\
& + (1-q_{t+1}(u,v_{[t+1]}))\frac{\partial \check{\psi}_{t+1}(u,\tilde{u};v_{[t+1]};0)}{\partial u}\Big|_{\tilde{u}=u}\Big]\frac{\bar{F}(u)}{f(u)} \\
=& \beta_t^1(u,v_{[t]})\omega_t(v_{[t]}) + \beta_t^2(u,v_{[t]})\mathbb{E}_{v_{t+1}}[q_{t+1}(u,v_{[t+1]})l_{t+1}(u,v_{[t+1]};1) + (1-q_{t+1}(u,v_{[t+1]}))l_{t+1}(u,v_{[t+1]};0)] - s\mathbb{1}_{\{t>0\}}.
\end{aligned}$$

Now we are ready to reformulate the seller's revenue function:

$$\begin{aligned}
\mathbb{E}_u[\Phi_0(u)] &= \mathbb{E}_u\big[q_0(u)\varphi_0(u;1) + (1-q_0(u))\varphi_0(u;0)\big] \\
&= \mathbb{E}_u\big[q_0(u)\varphi_0(u;1) + (1-q_0(u))\varphi_0(u;0) + q_0(u)\check{\psi}_0(u;1) + (1-q_0(u))\check{\psi}_0(u;0) - \check{\Psi}_0(u,u)\big] \\
&= \mathbb{E}_u\big[q_0(u)h_0(u;1) + (1-q_0(u))h_0(u;0) - \check{\Psi}_0(u,u)\big].
\end{aligned}$$

The convexity of $\check{\Psi}_0(u,u)$ implies that it is absolutely continuous in u and hence

$$\mathbb{E}_u[\check{\Psi}_0(u,u)] = \mathbb{E}_u\Big[\check{\Psi}_0(0,0) + \int_0^u \frac{\partial \check{\Psi}_0(\nu,\tilde{u})}{\partial \nu}\Big|_{\tilde{u}=\nu} d\nu\Big] = \check{\Psi}_0(0,0) + \mathbb{E}_u\Big[\frac{\partial \check{\Psi}_0(u,\tilde{u})}{\partial u}\Big|_{\tilde{u}=u}\frac{\bar{F}(u)}{f(u)}\Big],$$

where the last equality holds by integration by parts. Finally, using the equations for $\mathbb{E}_u[\check{\Psi}_0(u,u)]$ and $\frac{\partial \check{\Psi}_0(u,\tilde{u})}{du}\Big|_{\tilde{u}=u}$ derived in (D.4), we have

$$
\begin{aligned}
&\mathbb{E}_u[\Phi_0(u)]\\
=&\mathbb{E}\left[q_0(u)h_0(u;1)+(1-q_0(u))h_0(u;0)-\check{\Psi}_0(u,u)\right]\\
=&\mathbb{E}\left[q_0(u)h_0(u;1)+(1-q_0(u))h_0(u;0)-\frac{\partial \check{\Psi}_0(u,\tilde{u})}{du}\Big|_{\tilde{u}=u}\frac{\bar{F}(u)}{f(u)}\right]-\check{\Psi}_0(0,0)\\
=&\mathbb{E}\left[q_0(u)h_0(u;1)+(1-q_0(u))h_0(u;0)-q_0(u)\frac{\partial \check{\psi}_0(u,\tilde{u};1)}{du}\Big|_{\tilde{u}=u}\frac{\bar{F}(u)}{f(u)}-(1-q_0(u))\frac{\partial \check{\psi}_0(u,\tilde{u};0)}{du}\Big|_{\tilde{u}=u}\frac{\bar{F}(u)}{f(u)}\right]-\check{\Psi}_0(0,0)\\
=&\mathbb{E}\left[q_0(u)l_0(u;1)+(1-q_0(u))l_0(u;0)\right]-\check{\Psi}_0(0,0). \quad \square
\end{aligned}
$$

D.2. Proof of Lemma A1

By taking derivative of $\phi_1(\lambda(\tilde{u}))$ over $\tilde{u}$, we have

$$\frac{d\phi_1(\lambda(\tilde{u}))}{d\tilde{u}}=\phi_1'(\lambda(\tilde{u}))\lambda'(\tilde{u})=\xi_1(\lambda(\tilde{u}))(1-p'(\tilde{u})).$$

Then, by the definition of $D(\tilde{u})$,

$$\frac{dD(\tilde{u})}{d\tilde{u}}=\frac{d\phi_1(\lambda(\tilde{u}))}{d\tilde{u}}-\xi_1(\lambda(\tilde{u}))=-p'(\tilde{u})\xi_1(\lambda(\tilde{u})).$$

Note that $p(\tilde{u})$ is decreasing in $\tilde{u}$ by our assumption of increasing failure rate of $F(\cdot)$ and hence $D(\tilde{u})$ is increasing in $\tilde{u}$ (one can also easily see that $p(\tilde{u})+D(\tilde{u})$ is decreasing in $\tilde{u}$).

