E-Companion

"***Audit and Remediation Strategies in the Presence of Evasion Capabilities***"

by Shouqiang Wang, Francis de Véricourt, and Peng Sun

Appendix A: Definitions in Section 3 and Proofs in Section 4

We first formally define the audit policy by adopting Definition A.1 from Wang et al. (2016).

DEFINITION A.1 (AUDIT POLICY). Let dt and δ_t denote the usual Lebesgue measure and Dirac measure on time horizon $t \in [0,\infty)$, respectively. We call $\{q_t^n \in [0,\infty) : t \geq 0\}$ and $\{q_t^m \in [0,1] : t \geq 0\}$ an *intensity audit policy* and an *impulsive audit policy*, respectively, if

1. the process q_t^n and q_t^m are $\mathscr{F}_t$-predictable, where $\mathscr{F}_t$ is the filtration generated by N_t;
2. the measure $\mu(dt) := q_t^n dt + q_t^m \delta_t$ satisfies
$$\int_0^t \mu(ds) < \infty, \quad t \geq 0; \text{ and} \tag{A.1}$$
3. the measure $\mu(dt)$ consists of an $\mathscr{F}_t$-predictable compensator (e.g., Brémaud 1981, Lipster and Shiryaev 2010) for the counting process N_t, i.e.,
$$\mathbb{E}\left[\int_0^\infty X_t dN_t\right] = \mathbb{E}\left[\int_0^\infty X_t \mu(dt)\right] \tag{A.2}$$

for any bounded $\mathscr{F}_t$-predictable process X_t. $\square$

Let random variable $Y_t \in \{0,1\}$ denote the result of an audit conducted at time t, with $Y_t = 0$ if an issue is detected and $Y_t = 1$ otherwise. Thus, the random process $Z_t = \Pi_{\tau \in \mathcal{I}_t} Y_\tau \in \{0,1\}$ denotes whether the agent survive the principal's audits by time t. Thus, Z_t starts with value 1 representing no detection by an audit up to time t and jumps down to value 0 once an audit detects an issue at time t. Furthermore, let the ternary process $H_t \in \{0, \pm 1\}$ denote whether the agent has taken any action: $H_t = 0$ before the agent makes any action (i.e., for all $t \leq \sigma(T)$); H_t enters the absorbing state 1 once the agent takes an evasive action (and the auditing accuracy reduces to β); H_t enters the absorbing state -1 once the agent conducts self-correction (and the auditing accuracy reduces to 0 and $Z_s := 1$ for all $s \geq t$). Using these notation, we have

$$\mathbb{P}[Y_t = 0 \mid T > t] = 0, \ \mathbb{P}[Y_t = 0 \mid T \leq t, Z_t = 1, H_t = 0] = 1, \text{ and } \mathbb{P}[Y_t = 0 \mid T \leq t, Z_t = 1, H_t = 1] = \beta. \tag{A.3}$$

In particular, having taken an evasive action at time t (i.e., $H_t = 1$), the agent's expected discounted cost onwards under policy $\mathcal{P} := (F_t, P_t, Q_t)_{t \in [0,\infty)}$ can be written as

$$U_t := \mathbb{E}\left[-\int_t^\infty e^{-\theta(\zeta - t)} F_\zeta dZ_\zeta \,\middle|\, T \leq t, Z_t = 1, H_t = 1, \mathcal{I}_t, \mathcal{P}\right]. \tag{A.4}$$

Correspondingly, the principal's expected total cost after the agent takes an evasive action from t onwards can be similarly computed as

$$V_t := \mathbb{E}\left[\int_t^\infty e^{-\theta(\zeta - t)} \Big\{ Z_\zeta \left(c d\zeta + k dN_\zeta\right) - (r - F_\zeta) dZ_\zeta \Big\} \,\middle|\, T \leq t, Z_t = 1, H_t = 1, \mathcal{I}_t, \mathcal{P}\right]. \tag{A.5}$$

If a self-correction is conducted by time t, the principal will only incur auditing cost afterwards indefinitely (because no detection will ever occur), resulting an expected total cost of

$$W_t := \mathbb{E}\left[\int_t^\infty e^{-\theta(\zeta - t)} k dN_\zeta \,\middle|\, T \leq t, Z_t = 1, H_t = -1, \mathcal{I}_t, \mathcal{P}\right]. \tag{A.6}$$

Proof of Theorem 1. Denote $\widehat{U}_t$ and $\widehat{V}_t$ (resp., $\widehat{W}_t$) as the agent's and the principal's expected discounted cost from time t onwards under the policy $\widehat{\mathcal{P}} := \left(\widehat{F}_t, \widehat{P}_t, \widehat{Q}_t\right)_{t\in[0,\infty)}$ after the agent takes evasive action (resp., self-correction) at time t. Also, denote the agent's corresponding best response strategy as $\widehat{\sigma}^\star(\cdot)$, the optimal stopping time to take action (i.e., disclosure, evasion, or self-correction). According to (A.4), (A.5) and (A.6),

$$\widehat{U}_t := \mathbb{E}\left[-\int_t^\infty e^{-\theta(\zeta-t)}\widehat{F}_\zeta dZ_\zeta \,\middle|\, T\le t, Z_t=1, H_t=1, \mathcal{I}_t, \widehat{\mathcal{P}}\right], \tag{A.7}$$

$$\widehat{V}_t := \mathbb{E}\left[\int_t^\infty e^{-\theta(\zeta-t)}\left\{Z_\zeta\left(cd\zeta + kdN_\zeta\right) - (r-\widehat{F}_\zeta)dZ_\zeta\right\} \,\middle|\, T\le t, Z_t=1, H_t=1, \mathcal{I}_t, \widehat{\mathcal{P}}\right], \quad \text{and} \tag{A.8}$$

$$\widehat{W}_t := \mathbb{E}\left[\int_t^\infty e^{-\theta(\zeta-t)}kdN_\zeta \,\middle|\, T\le t, Z_t=1, H_t=-1, \mathcal{I}_t, \widehat{\mathcal{P}}\right]. \tag{A.9}$$

Now we construct an alternative policy $\mathcal{P} := (F_t, P_t, Q_t)_{t\in[0,\infty)}$ by letting $F_t := \widehat{F}_t$, $Q_t := \widehat{Q}_t$, and,

$$\begin{aligned} P_t := &\min_{\widehat{\sigma}\ge t}\mathbb{E}\left[e^{-\theta(\widehat{\sigma}-t)}\min\left\{\widehat{P}_{\widehat{\sigma}}, h+\widehat{U}_{\widehat{\sigma}}, r\right\}Z_{\widehat{\sigma}} - \int_t^{\widehat{\sigma}} e^{-\theta(\zeta-t)}\widehat{F}_\zeta dZ_\zeta \,\middle|\, T\le t, Z_t=1, H_t=0, \mathcal{I}_t, \widehat{\mathcal{P}}\right], \\ = &\mathbb{E}\left[e^{-\theta(\widehat{\sigma}^\star(t)-t)}\min\left\{\widehat{P}_{\widehat{\sigma}^\star(t)}, h+\widehat{U}_{\widehat{\sigma}^\star(t)}, r\right\}Z_{\widehat{\sigma}^\star(t)} - \int_t^{\widehat{\sigma}^\star(t)} e^{-\theta(\zeta-t)}\widehat{F}_\zeta dZ_\zeta \,\middle|\, T\le t, Z_t=1, H_t=0, \mathcal{I}_t, \widehat{\mathcal{P}}\right], \quad \forall \mathcal{I}_t, \end{aligned} \tag{A.10}$$

namely P_t is the agent's minimum expected discounted cost from t onwards under policy $\widehat{\mathcal{P}}$, given that the issue has emerged ($T\le t$), no disclosure nor detection has occurred ($Z_t=1$), and the agent has not yet taken any evasive action ($H_t=0$). Under this specification of $\mathcal{P}$, we immediately have $U_t=\widehat{U}_t$ by (A.4) and (A.7), $V_t=\widehat{V}_t$ by (A.5) and (A.8), and $W_t=\widehat{W}_t$ by (A.6) and (A.9), for all $\mathcal{I}_t$.

Now we demonstrate that the above-defined policy $\mathcal{P}$ satisfies the following properties.

Property 1: $\mathcal{P}$ is well defined (i.e., $P_t\le F$) and, in particular, $P_t\le \min\{h+U_t, r\}$ for all t, suggesting that the agent always weakly prefers disclosure to evasion and self-correction, namely (6). Indeed, it is obvious that F_t and Q_t is, by construction, well defined. By the definition of P_t in (A.10), we immediately note that, since $\widehat{P}_t\le F$ and $\widehat{F}_t\le F$,

$$P_t \le F\mathbb{E}\left[e^{-\theta(\widehat{\sigma}^\star(t)-t)}Z_{\widehat{\sigma}} - \int_t^{\widehat{\sigma}^\star(t)} e^{-\theta(\zeta-t)}dZ_\zeta \,\middle|\, T\le t, Z_t=1, H_t=0, \mathcal{I}_t, \widehat{\mathcal{P}}\right] \le F.$$

The property that $P_t\le\min\{h+U_t, r\}$ follows from the optimality of $\widehat{\sigma}^\star$ in (A.10): $P_t\le \min\left\{\widehat{P}_t, h+\widehat{U}_t, r\right\}\le \min\{h+U_t, r\}$.

