E-Companion

EC.1. Proof of Theorem 1

Before proceeding to the formal proof, we first introduce several new definitions. Let

$$Y(t) := (Z(t), Q_1(t), Q_2(t), \ldots, Q_K(t))$$

denote the system state, where $Z(t) \in \mathbb{Z}^+$ represents the number of customers in service and $Q_j(t) \in \mathbb{Z}^+$ represents the number of customers of class j in the queue, with $\mathbb{Z}^+$ denoting the set of non-negative integers. Note that the state descriptor $Y(t)$ may contain information redundancy due to the non-idling requirement. However, this redundancy serves to facilitate the subsequent presentation. We assume that the patience time for each class follows an exponential distribution with mean $1/\rho_k$, so that $R_k(t) = N_k\left(\rho_k \int_0^t Q_k(u)\mathrm{d}u\right)$ where N_k is a unit-rate Poisson process for each k and independent with each other. With this assumption, the process Y is a controlled Markov chain on a countable state space, denoted as $\mathcal{X}$.

Let $\mathcal{A}$ be a finite set consisting of all admissible scheduling decisions. We define $\mathcal{B} := [-1, \infty]$ as the set of admissible perturbations from nature (we stipulate $0 \ln 0 = 0$). Furthermore, we let Π^{RS} and Ψ^{RS} denote the sets of all admissible random stationary policies for scheduling and nature's perturbation, respectively. To be specific, a random stationary policy $\pi \in \Pi^{RS}(\psi \in \Psi^{RS})$ is a function such that for each $y \in \mathcal{X}$, $\pi(y, \cdot)(\psi(y, \cdot))$ is a probability measure over the Borel set generated by $\mathcal{A}(\mathcal{B})$. In addition, we define Π^S and Ψ^S as the sets of corresponding stationary policies. However, for notational convenience, $\pi \in \Pi^S(\psi \in \Psi^S)$ is represented by a mapping $\mathcal{X} \mapsto \mathcal{A}(\mathcal{X} \mapsto \mathcal{B})$.

Our proof is consisted of two main steps. First, we study two restricted problems that correspond to constraint and penalty problems, with the assumption that admissible policies are random stationary. Specifically, the two restricted problems are defined as follows:

$$\mathfrak{F}^{RS}(\gamma) = \inf_{\pi \in \Pi^{RS}} \sup_{\psi \in \Psi^{RS}} \bar{C}(\pi, \psi), \quad \text{subject to } \bar{\mathcal{R}}(\pi, \psi) \le \gamma, \tag{EC.1}$$

$$\mathfrak{G}^{RS}(\alpha) = \inf_{\pi \in \Pi^{RS}} \sup_{\psi \in \Psi^{RS}} \bar{C}(\pi, \psi) - \alpha \bar{\mathcal{R}}(\pi, \psi), \tag{EC.2}$$

where $\gamma > 0$ and $\alpha \ge 0$, and $\bar{C}$ and $\bar{\mathcal{R}}$ are some functionals which will be defined in the next paragraph. We establish a similar relation as shown in Theorem 1 for the two restricted problems in Proposition EC.1. Second, we relax the aforementioned assumption and provide a full-fledged proof of Theorem 1.

Let us first define an ergodic occupation measure μ_ψ^π under policy (π, ψ), which is

$$\mu_\psi^\pi(y, a, \mathrm{d}s) = \mu(y)\pi(y, a)\psi(y, \mathrm{d}s), \tag{EC.3}$$

where μ is the corresponding invariant measure under (π, ψ), and $\pi \in \Pi^{RS}$ and $\psi \in \Psi^{RS}$. Let $c : \mathcal{X} \mapsto \mathbb{R}$ be a function that $c(y) = \langle (0, c_1\rho_1, c_2\rho_2, \ldots, c_K\rho_K), y\rangle$. Then, according to the ergodic theory, the long-run average abandonment cost under policy (π, ψ) is almost surely equal to $\bar{C}(\pi, \psi) := \int_{\mathcal{X}\times\mathcal{A}\times\mathcal{B}} c\mathrm{d}\mu_\psi^\pi = \sum_{y\in\mathcal{X}} c(y)\mu(y)$. Similarly, let $r : \mathcal{X}\times\mathcal{B} \mapsto \mathbb{R}$ be defined as $r(y, s) := \tilde{f}(s, \langle (1, 0, \ldots, 0), y\rangle)$ so that the long-run average relative entropy is almost surely equal to $\bar{\mathcal{R}}(\pi, \psi) := \int_{\mathcal{X}\times\mathcal{A}\times\mathcal{B}} r\mathrm{d}\mu_\psi^\pi = \sum_{y\in\mathcal{X}} \mu(y) \int_{\mathcal{B}} r(y, s)\psi(y, \mathrm{d}s)$.

Consider the set $\mathcal{M}$, which consists of all ergodic occupation measures with $\pi \in \Pi^{RS}$ and $\psi \in \Psi^{RS}$. Let $\mathcal{M}_\gamma := \{\mu_\psi^\pi \in \mathcal{M} : \bar{\mathcal{R}}(\pi, \psi) \le \gamma\}$ be the set of ergodic occupation measures that satisfy the constraint in (EC.1). The following lemma will be helpful in proving Proposition EC.1.

Lemma EC.1. *The set $\mathcal{M}$ is compact, and $\mathcal{M}_\gamma$ is closed, and thus also compact.*

Proof. Based on Theorem A.1 in (Feinberg and Shwartz 2002, Chapter 11, pp. 370-371), $\mathcal{M}$ is compact if and only if it is tight. We now demonstrate that $\mathcal{M}$ is tight by analyzing the stationary states of an alternative system. Let $\tilde{Y}(t) := (\tilde{Z}(t), \tilde{Q}_1(t), \ldots, \tilde{Q}_K(t))$ represent a system with the initial condition $\tilde{Z}(0) = n$ and $\tilde{Q}_k(0) = Q_k(0)$ almost surely for all k, but the service time of each server is ∞. Then it is not difficult to argue that $Q_i(t) \overset{s.t.}{\le} \tilde{Q}_i(t)$ for all i and t, where $\overset{s.t.}{\le}$ denotes stochastic dominance, under any admissible policies (π, ψ). Letting $t \to \infty$, we have $Q_i(\infty) \overset{s.t.}{\le} \tilde{Q}_i(\infty)$. In addition, $\tilde{Q}_i(\infty)$ follows a Poisson distribution, which can be calculated as follows. First, the evolution of $\tilde{Q}_i$ is given by $\tilde{Q}_i(t) = \tilde{Q}_i(0) + A_i(t) - N_i(\rho_i \int_0^t \tilde{Q}_i(u)du)$, where $A_i(t)$ is a Poisson process with rate $a_i\lambda$. Then each $\tilde{Q}_i(t)$ is itself a continuous-time Markov chain with generator matrix $P(j, j+1) = a_i\lambda$, $P(j, j-1) = j\rho_i$, and $P(j, j) = -(a_i\lambda + j\rho_i)$, with all other entries being 0. One can then show that there exists a unique vector $\tilde{\mu}$ satisfying $\tilde{\mu}^T P = 0$ and $\langle \tilde{\mu}, \mathbf{e}\rangle = 1$, where $\mathbf{e}$ is the all-ones vector. More precisely, we have a recursive relation $\tilde{\mu}_n/\tilde{\mu}_{n-1} = a_i\lambda/(n\rho_i)$ such that $\tilde{\mu}_n = [(a_i\lambda/\rho_i)^n\tilde{\mu}_0]/n!$. By the condition $\langle \tilde{\mu}, \mathbf{e}\rangle = 1$, we have $\tilde{\mu}_0 = e^{-a_i\lambda/\rho_i}$. Hence, $\tilde{\mu}$ can be regarded as the probability mass function of a Poisson distribution with rate $a_i\lambda/\rho_i$. In addition, we have $\mathbb{E}[\tilde{Q}_i(\infty)] = a_i\lambda/\rho_i$.

Let $K_j := \{(z, q_1, \ldots, q_K) : z \le n, q_i \le j, \forall i = 1, \ldots, K\}$. Note that K_j is a compact set for each j. Then, for each fixed j, we have $\mu_{\tilde{\psi}}^{\tilde{\pi}}(K_j, \mathcal{A}, \mathcal{B}) = \tilde{\mu}(K_j)$ under any $\left(\tilde{\pi}, \tilde{\psi}\right)$. By the Poisson distribution, for any fixed $\epsilon > 0$, we can choose a sufficiently large j such that $\tilde{\mu}(K_j) > 1 - \epsilon$. Since $\tilde{\mu}$ stochastically dominates any other measure, it follows that for any (π, ψ), we have $\mu_\psi^\pi(K_j, \mathcal{A}, \mathcal{B}) = \mu(K_j) \ge \tilde{\mu}(K_j) > 1 - \epsilon$. It follows that the family of measures $\mathcal{M}$ is tight. Moreover, it is easy to see that $\bar{\mathcal{R}}$ is continuous with respect to μ_ψ^π in the topology of weak convergence. Hence, the set $\mathcal{M}_\gamma$ is closed. Since a closed subset of a compact set is compact, the proof is done. □

Having established Lemma EC.1, which shows that the optimizer in $\mathfrak{F}(\gamma)$ can be achieved, we are now prepared to prove the following key result.