(a) For any $u\in[0,1]$ and $\tilde{u}\in[\underline{u},1]$, we show $w_0(u,\tilde{u})$ is decreasing in $\tilde{u}$ for $\tilde{u}\geq u$ by distinguishing between two cases.

- If $\tilde{u}\in[\hat{u},1]$, since $D(\tilde{u})=D(\hat{u})$ and $p(\tilde{u})=p(\hat{u})$, we have $w_0(u,\tilde{u})=w_0(u,\hat{u})$ and $\phi_t(u-p(\tilde{u}),v_{[t-1]})=\phi_t(u-p(\hat{u}),v_{[t-1]})$, which are constants in $\tilde{u}$.
- If $\tilde{u}\in[\underline{u}\vee u,\hat{u})$, we have $u-p(\tilde{u})\leq\hat{u}-p(\hat{u})=\lambda(\hat{u})$. Then by the definition of $\hat{u}$ and the fact that $\phi_t(u,v_{[t-1]})-u$ is decreasing in u by Lemma 1(a), we have $\phi_1(u-p(\tilde{u}))\geq u-p(\tilde{u})$. Then

$$\frac{\partial w_0(u,\tilde{u})}{\partial\tilde{u}}=\frac{\partial\phi_1(u-p(\tilde{u}))}{\partial\tilde{u}}-\frac{dD(\tilde{u})}{d\tilde{u}}=p'(\tilde{u})[\xi_1(\lambda(\tilde{u}))-\xi_1(u-p(\tilde{u}))].$$

  Since $\xi_1(x)$ is increasing in x by Lemma 1(b) and due to $u\leq\tilde{u}$, we have $\xi_1(\lambda(\tilde{u}))\geq\xi_1(u-p(\tilde{u}))$ and by $p'(\tilde{u})\leq 0$, it follows that $\frac{\partial w_0(u,\tilde{u})}{\partial\tilde{u}}\leq 0$.

(b) For $\tilde{u}<u$,

- If $u-p(\tilde{u})\geq\phi_1(u-p(\tilde{u}))$, then

$$\frac{\partial w_0(u,\tilde{u})}{\partial\tilde{u}}=-p'(\tilde{u})+p'(u)\xi_1(\lambda(\tilde{u}))\geq 0,$$

  where the last inequality holds because $p'(\tilde{u})\leq 0$ and $\xi_1(\lambda(\tilde{u}))\leq 1$ by Lemma 1 (b).

- If $u - p(\tilde{u}) < \phi_1(u - p(\tilde{u}))$, then

$$\frac{\partial w_0(u,\tilde{u})}{\partial \tilde{u}} = \frac{\partial \phi_1(u-p(\tilde{u}))}{\partial \tilde{u}} - \frac{dD(\tilde{u})}{d\tilde{u}} = p'(\tilde{u})[\xi_1(\lambda(\tilde{u})) - \xi_1(u-p(\tilde{u}))].$$

For $\tilde{u} < u$, we have $\xi_1(\lambda(\tilde{u})) \le \xi_1(u-p(\tilde{u}))$ and consequently $\frac{\partial w_0(u,\tilde{u})}{\partial \tilde{u}} \ge 0$.[11] □

D.3. Proof of Lemma B2

It is sufficient to show that there is no loss of optimality for the seller to focus on deterministic message generating policies: $\hat{\mu}_t : R_t \times \bar{\mathcal{H}}_t \to M_t$. Indeed, under a deterministic message generating policy, the message history $m^t \in M^t$ is uniquely determined by the reporting, allocation and the customer's action history $(r^t, x^{t-1}, a^{t-1}) \in R^t \times X^{t-1} \times A^{t-1}$ via the recursion $m_\tau = \hat{\mu}_\tau(r^\tau, m^{\tau-1}, x^{\tau-1}, a^{\tau-1})$ for $\tau = 0, 1, ..., t$. Hence, it is sufficient to use $R^{t-1} \times X^{t-1} \times A^{t-1}$ to summarize the set of public history $\bar{\mathcal{H}}_t$.