Property 2: Prompt disclosure is the agent's best response to $\mathcal{P}$ (in the sense of weakly dominant strategy). Indeed, we have, for all $s\ge t$,

$$\begin{aligned} P_t = &\mathbb{E}\left[e^{-\theta(\widehat{\sigma}^\star(t)-t)}\min\left\{\widehat{P}_{\widehat{\sigma}^\star(t)}, h+\widehat{U}_{\widehat{\sigma}^\star(t)}, r\right\}Z_{\widehat{\sigma}^\star(t)} - \int_t^{\widehat{\sigma}^\star(t)} e^{-\theta(\zeta-t)}\widehat{F}_\zeta dZ_\zeta \,\middle|\, T\le t, Z_t=1, H_t=0, \mathcal{I}_t, \widehat{\mathcal{P}}\right] \\ \le &\mathbb{E}\left[e^{-\theta(\widehat{\sigma}^\star(s)-t)}\min\left\{\widehat{P}_{\widehat{\sigma}^\star(s)}, h+\widehat{U}_{\widehat{\sigma}^\star(s)}, r\right\}Z_{\widehat{\sigma}^\star(s)} - \int_t^{\widehat{\sigma}^\star(s)} e^{-\theta(\zeta-t)}\widehat{F}_\zeta dZ_\zeta \,\middle|\, T\le t, Z_t=1, H_t=0, \mathcal{I}_t, \widehat{\mathcal{P}}\right] \\ = &\mathbb{E}\left[e^{-\theta(s-t)}Z_sP_s - \int_t^s e^{-\theta(\zeta-t)}\widehat{F}_\zeta dZ_\zeta \,\middle|\, T\le t, Z_t=1, H_t=0, \mathcal{I}_t, \widehat{\mathcal{P}}\right] \\ = &\mathbb{E}\left[e^{-\theta(s-t)}Z_sP_s - \int_t^s e^{-\theta(\zeta-t)}F_\zeta dZ_\zeta \,\middle|\, T\le t, Z_t=1, H_t=0, \mathcal{I}_t, \mathcal{P}\right], \end{aligned} \tag{A.11}$$

where the first equality follows from the construction of P_t in (A.10), the first inequality follows from the optimality of $\widehat{\sigma}^\star$ in (A.10), the second equality follows by splitting the time interval $[t,\widehat{\sigma}^\star(s)]$ into $[t,s]$ and $[s,\widehat{\sigma}^\star(s)]$ and the construction of P_s in (A.10), and the last equality follows from the construction that $F_t=\widehat{F}_t$. The right-hand side of (A.11) is nothing but the agent's total expected discounted cost of delaying the disclosure to any stopping time $s\geq t$ under $\mathcal{P}$ (by Property 1, there is no incentive to evade or self-correct at any point in time under $\mathcal{P}$). As such, the agent always prefers to disclose without delay. By taking unconditional expectation on both sides of (A.11) immediately yields $C_a(\mathcal{P},T)\leq C_a(\mathcal{P},\sigma)$ for all σ.

Property 3: The principal is not worse off under $\mathcal{P}$ than under $\widehat{\mathcal{P}}$. We first note that since $c\geq\theta r$ and $k\geq 0$, (A.8) implies that

$$\begin{aligned}
\widehat{V}_t=&\mathbb{E}\left[\int_t^\infty e^{-\theta(\zeta-t)}\Big\{Z_\zeta\left(cd\zeta+kdN_\zeta\right)-(r-\widehat{F}_\zeta)dZ_\zeta\Big\}\,\middle|\,T\leq t,Z_t=1,H_t=1,\mathcal{I}_t,\widehat{\mathcal{P}}\right]\\
\geq& r\mathbb{E}\left[\theta\int_t^\infty e^{-\theta(\zeta-t)}Z_\zeta d\zeta-\int_t^\infty e^{-\theta(\zeta-t)}dZ_\zeta\,\middle|\,T\leq t,Z_t=1,H_t=1,\mathcal{I}_t,\widehat{\mathcal{P}}\right]\\
&+\mathbb{E}\left[\int_t^\infty e^{-\theta(\zeta-t)}\widehat{F}_\zeta dZ_\zeta\,\middle|\,T\leq t,Z_t=1,H_t=1,\mathcal{I}_t,\widehat{\mathcal{P}}\right]\\
=&r-\widehat{U}_t,
\end{aligned}\tag{A.12}$$

where the last equality follows from the definition (A.4) and the direct calculation:

$$e^{-\theta(t'-t)}Z_{t'}-1=\int_t^{t'}e^{-\theta(\zeta-t)}dZ_\zeta-\theta\int_t^{t'}e^{-\theta(\zeta-t)}Z_\zeta d\zeta,\quad\text{for } Z_t=1,\ \text{and then let } t'\to\infty.\tag{A.13}$$

Then, by (4), the principal's expected cost under policy $\widehat{\mathcal{P}}$ is given by

$$\begin{aligned}
C\left(\widehat{\mathcal{P}},\widehat{\sigma}^\star\right)=&\mathbb{E}\Bigg[k\int_0^T e^{-\theta t}dN_t+c\int_T^{\widehat{\sigma}^\star(T)\wedge\widehat{\tau}(T)}e^{-\theta t}dt+\mathbb{1}_{\{\widehat{\sigma}^\star(T)>\widehat{\tau}(T)\}}e^{-\theta\widehat{\tau}(T)}\left(k+r-\widehat{F}_{\widehat{\tau}(T)}\right)\\
&+\mathbb{1}_{\{\widehat{\sigma}^\star(T)\leq\widehat{\tau}(T)\}}e^{-\theta\widehat{\sigma}^\star(T)}\Big\{\mathbb{1}_{\{\widehat{P}_{\widehat{\sigma}^\star(T)}\leq\min\{h+\widehat{U}_{\widehat{\sigma}^\star(T)},r\}\}}\left(r-\widehat{P}_{\widehat{\sigma}^\star(T)}\right)\\
&\qquad+\mathbb{1}_{\{r<\widehat{P}_{\widehat{\sigma}^\star(T)},r\leq h+\widehat{U}_{\widehat{\sigma}^\star(T)}\}}W_{\widehat{\sigma}^\star(T)}+\mathbb{1}_{\{h+\widehat{U}_{\widehat{\sigma}^\star(T)}<\min\{r,\widehat{P}_{\widehat{\sigma}^\star(T)}\}\}}V_{\widehat{\sigma}^\star(T)}\Big\}\,\Bigg|\,\widehat{\mathcal{P}},\widehat{\sigma}^\star\Bigg].\\
\geq&\mathbb{E}\Bigg[k\int_0^T e^{-\theta t}dN_t+c\int_T^{\widehat{\sigma}^\star(T)\wedge\widehat{\tau}(T)}e^{-\theta t}dt+\mathbb{1}_{\{\widehat{\sigma}^\star(T)>\widehat{\tau}(T)\}}e^{-\theta\widehat{\tau}(T)}\left(k+r-\widehat{F}_{\widehat{\tau}(T)}\right)\\
&+\mathbb{1}_{\{\widehat{\sigma}^\star(T)\leq\widehat{\tau}(T)\}}e^{-\theta\widehat{\sigma}^\star(T)}\Big\{\mathbb{1}_{\{\widehat{P}_{\widehat{\sigma}^\star(T)}\leq\min\{h+\widehat{U}_{\widehat{\sigma}^\star(T)},r\}\}}\left(r-\widehat{P}_{\widehat{\sigma}^\star(T)}\right)\\
&\qquad+\mathbb{1}_{\{r<\widehat{P}_{\widehat{\sigma}^\star(T)},r\leq h+\widehat{U}_{\widehat{\sigma}^\star(T)}\}}(r-r)+\mathbb{1}_{\{h+\widehat{U}_{\widehat{\sigma}^\star(T)}<\min\{r,\widehat{P}_{\widehat{\sigma}^\star(T)}\}\}}\left(r-\widehat{U}_{\widehat{\sigma}^\star(T)}\right)\Big\}\,\Bigg|\,\widehat{\mathcal{P}},\widehat{\sigma}^\star\Bigg]\\
\geq&\mathbb{E}\Bigg[k\int_0^T e^{-\theta t}dN_t+r\left(e^{-\theta T}-e^{-\theta\widehat{\sigma}^\star(T)\wedge\widehat{\tau}(T)}\right)+\mathbb{1}_{\{\widehat{\sigma}^\star(T)>\widehat{\tau}(T)\}}e^{-\theta\widehat{\tau}(T)}\left(r-\widehat{F}_{\widehat{\tau}(T)}\right)\\
&+\mathbb{1}_{\{\widehat{\sigma}^\star(T)\leq\widehat{\tau}(T)\}}e^{-\theta\widehat{\sigma}^\star(T)}\Big\{\mathbb{1}_{\{\widehat{P}_{\widehat{\sigma}^\star(T)}\leq\min\{h+\widehat{U}_{\widehat{\sigma}^\star(T)},r\}\}}\left(r-\widehat{P}_{\widehat{\sigma}^\star(T)}\right)\\
&\qquad+\mathbb{1}_{\{r<\widehat{P}_{\widehat{\sigma}^\star(T)},r\leq h+\widehat{U}_{\widehat{\sigma}^\star(T)}\}}(r-r)+\mathbb{1}_{\{h+\widehat{U}_{\widehat{\sigma}^\star(T)}<\min\{r,\widehat{P}_{\widehat{\sigma}^\star(T)}\}\}}\left(r-\widehat{U}_{\widehat{\sigma}^\star(T)}\right)\Big\}\,\Bigg|\,\widehat{\mathcal{P}},\widehat{\sigma}^\star\Bigg]\\
=&\mathbb{E}\Bigg[k\int_0^T e^{-\theta t}dN_t+e^{-\theta T}\Big\{r-\mathbb{1}_{\{\widehat{\sigma}^\star(T)>\widehat{\tau}(T)\}}e^{-\theta(\widehat{\tau}(T)-T)}\widehat{F}_{\widehat{\tau}(T)}\\
&-\mathbb{1}_{\{\widehat{\sigma}^\star(T)\leq\widehat{\tau}(T)\}}e^{-\theta(\widehat{\sigma}^\star(T)-T)}\Big\{\mathbb{1}_{\{\widehat{P}_{\widehat{\sigma}^\star(T)}\leq\min\{h+\widehat{U}_{\widehat{\sigma}^\star(T)},r\}\}}\widehat{P}_{\widehat{\sigma}^\star(T)}+\mathbb{1}_{\{r<\widehat{P}_{\widehat{\sigma}^\star(T)},r\leq h+\widehat{U}_{\widehat{\sigma}^\star(T)}\}}r\\
&\qquad+\mathbb{1}_{\{h+\widehat{U}_{\widehat{\sigma}^\star(T)}<\min\{r,\widehat{P}_{\widehat{\sigma}^\star(T)}\}\}}\widehat{U}_{\widehat{\sigma}^\star(T)}\Big\}\Big\}\,\Bigg|\,\widehat{\mathcal{P}},\widehat{\sigma}^\star\Bigg]\\
\geq&\mathbb{E}\Bigg[k\int_0^T e^{-\theta t}dN_t+e^{-\theta T}\Big\{r-\Big[\mathbb{1}_{\{\widehat{\sigma}^\star(T)>\widehat{\tau}(T)\}}e^{-\theta(\widehat{\tau}(T)-T)}\widehat{F}_{\widehat{\tau}(T)}
\end{aligned}$$

$$+\mathbb{1}_{\{\widehat{\sigma}^\star(T)\le\widehat{\tau}(T)\}}e^{-\theta(\widehat{\sigma}^\star(T)-T)}\min\left\{\widehat{P}_{\widehat{\sigma}^\star(T)},h+\widehat{U}_{\widehat{\sigma}^\star(T)},r\right\}\Big]\Big\}\,\Big|\,\widehat{\mathcal{P}},\widehat{\sigma}^\star\Big]$$

$$=\mathbb{E}\left[k\int_0^T e^{-\theta t}dN_t+e^{-\theta T}\left(r-P_T\right)\,\middle|\,\mathcal{P}\right]=C(\mathcal{P},T), \tag{A.14}$$

where the first inequality follows from (A.12) and the fact that $W_{\widehat{\sigma}^\star(T)}\ge r-r=0$, the second inequality follows from the fact that $k\ge 0$ and $c/\theta\ge r$, the second last equality follows from the construction of $\mathcal{P}$ (particularly P_t in (A.10)), and the last equality follows from (4) and by recognizing that the agent always prefers to disclose the issue under $\mathcal{P}$ once it occurs at T by Property 1 and 2 above.