PROPOSITION EC.1. *For any $\gamma \geq 0$, there exists $\pi^* \in \Pi^{RS}$ and $\psi^* \in \Psi^{RS}$ such that $\mathfrak{F}(\gamma) = \bar{C}(\pi^*, \psi^*)$.*

Furthermore, there exists an $\alpha^* \geq 0$, such that

$$\mathfrak{F}^{RS}(\gamma) = \min_{\alpha \geq 0} \left[\mathfrak{G}^{RS}(\alpha) + \gamma\alpha\right] = \mathfrak{G}^{RS}(\alpha^*) + \gamma\alpha^*.$$

Proof. By the compactness result from Lemma EC.1, we have

$$\mathfrak{F}^{RS}(\gamma) = \min_{\pi \in \Pi^{RS}} \max_{\left\{\psi \in \Psi^{RS} : \bar{\mathcal{R}}(\pi,\psi) \leq \gamma\right\}} \bar{C}(\pi, \psi).$$

Furthermore, there exist π^* and ψ^* such that $\mathfrak{F}^{RS}(\gamma) = \bar{C}(\pi^*, \psi^*)$.

The convexity of the relative entropy from the induced measure to the original measure implies that $\bar{\mathcal{R}}$ is convex with respect to ψ (see, e.g., Proof of Proposition 4.1 (Lim and Shanthikumar 2007)). Additionally, since $\bar{C}$ is linear in ψ by definition, we have

$$\begin{aligned}
\mathfrak{F}^{RS}(\gamma) &= \min_{\pi \in \Pi^{RS}} \min_{\alpha \geq 0} \max_{\psi \in \Psi^{RS}} \ \bar{C}(\pi, \psi) - \alpha\bar{\mathcal{R}}(\pi, \psi) + \alpha\gamma \\
&= \min_{\alpha \geq 0} \min_{\pi \in \Pi^{RS}} \max_{\psi \in \Psi^{RS}} \ \bar{C}(\pi, \psi) - \alpha\bar{\mathcal{R}}(\pi, \psi) + \alpha\gamma \\
&= \min_{\alpha \geq 0} \ \mathfrak{G}^{RS}(\alpha) + \alpha\gamma.
\end{aligned}$$

By definition,

$$\mathfrak{G}^{RS}(\alpha) = \min_{\pi \in \Pi^{RS}} \ \max_{\psi \in \Psi^{RS}} \ \bar{C}(\pi, \psi) - \alpha\bar{\mathcal{R}}(\pi, \psi) \geq \min_{\pi \in \Pi^{RS}} \ \bar{C}(\pi, p) \geq 0,$$

where p represents the strategy of "no perturbation" such that $p(y, \mathrm{d}s) = \delta_0(\mathrm{d}s)$ for all $y \in \mathcal{X}$ ($\delta_{x_0}(\cdot)$ is the delta measure concentrated on x_0), and the inequality is due to $\bar{\mathcal{R}}(\pi, p) = 0$ for all π. Thus, $\mathfrak{G}^{RS}(\alpha)$ is lower bounded, and for any $\gamma > 0$, $\lim_{\alpha \to \infty} \mathfrak{G}^{RS}(\alpha) + \alpha\gamma = \infty$. This observation implies that $\min_{\alpha \geq 0} \ \mathfrak{G}^{RS}(\alpha) + \alpha\gamma$ can be achieved at some $\alpha^* < \infty$. □

Next, we prove the case where both $\pi = (\pi_t)_t$ and $\psi = (\psi_t)_t$ are any admissible policies (possibly nonstationary), such that the decision is now assumed to be made based on the full history before time t, instead of the state vector $y \in \mathcal{X}$. The corresponding constraint problem is

$$\mathfrak{F}(\gamma) = \inf_{\pi} \ \sup_{\psi} \ \limsup_{t \to \infty} \frac{1}{t} \mathbb{E}^{\psi}\left[c_k R_k(t)\right] \quad \text{subject to} \ \limsup_{t \to \infty} \frac{1}{t} \mathbb{E}^{\psi}\left[\mathcal{R}(t)\right] \leq \gamma.$$

We introduce the concept of non-stationary occupation measure $\mu_t^{(\psi,\pi)}$, defined as

$$\mu_t^{(\psi,\pi)}(y, a, \mathrm{d}s) = \frac{1}{t} \mathbb{E}^{\psi}\left[\int_0^t \mathbf{1}_{\{Y(u)=y, A(u)=a, B(u) \in \mathrm{d}s\}} \mathrm{d}u\right],$$

where $A(t)$ and $B(t)$ represent the action taken by the DM and nature at time t, respectively. We have the following lemma.

LEMMA EC.2. *For any sequence of $\mu_t^{(\psi,\pi)}$ that satisfies the constraint in* (9)*, there exists a convergent subsequence of $\mu_t^{(\psi,\pi)}$ with its limit lying in $\mathcal{M}_\gamma$. Moreover, every such convergent subsequence $\mu_t^{(\psi,\pi)}$ also converges to $\mathcal{M}_\gamma$.*

Proof. Let us begin by noting that if $\lim_{t\to\infty}\mu_t^{(\psi,\pi)}$ exists, $\lim_{t\to\infty}\mu_t^{(\psi,\pi)}\in\mathcal{M}$ by the same stochastic order arguments used in the proof of Lemma EC.1. Next, let $\mu_t^{(\psi,\pi)}$ be any sequence that satisfies the constraint in (9), such that

$$\limsup_{t\to\infty}\frac{1}{t}\mathbb{E}^{\psi}\left[\mathcal{R}(t)\right]=\limsup_{t\to\infty}\int_{\mathcal{X}\times\mathcal{A}\times\mathcal{B}} r\mathrm{d}\mu_t^{(\psi,\pi)}\le\gamma.$$

Since $\mathcal{M}$ is a compact set, we can find a convergent subsequence $(t_n)_n$ such that $\mu_{t_n}^{(\psi,\pi)}$ converges. Therefore, we have

$$\lim_{n\to\infty}\int r d\mu_{t_n}^{(\psi,\pi)}\le\limsup_{t\to\infty}\int r d\mu_t^{(\psi,\pi)}=\limsup_{t\to\infty}\frac{1}{t}\mathbb{E}^{\psi}\left[\mathcal{R}(t)\right]\le\gamma. \tag{EC.4}$$

This shows that $\mu_\infty^{(\psi,\pi)}:=\lim_{n\to\infty}\mu_{t_n}^{(\psi,\pi)}\in\mathcal{M}_\gamma$. Since (EC.4) holds for any convergent subsequences satisfying the constraint in (9), the second claim follows. □

We are now ready to prove Theorem 1. The importance of Lemma EC.2 lies in the fact that we can determine the optimality of the constraint problem among all admissible policies simply by considering all random stationary policies. Moreover, according to Theorem 11.6 in (Feinberg and Shwartz 2002, Chapter 11), we know that the optimal policies require at most one randomization. However, we can do even better: the optimal policies are actually stationary policies.

Proof of Theorem 1. Combining Proposition EC.1 and Lemma EC.2, we can determine the optimality of the constraint problem among all admissible policies by only considering random stationary policies. Specifically, we have

$$\mathfrak{F}(\gamma)=\mathfrak{F}^{RS}(\gamma)=\mathfrak{G}^{RS}(\alpha^*)+\gamma\alpha^*,$$

where $\alpha^*\ge 0$ is some constant. Theorem 11.6 in (Feinberg and Shwartz 2002, Chapter 11) implies that an optimal policy subject to one constraint can be found with at most one randomization. Let ψ^* be the optimal nature's perturbation policy (with one randomization). Next, let π^* be an optimal scheduling rule, which can be achieved by a stationary rule due to the standard theory of Markov decision processes. Let $P(\pi^*,\psi^*)$ be the infinitesimal generator matrix under policy (π^*,ψ^*) and let $P_y(\pi^*,\psi^*)$ be the row corresponding to state y. Then we know there exists some relative value function V and long-run average cost η under the policy (π^*,ψ^*) satisfies the equation

$$\begin{aligned}&\min_{\pi}\max_{\psi}\left\{P_y(\pi,\psi)V+c(y)-\alpha^*\int_{\mathcal{B}}r(y,u)\psi(y,\mathrm{d}u)\right\}\\ =&P_y(\pi^*,\psi^*)V+c(y)-\alpha^*\int_{\mathcal{B}}r(y,u)\psi^*(y,\mathrm{d}u)=\eta,\quad\text{for all}\quad y\in\mathcal{X}.\end{aligned} \tag{EC.5}$$

Following Lemma 11.2 (Feinberg and Shwartz 2002, Chapter 11), if ψ^* is represented by

$$\psi^*(y,\mathrm{d}u)=\frac{\beta\mu_1(y)}{\beta\mu_1(y)+(1-\beta)\mu_2(y)}\delta_{\psi_1(y)}(\mathrm{d}u)+\frac{(1-\beta)\mu_2(y)}{\beta\mu_1(y)+(1-\beta)\mu_2(y)}\delta_{\psi_2(y)}(\mathrm{d}u) \tag{EC.6}$$

for some $\psi_1, \psi_2 \in \Psi^S$ and some $\beta \in [0,1]$, then $\mu^* = \beta\mu_1 + (1-\beta)\mu_2$ and $\mu^{\pi^*}_{\psi^*} = \beta\mu^{\pi^*}_{\psi_1} + (1-\beta)\mu^{\pi^*}_{\psi_2}$. We use μ^*, μ_1, and μ_2 ($\mu^{\pi^*}_{\psi^*}$, $\mu^{\pi^*}_{\psi_1}$, and $\mu^{\pi^*}_{\psi_2}$) to represent the stationary distributions (occupation measures, defined in (EC.3)) of the Markov chain under ψ^*, ψ_1 and ψ_2, respectively.