Now consider any random message generating policies: $\hat{\mu}_t : R_t \times \bar{\mathcal{H}}_t \to \Delta(M_t)$. With a slight abuse of notation, we use $\hat{\mu}_t(m|(r_t, \bar{h}_t))$ to denote the probability of sending message $m \in M_t$ conditioning on the history $(r_t, \bar{h}_t)$. Given a public history $\bar{h}_t \in \bar{\mathcal{H}}_t$, a report r_t and allocation realization x_t in period t, we let $\hat{\varphi}_t((x_t, r_t, \bar{h}_t))$ denote the seller's revenue-to-go before the customer's action averaged over the possible random messages sent in period t. We let $\hat{\Phi}_t((r_t, \bar{h}_t)) = \mathbb{E}_{x_t}[\hat{\varphi}_t((x_t, r_t, \bar{h}_t))|x_{t-1}]$ further denote the average over the random allocation. It then follows from our definition that $\hat{\varphi}_t$ satisfies

$$\begin{aligned}\hat{\varphi}_t((x_t, r_t, \bar{h}_t)) &= \hat{z}_t((r_t, \bar{h}_t)) + \mathbb{E}\left[\mathbb{1}_{\{\hat{\eta}_t^*(h_t)=2\}} \cdot \hat{\Phi}_{t+1}((\hat{\delta}_{t+1}^*(h_{t+1}'), \bar{h}_{t+1}))\right], \\ &= \hat{z}_t((r_t, \bar{h}_t)) + \sum_{m_t \in M_t} \mathbb{E}\left[\mathbb{1}_{\{\hat{\eta}_t^*(h_t)=2\}} \cdot \hat{\Phi}_{t+1}((\hat{\delta}_{t+1}^*(h_{t+1}'), \bar{h}_{t+1})) \middle| m_t\right] \mu_t(m_t|(r_t, \bar{h}_t)),\end{aligned} \tag{D.5}$$

with $\hat{\varphi}_{T+1}(\cdot) = 0$. The expectation in the first equation in (D.5) is taken with respect to both the messages and the unobservable private signals. It is clear that the function in (D.5) is linear in $\mu_t(m_t|(r_t, \bar{h}_t))$ and hence there is no loss of optimality in focusing on deterministic message generating policies. □

D.4. Proof of Lemma C3

(a) The proof is similar to that of Lemma 1(a) and hence omitted.

[11] One can observe from our proof that as opposed to the typical assumption imposed in the static screening problem that the virtual valuation function $\lambda(u)$ being increasing, we need the stronger condition that $p(u) = \frac{\bar{F}(u)}{f(u)}$ is decreasing in u here to ensure the quasi-concavity of $w_0(u, \tilde{u})$, i.e., the type-u customer does not have incentive to take the contract designed for type $\tilde{u}$. In dynamic context, one usually needs such stronger conditions to ensure the implementability of the allocation rule derived from the relaxed problem (4) (see also Courty and Li 2000, Eső and Szentes 2007).

(b) We prove via induction. At period $t=T$, because $\phi_{T+1}(u,\varepsilon_{[T]})=0$,

$$\begin{aligned}\eta_T(u,\varepsilon_{[T-1]}) &= \mathbb{E}_{\varepsilon_T}\left[\mathbb{1}_{\left\{u\geq \gamma u+\omega_T(\varepsilon_{[T]})\right\}}+\mathbb{1}_{\left\{\gamma u+\omega_T(\varepsilon_{[T]})>u\right\}}\gamma\right]\\ &= \mathbb{E}_{\varepsilon_T}\left[\gamma+(1-\gamma)\mathbb{1}_{\left\{u\geq \gamma u+\omega_T(\varepsilon_{[T]})\right\}}\right]\\ &= \gamma+(1-\gamma)\xi_T(u,\varepsilon_{[T-1]}).\end{aligned}$$