Property 4: It is optimal for the principal to set $F_t:=F$ for all $t\ge 0$, which immediately yields the recursive representation of (A.4) and (A.11) in (7), (8), (9) and (10) by following a similar derivation as in Lemma 1 of Wang et al. (2016). Indeed, we note that the variable F_t is absent from the principal's expected discounted cost (A.14). Therefore, it is optimal for the principal to relax the constraints (A.11) and (6) to the extent that is allowed. The limited liability constraint $F_t\le F$ hence suggests the optimality of setting $F_t:=F$.

Property 5: Policy $\mathcal{P}$ and policy $\widehat{\mathcal{P}}$ are payoff-equivalent to the agent, i.e., $C_a(\mathcal{P},T)=C_a(\widehat{\mathcal{P}},\widehat{\sigma}^\star)$. This is because, by construction in (A.10), P_t is the agent's minimum expected discounted cost from t onwards under policy $\widehat{\mathcal{P}}$ by following $\widehat{\sigma}^\star$ as the response. On the other hand, by Properties 1 and 2, the agent will always promptly disclose at time T under policy $\mathcal{P}$ and hence incur the same expected cost P_t, leading to the conclusion. □

LEMMA A.1. *The optimal policy must satisfy* $0\le\beta P_t\le U_t\le P_t\le\bar{h}$.

Proof of Lemma A.1. To show $P_t\ge 0$ in the optimum, we first note that $P_t:=0$ and $U_t:=0$ for all $t\ge 0$ always satisfy (6)–(10) with $q_t^m=q_t^n=0$, which results in no auditing cost. Therefore, any policy with $P_t<0$ is dominated by the policy with $P_t=U_t=0$.

To see $\beta P_t\le U_t$, we first note that by definition (A.4), $U_t\in[0,F]$. Denote $D_t=U_t-\beta P_t$ and then, by (7)-(10), we have

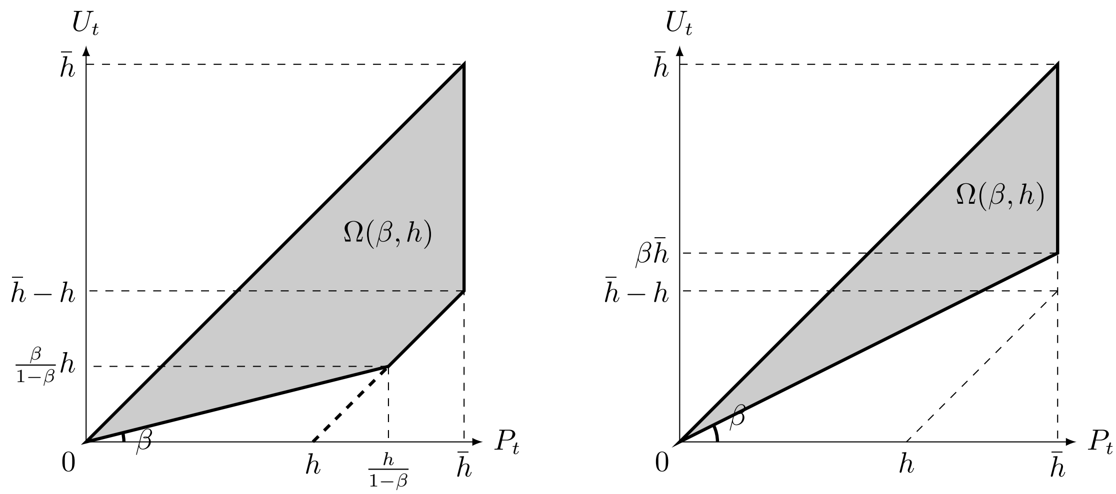
$$\begin{aligned}&(1-q_t^m)(D_{t+}-D_t)\le -q_t^m\left[(1-\beta)U_{t+}^I-D_t\right],\quad \text{for } q_t^m>0,\quad\text{and}\\ &D_{t+}\le D_t,\quad\text{or}\quad \frac{dD_t}{dt}\le(\theta+q_t^n)D_t-q_t^n(1-\beta)U_{t+}^I,\quad \text{for } q_t^m=0.\end{aligned}$$

Thus, if $D_{t_0}<0$ for some t_0, then D_t will be decreasing in $t\ge t_0$ and $D_t\to-\infty$ as $t\to\infty$, which must imply that $P_t\to\infty$, leading to a contradiction.

Finally, to see that $U_t\le P_t$, it suffice to argue that a policy with $P_t:=U_t$ for all $t\ge 0$ is incentive feasible, because it dominates any policy with $P_t<U_t$ (as the principal would like to elicit a payment P_t as high as possible). Indeed, it is obvious that $P_t:=U_t$ satisfies (6); and (7)-(8) imply

$$\begin{aligned}U_t&=(1-q_t^m)U_{t+}+q_t^m\left(\beta F+(1-\beta)U_{t+}^I\right)\le(1-q_t^m)U_{t+}+q_t^mF,\quad\text{for}\ \ q_t^m>0,\quad\text{and}\\ \frac{dU_t}{dt}&=\theta U_t-q_t^n\left[\beta F+(1-\beta)U_{t+}^I-U_t\right]\ge\theta U_t-q_t^n\left[F-U_t\right],\quad\text{for}\ \ q_t^m=0.\end{aligned}$$

Thus, $P_t:=U_t$ also satisfies (9)-(10). Finally, $P_t\le\min\{r,F\}$ follows from the limited liability constraint and (6). □



(a) $h < (1-\beta)\bar{h}$.

(b) $h \ge (1-\beta)\bar{h}$.

Figure A.1 Feasible range of (P_t, U_t) (shaded area).

Remark A.1. Intuitively, because evasion reduces the audit effectiveness, the threat that the principal will be able to impose on the evading agent, namely U_t, cannot exceed the payment she is able to charge from the agent without any evasion, namely P_t. While an audit after an evasion can only detect the issue with probability β per each audit, repeated audits allow the principle to impose a threat U_t higher than βP_t.

Together with (6), Lemma A.1 allows us to narrow down the feasible region of (P_t, U_t) to $\Omega(h,\beta) := \left\{(p,u) : 0 \le \beta p \le u \le p \le \bar{h},\ p \le h + u\right\}$, which is illustrated in Figure A.1. As will become evident later, the boundary of $\Omega(h,\beta)$ critically determines the binding constraints in the optimal policy, and hence play an important role in shaping the optimal policy. In particular, as can be seen from Figure 1(b), the obedience constraint (6) will never be active when either evasion is too costly or less effective so that $h \ge (1-\beta)\bar{h}$, suggesting the irrelevance of the moral hazard issue due to the agent's evasion. Indeed, Theorem 3 characterizes the *exact* condition $h \ge \widehat{h}(\beta)$, under which the principal's optimal policy does not bind the obedience constraint (6). For $h < \widehat{h}(\beta)$, however, (P_t, U_t) can move in a plethora of trajectories in the feasible region of Figure 1(a), making the characterization of the optimal policy extremely challenging. □

Appendix B: Proofs in Sections 5 and 6

Lemma B.1 **(Verification of Optimality)**. *For a constant $B \le F$, the policy $\mathcal{P} := (P_t, Q_t)_{t\in[0,\infty)}$ solves*

$$\frac{\lambda}{\theta+\lambda} r + \min_{\mathcal{P}=(P_t,Q_t)_{t\in[0,\infty)}} \mathbb{E}\left[k\int_0^T e^{-\theta t} dN_t - e^{-\theta T} P_T \,\middle|\, \mathcal{P}\right], \quad \text{subject to } P_t \le B, \text{ (9) and (10)}, \tag{B.1}$$

if the principal's cost-to-go function under $\mathcal{P}$,

$$C(\mathcal{I}_t, t) := \frac{\lambda}{\theta+\lambda} r + \mathbb{E}\left[k\int_t^T e^{-\theta(\zeta - t)} dN_\zeta - e^{-\theta(T-t)} P_T \,\middle|\, \mathcal{I}_t, t \le T\right], \tag{B.2}$$

satisfies the following properties:

Property 1: $C(\mathcal{I}_t, t)$ depends on $(\mathcal{I}_t, t)$ only through P_t and can hence be denoted as $C(\mathcal{I}_t, t) = C(P_t)$. That is, $C(\mathcal{I}_t, t) = C\left(\widehat{\mathcal{I}}_{\hat{t}}, \hat{t}\right)$ for any $(\mathcal{I}_t, t)$ and $\left(\widehat{\mathcal{I}}_{\hat{t}}, \hat{t}\right)$ such that $P_t(\mathcal{I}_t) = P_{\hat{t}}(\widehat{\mathcal{I}}_{\hat{t}})$.

Property 2: $C(p)$ is bounded, non-decreasing, and continuously differentiable in $p \in [0,B]$.