If $\psi_1 = \psi_2$ or $\beta \in \{0,1\}$, we are done. Assume $\psi_1 \neq \psi_2$, and $0 < \beta < 1$. We divide the analysis into three different scenarios based on the resulting occupation measure: both $\mu^{\pi^*}_{\psi_1}$ and $\mu^{\pi^*}_{\psi_2}$ are infeasible, both $\mu^{\pi^*}_{\psi_1}$ and $\mu^{\pi^*}_{\psi_2}$ are feasible, one of $\mu^{\pi^*}_{\psi_1}$ and $\mu^{\pi^*}_{\psi_2}$ is feasible and the other is not, in the set $\mathcal{M}_\gamma$.

In the first scenario, if both $\mu^{\pi^*}_{\psi_1}$ and $\mu^{\pi^*}_{\psi_2}$ are infeasible, then $\mu^{\pi^*}_{\psi^*}$ must also be infeasible since $\mathcal{M}_\gamma$ is convex. Hence, the first scenario never happens. For the second scenario, we observe that

$$\bar{C}(\pi^*, \psi^*) = \int c\mathrm{d}\mu^{\pi^*}_{\psi^*} = \beta \int c\mathrm{d}\mu^{\pi^*}_{\psi_1} + (1-\beta)\int c\mathrm{d}\mu^{\pi^*}_{\psi_2} = \beta\bar{C}(\pi^*, \psi_1) + (1-\beta)\bar{C}(\pi^*, \psi_2).$$

If $\bar{C}(\pi^*, \psi_1) \neq \bar{C}(\pi^*, \psi_2)$, $\bar{C}(\pi^*, \psi^*) = \max\left\{\bar{C}(\pi^*, \psi_1), \bar{C}(\pi^*, \psi_2)\right\}$ must hold, due to the optimality of ψ^*. But this contradicts to $0 < \beta < 1$, so this case cannot exist. If $\bar{C}(\pi^*, \psi_1) = \bar{C}(\pi^*, \psi_2)$, then we can choose either $\beta = 0$ or $\beta = 1$ to achieve the same objective value.

Finally, in the third scenario, without loss of generality, let us assume $\mu^{\pi^*}_{\psi_1}$ is feasible but $\mu^{\pi^*}_{\psi_2}$ is not. We check the equation (EC.5). By (EC.6), it can be derived that

$$P_y(\pi^*, \psi^*) = \frac{\beta\mu_1(y)}{\beta\mu_1(y) + (1-\beta)\mu_2(y)} P_y(\pi^*, \psi_1) + \frac{(1-\beta)\mu_2(y)}{\beta\mu_1(y) + (1-\beta)\mu_2(y)} P_y(\pi^*, \psi_2).$$

Let

$$\psi_3(y) = \frac{\beta\mu_1(y)}{\beta\mu_1(y) + (1-\beta)\mu_2(y)} \psi_1(y) + \frac{(1-\beta)\mu_2(y)}{\beta\mu_1(y) + (1-\beta)\mu_2(y)} \psi_2(y), \tag{EC.7}$$

which compared to (EC.6), is a stationary policy of a convex combination of $\psi_1(y)$ and $\psi_2(y)$. We aim to prove

$$P(\pi^*, \psi^*) = P(\pi^*, \psi_3). \tag{EC.8}$$

But this follows from the linearity of each component of $P(\pi, \psi)$ with respect to the nature's perturbation rate. For example, consider the state $y = (n, 0, \ldots, 0)$. Then transition rate from y to $y + e_j$ (e_j represents the vector with j-th component 1 and others 0) for $j = 2, \ldots, K+1$ is

$$\begin{aligned}
P(\pi^*, \psi_3)(y, y+e_j) &= a_{j-1}\lambda = \frac{\beta\mu_1(y)}{\beta\mu_1(y) + (1-\beta)\mu_2(y)} a_{j-1}\lambda + \frac{(1-\beta)\mu_2(y)}{\beta\mu_1(y) + (1-\beta)\mu_2(y)} a_{j-1}\lambda \\
&= \frac{\beta\mu_1(y)}{\beta\mu_1(y) + (1-\beta)\mu_2(y)} P_y(\pi^*, \psi_1)(y, y+e_j) + \frac{(1-\beta)\mu_2(y)}{\beta\mu_1(y) + (1-\beta)\mu_2(y)} P_y(\pi^*, \psi_2)(y, y+e_j),
\end{aligned}$$

and transition rate from y to $y - e_1$ is

$$\begin{aligned}
P(\pi^*, \psi_3)(y, y-e_1) &= \left(1 + \frac{\beta\mu_1(y)}{\beta\mu_1(y) + (1-\beta)\mu_2(y)} \psi_1(y) + \frac{(1-\beta)\mu_2(y)}{\beta\mu_1(y) + (1-\beta)\mu_2(y)} \psi_2(y)\right) \mu n \\
&= \frac{\beta\mu_1(y)}{\beta\mu_1(y) + (1-\beta)\mu_2(y)} (1 + \psi_1(y))\mu n + \frac{(1-\beta)\mu_2(y)}{\beta\mu_1(y) + (1-\beta)\mu_2(y)} (1 + \psi_2(y))\mu n \\
&= \frac{\beta\mu_1(y)}{\beta\mu_1(y) + (1-\beta)\mu_2(y)} P(\pi^*, \psi_1)(y, y-e_1) + \frac{(1-\beta)\mu_2(y)}{\beta\mu_1(y) + (1-\beta)\mu_2(y)} P(\pi^*, \psi_2)(y, y-e_1).
\end{aligned}$$

The other quantities can be checked in the same fashion.

Using the strict convexity of $r(y,z)$ with respect to z, and Jensen' inequality, we can conclude that there exists some y (the states requiring randomization) such that

$$\begin{aligned}\int_{\mathcal{B}} r(y,u)\psi^*(y,\mathrm{d}u) &= \frac{\beta\mu_1(y)}{\beta\mu_1(y)+(1-\beta)\mu_2(y)} r(y,\psi_1(y)) + \frac{(1-\beta)\mu_2(y)}{\beta\mu_1(y)+(1-\beta)\mu_2(y)} r(y,\psi_2(y)) \\ &> r(y,\psi_3(y)).\end{aligned} \tag{EC.9}$$

Combining (EC.8) and (EC.9), we have if $\alpha^* > 0$, then

$$P_y(\pi^*,\psi^*)V + c(y) - \alpha^* r(x,\psi^*(y)) < P_y(\pi^*,\psi_3)V(y) + c(y) - \alpha^* r(x,\psi_3(y)),$$

which implies ψ^* does not satisfy the optimality condition in (EC.5), contradicting to ψ^* being an optimal policy. On the other hand, if $\alpha^* = 0$, we can treat $\mathfrak{F}(\gamma)$ as an unconstrained problem, so that the optimal stationary policy exists.

We have shown that by solving $\mathfrak{G}^{RS}(\alpha^*)$, one can always find a stationary policy (π^*,ψ^*) which is an optimal solution in the constrained problem $\mathfrak{F}(\gamma)$, and $\mathfrak{G}^{RS}(\alpha^*) = \mathfrak{G}^{S}(\alpha^*)$. Finally, $\mathfrak{G}^{S}(\alpha^*) = \mathfrak{G}(\alpha^*)$ follows from a standard argument of Bellman's principle of optimality, so it is omitted. □

EC.2. Derivation of the SDG

We first describe how to reach the DCP as introduced in Kim et al. (2018). To this end, we invoke the flow conservation law to get

$$X(t) = X(0) + \sum_k (A_k(t) - R_k(t)) - D(t), \tag{EC.10}$$

where $X(0)$ is the number of customers at time zero centered by n, A_k count the class k arrivals, and D tracks the number of service completions and can be further expressed as

$$D(t) = N^{(d)}\left(\int_0^t \mu Z(u)\mathrm{d}u\right), \tag{EC.11}$$

where $N^{(d)}$ is a unit-rate Poisson Process and $Z(t)$ denotes the number of customers in service (or busy servers) at time t. It is worth emphasizing that $[X(t)]^+$ and $[X(t)]^-$ represent the number of waiting customers and the number of idling servers at time t, respectively.