Suppose $\eta_{t+1}(u,\varepsilon_{[t]})=\gamma+(1-\gamma)\xi_{t+1}(u,\varepsilon_{[t]})$, by the definition of $\phi_t(u,\varepsilon_{[t-1]})$, we have

$$\phi_t(u,\varepsilon_{[t-1]})-\gamma u=\mathbb{E}_{\varepsilon_t}\left[(1-\gamma)u\vee\omega_t(\varepsilon_{[t]})\vee(\phi_{t+1}(u,\varepsilon_{[t]})-\gamma u)\right]-s,$$

and therefore,

$$\begin{aligned}\eta_t(u,\varepsilon_{[t-1]}) &= \gamma+\mathbb{E}\left[(1-\gamma)\mathbb{1}_{\left\{u\geq(\gamma u+\omega_t(\varepsilon_{[t]}))\vee\phi_{t+1}(u,\varepsilon_{[t]})\right\}}+\mathbb{1}_{\left\{\phi_{t+1}(u,\varepsilon_{[t]})>u\vee(\gamma u+\omega_t(\varepsilon_{[t]}))\right\}}(\eta_{t+1}(u,\varepsilon_{[t]})-\gamma)\right]\\ &= \gamma+\mathbb{E}\left[(1-\gamma)\mathbb{1}_{\left\{u\geq(\gamma u+\omega_t(\varepsilon_{[t]}))\vee\phi_{t+1}(u,\varepsilon_{[t]})\right\}}+(1-\gamma)\mathbb{1}_{\left\{\phi_{t+1}(u,\varepsilon_{[t]})>u\vee(\gamma u+\omega_t(\varepsilon_{[t]}))\right\}}\xi_{t+1}(u,\varepsilon_{[t]})\right]\\ &= \gamma+(1-\gamma)\xi_t(u,\varepsilon_{[t-1]}),\end{aligned}$$

where the second equality follows from the induction hypothesis, and the last equality holds by the definition of $\xi_t(u,\varepsilon_{[t-1]})$. □

D.5. Proof of Theorem 2

When $\gamma=1$, the revenue upper bound is

$$\bar{\Phi}=\mathbb{E}_u[\lambda(u)+0\vee\phi_1(0)]-\phi_1(0)=(0-\phi_1(0))^+,$$

The optimal mechanism can be implemented by a posted price contract with price $(0-\phi_1(0))^+$. The incentive compatibility constraint is satisfied automatically. Then it remains for us to check individual rationality constraint. Under such contract, the customer's net utility is:

$$(u-(0-\phi_1(0))^+)\vee\check{\phi}_1(u),$$

where $\check{\phi}_t(u,\varepsilon_{[t-1]})=\mathbb{E}[(u-(0-\phi_1(0))^+)\vee(u+\omega_t(\varepsilon_{[t]}))\vee\check{\phi}_{t+1}(u,\varepsilon_{[t]})]-s$. Note that $-(0-\phi_1(0))^+<0<\omega_t(\varepsilon_{[t]})$, we then have $\check{\phi}_t(u,\varepsilon_{[t-1]})=u+\nu_t(\varepsilon_{[t-1]})$. It follows that the customer's utility is:

$$(u-(0-\phi_1(0))^+)\vee\check{\phi}_1(u)=(u-(0-\phi_1(0))^+)\vee(u+\phi_1(0))=u+\phi_1(0)=\hat{\phi}_1(u), \tag{D.6}$$

Therefore, the posted price contract with price $(0-\phi_1(0))^+$ is incentive compatible. Finally, we show that such contract achieves the revenue upper bound. If $0\leq\phi_1(0)$, by (D.6), the agent will not purchase and the revenue is $0=\bar{\Phi}$; if $0>\phi_1(0)$, by (D.6), the agent will purchase immediately and pay price $-\phi_1(0)$ and thus the revenue is $-\phi_1(0)=\bar{\Phi}$. In summary, the posted price contract with price $(0-\phi_1(0))^+$ is optimal.

When $0 \le \gamma < 1$, we first show that under the optimal mechanism $\mathcal{M}^*$, customer with valuation u indeed behaves as stated in Theorem 2. We then show the revenue collected under such behavior achieves the revenue upper bound $\bar{\Phi}$.