Property 3: $C(p)$ satisfies

$$\lambda(r-p)-(\theta+\lambda)C(p)+\mathcal{N}C(p)\ge 0 \quad \textit{and} \quad \mathcal{M}C(p)-C(p)\ge 0, \quad \textit{for } p\in[0,B], \tag{B.3}$$

where the functional operators $\mathcal{N}$ and $\mathcal{M}$ are defined as

$$\mathcal{N}C(p) := \min_{\substack{p_+^I\le B\\ q^n\ge 0, z\ge 0}} q^n\left[k+C\left(p_+^I\right)-C(p)\right]+\{\theta p-q^n[F-p]+z\}\frac{dC(p)}{dp}, \quad \textit{and} \tag{B.4}$$

$$\begin{aligned}\mathcal{M}C(p) := &\min_{\substack{p_+^I\le B\\ q^m\in[0,1], z\ge 0}} \quad q^m\left(k+C\left(p_+^I\right)\right)+(1-q^m)C(p_+)\\ &\text{subject to} \quad p=(1-q^m)p_+ + q^m F - z.\end{aligned} \tag{B.5}$$

Proof of Lemma B.5. By item (3) of Theorem 1, P_t is a controlled piecewise deterministic process and hence Markovian (Davis 1993) with $\left(P_{t+}^I, Q_t, z_t\right)$ as the control variables, which uniquely determine the evolution of P_t according to (9) and (10). Therefore, for any $\mathcal{I}_{t_0}$ and $\widehat{\mathcal{I}}_{t_0}$ that yields the same $P_{t_0}=p$, $(P_t)_{t\ge t_0}$ will follow the same trajectory under the same control $\left(P_{t+}^I, Q_t, z_t\right)_{t\ge t_0}$. We further note that the principal's *optimal* cost-to-go function (in the current value) starting from any $(\mathcal{I}_{t_0}, t_0)$ (assuming $t_0\le T$) can be rewritten as

$$\begin{aligned}\widehat{C}^\star(\mathcal{I}_{t_0},t_0) := \frac{\lambda}{\lambda+\theta}r + &\min_{\left(P_{t+}^I,Q_t,z_t\right)_{t\ge t_0}} e^{(\lambda+\theta)t_0}\mathbb{E}\left[\int_{t_0}^\infty e^{-(\lambda+\theta)t}\left(kq_t^n-\lambda P_t\right)dt + \sum_{t\ge t_0, q_t^m>0} e^{-(\lambda+\theta)t}kq_t^m \,\middle|\, \mathcal{I}_{t_0}\right]\\ &\text{subject to } P_t\le B, \text{ (9) and (10)},\end{aligned}$$

where we use the property of T being exponential distribution and the definition of audits. As such, the objective function above depends on $\mathcal{I}_t$ only through P_t and Q_t.

Therefore, for any $\mathcal{I}_{t_0}$ and $\widehat{\mathcal{I}}_{t_0}$ that yields the same $P_{t_0}=p$, $\widehat{C}^\star(\mathcal{I}_{t_0},t_0)=\widehat{C}^\star(\widehat{\mathcal{I}}_{t_0},t_0)=:\widetilde{C}^\star(p,t_0)$ is given by

$$\begin{aligned}\widetilde{C}^\star(p,t_0) := \frac{\lambda}{\lambda+\theta}r + &\min_{\left(P_{t+}^I,Q_t,z_t\right)_{t\ge t_0}} e^{(\lambda+\theta)t_0}\mathbb{E}\left[\int_{t_0}^\infty e^{-(\lambda+\theta)t}\left(kq_t^n-\lambda P_t\right)dt + \sum_{t\ge t_0, q_t^m>0} e^{-(\lambda+\theta)t}kq_t^m \,\middle|\, P_{t_0}=p\right]\\ &\text{subject to } P_t\le B, \text{ (9) and (10)}.\end{aligned} \tag{B.6}$$

That is, (P_t,t) can be used as the (payoff-relevant) state variables for the principal's problem.

By a time shifting, we can further rewrite (B.6) as follows

$$\begin{aligned}\widetilde{C}^\star(p,t_0) = \frac{\lambda}{\lambda+\theta}r + &\min_{\left(P_{(t+t_0)+}^I,Q_{t+t_0},z_{t+t_0}\right)_{t\ge 0}} \mathbb{E}\left[\int_0^\infty e^{-(\lambda+\theta)t}\left(kq_{t+t_0}^n-\lambda P_{t+t_0}\right)dt + \sum_{t\ge 0, q_{t+t_0}^m>0} e^{-(\lambda+\theta)t}kq_{t+t_0}^m \,\middle|\, P_{t_0}=p\right]\\ &\text{subject to } P_{t+t_0}\le B; \quad P_{t+t_0}=(1-q_{t+t_0}^m)P_{(t+t_0)+}+q_{t+t_0}^m F - z_{t+t_0}, \text{ if } q_{t+t_0}^m>0; \text{and}\\ &P_{t+t_0}=P_{(t+t_0)+}-z_{t+t_0}, \text{ or } \frac{dP_{t+t_0}}{dt}=\theta P_{t+t_0}-q_{t+t_0}^n\left[F-P_{t+t_0}\right]+z_{t+t_0}, \text{ if } q_{t+t_0}^m=0.\end{aligned}$$

By the virtue of the Markovian property of P_t, we immediately see that $\widetilde{C}^\star(p,t_0)=\widetilde{C}^\star(p,0)=C^\star(p)$, where

$$C^\star(p) := \frac{\lambda}{\lambda+\theta}r + \min_{\left(P_{t+}^I,Q_t,Z_t\right)_{t\ge 0}} \mathbb{E}\left[\int_0^\infty e^{-(\lambda+\theta)t}\left(kq_t^n-\lambda P_t\right)dt + \sum_{t\ge 0, q_t^m>0} e^{-(\lambda+\theta)t}kq_t^m \,\middle|\, P_0=p\right]$$

$$\text{subject to } P_t \le B, \quad P_t = (1-q_t^m)P_{t+} + q_t^m F - z_t, \text{ if } q_t^m > 0, \text{and}$$
$$P_t = P_{t+} - z_t, \quad \text{or} \quad \frac{dP_t}{dt} = \theta P_t - q_t^n\left[F - P_t\right] + z_t, \text{ if } q_t^m = 0. \tag{B.7}$$

That is, the optimal cost-to-go function is in fact time-homogeneous.

Now suppose that the principal's current-value cost-to-go function $C(\mathcal{I}_t, t)$ satisfies the three properties listed in the lemma; in particular, $C(\mathcal{I}_t, t) = C(P_t)$. Thus, the optimality of $C^\star(p)$ immediately implies that $C(p) \ge C^\star(p)$ for all $p \le B$. We now demonstrate that $C(p) \le C^\star(p)$ also holds for all $p \le B$, which immediately implies that the policy $\mathcal{P}$ is optimal for the principal.

Let $\left(\widetilde{P}^I_{t+}, \widetilde{Q}_t, \widetilde{z}_t\right)_{t\ge 0}$ be an arbitrary admissible policy, under which the transfer trajectory $\widetilde{P}_t$ is given by

$$\left.\begin{array}{ll} \widetilde{P}_t = (1-\tilde{q}_t^m)\widetilde{P}_{t+} + \tilde{q}_t^m F - \tilde{z}_t, & \text{if} \quad t \in \widetilde{\Gamma} \\ \frac{d\widetilde{P}_t}{dt} = \theta \widetilde{P}_t - \tilde{q}_t^n\left[F - \widetilde{P}_t\right] + \tilde{z}_t, & \text{if} \quad t \notin \widetilde{\Gamma} \end{array}\right\} \quad \forall t \notin \mathcal{I}_{[0,\infty)}, \quad \text{and } \widetilde{P}_t \text{ is reset to } \widetilde{P}^I_{t+}, \text{ for } t \in \mathcal{I}_{[0,\infty)}, \tag{B.8}$$

where $\widetilde{\Gamma} := \{t \ge 0 : \tilde{q}_t^m > 0 \text{ or } P_{t+} > P_t\} = \{\nu_1, \nu_2, \cdots\}$ with $\nu_0 = 0$. Therefore, by Davis (1993, Theorem 31.3 and 31.9), we have[16]

$$\begin{aligned}
&\mathbb{E}^p\left[e^{-(\lambda+\theta)\nu_j} C\left(\widetilde{P}_{\nu_j}\right) - e^{-(\lambda+\theta)\nu_{j-1}} C\left(\widetilde{P}_{\nu_{j-1}+}\right)\right] \\
=&\mathbb{E}^p\left[\int_{\nu_{j-1}}^{\nu_j} e^{-(\lambda+\theta)t}\left\{\theta\widetilde{P}_t - \tilde{q}_t^n\left[F - \widetilde{P}_t\right] + \tilde{z}_t^n\right\}\frac{dC\left(\widetilde{P}_t\right)}{dt}dt\right] \\
&+\mathbb{E}^p\left[\int_{\nu_{j-1}}^{\nu_j} e^{-(\lambda+\theta)t}\left\{\tilde{q}_t^n\left[C\left(\widetilde{P}^I_{t+}\right) - C\left(\widetilde{P}_t\right)\right] - (\lambda+\theta)C\left(\widetilde{P}_t\right)\right\}dt\right] \\
\ge&\mathbb{E}^p\left[\int_{\nu_{j-1}}^{\nu_j} e^{-(\lambda+\theta)t}\left(-k\tilde{q}_t^n + \mathcal{N}C(\widetilde{P}_t) - (\lambda+\theta)C\left(\widetilde{P}_t\right)\right)dt\right] \\
\ge& -\mathbb{E}^p\left[\int_{\nu_{j-1}}^{\nu_j} e^{-(\lambda+\theta)t}\left(\lambda(r - \widetilde{P}_t) + k\tilde{q}_t^n\right)dt\right],
\end{aligned} \tag{B.9}$$

where the first inequality follows from the definition of operator $\mathcal{N}$ in(B.4) and the second inequality follows from the first inequality of(B.3).