Towards establishing the diffusion approximation for $X(t;\pi)$, we can apply the strong approximation for renewal processes to A_k and D, as well as the well-known approximation $Z(t) \approx n$ (due to the functional law of large numbers (FLLN) for the number-in-system process) to obtain

$$\begin{aligned} A_k(t) &= a_k\lambda t + \sqrt{a_k\lambda}\hat{A}_k(t) + \epsilon_i^a(t) \quad \text{for} \quad i = 1,\dots,K, \\ D(t) &= \int_0^t \mu(n - [X(u)]^-)\mathrm{d}u + \sqrt{\lambda}\hat{S}(t) + \epsilon^d(t), \end{aligned} \tag{EC.12}$$

where $\hat{A}_k$ and $\hat{S}$ are independent standard Brownian motions, and ϵ_i^a and ϵ^d are error terms arising from the strong approximation, being an order of magnitude smaller than $\sqrt{\lambda}$ over any finite time horizon. Next, to approximate R_k, we recall that $Q_k(t)$ represents the number of class k customers waiting at time t under the scheduling rule π. By applying a sample path version of Little's law known as the snapshot principle (Reed and Tezcan 2012, Kim and Ward 2013), we get

$$R_k(t) \approx \lambda \int_0^t \zeta_k \left(\frac{Q_k(u)}{\lambda} \right) \mathrm{d}u. \tag{EC.13}$$

By substituting (EC.12) into (EC.10), ignoring the error terms, and replacing R_k in (EC.10) with the right-hand side of (EC.13), we obtain the diffusion approximation

$$\hat{X}(t) = \hat{X}(0) - \beta\sqrt{\lambda\mu}t + \mu \int_0^t [\hat{X}(u)]^- \mathrm{d}u - \lambda \sum_k \int_0^t \zeta_k \left(\frac{\hat{Q}_k(u)}{\lambda} \right) \mathrm{d}u + \sqrt{2\lambda} B(t), \tag{EC.14}$$

where $\hat{Q}_k$ replaces Q_k to mean the diffusion approximation for the class k queue length process.

The derivation of the SDG relies on very similar arguments. In particular, when nature adopts a perturbation strategy θ, we have

$$D(t) = n\mu t + \sqrt{\lambda} \int_0^t \theta(u) \mathrm{d}u - \mu \int_0^t [X(u)]^- \mathrm{d}u + \sqrt{\lambda}\hat{S}(t) + \epsilon^d(t),$$

where the second term on the right-hand side arises due to the belief distortion and $\hat{S}$ is a standard Brownian motion. Thus, following the steps taken towards the DCP, we obtain (14). The derivation of (16) is also straightforward, relying on (EC.13) and the previous FLLN for $Z(t)$.

EC.3. Proof of Theorem 2.

Construction of w on $[0,\infty)$. Here we analyze the situation on the domain $[0,\infty)$. In particular, we aim to establish the following proposition.

PROPOSITION EC.2. *(i) For each $\alpha < c_*$, there exists a unique continuous function w_α on $[0,\infty)$ and a constant η_α such that w_α satisfies*

$$\lambda w_\alpha'(x) - \beta\sqrt{\lambda\mu} w_\alpha(x) + \lambda\phi(x, w_\alpha(x)) + \lambda g(w_\alpha(x)) = \eta_\alpha, \tag{EC.15}$$

subject to boundary conditions $w_\alpha(0) = \alpha$ and $\lim_{x\to\infty} w_\alpha(x) = c_*$; moreover, $w_\alpha(\cdot)$ is strictly increasing on $[0,\infty)$. (ii) If letting χ_+ be defined via $\chi_+(\alpha) = \eta_\alpha$, then χ_+ is continuous and $\lim_{x\to c_*} \chi_+(x) = -\beta\sqrt{\lambda\mu}(c_*) + \lambda g(c_*)$.

In what follows, we keep the value of α fixed, and for each η, let ξ_η be the solution to

$$\lambda \xi_\eta'(x) - \beta\sqrt{\lambda\mu}\xi_\eta(x) + \lambda\phi(x, \xi_\eta(x)) + \lambda g(\xi_\eta(x)) = \eta \tag{EC.16}$$

on $[0, \omega_\eta)$, subject to the boundary condition $\xi_\eta(0) = \alpha$, where ω_η is the explosion time. In the next several lemmas, we fix the value of α and analyze the solution family $\{\xi_\eta\}$.

LEMMA EC.3. *If $\eta_1 > \eta_2$, then $\xi_{\eta_1} > \xi_{\eta_2}$ holds true for all $0 < x < (\omega_{\eta_1} \wedge \omega_{\eta_2})$.*

Proof of Lemma EC.3. From $\eta_1 > \eta_2$, it follows that $\xi'_{\eta_1}(0) > \xi'_{\eta_2}(0)$. Hence, there exists an interval $(0,\delta)$ such that $\xi_{\eta_1}(x) > \xi_{\eta_2}(x)$ for all $x \in (0,\delta)$. Suppose that the assertion is not true. Then there exists some point x_0 such that $0 < x_0 < (\omega_{\eta_1} \wedge \omega_{\eta_2})$, $\xi_{\eta_1}(x_0) = \xi_{\eta_2}(x_0)$ and $\xi_{\eta_1}(x) > \xi_{\eta_2}(x)$ for all $x \in (0, x_0)$; moreover, $\xi'_{\eta_1}(x_0) \le \xi'_{\eta_2}(x_0)$. By (EC.16),

$$\lambda\xi'_{\eta_1}(x_0) - \lambda\xi'_{\eta_2}(x_0) = \eta_1 - \eta_2.$$

By our hypothesis, the left-hand side of the above is non-positive while the right-hand side is strictly positive. This leads to a contradiction; hence the proof is complete. □

LEMMA EC.4. *Any limit points of $\xi_\eta(x)$ as $x \to \omega_\eta$ must be ∞, c_*, or $-\infty$.*

Proof of Lemma EC.4. For the purpose of illustration, we reproduce the definition of function $\phi(x,y)$ here:

$$\phi(x,y) = \inf_{q \in \mathcal{A}(x)} \sum_k (c_k - y)\zeta_k\left(\frac{q_k}{\lambda}\right).$$

From the definition of $\zeta_k(x)$, we know that $\lim_{x\to\infty}\zeta_k(x) = \infty$ for each $k \in \{1,\ldots,K\}$. First, for the case $\omega_\eta < \infty$, $\lim_{x\to\omega_\eta}\xi_\eta(x)$ is either $-\infty$ or ∞. Next, consider the case $\omega_\eta = \infty$. Suppose without loss of generality that $\lim_{x\to\infty}\xi_\eta(x)$ exists and $\lim_{x\to\infty}\xi_\eta(x) = u \in \mathbb{R}$. If $u > c_*$, from the definition of $\phi(\cdot,\cdot)$, we have $\lim_{x\to\infty}\phi(x,\xi_\eta(x)) = -\infty$. Then $\xi'_\eta(x) \to \infty$ as $x \to \infty$ from (EC.16), which is a contradiction to $\lim_{x\to\infty}\xi_\eta(x) = u$. If $u < c_*$, similar arguments lead to $\xi'_\eta(x) \to -\infty$ as $x \to \infty$. Hence, the proof is complete. □

LEMMA EC.5. *Define $S_- := \{\eta : \sup_{x>0}\xi_\eta(x) < c_*\}$. If $\eta \in S_-$, $\xi'_\eta(b) < 0$ for some $b \ge 0$. Besides, $\xi_\eta(x)$ decreases monotonically to $-\infty$ on (b,ω_η).*

Proof of Lemma EC.5. Observing the definition of $\phi(\cdot,\cdot)$, we know that $\phi(x,y)$ strictly increases in x if $y < c_*$ and strictly decreases in x if $y > c_*$. Besides, $\phi(x,c_*) = 0$ for any x.