We let $w_t(u,\tilde{u})$ be the utility-to-go at period $t, t = 0, 1, ..., T$ of a customer who has valuation $u \in [0,1]$ but chooses the option contract $(D(\tilde{u}), p(\tilde{u})), \tilde{u} \in [\underline{u}, 1]$. Then, by definition,

$$w_0(u,\tilde{u}) = \big(u - (1-\gamma)p(\tilde{u})\big) \vee w_1(u,\tilde{u}) - D(\tilde{u}),$$

and for $t = 1, ..., T$,

$$w_t(u,\tilde{u};\varepsilon_{[t-1]}) = \mathbb{E}_{\varepsilon_t}\left[\big(u-(1-\gamma)p(\tilde{u})\big) \vee (\gamma u + \omega_t(\varepsilon_{[t]})) \vee w_{t+1}(u,\tilde{u};\varepsilon_{[t]})\right] - s,$$

where we let $w_{T+1}(u,\tilde{u};\varepsilon_{[T]}) = 0$. Next, we show $w_t(u,\tilde{u};\varepsilon_{[t-1]}) = \phi_t(u - p(\tilde{u});\varepsilon_{[t-1]}) + \gamma p(\tilde{u})$ by induction. At period $t = T$, we have

$$\begin{aligned} w_T(u,\tilde{u};\varepsilon_{[T-1]}) &= \mathbb{E}_{\varepsilon_T}\left[\big(u-(1-\gamma)p(\tilde{u})\big) \vee (\gamma u + \omega_T(\varepsilon_{[T]}))\right] - s \\ &= \mathbb{E}_{\varepsilon_T}\left[\big(u-p(\tilde{u})\big) \vee (\gamma(u-p(\tilde{u})) + \omega_T(\varepsilon_{[T]}))\right] + \gamma p(\tilde{u}) - s = \phi_T(u-p(\tilde{u});\varepsilon_{[T-1]}) + \gamma p(\tilde{u}). \end{aligned}$$

Suppose $w_{t+1}(u,\tilde{u};\varepsilon_{[t]}) = \phi_{t+1}(u-p(\tilde{u});\varepsilon_{[t]}) + \gamma p(\tilde{u})$, then

$$\begin{aligned} w_t(u,\tilde{u};\varepsilon_{[t-1]}) &= \mathbb{E}_{\varepsilon_t}\left[(u-(1-\gamma)p(\tilde{u})) \vee (\gamma u + \omega_t(\varepsilon_{[t]})) \vee w_{t+1}(u,\tilde{u};\varepsilon_{[t]})\right] - s \\ &= \mathbb{E}_{\varepsilon_t}\left[(u-(1-\gamma)p(\tilde{u})) \vee (\gamma u + \omega_t(\varepsilon_{[t]})) \vee (\phi_{t+1}(u-p(\tilde{u});\varepsilon_{[t]}) + \gamma p(\tilde{u}))\right] - s \\ &= \mathbb{E}_{\varepsilon_t}\left[(u-p(\tilde{u})) \vee (\gamma(u-p(\tilde{u})) + \omega_t(\varepsilon_{[t]})) \vee \phi_{t+1}(u-p(\tilde{u});\varepsilon_{[t]})\right] + \gamma p(\tilde{u}) - s \\ &= \phi_t(u-p(\tilde{u});\varepsilon_{[t-1]}) + \gamma p(\tilde{u}), \end{aligned}$$

where the second equality holds by induction hypothesis. Lemma D5 below summarizes the properties of $w_0(u,\tilde{u})$.

Lemma D5. *For any $u \in [0,1]$ and $\tilde{u} \in [\underline{u},1]$, $w_0(u,\tilde{u})$ is decreasing in $\tilde{u}$ for $\tilde{u} \ge u$; and is increasing in $\tilde{u}$ for $\tilde{u} < u$.*

Proof of Lemma D5: By the definition of $\eta_1(u)$ and $D(\tilde{u})$, we have

$$\frac{dD(\tilde{u})}{d\tilde{u}} = \gamma p'(\tilde{u}) + \frac{d\phi_1(\lambda(\tilde{u}))}{d\tilde{u}} - \eta_1(\lambda(\tilde{u})) = \gamma p'(\tilde{u}) + (1-p'(\tilde{u}))\eta_1(\lambda(\tilde{u})) - \eta_1(\lambda(\tilde{u})) = \gamma p'(\tilde{u}) - p'(\tilde{u})\eta_1(\lambda(\tilde{u})).$$

(a) For any $u \in [0,1]$ and $\tilde{u} \in [\underline{u},1]$, we show $w_0(u,\tilde{u})$ is decreasing in $\tilde{u}$ for $\tilde{u} \ge u$ by distinguishing between two cases.

- If $\tilde{u} \in [\hat{u}, 1]$, since $D(\tilde{u}) = D(\hat{u})$ and $p(\tilde{u}) = p(\hat{u})$, we have $w_0(u,\tilde{u}) = w_0(u,\hat{u})$ and $\phi_t(u - p(\tilde{u})) = \phi_t(u - p(\hat{u}))$, which are constants in $\tilde{u}$.