We then observe that, between two consecutive intervention time epochs,

$$\begin{aligned}
&\mathbb{E}^p\left[e^{-(\lambda+\theta)\nu_j}\left(C\left(\widetilde{P}_{\nu_j+}\right) - C\left(\widetilde{P}_{\nu_j}\right)\right)\right] \\
=&\mathbb{E}^p\left[\mathbb{E}^p\left[e^{-(\lambda+\theta)\nu_j}\left(C\left(\widetilde{P}_{\nu_j+}\right) - C\left(\widetilde{P}_{\nu_j}\right)\right)\,\middle|\,\widetilde{P}_{\nu_j}\right]\right] \\
=&\mathbb{E}^p\left[\mathbb{E}^p\left[e^{-(\lambda+\theta)\nu_j}\left(\tilde{q}_{\nu_j}^m C\left(\widetilde{P}^I_{\nu_j+}\right) + (1-\tilde{q}_{\nu_j}^m)C\left(\widetilde{P}_{\nu_j+}\right) - C\left(\widetilde{P}_{\nu_j}\right)\right)\,\middle|\,\widetilde{P}_{\nu_j}\right]\right] \\
\ge&\mathbb{E}^p\left[\mathbb{E}^p\left[e^{-(\lambda+\theta)\nu_j}\left(-k\tilde{q}_{\nu_j}^m + \mathcal{M}C(\widetilde{P}_{\nu_j}) - C(\widetilde{P}_{\nu_j})\right)\,\middle|\,\widetilde{P}_{\nu_j}\right]\right] \\
\ge& -\mathbb{E}^p\left[e^{-(\lambda+\theta)\nu_j}k\tilde{q}_{\nu_j}^m\right],
\end{aligned} \tag{B.10}$$

where the first equality follows from the tower rule of expectation operator, the second one from the fact that $\widetilde{P}_{\nu_j}$ is rest to $\widetilde{P}^I_{\nu_j+}$ with probability $\tilde{q}_{\nu_j}^m$ (in which case an audit takes place) and to $\widetilde{P}_{\nu_j+}$ with probability $1 - \tilde{q}_{\nu_j}^m$ (in which case no audit takes place) subject to the first constraint in(B.8), the first inequality from the definition of operator $\mathcal{M}$ in(B.5) and the second inequality from the second inequality in(B.3).

[16] We denote $\mathbb{E}^p[\cdot] := \mathbb{E}\left[\cdot\,\middle|\,\widetilde{P}_0 = p\right]$.

For any $p \le B$ and $n = 1, 2, \ldots$, we can then make the following decomposition:

$$\begin{aligned} & C(p) - \mathbb{E}^p\left[e^{-(\lambda+\theta)\nu_n} C\left(P_{\nu_n}\right)\right] \\ = & \sum_{j=1}^{n} \mathbb{E}^p\left[e^{-(\lambda+\theta)\nu_{j-1}} C\left(\widetilde{P}_{\nu_{j-1}+}\right) - e^{-(\lambda+\theta)\nu_j} C\left(\widetilde{P}_{\nu_j}\right)\right] + \sum_{j=0}^{n-1} \mathbb{E}^p\left[e^{-(\lambda+\theta)\nu_j}\left(C\left(\widetilde{P}_{\nu_j}\right) - C\left(\widetilde{P}_{\nu_j+}\right)\right)\right], \\ \le & \mathbb{E}^p\left[\int_0^{\nu_n} e^{-(\lambda+\theta)t}\left(\lambda(r - \widetilde{P}_t) + k\tilde{q}_t^n\right) dt + \sum_{j=0}^{n-1} e^{-(\lambda+\theta)\nu_j} k\tilde{q}_{\nu_j}^m\right], \end{aligned} \tag{B.11}$$

where the last inequality follows from(B.9) and (B.10).

Since $\lim_{n\to\infty} \nu_n = \infty$ with probability one by the admissibility of policy $\left(\widetilde{P}_{t+}^I, \widetilde{Q}_t, \widetilde{z}_t\right)_{t\ge 0}$ and $C(\cdot)$ is bounded, letting n go to infinity in (B.11) yields

$$C(p) \le \frac{\lambda}{\lambda+\theta} r + \mathbb{E}^p\left[\int_0^{\infty} e^{-(\lambda+\theta)t}\left(k\tilde{q}_t^n - \lambda\widetilde{P}_t\right) dt + \sum_{j=0}^{\infty} e^{-(\lambda+\theta)\nu_j} k\tilde{q}_{\nu_j}^m\right],$$

which, by (B.7), implies $C(p) \le C^\star(p)$ for $p \le B$ due to the arbitrariness of $\left(\widetilde{P}_{t+}^I, \widetilde{Q}_t, \widetilde{z}_t\right)_{t\ge 0}$. $\square$

LEMMA B.2. *There exists a unique solution $t^\star > 0$ to (12), which is increasing in $h \in [0, \bar{h}]$.*

Proof. Let $f(t) := \theta F + \lambda(F - h) + \lambda h e^{-(\lambda+\theta)t} - (\lambda+\theta)(k+F)e^{-\lambda t}$. Then, (12) is equivalent to $f(t^\star) = 0$. The existence and uniqueness of $t^\star$ thus follow from the straightforward verification that $f(0) = -(\lambda+\theta)k < 0$, $f(\infty) = \theta F + \lambda(F - h) > 0$, and $f(t)$ is increasing:

$$f'(t) = \lambda(\lambda+\theta)e^{-\lambda t}\left(k + F - he^{-\theta t}\right) \ge \lambda(\lambda+\theta)e^{-\lambda t}(k + F - h) > 0.$$

To see that $t^\star$ is increasing in h, it suffices to note that $f(t)$ is decreasing in h for any given t. $\square$

LEMMA B.3. *The policies prescribed in Theorems 2 and 3 both satisfy (9) and (10), and furthermore, satisfy $P_t^\star \le h$ and $P_t^\star \le \bar{h}$, respectively.*

Proof. Under the policy prescribed in Theorem 2, it is straightforward to verify:

- For any $t \in (\tau_{i-1}, \tau_{i-1} + t^\star]$ and i, (13) implies that $P_t^\star \le h$ and

$$\frac{dP_t^\star}{dt} = \theta h e^{\theta(t - \tau_i - t^\star)} = \theta P_t^\star,$$

which is essentially the binding constraint (10) by noticing $q_t^{n\star} := 0$ during $(\tau_{i-1}, \tau_{i-1} + t^\star]$.

- For any $t \in (\tau_{i-1} + t^\star, \tau_i]$ and i, (14) and $P_t^\star := h$ imply that

$$\frac{dP_t^\star}{dt} = 0 = \theta h - \frac{\theta h}{F - h}(F - h) = \theta P^\star - q^{n\star}(F - P_t^\star),$$

which is again essentially the binding constraint (10).

Under the policy prescribed in Theorem 3, it is also straightforward to verify:

- For any $t \in (\tau_{i-1}, \tau_{i-1} + t^\circ]$ and i, (**??**) implies that $P_t^\star \le \bar{h}$ and

$$\frac{dP_t^\star}{dt} = \theta \bar{h} e^{-\theta(\tau_i - t)} = \theta P_t^\star,$$

which is essentially the binding constraint (10) as $q_t^{n\star} := 0$. In particular, if $r \ge F$, we have $\tau_i = \tau_{i-1} + t^\circ$ and $P_{\tau_i}^\star = F$. Thus, (9) holds with equality at those impulsive audit epochs τ_i.

- If $r < F$, for any $t \in (\tau_{i-1} + t^\circ, \tau_i]$ and i, (16) and $P_t^\star := r$ imply that

$$\frac{dP_t^\star}{dt} = 0 = \theta r - \frac{\theta r}{F - r}(F - r) = \theta P^\star - q^{n\star}(F - P_t^\star),$$

which is again essentially the binding constraint (10).

LEMMA B.4. *The current-value cost-to-go functions under the policies prescribed in Theorems 2 and 3 depend on the past history $(\mathcal{I}_t, t)$ only through $P_t^\star$ and are given by a bounded, strictly convex increasing, and continuously differentiable function*

$$C(p) = \frac{\lambda}{\lambda+\theta} r + \kappa^\star p^{\frac{\lambda+\theta}{\theta}} - p, \text{ for } p \in [\underline{p}^\star, h], \text{ with } \kappa^\star = \frac{1}{h^{\frac{\lambda+\theta}{\theta}}} \frac{k + F - he^{-\theta t^\star}}{\frac{\lambda+\theta}{\theta}\frac{F-h}{h} + 1 - e^{-(\lambda+\theta)t^\star}} \text{ and } \underline{p}^\star = he^{-\theta t^\star}, \text{ and} \tag{B.12}$$

$$C(p) = \frac{\lambda}{\lambda+\theta} r + \kappa^\star p^{\frac{\lambda+\theta}{\theta}} - p, \text{ for } p \in [\underline{p}^\star, \bar{h}], \text{ with } \kappa^\star = \frac{1}{\bar{h}^{\frac{\lambda+\theta}{\theta}}} \frac{k + F - \bar{h}e^{-\theta t^\circ}}{\frac{\lambda+\theta}{\theta}\frac{F-\bar{h}}{\bar{h}} + 1 - e^{-(\lambda+\theta)t^\circ}} \text{ and } \underline{p}^\star = \bar{h}e^{-\theta t^\circ}, \text{ respectively.} \tag{B.13}$$

In particular, both functions satisfy

$$C(\underline{p}^\star) = \frac{\lambda}{\lambda+\theta}(r - \underline{p}^\star) \quad \text{and} \quad \frac{dC(\underline{p}^\star)}{dp} = \frac{\lambda+\theta}{\theta}\kappa^\star (\underline{p}^\star)^{\frac{\lambda}{\theta}} - 1 = 0. \tag{B.14}$$

Proof. It is clear that the policies in Theorems 2 and 3 are prescribed purely as a function of $P_t^\star$, and hence its current-value cost-to-go function depends on $(\mathcal{I}_t, t)$ only through $P_t^\star$. Let $B = h$ and $B = F$ under policies prescribed in Theorems 2 and 3, respectively. Then, (13) and $P_t^\star = Fe^{-\theta(\tau_i - t)}$ imply that $P_t^\star$ evolves deterministically according to $P_t^\star = pe^{\theta t}$ staring from any $P_0^\star = p$ before reaching the threshold B, which takes $\tau(p) := \frac{1}{\theta}\ln\frac{B}{p}$ amount of time, i.e., $P_{\tau(p)}^\star = B$. No audit ($q_t^{m\star} = q_t^{n\star} := 0$) is conducted between $[0, \tau(p))$. Therefore, we compute the cost-to-go function as follows:

$$\begin{aligned}
C(p) =& \mathbb{E}\left[\int_0^\infty e^{-(\lambda+\theta)t}\left(kq_t^{n\star} + \lambda(r - P_t^\star)\right)dt + \sum_{t\geq 0, q_t^{m\star}>0} e^{-(\lambda+\theta)t} k q_t^{m\star} \,\middle|\, P_0^\star = p\right] \\
=& \int_0^{\tau(p)} \lambda e^{-(\lambda+\theta)t}\left[r - pe^{\theta t}\right]dt + e^{-(\lambda+\theta)\tau(p)}C(B) \\
=& \frac{\lambda}{\lambda+\theta} r \left(1 - e^{-(\lambda+\theta)\tau(p)}\right) - p\left(1 - e^{-\lambda\tau(p)}\right) + e^{-(\lambda+\theta)\tau(p)}C(B) \\
=& \frac{\lambda r}{\lambda+\theta} + \left[C(B) - \frac{\lambda r}{\lambda+\theta} + B\right]\left(\frac{p}{B}\right)^{\frac{\lambda+\theta}{\theta}} - p.
\end{aligned} \tag{B.15}$$