First, since $\sup_{x>0}\xi(x) < c_*$, by Lemma EC.4, we know that $\lim_{x\to w_\eta}\xi_\eta(x) = -\infty$. Thus, by the continuity of $\xi_\eta(x)$, there exists $b \ge 0$ such that $\xi'_\eta(b) < 0$. Now we prove the second claim: if $\eta \in S_-$ and $\xi'_\eta(b) < 0$, then $\xi'_\eta(x) < 0$ on (b,ω_η). By way of contradiction, if this claim is not true, then there exists x_0 and x_1 such that $b \le x_0 < x_1$ satisfying $\xi_\eta(x_0) = \xi_\eta(x_1) < c_*$, $\xi'_\eta(x_0) \le 0$ and $\xi'_\eta(x_1) \ge 0$. By (EC.16), we have

$$\xi'_\eta(x_0) - \xi'_\eta(x_1) = -(\phi(x_0,\xi_\eta(x_0)) - \phi(x_1,\xi_\eta(x_1))),$$

which is a contradiction since the left-hand side is non-positive but the right-hand side is strictly positive. The proof is complete. □

LEMMA EC.6. *Define $S_+ := \mathbb{R} \setminus S_-$. If $\eta \in S_+$, $\xi_\eta(x)$ strictly increases on $(0,\omega_\eta)$.*

Proof of Lemma EC.6. We begin by arguing that if there exists some b such that $\xi_\eta(b) \geq c_*$, then $\xi'_\eta(b) < 0$ is impossible. By way of contradiction, suppose such b exists. Since $\xi_\eta(0) < c_*$ and $\xi_\eta(x)$ is continuous, we know there exists a $x_0 \in (0,b)$ such that $\xi_\eta(x_0) = \xi_\eta(b) \geq c_*$, $\xi'_\eta(b) < 0$, and $\xi'_\eta(x_0) \geq 0$. However, from (EC.16) we have

$$\xi'_\eta(x_0) - \xi'_\eta(b) = -(\phi(x_0, \xi_\eta(x_0)) - \phi(b, \xi_\eta(b))),$$

which is a contradiction since the left-hand side is positive while the right-hand side is non-positive (zero only if $\xi_\eta(b) = c_*$).

Let $r := \inf\{x > 0 : \xi_\eta(x) \geq c_*\}$. We show $\xi'_\eta(x) \geq 0$ for $x \in (0,r)$. Now suppose there exists some b such that $\xi'(b) < 0$ for some $b \in (0,r)$. Since $\xi_\eta(x) < c_*$ on $(0,r)$, by the same arguments in Lemma EC.5, $\xi_\eta(x)$ decreases on (b, w_η), so $\sup \xi_\eta(x) < c_*$. But this is a contradiction to the definition of S_+.

After proving $\xi'_\eta(x) \geq 0$ on $(0, \omega_\eta)$, the rest is to show that the increase of $\xi_\eta(x)$ is strict: for any $0 \leq x_0 < x_1 < \omega_\eta$, $\xi_\eta(x_0) = \xi_\eta(x_1)$ is impossible. Suppose this claim is not true. Then $\xi_\eta(x) = \xi_\eta(x_0)$ for all $x \in [x_0, x_1]$, because otherwise there must be some $t \in (x_0, x_1)$ such that $\xi'_\eta(t) < 0$ which is impossible based on the aforementioned analysis. Thus, we can find t_1 and t_2 such that $x_0 < t_1 < t_2 < x_1$ satisfying $\xi'_\eta(t_1) = \xi'_\eta(t_2) = 0$ and $\xi_\eta(t_1) = \xi_\eta(t_2)$. By (EC.16), we have:

$$\phi(t_0, \xi_\eta(t_0)) - \phi(t_1, \xi_\eta(t_1)) = 0,$$

which cannot hold if $\xi_\eta(t_1) \neq c_*$. On the other hand, if $\xi_\eta(t_1) = c_*$, the only η satisfying (EC.16) is $\eta_0 := -\beta\sqrt{\lambda\mu}c_* + \lambda g(c_*)$. Then $\xi_{\eta_0}(x)$ on (x_0, x_1) coincides with function $f(x) \equiv c_*$ for $x \in \mathbb{R}$ which is also a solution to (EC.16). By the uniqueness of solutions to ODE with initial values, $f(x) \equiv 0$ on $[0, \infty)$ is the unique solution. But this contradicts our assumption that $\alpha < c_*$. Hence we have proved $\xi_\eta(x)$ strictly increases on $(0, \omega_\eta)$. $\square$

Lemma EC.7. *Both S_+ and S_- are non-empty sets.*

Proof of Lemma EC.7. First, we argue that S_- is non-empty. From (EC.16) we have $\lambda\xi'_\eta(0) - \beta\sqrt{\lambda\mu}\alpha + \lambda g(\alpha) = \eta$. Thus, if $\eta < -\beta\sqrt{\lambda\mu}\alpha + \lambda g(\alpha)$, $\xi'_\eta(0) < 0$. Such η exists because the right-hand side of the inequality is a constant. By Lemma EC.5, $\xi_\eta(x)$ decreases on $[0, \omega_\eta)$, so $\sup \xi_\eta(x) = \alpha < c_*$.

We establish S_+ being non-empty by way of contradiction. Since $S_- \cap S_+ = \varnothing$ and $S_- \cup S_+ = \mathbb{R}$, assuming all $\eta \in S_-$, we first argue that d_η defined via

$$d_\eta := \sup\left\{x \geq 0 : \xi'_\eta(x) \geq 0\right\}$$

is greater than 1 for sufficiently large η. Note that if $d_\eta = \omega_\eta$, then the only possibility is that $d_\eta = \omega_\eta = \infty$, since otherwise both $\xi_\eta(\omega_\eta^-)$ and $\xi'_\eta(\omega_\eta^-)$ will be finite, contradicting the definition of the explosion time. Hence, we suppose without loss of generality that $d_\eta < \omega_\eta$. Then $\xi'_\eta(d_\eta) = 0$, and

$x = d_\eta$ is a local maximum. By Lemma EC.5, it holds that $\xi_\eta(x) \in (\alpha, c_*)$ for all $x \in (0, d_\eta)$. From (EC.16) it follows that $\eta \le K_1 d_\eta + K_2$ for some positive constants K_1 and K_2 that are independent of η and d_η. Therefore, there exists some η_1 such that $d_\eta > 1$ for all $\eta > \eta_1$. Now, combining the preceding result with (EC.16), we obtain

$$\lambda \xi_\eta'(x) \ge \eta - K \quad \text{for} \quad x \in [0,1],$$

where K is some constant independent of η. Integrating both sides from 0 to 1 yields

$$\lambda\left(\xi_\eta(1) - \alpha\right) \ge \eta - K.$$

Thus, we can find an η such that $\xi_\eta(1) \ge c_*$ for sufficiently large η, so $\eta \in S_+$. But this contradicts our assumption that all $\eta \in S_-$. Thus, the proof is complete. □

Proof of Proposition EC.2. To start, define $\eta_\alpha := \inf S_+$. From Lemma EC.4 and EC.7, we know that η_α is no less than any $\eta \in S_-$, so η_α is finite. Besides, from definition we know $\sup_{x\ge 0} \xi_{\eta_\alpha}(x) \ge c_*$, so $\eta_\alpha \in S_+$. We aim to show that ξ_{η_α} satisfies all the requirements in the proposition. We begin by arguing that

$$\xi_{\eta_\alpha}(x) < c_* \quad \text{for all} \quad 0 < x < \omega_{\eta_\alpha} \tag{EC.17}$$

Now, suppose for the sake of contradiction that $\xi_{\eta_\alpha}(x_1) > c_*$ for some $x_1 > 0$. Since ξ_{η_α} is continuous in η, there exists some $\epsilon > 0$ such that

$$\xi_\eta(x_1) > c_* \quad \text{for all} \quad \eta \in (\eta_\alpha - \epsilon, \eta_\alpha).$$

This contradicts the definition of η_α, so (EC.17) follows. We next argue that $\omega_{\eta_\alpha} = \infty$. Suppose by way of contradiction that ω_{η_α} is finite. Then by (EC.17), $\lim_{x\to\omega_{\eta_\alpha}} \xi_{\eta_\alpha}(x) = -\infty$, so $\sup_{x\ge 0} \xi_{\eta_\alpha}(x) < c_*$. However, this contradicts the fact that $\eta_\alpha \in S_+$, so ω_{η_α} must be infinity and $\lim_{x\to\infty} \xi_{\eta_\alpha}(x) = c_*$. The strictly increasing property of $\xi_{\eta_\alpha}(x)$ follows from Lemma EC.6. Hence, we have completed the proof for part (i).