- If $\tilde{u} \in [\underline{u} \vee u, \hat{u})$, we have $u - p(\tilde{u}) \leq \hat{u} - p(\hat{u})$. Then by Lemma C3(a) and the definition of $\hat{u}$, we have $u - p(\tilde{u}) \leq \phi_1(u - p(\tilde{u}))$, it follows that $u - p(\tilde{u}) + \gamma p(\tilde{u}) \leq \phi_1(u - p(\tilde{u})) + \gamma p(\tilde{u}) = w_1(u, \tilde{u})$. Then

$$w_0(u, \tilde{u}) = w_1(u, \tilde{u}) - D(\tilde{u}) = \phi_1(u - p(\tilde{u})) + \gamma p(\tilde{u}) - D(\tilde{u}),$$

and thus

$$\begin{aligned} \frac{\partial w_0(u, \tilde{u})}{\partial \tilde{u}} = \frac{\partial w_1(u, \tilde{u})}{\partial \tilde{u}} - \frac{\partial D(\tilde{u})}{\partial \tilde{u}} &= -p'(\tilde{u})\eta_1(u - p(\tilde{u})) + \gamma p'(\tilde{u}) - \gamma p'(\tilde{u}) + p'(\tilde{u})\eta_1(\lambda(\tilde{u})) \\ &= -p'(\tilde{u})[\eta_1(u - p(\tilde{u})) - \eta_1(\lambda(\tilde{u}))] \end{aligned}$$

Since $p'(\tilde{u}) \leq 0$ and $\eta_1(x)$ is increasing in x, then $\frac{\partial w_0(u, \tilde{u})}{\partial \tilde{u}} \leq 0$.

(b) For $\tilde{u} < u$,

- If $u - (1 - \gamma)p(\tilde{u}) \geq w_1(u, \tilde{u})$, then

$$\frac{\partial w_0(u, \tilde{u})}{\partial \tilde{u}} = -(1 - \gamma)p'(\tilde{u}) - \gamma p'(\tilde{u}) + p'(\tilde{u})\eta_1(\lambda(\tilde{u})) \geq 0,$$

where the last inequality holds because $p'(\tilde{u}) \leq 0$ and $\eta_1(\lambda(\tilde{u})) \leq 1$.

- If $u - (1 - \gamma)p(\tilde{u}) < w_1(u, \tilde{u})$, then

$$\frac{\partial w_0(u, \tilde{u})}{\partial \tilde{u}} = -p'(\tilde{u})[\eta_1(u - p(\tilde{u})) - \eta_1(\lambda(\tilde{u}))].$$

For $\tilde{u} < u$, we have $\eta_1(u - p(\tilde{u})) \geq \eta_1(\lambda(\tilde{u}))$ and thus $\frac{\partial w_0(u, \tilde{u})}{\partial \tilde{u}} \geq 0$.

The proof is completed. □

Next, we show that the customer will self-select the contract that is consistent with his valuation for the seller's product and the contract is individual rational by discussing among three cases.

- When $u \in [\hat{u}, 1]$, the customer's surplus from choosing the contract $(D(u), p(u))$ is the same as choosing $(D(\hat{u}), p(\hat{u}))$, i.e., $\omega_0(u, u) = \omega_0(u, \hat{u})$. It then suffices for us to show that $\omega_0(u, \hat{u}) \geq \omega_0(u, \tilde{u})$ for $\tilde{u} \in [\underline{u}, \hat{u}]$ and $\omega_0(u, u) \geq \hat{\phi}_1(0)$. By Lemma D5, we know that $\omega_0(u, \tilde{u}) \leq$ is increasing in $\tilde{u}$ for $\tilde{u} \leq u$. Hence,

$$\omega_0(u, \tilde{u}) \leq \omega_0(u, \hat{u}), \forall u \in [\underline{u}, \hat{u}].$$