Furthermore, direct calculation reveals

$$\frac{dC(p)}{dp} = \frac{1}{B}\frac{\lambda+\theta}{\theta}\left[C(B) - \frac{\lambda r}{\lambda+\theta} + B\right]\left(\frac{p}{B}\right)^{\lambda/\theta} - 1, \tag{B.16}$$

which is increasing in p. Therefore, $C(p)$ is strictly convex in p. It is also straightforward to verify

$$C(p) = -\frac{\theta}{\lambda+\theta} p \frac{dC(p)}{dp} + \frac{\lambda}{\lambda+\theta}(r - p) = \frac{\lambda}{\lambda+\theta}(r - p). \tag{B.17}$$

- Under the policy prescribed in Theorem 2, $B = h$ and an intensity audit with constant rate prescribed in (14) is used while maintaining $P_t^\star := h$, which suggests that

$$C(B) = C(h) = \int_0^\infty e^{-(\theta+\lambda+q^{n\star})t}\left\{\lambda(r - h) + q^{n\star}\left[k + C(\underline{p}^\star)\right]\right\}dt$$

$$=\frac{1}{\frac{\lambda+\theta}{q^{n\star}}+1}\left[\frac{\lambda}{q^{n\star}}(r-h)+k+C(he^{-\theta t^\star})\right]$$
$$=\frac{1}{\frac{\lambda+\theta}{\theta}\frac{F-h}{h}+1}\left\{\frac{\lambda}{\theta}\frac{F-h}{h}(r-h)+k+\frac{\lambda r}{\lambda+\theta}+\left[C(h)-\frac{\lambda r}{\lambda+\theta}+h\right]e^{-(\lambda+\theta)t^\star}-he^{-\theta t^\star}\right\},$$

where the first equality follows from the fact that $P_t^\star$ is reset to $\underline{p}^\star$ right after an audit and we use (B.15) to obtain the last equality. From the above equality, we can solve for

$$C(h)-\frac{\lambda r}{\lambda+\theta}+h=\frac{k+F-he^{-\theta t^\star}}{\frac{\lambda+\theta}{\theta}\frac{F-h}{h}+1-e^{-(\lambda+\theta)t^\star}},$$

which renders (B.15) to (B.12). Now substituting $\underline{p}^\star=he^{-\theta t^\star}$ into (B.16) yields

$$\frac{dC(\underline{p}^\star)}{dp}=\frac{dC(he^{-\theta t^\star})}{dp}=\frac{(\lambda+\theta)\left(k+F-he^{-\theta t^\star}\right)}{(\lambda+\theta)(F-h)+\theta h\left(1-e^{-(\lambda+\theta)t^\star}\right)}e^{-\lambda t^\star}-1 \tag{B.18}$$
$$=\frac{\lambda h\left(1-e^{-(\lambda+\theta)t^\star}\right)-(\lambda+\theta)\left[F-(k+F)e^{-\lambda t^\star}\right]}{(\lambda+\theta)(F-h)+\theta h\left(1-e^{-(\lambda+\theta)t^\star}\right)}=0,$$

where the last equality follows from the fact that $t^\star$ is the solution to (12). Thus, we obtain the second equation in (B.14), which, together with the convexity of $C(p)$, suggests that $C(p)$ is strictly increasing in $p\in[\underline{p}^\star,h]$. By (B.17), this also leads to the first equation of (B.14).

- Under the policy prescribed in Theorem 3, $B=\bar{h}$. If $r<F$, then $B=r$ and the proof follows the same argument as in the case of Theorem 2 above, where we replace h with r and $t^\star$ with t° given by (**??**). Otherwise ($r\geq F$), an impulsive audit is conducted at $P_t^\star=F$, suggesting that

$$C(F)=k+C(\underline{p}^\star)=k+C\left(Fe^{-\theta t^\circ}\right)=k+\frac{\lambda r}{\lambda+\theta}+\left(C(F)-\frac{\lambda r}{\lambda+\theta}+F\right)e^{-(\lambda+\theta)t^\circ}-Fe^{-\theta t^\circ},$$

where the first equality follows from the fact that $P_t^\star$ is reset to $\underline{p}^\star$ right after an audit and we use (B.15) to obtain the last equality. From the above equality, we can solve for

$$C(F)-\frac{\lambda r}{\lambda+\theta}+F=\frac{k+F-Fe^{-\theta t^\circ}}{1-e^{-(\lambda+\theta)t^\circ}},$$

which renders (B.15) to (B.13). Now substituting $\underline{p}^\star=Fe^{-\theta t^\circ}$ into (B.16) yields

$$\frac{dC(\underline{p}^\star)}{dp}=\frac{dC(Fe^{-\theta t^\circ})}{dp}=\frac{(\lambda+\theta)\left(k+F-Fe^{-\theta t^\circ}\right)}{\theta F\left(1-e^{-(\lambda+\theta)t^\circ}\right)}e^{-\lambda t^\circ}-1 \tag{B.19}$$
$$=\frac{\lambda F\left(1-e^{-(\lambda+\theta)t^\circ}\right)-(\lambda+\theta)\left[F-(k+F)e^{-\lambda t^\circ}\right]}{\theta F\left(1-e^{-(\lambda+\theta)t^\circ}\right)}=0,$$

where the last equality follows from the fact that t° is the solution to (**??**). Thus, we obtain the second equation in (B.14), which, together with the convexity of $C(p)$, suggests that $C(p)$ is strictly increasing in $p\in[\underline{p}^\circ,F]$. By (B.17), this also leads to the first equation of (B.14). ☐

Lemma B.5. *We extend the cost-to-go functions $C(p)$ in (B.12) and (B.13) by defining $C(p):=C\left(\underline{p}^\star\right)$ for $p\in\left[0,\underline{p}^\star\right]$. Then, the extended $C(p)$ satisfies (B.3).*

Proof of Lemma B.5. The extended cost-to-go functions $C(p)$ can be written as

$$C(p)=\begin{cases}\frac{\lambda}{\lambda+\theta}r+\kappa^\star p^{\lambda/\theta+1}-p, & \text{for} \quad p\in\left[\underline{p}^\star,B\right],\\ C\left(\underline{p}^\star\right), & \text{for} \quad p\in\left[0,\underline{p}^\star\right],\end{cases} \tag{B.20}$$

where $B=h$ and $B=\bar{h}$ under policies prescribed in Theorems 2 and 3, respectively, and $C\left(\underline{p}^\star\right)$ is given by (B.14).

We first show $\lambda p-(\theta+\lambda)C(p)+\mathcal{N}C(p)\geq 0$, where the functional operator $\mathcal{N}$ is defined in (B.4).

- For $p \le \underline{p}^\star$, $C(p)$ is a constant $C\left(\underline{p}^\star\right)$, which is the minimal value of $C(p)$ with $\frac{d}{dp}C(\underline{p}^\star) = 0$. Hence, by definition (B.4),

$$\begin{aligned}\lambda(r-p)-(\lambda+\theta)C(p)+\mathcal{N}C(p) =&\lambda(r-p)-(\lambda+\theta)C\left(\underline{p}^\star\right)+\min_{p_+^I\le B, q^n\ge 0} q^n\left[k+C(p_+^I)-C\left(\underline{p}^\star\right)\right]\\ =&\lambda(r-p)-(\lambda+\theta)C\left(\underline{p}^\star\right)+\min_{q^n\ge 0} kq^n=\lambda\left(\underline{p}^\star-p\right)\ge 0,\end{aligned}$$

where the last equality follows from $C\left(\underline{p}^\star\right)=\frac{\lambda}{\lambda+\theta}(r-\underline{p}^\star)$ according to (B.14).

- For $p\in\left(\underline{p}^\star, B\right]$, we have

$$\begin{aligned}&\lambda(r-p)-(\lambda+\theta)C(p)+\mathcal{N}C(p)\\ =&\lambda(r-p)-(\lambda+\theta)C(p)+\theta p\frac{dC(p)}{dp}+\min_{p_+^I\le B, q^n\ge 0} q^n\left[k+C\left(p_+^I\right)-C(p)+(p-F)\frac{dC(p)}{dp}\right]\\ =&\lambda(r-p)-(\lambda+\theta)C(p)+\theta p\frac{dC(p)}{dp}+\min_{q^n\ge 0} q^n\left[k+C(\underline{p}^\star)-C(p)+(p-F)\frac{dC(p)}{dp}\right],\end{aligned}\tag{B.21}$$

where the first equality follows from the definition (B.4) (with $z=0$ therein because $\frac{dC(p)}{dp}\ge 0$ by Lemma B.4), the second equality follows from the fact that $C(p)$ reaches its minimum value of $C\left(\underline{p}^\star\right)$ at $\underline{p}^\star$ by Lemma B.4.