To prove part (ii), we first argue that if $\alpha_1 > \alpha_2$, $w_{\alpha_1}(x) > w_{\alpha_2}(x)$ for each fixed $x > 0$. Suppose by way of contradiction that this claim is not true. Then there exists $x_0 > 0$ such that $w_{\alpha_1}(x_0) = w_{\alpha_2}(x_0)$ and $w_{\alpha_1}'(x_0) \le w_{\alpha_2}'(x_0)$. By the uniqueness of solutions to ODE, $w_{\alpha_1}'(x_0) \ne w_{\alpha_2}'(x_0)$, so $w_{\alpha_1}'(x_0) < w_{\alpha_2}'(x_0)$ and by (EC.15) we have $\eta_{\alpha_1} < \eta_{\alpha_2}$. Using similar arguments in Lemma EC.3, we know that $w_{\alpha_1}(x) < w_{\alpha_2}(x)$ on (x_0, ∞). Let x_n be a sequence that increases to ∞ as $n \to \infty$. By (EC.15)

$$\lim_{n\to\infty} \lambda(\phi(x_n, w_{\alpha_1}(x_n)) - \phi(x_n, w_{\alpha_2}(x_n))) = \eta_{\alpha_1} - \eta_{\alpha_2}.$$

It is plain to see that $\phi(x, y)$ strictly decreases in y, so the left-hand side is non-negative. This leads to a contradiction since the right-hand side is negative. Next, consider an increasing sequence $\{\alpha_n\}$

with α being the limit. For ease of notation, we write $(w_{\alpha_n}, \eta_{\alpha_n})$ as (w_n, η_n). We aim to show that $\eta_n \to \eta_\alpha$ as $n \to \infty$. The aforementioned arguments show that $\{w_n(x)\}$ is an increasing sequence and by our proof in part (i) $w_n(x) \le c_*$ for each fixed x. Hence $w_\infty(x) := \lim_{n\to\infty} w_n(x)$ is well defined for each fixed x. Integrating (EC.15) over $[0,x]$ yields

$$\lambda(w_n(x)-\alpha_n) = \beta\sqrt{\lambda\mu}\int_0^x w_n(u)\mathrm{d}y - \lambda\int_0^x \phi(y, w_n(y))\mathrm{d}y - \lambda\int_0^x g(w_n(y))\mathrm{d}y + \eta_n x.$$

Letting $n \to \infty$ gives us

$$\lambda(w_\infty(x)-\alpha) = \beta\sqrt{\lambda\mu}\int_0^x w_\infty(y)\mathrm{d}y - \lambda\int_0^x \phi(y, w_\infty(y))\mathrm{d}y - \lambda\int_0^x g(w_\infty(y))\mathrm{d}y + \eta_\infty x.$$

On the other hand, using the definition of (w_α, η_α) yields

$$\lambda(w_\alpha(x)-\alpha) = \beta\sqrt{\lambda\mu}\int_0^x w_\alpha(y)\mathrm{d}y - \lambda\int_0^x \phi(y, w_\alpha(y))\mathrm{d}y - \lambda\int_0^x g(w_\alpha(y))\mathrm{d}y + \eta_\alpha x.$$

By the uniqueness result in part (i), it follows that (w_∞, η_∞) must coincide with (w_α, η_α). The case where $\{\alpha_n\}$ decreases and converges to α can be analyzed similarly. We thus complete the proof of part (ii). □

Construction of w on $(-\infty, 0]$. We intend to establish the following proposition.

PROPOSITION EC.3. *(i) For each $\alpha > 0$, there exists a unique continuous function w_α on $(-\infty, 0]$ and a constant η_α such that w_α satisfies*

$$\lambda w_\alpha'(x) - \left(\mu x + \beta\sqrt{\lambda\mu}\right) w_\alpha(x) + \lambda g(w_\alpha(x)) = \eta_\alpha, \tag{EC.18}$$

subject to boundary conditions $w_\alpha(0) = \alpha$ and $\lim_{x\to-\infty} w_\alpha(x) = 0$; moreover, $w_\alpha(\cdot)$ is bounded and strictly increasing on $(-\infty, 0]$. (ii) If letting χ_- be defined via $\chi_-(\alpha) = \eta_\alpha$, then χ_- is continuous and $\lim_{x\to 0}\chi_-(x) = \lambda g(0)$.

By a slight abuse of notation, here and below we use ξ_η to denote the solution to

$$\lambda\xi_\eta'(x) - \left(\mu x + \beta\sqrt{\lambda\mu}\right)\xi_\eta(x) + \lambda g(\xi_\eta(x)) = \eta \tag{EC.19}$$

on $(-\omega_\eta, 0]$, subject to the boundary condition $\xi_\eta(0) = \alpha$, where $-\omega_\eta$ denotes the explosion time. Let $\bar\phi(x,y) := -\mu xy/\lambda$, then we can rewrite (EC.19) as

$$\lambda\xi_\eta'(x) - \beta\sqrt{\lambda\mu}\xi_\eta(x) + \lambda\bar\phi(x, \xi_\eta(x)) + \lambda g(\xi_\eta(x)) = \eta.$$

Note that $\bar\phi(x,y)$ is now strictly decreasing in x if $y > 0$ and strictly increasing in x if $y < 0$. Besides, $\bar\phi(x, 0) = 0$ for any x. The next several lemmas assume that the value of α is fixed. The proof of Proposition EC.3 relies on the following five lemmas. Proofs of these lemmas resemble those of Lemmas EC.3, EC.4, EC.5, EC.6 and EC.7 and are hence omitted.

LEMMA EC.8. *If $\eta_1 > \eta_2$, then $\xi_{\eta_1} < \xi_{\eta_2}$ holds true for all $-(\omega_{\eta_1} \wedge \omega_{\eta_2}) < x < 0$.*

LEMMA EC.9. *Any limit points of $\xi_\eta(x)$ as $x \to -\omega_\eta$ is ∞, 0, or $-\infty$.*

LEMMA EC.10. *Define $D_+ := \{\eta : \inf_{x<0} \xi_\eta(x) > 0\}$. If $\eta \in D_+$, $\xi'_\eta(b) < 0$ for some $b \leq 0$. Besides, $\xi_\eta(x)$ decreases monotonically on $(-\omega_\eta, b)$.*

LEMMA EC.11. *Define $D_- := \mathbb{R} \setminus D_+$. If $\eta \in D_-$, $\xi_\eta(x)$ strictly increases on $(-\omega_\eta, 0)$.*

LEMMA EC.12. *Both D_+ and D_- are non-empty sets.*

Proof of Proposition EC.3. To start, define $\eta_\alpha := \inf D_-$. From Lemma EC.9 and EC.12, we know that η_α is no less than any $\eta \in D_+$, so η_α is finite. Besides, from definition we know $\inf_{x \leq 0} \xi_{\eta_\alpha}(x) \leq 0$, so $\eta_\alpha \in D_-$. We aim to show that ξ_{η_α} satisfies all the requirements in the proposition. We begin by arguing that

$$\xi_{\eta_\alpha}(x) > 0 \quad \text{for all} \quad -\omega_{\eta_\alpha} < x < 0. \tag{EC.20}$$

Now, suppose for the sake of contradiction that $\xi_{\eta_\alpha}(x_1) < 0$ for some $x_1 < 0$. Since ξ_η is continuous in η, there exists some $\epsilon > 0$ such that

$$\xi_\eta(x_1) < 0 \quad \text{for all} \quad \eta \in (\eta_\alpha - \epsilon, \eta_\alpha).$$

This contradicts the definition of η_α, so (EC.20) follows. We next argue that $-\omega_{\eta_\alpha} = -\infty$. Suppose by way of contradiction that $-\omega_{\eta_\alpha}$ is finite. Then by (EC.20), $\lim_{x \to -\omega_{\eta_\alpha}} \xi_{\eta_\alpha}(x) = \infty$, so $\inf_{x \leq 0} \xi_{\eta_\alpha}(x) > 0$. However, this is impossible since $\inf_{x \leq 0} \xi_{\eta_\alpha}(x) \leq 0$. It follows that $-\omega_{\eta_\alpha}$ must be negative infinity and $\lim_{x \to -\infty} \xi_{\eta_\alpha}(x) = 0$. The strictly increasing property of $\xi_{\eta_\alpha}(x)$ follows from Lemma EC.11. We thus have completed the proof of part (i).

To prove part (ii), we first argue that if $\alpha_1 > \alpha_2$, $w_{\alpha_1}(x) > w_{\alpha_2}(x)$ for each fixed $x < 0$. Suppose by way of contradiction that this claim is not true. Then there exists $x_0 < 0$ such that $w_{\alpha_1}(x_0) = w_{\alpha_2}(x_0)$ and $w'_{\alpha_1}(x_0) \geq w'_{\alpha_2}(x_0)$. However, by the uniqueness of solutions to ODE, $w'_{\alpha_1}(x_0) \neq w'_{\alpha_2}(x_0)$, so $w'_{\alpha_1}(x_0) > w'_{\alpha_2}(x_0)$ and by (EC.18) we have $\eta_{\alpha_1} > \eta_{\alpha_2}$. Using the similar arguments in Lemma EC.3, we know that $w_{\alpha_1}(x) < w_{\alpha_2}(x)$ on $(-\infty, x_0)$. Let x_n be a sequence that decreases to $-\infty$ as $n \to \infty$. By (EC.18)

$$\lim_{n \to \infty} \lambda(\bar{\phi}(x_n, w_{\alpha_1}(x_n)) - \bar{\phi}(x_n, w_{\alpha_2}(x_n))) = \eta_{\alpha_1} - \eta_{\alpha_2}.$$