In addition, because $u - p(\hat{u}) \geq \hat{u} - p(\hat{u}) = \lambda(\hat{u})$, we have $u - p(\hat{u}) \geq \phi_1(u - p(\hat{u}))$, i.e., $u - (1 - \gamma)p(\hat{u}) \geq w_1(u, \hat{u})$ by Lemma C3(a), implying that the customer with valuation $u \in [\hat{u}, 1]$ will exercise the option immediately without any search. Then

$$\begin{aligned} w_0(u, \hat{u}) &= u - (1 - \gamma)p(\hat{u}) - D(\hat{u}) \\ &= u - (1 - \gamma)p(\hat{u}) - \phi_1(\lambda(\hat{u})) - \gamma p(\hat{u}) + \phi_1(0) + \int_0^{\hat{u}} \eta_1(\lambda(\nu))d\nu \\ &= u - \hat{u} + \phi_1(0) + \int_0^{\hat{u}} \eta_1(\lambda(\nu))d\nu \\ &\geq u - \hat{u} + \gamma\hat{u} + \phi_1(0) \geq \gamma u + \phi_1(0) = \hat{\phi}_1(u), \end{aligned}$$

where the first inequality holds $\eta_1(\lambda(\nu)) \geq \gamma$ and the last inequality is from the fact that $u \geq \hat{u}$.

- When $u \in [\underline{u}, \hat{u})$, by Lemma D5, $\omega_0(u,u) \geq \omega_0(u,\tilde{u}), \forall \tilde{u} \in [\underline{u}, 1]$, implying that it is optimal for the customer to self-select the contract that is consistent with his valuation. In addition,

$$\begin{aligned}\omega_0(u,u) &= \lambda(u) \vee \phi_1(\lambda(u)) + \gamma p(u) - D(u)\\ &= \phi_1(\lambda(u)) + \gamma p(u) - D(u)\\ &= \phi_1(0) + \int_0^u \eta_1(\lambda(\nu))d\nu \geq \phi_1(0) + \gamma u = \hat{\phi}_1(u)\end{aligned}$$

where the inequality holds because $\eta_1(\lambda(\nu)) \geq \gamma$.

For customer with valuation $u \in [\underline{u}, \hat{u})$, since $\phi_1(\lambda(u)) \geq \lambda(u)$, the customer will continue searching at the initial period. Then the customer's utility to go at each period is:

$$\begin{aligned}&w_t(u,u;\varepsilon_{[t-1]}) = \phi_t(\lambda(u),\varepsilon_{[t-1]}) + \gamma p(u)\\ &\qquad \geq \hat{\phi}_t(\lambda(u),\varepsilon_{[t-1]}) + \gamma p(u) = \gamma\lambda(u) + \nu_t(\varepsilon_{[t-1]}) + \gamma p(u) = \gamma u + \nu_t(\varepsilon_{[t-1]}) = \hat{\phi}_t(u,\varepsilon_{[t-1]}),\end{aligned}$$

where the inequality holds because $\lambda(u) \geq 0$. This implies that the mechanism satisfies the interim phases individual rationality constraint.

- When $0 \leq u < \underline{u}$, by Lemm D5, $w_0(u,\tilde{u})$ is decreasing in $\tilde{u}$ for $\tilde{u} \geq u$ with $\tilde{u} \in [\underline{u}, 1]$, then we have

$$\begin{aligned}w_0(u,\tilde{u}) \leq w_0(u,\underline{u}) &= \phi_1(u - p(\underline{u})) + \gamma p(\underline{u}) - D(\underline{u})\\ &= \phi_1(u - p(\underline{u})) + \gamma p(\underline{u}) - \phi_1(\lambda(\underline{u})) - \gamma p(\underline{u}) + \phi_1(0) + \int_0^{\underline{u}} \eta_1(\lambda(\nu))d\nu\\ &= \phi_1(u - p(\underline{u})) - \phi_1(\lambda(\underline{u})) + \phi_1(0) + \gamma u \leq \hat{\phi}_1(u).\end{aligned}$$

Here the first equality holds by $u - p(\underline{u}) \leq \lambda(\underline{u}) \leq \lambda(\hat{u})$ and Lemma C3(a), the third equality holds because $\eta_1(\lambda(\nu)) = \gamma$ when $\nu \leq \underline{u}$, and the inequality holds because $\phi_1(u)$ is increasing in u. Therefore, when $u < \underline{u}$, the customer searches without purchasing from the seller.