For $p\in\left(\underline{p}^\star, B\right]$, since $C\left(\underline{p}^\star\right)=\frac{\lambda}{\lambda+\theta}(r-\underline{p}^\star)$ by (B.14), direct calculation reveals

$$\begin{aligned}&k+C\left(\underline{p}^\star\right)-C(p)+(p-F)\frac{dC(p)}{dp}\\ =&k+\frac{\lambda}{\lambda+\theta}\left(r-\underline{p}^\star\right)-\frac{\lambda}{\lambda+\theta}r-\kappa^\star p^{\lambda/\theta+1}+p+(p-F)\left[\frac{(\lambda+\theta)}{\theta}\kappa^\star p^{\lambda/\theta}-1\right]\\ =&k+F-\frac{\lambda}{\lambda+\theta}\underline{p}^\star-\frac{\kappa^\star}{\theta}\,p^{\lambda/\theta}\left[(\lambda+\theta)F-\lambda p\right]\\ \ge&k+F-\frac{\lambda}{\lambda+\theta}\underline{p}^\star-\frac{\kappa^\star}{\theta}B^{\lambda/\theta}\left[(\lambda+\theta)F-\lambda B\right],\end{aligned}\tag{B.22}$$

where the last inequality follows by letting $p=B$ because the function $p^{\lambda/\theta}\left[(\lambda+\theta)F-\lambda p\right]$ is decreasing in p:

$$\frac{d}{dp}\left\{p^{\lambda/\theta}\left[(\lambda+\theta)F-\lambda p\right]\right\}=\frac{\lambda(\lambda+\theta)}{\theta}p^{\lambda/\theta-1}(F-p)\ge 0\quad\text{for } p\le B\le F.$$

—In the case of (B.12), we have $B=h$ and hence

$$k+F-\frac{\lambda}{\lambda+\theta}\underline{p}^\star-\frac{\kappa^\star}{\theta}B^{\lambda/\theta}\left[(\lambda+\theta)F-\lambda B\right]\tag{B.23}$$

$$=k+F-\frac{\lambda}{\lambda+\theta}he^{-\theta t^\star}-\frac{k+F-he^{-\theta t^\star}}{\frac{\lambda+\theta}{\theta}\frac{F-h}{h}+1-e^{-(\lambda+\theta)t^\star}}\left[\frac{\lambda+\theta}{\theta}\frac{F}{h}-\frac{\lambda}{\theta}\right]$$

$$=\frac{1}{\lambda+\theta}\frac{e^{-\theta t^\star}}{\frac{\lambda+\theta}{\theta}\frac{F-h}{h}+1-e^{-(\lambda+\theta)t^\star}}\left[(\lambda+\theta)\left[F-(k+F)\,e^{-\lambda t^\star}\right]-\lambda h\left(1-e^{-(\lambda+\theta)t^\star}\right)\right]=0,\tag{B.24}$$

where the last equality follows from the fact that $t^\star$ is the solution to (12).

—In the case of (B.13), we have $B=\bar{h}$ and hence

$$k+F-\frac{\lambda}{\lambda+\theta}\underline{p}^\star-\frac{\kappa^\star}{\theta}B^{\lambda/\theta}\left[(\lambda+\theta)F-\lambda B\right]\tag{B.25}$$

$$=k+F-\frac{\lambda}{\lambda+\theta}\bar{h}e^{-\theta t^\circ}-\frac{k+F-\bar{h}e^{-\theta t^\circ}}{\frac{\lambda+\theta}{\theta}\frac{F-\bar{h}}{\bar{h}}-e^{-(\lambda+\theta)t^\circ}}\left[\frac{\lambda+\theta}{\theta}\frac{F}{\bar{h}}-\frac{\lambda}{\theta}\right]$$

$$=\frac{1}{\lambda+\theta}\frac{e^{-\theta t^\circ}}{\frac{\lambda+\theta}{\theta}\frac{F-\bar{h}}{\bar{h}}-e^{-(\lambda+\theta)t^\circ}}\left[(\lambda+\theta)\left[F-(k+F)\,e^{-\lambda t^\circ}\right]-\lambda\bar{h}\left(1-e^{-(\lambda+\theta)t^\circ}\right)\right]=0,\tag{B.26}$$

where the last equality follows from the fact that t° is the solution to (**??**).

Combining (B.21) with (B.24) and (B.26), we must have $\min_{q^n\geq 0} q^n \left[k + C(\underline{p}^\star) - C(p) + (p-F)\frac{dC(p)}{dp}\right] = 0$, reducing (B.21) to

$$\begin{aligned}\lambda(r-p)-(\lambda+\theta)C(p)+\mathcal{N}C(p) &= \lambda(r-p)-(\lambda+\theta)C(p)+\theta p\frac{dC(p)}{dp}\\ &= \lambda(r-p)-(\lambda+\theta)\left[\frac{\lambda}{\lambda+\theta}r+\kappa^\star p^{\lambda/\theta+1}-p\right]+\theta p\left[\frac{(\lambda+\theta)}{\theta}\kappa^\star p^{\lambda/\theta}-1\right]=0.\end{aligned}$$

That is, $\lambda(r-p)-(\theta+\lambda)C(p)+\mathcal{N}C(p)\geq 0$ holds with equality for $p\in[\underline{p}^\star, B]$.

We then show $\mathcal{M}C(p) - C(p) \geq 0$. The functional operator $\mathcal{M}$ is defined by(B.5) and can be rewritten as

$$\mathcal{M}C(p) := \min_{q^m\in[0,1]} \Upsilon(q^m \mid p) := q^m\left(k + C\left(\underline{p}^\star\right)\right) + (1-q^m)C\left(\max\left\{\frac{p-q^mF}{1-q^m}, \underline{p}^\star\right\}\right), \tag{B.27}$$

where we use the fact that $C(p)$ is increasing in $p\geq \underline{p}^\star$ and reaches its minimal value of $C\left(\underline{p}^\star\right)$ at $p=\underline{p}^\star$ by Lemma B.4.

- For $p\leq \underline{p}^\star$, the minimality of $\underline{p}^\star$ suggests that

$$\begin{aligned}C(p) := C\left(\underline{p}^\star\right) &\leq q^m\left(k+C\left(\underline{p}^\star\right)\right)+(1-q^m)C\left(\underline{p}^\star\right)\\ &\leq q^m\left(k+C\left(\underline{p}^\star\right)\right)+(1-q^m)C\left(\max\left\{\frac{p-q^mF}{1-q^m}, \underline{p}^\star\right\}\right),\ \forall q^m\in[0,1].\end{aligned}$$

Therefore, by definition(B.27), we must have $\mathcal{M}C(p)-C(p)\geq 0$.

- We now consider $p\in\left(\underline{p}^\star, B\right]$. If $\frac{p-q^mF}{1-q^m}\leq \underline{p}^\star$, or, equivalently, $q^m\geq (p-\underline{p}^\star)/(F-\underline{p}^\star)$, then $\Upsilon(q^m\mid p)$ reduces to

$$\Upsilon(q^m\mid p) = kq^m + C\left(\underline{p}^\star\right),$$

which is obviously increasing in q^m. Thus, we can restrict the search for the minimizer of $\Upsilon(q^m\mid p)$ within $q^m\leq (p-\underline{p}^\star)/(F-\underline{p}^\star)$, or equivalently, $\hat{p}:=\frac{p-q^mF}{1-q^m}\geq \underline{p}^\star$, in which case

$$\Upsilon(q^m\mid p) = q^m\left[k+C\left(\underline{p}^\star\right)\right]+(1-q^m)C\left(\frac{p-q^mF}{1-q^m}\right),$$

which is also increasing in q^m because its derivative with respect to q^m can be calculated as

$$\frac{d}{dq^m}\Upsilon(q^m\mid p) = k + C\left(\underline{p}^\star\right) - C(\hat{p}) + (\hat{p}-F)\frac{dC(\hat{p})}{dp}\geq 0,$$

where the last inequality follows from the same argument as in (B.21), (B.24) and (B.26). Altogether, we have shown that $\Upsilon(q^m\mid p)$ is monotonically increasing in q^m, and hence, by (B.27), $\mathcal{M}C(p) = \Upsilon(0\mid p) = C(p)$, implying that $\mathcal{M}C(p)-C(p)\geq 0$ holds with equality for $p\in\left(\underline{p}^\star, B\right]$. □

Proof of Theorem 2. By definition in A.3, if $\beta=0$, then we have $\mathbb{P}[Z_\zeta = 1\mid T\leq t, Z_t=1, H_t=1]=1$ for all $\zeta\geq t$. Therefore, it follows from the definition of U_t in (A.4) that $U_t:=0$ for all $t\geq 0$. As a result, the constraint (6) reduces to $P_t\leq h$ and hence the principal's problem (11) reduces to (B.1) with $B=h$. Then, Lemmas B.4 and B.5 immediately imply the optimality of the policy prescribed in Theorem 2. □

Proof of Corollary 1. The principal's cost $C^\star$ immediately follows from the first equation in (B.14) with $\underline{p}^\star = he^{-\theta t^\star}$. To compute the agent's cost, we denote the agent's cost-to-go function as $C_a(p) := \mathbb{E}\left[e^{-\theta(T-t)}P_T^\star \,\middle|\, P_t^\star = p, t<T\right]$. Then, letting $r=k=0$ in (B.12) yields $-C_a(p)$, namely

$$C_a(p) = p - \frac{1}{h^{\frac{\lambda+\theta}{\theta}}}\frac{F-he^{-\theta t^\star}}{\frac{\lambda+\theta}{\theta}\frac{F-h}{h}+1-e^{-(\lambda+\theta)t^\star}}p^{\frac{\lambda+\theta}{\theta}}. \tag{B.28}$$

Thus, we can easily obtain the agent's cost as follows:

$$\begin{aligned} C_a^\star = C_a(\underline{p}^\star) =& \underline{p}^\star - \frac{1}{h^{\frac{\lambda+\theta}{\theta}}} \frac{F - he^{-\theta t^\star}}{\frac{\lambda+\theta}{\theta}\frac{F-h}{h} + 1 - e^{-(\lambda+\theta)t^\star}} \left(\underline{p}^\star\right)^{\frac{\lambda+\theta}{\theta}} \\ =& \left[1 - \frac{\theta}{\lambda+\theta}\frac{F - he^{-\theta t^\star}}{k + F - he^{-\theta t^\star}}\right]\underline{p}^\star = \frac{\lambda}{\lambda+\theta} he^{-\theta t^\star} + \frac{\theta}{\lambda+\theta}\frac{khe^{-\theta t^\star}}{k+F-he^{-\theta t^\star}}, \end{aligned}$$

where the second equality follows from the second equation in (B.14) and the third equality follows from $\underline{p}^\star = he^{-\theta t^\star}$. According to (A.14), we then have

$$\frac{\lambda}{\lambda+\theta}(r - \underline{p}^\star) = C^\star = \frac{\lambda}{\theta+\lambda} r + \underbrace{\mathbb{E}\left[k\int_0^T e^{-\theta t} dN_t \,\middle|\, Q^\star\right]}_{A^\star} - \underbrace{\mathbb{E}\left[e^{-\theta T} P_T^\star\right]}_{C_a^\star}$$

which immediately implies that $A^\star = C_a^\star - \frac{\lambda}{\lambda+\theta} he^{-\theta t^\star} = \frac{\theta}{\lambda+\theta}\frac{khe^{-\theta t^\star}}{k+F-he^{-\theta t^\star}}$. □

Proof of Theorem 3. We first note that (B.1) with $B = \bar{h}$ is a relaxed problem of (11) with constraints (6)–(8) ignored, whose solution is given by the policy prescribed in Theorem 3, as implied by Lemmas B.4 and B.5.