From the definition of $\bar{\phi}(\cdot, \cdot)$, one could easily show that $\bar{\phi}(x, y)$ is increasing in y when $x < 0$, so the left-hand side is non-positive. This leads to a contradiction since the right-hand side is positive. Next, consider an increasing sequence $\{\alpha_n\}$ with α being the limit. For ease of notation, we write $(w_{\alpha_n}, \eta_{\alpha_n})$ as (w_n, η_n). We aim to show that $\eta_n \to \eta_\alpha$ as $n \to \infty$. The aforementioned arguments show that $\{w_n(x)\}$ is an increasing sequence and $w_n(x) \le \alpha$ for each fixed x. Hence $w_\infty(x) := \lim_{n\to\infty} w_n(x)$ is well defined for each fixed x. Integrating (EC.18) over $[x,0]$ yields

$$\lambda(\alpha_n - w_n(x)) = \int_x^0 (\beta\sqrt{\lambda\mu} + \mu y) w_n(y)\mathrm{d}y - \lambda \int_x^0 g(w_n(y))\mathrm{d}y - \eta_n x.$$

Sending $n \to \infty$ in the above identify gives us

$$\lambda(\alpha - w_\infty(x)) = \int_x^0 (\beta\sqrt{\lambda\mu} + \mu y) w_\infty(y)\mathrm{d}y - \lambda \int_x^0 g(w_\infty(y))\mathrm{d}y - \eta_\infty x.$$

On the other hand, using the definition of (w_α, η_α) yields to

$$\lambda(\alpha - w_\alpha(x)) = \int_x^0 (\beta\sqrt{\lambda\mu} + \mu y) w_\alpha(y)\mathrm{d}y - \lambda \int_x^0 g(w_\alpha(y))\mathrm{d}y - \eta_\alpha x.$$

By the uniqueness result in part (i), it follows that (w_∞, η_∞) must coincide with (w_α, η_α). The case where $\{\alpha_n\}$ decreases and converges to α can be analyzed in exactly the same way. Hence we complete the proof of part (ii). □

With the two propositions, namely, Propositions EC.2 and EC.3 at hand, we are ready to complete the proof of Theorem 2.

Proof of Theorem 2. Recall the two functions χ_+ and χ_- introduced in Propositions EC.2 and EC.3, respectively. Clearly, we have

$$\chi_+(0) = \lambda w_0'(x) + \lambda g(0) > \lambda g(0) = \chi_-(0)$$

and

$$\chi_+(c_*) = -\beta\sqrt{\lambda\mu} c_* + \lambda g(c_*) < \lambda w_c'(0) - \beta\sqrt{\lambda\mu} c_* + \lambda g(c_*) = \chi_-(c_*).$$

By part (ii) of Proposition EC.2 and part (ii) of Proposition EC.3, we know that both χ_+ and χ_- are continuous functions on their respective domains. Thus, we can use the intermediate value theorem to conclude that there exists $\alpha^\star \in (0, c_*)$ such that $\chi_+(\alpha^\star) = \chi_-(\alpha^\star)$. Let w be w_α as in Proposition EC.2 with $\alpha = \alpha^\star$ on $[0,\infty]$ and w_α as in Proposition EC.3 with $\alpha = \alpha^\star$ on $(-\infty, 0]$. Also, let $\eta := \chi_+(\alpha^\star) = \chi_-(\alpha^\star)$. By our construction, $(w_{\alpha^\star}, \eta)$ satisfies all the requirements as stipulated in Theorem 2; hence the proof is complete. □

EC.4. Proof of Theorem 3

Proof of Theorem 3. Let v be any function such that $v' = w$. Also, denote by $\hat{Q}$ an arbitrary stationary Markov control and $\hat{X}$ the associated controlled process. Suppose that $\hat{X}(0) = x$. By applying the Itô's formula, we get

$$\begin{aligned} v\left(\hat{X}(t)\right) = \int_0^t \left(-\beta\sqrt{\lambda\mu} - \sqrt{\lambda}\theta(u) + \mu[\hat{X}(u)]^- - \lambda \sum_k \zeta_k \left(\frac{\hat{Q}_k(u)}{\lambda}\right)\right) v'\left(\hat{X}(u)\right)\mathrm{d}u \\ + v(x) + \lambda \int_0^t v''\left(\hat{X}(u)\right)\mathrm{d}u + \sqrt{2\lambda}\int_0^t v'\left(\hat{X}(u)\right)\mathrm{d}B(u). \end{aligned} \tag{EC.21}$$

From the definition of ϕ it follows that

$$\phi\left(\hat{X}(t), v'\left(\hat{X}(t)\right)\right) \leq \sum_k c_k \zeta_k \left(\frac{\hat{Q}_k(t)}{\lambda}\right) - v'\left(\hat{X}(t)\right) \sum_k \zeta_k \left(\frac{\hat{Q}_k(t)}{\lambda}\right) \quad \text{for all} \quad t. \tag{EC.22}$$

To proceed, define

$$\theta^{\#}(x) := \arg\max_{\theta} \left\{-v'(x)(\theta/\sqrt{\lambda}) - \alpha f(\theta/\sqrt{\lambda})\right\}. \tag{EC.23}$$

Clearly, $\theta^{\#}$ is one of nature's admissible controls. From the definition of g, it also follows that

$$g\left(\hat{X}(t)\right) = -v'\left(\hat{X}(t)\right)\left(\theta^{\#}\left(\hat{X}(t)\right)/\sqrt{\lambda}\right) - \alpha f\left(\theta^{\#}\left(\hat{X}(t)\right)/\sqrt{\lambda}\right). \tag{EC.24}$$

Now, combining (EC.21), (EC.22) and (EC.24) yields

$$\int_0^t \left[\left(\mu[\hat{X}(u)]^- - \beta\sqrt{\lambda\mu} + \lambda\phi\left(\hat{X}(u), v'\left(\hat{X}(u)\right)\right) + \lambda g\left(\hat{X}(u)\right)\right) v'\left(\hat{X}(u)\right) + \lambda v''\left(\hat{X}(u)\right)\right] \mathrm{d}u$$
$$\leq v\left(\hat{X}(t)\right) - v(x) + \lambda \int_0^t \left(\sum_k c_k \zeta_k \left(\frac{\hat{Q}_k(u)}{\lambda}\right) - \alpha f\left(\frac{\theta^{\#}\left(\hat{X}(u)\right)}{\sqrt{\lambda}}\right)\right) \mathrm{d}u - \sqrt{2\lambda}\int_0^t v'\left(\hat{X}(u)\right) \mathrm{d}B(u).$$

In view of (19), we deduce using the above inequality that

$$\eta t \leq v\left(\hat{X}(t)\right) - v(x) + \lambda \int_0^t \left(\sum_k c_k \zeta_k \left(\frac{\hat{Q}_k(u)}{\lambda}\right) - \alpha f\left(\frac{\theta^{\#}\left(\hat{X}(u)\right)}{\sqrt{\lambda}}\right)\right) \mathrm{d}u - \sqrt{2\lambda}\int_0^t v'\left(\hat{X}(u)\right) \mathrm{d}B(u).$$

Taking expectation on both sides, dividing both sides by t, and then sending $t \to \infty$, we obtain

$$\eta \leq \limsup_{t\to\infty} \frac{1}{t}\mathbb{E}^{\theta^{\#}}\left[v\left(\hat{X}(t)\right)\right] + \limsup_{t\to\infty}\frac{\lambda}{t}\mathbb{E}^{\theta^{\#}}\left[\int_0^t \left(\sum_k c_k\zeta_k\left(\frac{\hat{Q}_k(u)}{\lambda}\right) - \alpha f\left(\frac{\theta^{\#}\left(\hat{X}(u)\right)}{\sqrt{\lambda}}\right)\right)\mathrm{d}u\right].$$

The first assertion of the theorem then follows from the following lemma (whose proof is deferred to after the current proof) and the Lipschitz continuity of v plus the fact that $\theta^{\#}$ is an admissible strategy.