Given that customer indeed behaves as stated in Theorem 2, we know from the definition of $\hat{u}$ that for customer with valuation $u \in [\underline{u}, \hat{u}]$, he will return to purchase the product with probability $\xi_1(\lambda(u)) = \frac{\eta_1(\lambda(u)) - \gamma}{1-\gamma}$ and for customer with valuation $u \in (\hat{u}, 1]$, he will purchase seller's product without any search at period 0. Therefore, the revenue that the seller can achieve is

$$\begin{aligned}&\int_{\underline{u}}^{\hat{u}} [D(u) + \tfrac{\eta_1(\lambda(u)) - \gamma}{1-\gamma}(1-\gamma)p(u)]dF(u) + \int_{\hat{u}}^1 [D(\hat{u}) + p(\hat{u})]dF(u)\\ &= \int_{\underline{u}}^{\hat{u}} [\phi_1(\lambda(u)) - \phi_1(0) + \gamma p(u) + (\eta_1(\lambda(u)) - \gamma)p(u)]dF(u) - \int_{\underline{u}}^{\hat{u}} \int_0^u \eta_1(\lambda(\nu))d\nu dF(u) + \int_{\hat{u}}^1 [D(\hat{u}) + p(\hat{u})]dF(u)\\ &= \int_{\underline{u}}^{\hat{u}} [\phi_1(\lambda(u)) - \phi_1(0) + \eta_1(\lambda(u))p(u)]dF(u) + \bar{F}(\hat{u}) \int_0^{\hat{u}} \eta_1(\lambda(u))du - \int_{\underline{u}}^{\hat{u}} \eta_1(\lambda(u))p(u)dF(u)\\ &\quad - \bar{F}(\underline{u}) \int_0^{\underline{u}} \eta_1(\lambda(\nu))d\nu + \int_{\hat{u}}^1 [D(\hat{u}) + p(\hat{u})]dF(u)\end{aligned}$$

$$
\begin{aligned}
&= \int_{\underline{u}}^{\hat{u}} [\phi_1(\lambda(u)) - \phi_1(0)] dF(u) + \bar{F}(\hat{u}) \left(\int_0^{\hat{u}} \eta_1(\lambda(u)) du + D(\hat{u}) + p(\hat{u}) \right) - \bar{F}(\underline{u}) \int_0^{\underline{u}} \eta_1(\lambda(\nu)) d\nu \\
&= \int_{\underline{u}}^{\hat{u}} [\phi_1(\lambda(u)) - \phi_1(0)] dF(u) + \bar{F}(\hat{u}) \Big(\int_0^{\hat{u}} \eta_1(\lambda(u)) du + \phi_1(\lambda(\hat{u})) + \gamma p(\hat{u}) - \phi_1(0) - \int_0^{\hat{u}} \eta_1(\lambda(u)) du + (1-\gamma) p(\hat{u}) \Big) \\
&\quad - \gamma \underline{u} \bar{F}(\underline{u}) \\
&= \int_{\underline{u}}^{\hat{u}} [\phi_1(\lambda(u)) - \phi_1(0)] dF(u) + \bar{F}(\hat{u})(\hat{u} - \phi_1(0)) + \int_0^{\underline{u}} \gamma \lambda(u) dF(u) \\
&= \int_0^{\underline{u}} [\phi_1(\lambda(u)) - \phi_1(0)] dF(u) + \int_{\underline{u}}^{\hat{u}} [\phi_1(\lambda(u)) - \phi_1(0)] dF(u) + \bar{F}(\hat{u})(\hat{u} - \phi_1(0)) \\
&= \bar{\Phi},
\end{aligned}
$$

where the second equality holds by integration by parts that

$$
\begin{aligned}
\int_{\underline{u}}^{\hat{u}} \int_0^u \eta_1(\lambda(\nu)) d\nu dF(u) &= \int_0^u \eta_1(\lambda(\nu)) d\nu \Big|_{u=\underline{u}}^{\hat{u}} - \int_{\underline{u}}^{\hat{u}} \eta_1(\lambda(u)) F(u) du \\
&= \bar{F}(\hat{u}) \int_0^{\hat{u}} \eta_1(\lambda(u)) du - \int_{\underline{u}}^{\hat{u}} \eta_1(\lambda(u)) p(u) dF(u) - \bar{F}(\underline{u}) \int_0^{\underline{u}} \eta_1(\lambda(\nu)) d\nu,
\end{aligned}
$$

the fourth equality holds because $\eta_1(\lambda(\nu)) = \gamma$ for $\nu \le \underline{u}$ and $\phi_1(\lambda(\hat{u})) = \lambda(\hat{u})$, the second to the last equality comes from the fact that $\phi_1(\lambda(u)) = \gamma\lambda(u) + \nu_1 == \gamma\lambda(u) + \phi_1(0)$ when $u \le \underline{u}$.

This establishes that $\mathcal{M}^*$ achieves the revenue upper bound $\bar{\Phi}$ and concludes our proof. □