LEMMA EC.13. *Regardless of the choice of $\hat{Q}$, $\limsup_{t\to\infty} t^{-1}\mathbb{E}^{\theta^{\#}}\left[\hat{X}(t)\right] = 0$ for $\theta^{\#}$ specified via* (EC.23)*.*

Towards establishing the second assertion, note that all the preceding inequalities hold as an equality when $\hat{Q}$ is replaced with $q^\star$. In particular,

$$\eta = \limsup_{t\to\infty} \frac{1}{t}\mathbb{E}^{\theta^{\#}}\left[v\left(\hat{X}^\star(t)\right)\right] + \limsup_{t\to\infty}\frac{\lambda}{t}\mathbb{E}^{\theta^{\#}}\left[\int_0^t \left(\sum_k c_k\zeta_k\left(\frac{q_k^\star(u)}{\lambda}\right) - \alpha f\left(\frac{\theta^{\#}\left(\hat{X}(u)\right)}{\sqrt{\lambda}}\right)\right)\mathrm{d}u\right],$$

where we have used $\hat{X}^\star$ to denote the controlled process under the policy $q^\star$. Again, by Lemma EC.13 and the Lipschitz continuity of v, we have that $\limsup_{t\to\infty} t^{-1}\mathbb{E}^{\theta^{\#}}\left[v\left(\hat{X}^\star(t)\right)\right] = 0$. Therefore, to prove the second assertion, it suffices to argue that $\theta^{\#}$ is nature's best response to $q^\star$. But that follows from yet another standard verification argument plus a straightforward application of the Itô's formula. □

Proof of Lemma EC.13. First, using the uniform boundedness of the function w, one can easily argue that there exists some $M>0$ such that $-M \leq \theta^{\#} \leq 0$. Our goal is to seek two processes $\hat{Y}_1$ and $\hat{Y}_2$ such that

$$\hat{Y}_1(t) \overset{s.t.}{\leq} \hat{X}(t) \overset{s.t.}{\leq} \hat{Y}_2(t). \tag{EC.25}$$

If both $\hat{Y}_1$ and $\hat{Y}_2$ have a stationary distribution with a finite mean when nature chooses $\theta^{\#}$, then we are done with the proof. Towards that goal, let $\hat{Y}_1$ be a non-positive Ornstein–Uhlenbeck (OU) process with reflection at the origin:

$$\hat{Y}_1(t) = \hat{X}(0) - \beta\sqrt{\lambda\mu}t - \mu\int_0^t \hat{Y}_1(u)\mathrm{d}u + \sqrt{2\lambda}B(t) - U(t),$$

where U denotes the regulator that keeps $\hat{Y}_1$ non-positive at all times. Similarly, we define $\hat{Y}_2$ to be a non-negative OU process with reflection at the origin:

$$\hat{Y}_2(t) = \hat{X}(0) + \sqrt{\lambda}(M - \beta\sqrt{\lambda\mu})t - b\int_0^t \hat{Y}_2(u)\mathrm{d}u + \sqrt{2\lambda}B(t) + L(t),$$

where L denotes the regulator that keeps $\hat{Y}_2$ non-negative at all times and b is as in Assumption 1. The first stochastic relation in (EC.25) is immediate by our construction. The second stochastic relation in (EC.25) follows from Assumption 1. The processes $\hat{Y}_1$ and $\hat{Y}_2$ have a stationary distribution with a finite mean (when nature chooses $\theta^{\#}$) since a reflected OU process is known to have a stationary distribution that is truncated normal. This concludes the proof. $\square$

EC.5. Model Uncertainty About Demand Realization

If the model uncertainty relates to the arrival processes, then nature's function is to influence the law regulating customer arrivals, changing the arrival rate of class k customers from λ_k to $(1+\psi_k(t))\lambda_k$, where ψ_k is a belief distortion process similar to ψ in the main paper. Letting $\theta_k(t) := \psi_k(t)/\sqrt{\lambda}$ and $\boldsymbol{\theta} := (\theta_1, \ldots, \theta_K)$ and replicating the steps taken to derive the SDG in Section 4, we can derive the diffusion approximation for the resulting stochastic game, where the DM seeks to minimize

$$\sup_{\boldsymbol{\theta}} \limsup_{t\to\infty} \frac{\lambda}{t}\mathbb{E}^{\boldsymbol{\theta}}\left[\sum_k c_k \int_0^t \zeta_k\left(\frac{\hat{Q}_k(u)}{\lambda}\right)\mathrm{d}u - \alpha\int_0^t \check{f}\left(\frac{\boldsymbol{\theta}(u)}{\sqrt{\lambda}}\right)\mathrm{d}u\right],$$

subject to

$$\hat{X}(t) = \hat{X}(0) - \beta\sqrt{\lambda\mu}t + \sqrt{\lambda}\sum_k\int_0^t a_k\theta_k(u)\mathrm{d}u + \mu\int_0^t [\hat{X}(u)]^-\mathrm{d}u - \lambda\sum_k\int_0^t \zeta_k\left(\frac{\hat{Q}_k(u)}{\lambda}\right)\mathrm{d}u + \sqrt{2\lambda}B(t)$$

and (15), where $\check{f}$ is defined as $\check{f}(\boldsymbol{x}) := \sum_k a_k((1+x_k)\ln(1+x_k) - x_k)$. The specific form of $\check{f}$ follows from the fact that changing the arrival rate of class k customers from λ_k to $(1+\psi_k(t))\lambda_k$ is tantamount to changing the probability low from $\mathbb{P}$ to a new one defined by

$$\left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} = \exp\left\{\sum_k\int_0^t \ln(1+\psi_k(u))\mathrm{d}A(u) - \psi_k(u)\lambda_k\mathrm{d}u\right\},$$

plus the usage of relative entropy to quantify the distance from $\mathbb{Q}$ to $\mathbb{P}$.

The Bellman-Isaacs equation can be written as

$$\lambda v''(x) + \left(\mu x^- - \beta\sqrt{\lambda\mu}\right) v'(x) + \lambda\phi(x, v'(x)) + \lambda\check{g}(v'(x)) = \eta, \tag{EC.26}$$

where $\check{g}$ is given by

$$\check{g}(y) := \sup_{\boldsymbol{\theta}} \left\{ (y/\sqrt{\lambda}) \sum_k a_k \theta_k - \alpha \check{f}(\boldsymbol{\theta}/\sqrt{\lambda}) \right\}.$$

One could easily argue that $\check{g}(y)$ satisfies the local Lipschitz continuity condition. Thus, since the proof of Theorem 2 only relies on the local Lipschitz continuity of g, we conclude that Equation (EC.26) (with appropriate boundary conditions) has a solution as well.

EC.6. Quadratic Search Algorithm (QSA)

We introduce a quadratic search algorithm to search for the best possible policy generated through our modeling framework. The motivation for designing this algorithm is that when the generated policies are not of threshold types, it is impossible to exhaustively evaluate all alternative policies corresponding to different α, since there are infinitely many α that can be chosen. The quadratic search algorithm is a heuristic that helps find the best possible policy by iteratively adjusting the "shape" of quadratic curves using simulated data to search for a new candidate α and then run the simulation program using the policy generated through solving the penalty problem under the new α. The detailed description of the algorithm is given as follows.

Algorithm 1 Quadratic Search Algorithm

Input: MAXCOUNTS, LB_α, δ, $\hat{\gamma}$

1: Choose $\alpha_1, \alpha_2, \alpha_3$ with $\alpha_1 < \alpha_2 = \alpha(\hat{\gamma}) < \alpha_3$ and calculate corresponding policies q_1, q_2, q_3.
2: Estimate long-run average abandonment cost $\hat{J}_1, \hat{J}_2, \hat{J}_3$ based on simulation programs using q_1, q_2, q_3, respectively.
3: Fit the quadratic curve $y = ax^2 + bx + c$ using data $(\alpha_1, \alpha_2, \alpha_3)$ and $(\hat{J}_1, \hat{J}_2, \hat{J}_3)$.
4: **if** $a > 0$ **then**
5:     $\alpha \leftarrow -b/(2a)$
6: **else**
7:     **if** $\hat{J}_1 > \hat{J}_3$ **then**
8:         $\alpha \leftarrow (1+\delta)\alpha_3$
9:     **else**
10:         $\alpha \leftarrow (1-\delta)\alpha_1$
11:     **end if**
12: **end if**
13: **if** $\alpha < 0$ **then**
14:     $\alpha \leftarrow \mathrm{LB}_\alpha$
15: **end if**
16: Calculate policy q based on α and estimate $\hat{J}$.
17: **if** $\alpha < \alpha_1$ **then**
18:     $\alpha_3 \leftarrow \alpha_2,\ \alpha_2 \leftarrow \alpha_1,\ \alpha_1 \leftarrow \alpha;\ \hat{J}_3 \leftarrow \hat{J}_2,\ \hat{J}_2 \leftarrow \hat{J}_1,\ \hat{J}_1 \leftarrow \hat{J}$
19: **else if** $\alpha \geq \alpha_3$ **then**
20:     $\alpha_1 \leftarrow \alpha_2,\ \alpha_2 \leftarrow \alpha_3,\ \alpha_3 \leftarrow \alpha;\ \hat{J}_1 \leftarrow \hat{J}_2,\ \hat{J}_2 \leftarrow \hat{J}_3,\ \hat{J}_3 \leftarrow \hat{J}$
21: **else**
22:     Find i such that $\alpha_i \leq \alpha < \alpha_{i+1}$. $\alpha_{-2i+5} \leftarrow \alpha_2,\ \alpha_2 \leftarrow \alpha;\ \hat{J}_{-2i+5} \leftarrow \hat{J}_2,\ \hat{J}_2 \leftarrow \hat{J}$
23: **end if**
24: **repeat**
25:     Steps 2 - 23, $\mathrm{COUNTS} \leftarrow \mathrm{COUNTS} + 1$
26: **until** $\min\{(\alpha_2 - \alpha_1), (\alpha_3 - \alpha_2)\} \leq \epsilon$ or $\mathrm{COUNTS} > \mathrm{MAXCOUNTS}$
27: **return** $\alpha_i, \hat{J}_i$, where $i = \min\left(\arg\min\left\{\hat{J}_1, \hat{J}_2, \hat{J}_3\right\}\right)$