We now complete the proof of Theorem 3 by showing that this policy automatically satisfies the ignored constraints (6)–(8) for $h \geq \widehat{h}(\beta)$ with $\widehat{h}(\beta)$ given by (15). To that end, we construct below a cyclical process $U_t^\star$ according to (7) and (8) under the policy prescribed in Theorem 3. Denote $\underline{u}^\star := U_0^\star = U_{\tau_i+}^\star$ for all i. For any $t \in (\tau_{i-1}, \tau_{i-1} + t^\circ)$, because $q_t^{n\star} = q_t^{m\star} = 0$, we have $U_t^\star = \underline{u}^\star e^{\theta(t-\tau_{i-1})}$ (because $\frac{dU_t^\star}{dt} = \theta U_t^\star$ according to (8)). This implies that $U_{\tau_i}^\star = \underline{u}^\star e^{\theta t^\circ}$.

- If $r \geq F$, then $\tau_i = \tau_{i-1} + t^\circ$ (i.e., $q_{\tau_{i-1}+t^\circ}^{m\star} = 1$) and (7) implies that

$$\underline{u}^\star e^{\theta t^\circ} = U_{\tau_i}^\star = \beta F + (1-\beta)\underline{u}^\star. \tag{B.29}$$

- If $r < F$, then for $t \in [\tau_{i-1} + t^\circ, \tau_i]$, we have $q_t^{m\star} = 0$, $q_t^{n\star} = \frac{\theta r}{F - r}$ and $P_t^\star = r$ by (16). Thus, (8) implies that

$$\begin{aligned} \frac{dU_t^\star}{dt} =& \theta U_t^\star - \frac{\theta r}{F-r}\left[\beta F + (1-\beta)\underline{u}^\star - U_t^\star\right] \\ =& \frac{\theta}{F-r}\left\{F U_t^\star - r\left[\beta F + (1-\beta)\underline{u}^\star\right]\right\}, \end{aligned}$$

which implies that $U_t^\star$ would be unbounded for $t \in [\tau_{i-1} + t^\circ, \tau_i]$ if $\frac{dU_t^\star}{dt} \gtrless 0$ (i.e., $U_t^\star \gtrless r/F\left[\beta F + (1-\beta)\underline{u}^\star\right]$). Thus, we must have $\frac{dU_t^\star}{dt} = 0$ for $t \in [\tau_{i-1} + t^\circ, \tau_i]$, implying

$$U_t^\star := U_{\tau_{i-1}+t^\circ}^\star = \underline{u}^\star e^{\theta t^\circ} = r/F\left[\beta F + (1-\beta)\underline{u}^\star\right], \text{ for } t \in [\tau_{i-1} + t^\circ, \tau_i]. \tag{B.30}$$

Combining (B.29) and (B.30) yields

$$\underline{u}^\star e^{\theta t^\circ} = \frac{\bar{h}}{F}\left[\beta F + (1-\beta)\underline{u}^\star\right] \quad \Leftrightarrow \quad \underline{u}^\star = \frac{\beta F \bar{h}}{F e^{\theta t^\circ} - (1-\beta)\bar{h}}. \tag{B.31}$$

Thus, for $t \in (\tau_{i-1}, \tau_{i-1} + t^\circ]$, (**??**), together with (B.31), implies that

$$\begin{aligned} P_t^\star - U_t^\star =& \left[\bar{h} - \underline{u}^\star e^{\theta t^\circ}\right] e^{-\theta(\tau_{i-1}+t^\circ - t)} \\ \leq& \left[\bar{h} - \underline{u}^\star e^{\theta t^\circ}\right] = \frac{(1-\beta)\left[F - \bar{h}e^{-\theta t^\circ}\right]}{F - \bar{h}(1-\beta)e^{-\theta t^\circ}}\bar{h} = \widehat{h}(\beta), \end{aligned}$$

which is bounded from above by h if $h \ge \widehat{h}(\beta)$ with $\widehat{h}(\beta)$ given by (15). Therefore, the ignored constraint (6) is indeed automatically satisfied. Finally, we note that

$$\frac{U_t^\star}{P_t^\star} = \frac{\underline{u}^\star e^{\theta(t-\tau_{i-1})}}{\bar{h}e^{-\theta(\tau_i - t)}} = \frac{\beta F}{F-(1-\beta)\bar{h}e^{-\theta t^\circ}}. \quad \square$$

Proof of Corollary 2. The derivation of the principal's cost $C^\star$, auditing cost $A^\star$ and the agent's cost $C_a^\star$ follows a similar argument as in the proof of Corollary 1 by replacing $\underline{p}^\star = he^{-\theta t^\star}$ with $\underline{p}^\star = \bar{h}e^{-\theta t^\circ}$. $\square$

Appendix C: Proofs in Section 7

We first outline the proof strategy in the following remark.

REMARK C.1. First, we establish Lemma C.1, which allows us to convert (18) to the following problem

$$\frac{\lambda}{\theta+\lambda} r + \min_{\gamma \ge 0, (P_t, Q_t)_{t\in[0,\infty)}} \mathbb{E}\left[k\int_0^T e^{-\theta t} dN_t - e^{-\theta T} P_T \,\middle|\, \mathcal{P}\right], \text{ subject to (C.3)–(C.8)}, \tag{C.1}$$

whose solution is denoted as $\{\gamma^\star, P_t^\star, Q_t^\star\}$. Then, $(P_t^\star, Q_t^\star)$ together with $U_t^\star := \gamma^\star P_t^\star$ satisfy (6)–(10), thus obtaining the complete solution to (18). To that end, we solve (C.1) using the following two steps.

1. For any given γ satisfying (C.3), we first solve the following one-dimensional stochastic control in P_t using the verification approach:

$$c_\gamma^\star := \frac{\lambda}{\theta+\lambda} r + \min_{(P_t, Q_t)_{t\in[0,\infty)}} \mathbb{E}\left[k\int_0^T e^{-\theta t} dN_t - e^{-\theta T} P_T \,\middle|\, \mathcal{P}\right], \text{ subject to (C.4)–(C.8)}. \tag{C.2}$$

As characterized by Lemma C.4, the optimal policy for (C.2) exhibits a cyclical structure, which allows us to explicitly compute the principal's cost-to-go function. All key policy parameters enter the cost-to-go function through the objective function in (C.18). Thus, the stochastic control problem is turned into a one-dimensional static optimization problem (C.18) in Lemma C.3, in which the decision variable x corresponds to an exponential transformation of the deterministic phase of each cycle t, i.e., $x = e^{-\theta t}$.

2. We then optimize $c_\gamma^\star$ over γ satisfying (C.3) to identify the optimal $\gamma^\star$. The two-dimensional static optimization problem (19) essentially combines the one-dimensional static optimization problem (C.18) from the previous step and the optimization of $c_\gamma^\star$ over γ in this step. The optimal policy solving (C.2) for $\gamma^\star$ is the solution to (C.1). $\square$

LEMMA C.1. *Under $U_t = \gamma P_t$ for all $t \ge 0$, constraints (6)–(10) imply*

$$\beta \le \gamma \le 1 - h/\bar{h}, \tag{C.3}$$

$$0 \le P_t \le \frac{h}{1-\gamma}, \tag{C.4}$$

$$\frac{1}{1-\beta}\left(\frac{h}{1-\gamma} - \frac{\beta}{\gamma}F\right)^+ \le P_{t+}^I \le \min\left\{\frac{h}{1-\gamma}, \frac{1-\beta/\gamma}{1-\beta}F, r\right\}. \tag{C.5}$$

$$P_t = (1-q_t^m)P_{t+} + q_t^m F - z_t, \quad \textit{for} \;\; q_t^m > 0, \tag{C.6}$$

$$\frac{dP_t}{dt} = \theta P_t - q_t^n (F - P_t) + z_t, \quad \textit{for} \;\; q_t^m = 0, \quad \textit{and} \tag{C.7}$$

$$z_t = \begin{cases} q_t^m \left[(1-\beta/\gamma)F - (1-\beta)P_{t+}^I\right], & \textit{for } q_t^m > 0, \\ q_t^n \left[(1-\beta/\gamma)F - (1-\beta)P_{t+}^I\right], & \textit{for } q_t^m = 0. \end{cases} \tag{C.8}$$

Conversely, for any γ, P_t and Q_t satisfying (C.3)–(C.8), the pair (P_t, U_t) with $U_t := \gamma P_t$, together with Q_t, for all $t \ge 0$ satisfies (6)–(10).

Proof of Lemma C.1. By Lemma A.1 (and its proof), we must have $\gamma \geq \beta$, and further by (6), we have $0 \leq P_t - U_t \leq h$, which immediately implies (C.4) given that $U_t = \gamma P_t$. Since $P_t \leq \bar{h}$, we can restrict to $h/(1-\gamma) \leq \bar{h}$, namely $\gamma \leq 1 - h/\bar{h}$. Altogether, we have (C.3).

Under $U_t = \gamma P_t$, P_t must be continuous between audits because so is U_t by (7)-(8). Further, (9) and (10), which can be written as (C.6) and (C.7), imply

$$U_t = (1-q_t^m)U_{t+} + q_t^m \gamma F - \gamma z_t, \quad \text{for } q_t^m > 0, \quad \text{and} \tag{C.9}$$

$$\frac{dU_t}{dt} = \theta U_t - q_t^n(\gamma F - U_t) + \gamma z_t, \quad \text{for } q_t^m = 0, \tag{C.10}$$

where $z_t \geq 0$ is the slack variable. By contrasting (C.9)-(C.10) with (7)-(8), we must have

$$\gamma z_t = q_t^m\left[(\gamma-\beta)F - (1-\beta)U_{t+}^I\right] = q_t^m\left[(\gamma-\beta)F - (1-\beta)\gamma P_{t+}^I\right] \geq 0, \text{ for } q_t^m > 0, \text{ and} \tag{C.11}$$

$$\gamma z_t = q_t^n\left[(\gamma-\beta)F - (1-\beta)U_{t+}^I\right] = q_t^n\left[(\gamma-\beta)F - (1-\beta)\gamma P_{t+}^I\right] \geq 0, \text{ for } q_t^m = 0, \tag{C.12}$$

which immediately yield (C.8) and, together with (C.4), yield the upper bound on P_{t+}^I in (C.5). To show the lower bound on P_{t+}^I in (C.5), we note (C.4) requires (7)-(8) at $P_t = \frac{h}{1-\gamma}$ to satisfy

$$\frac{h}{1-\gamma} = (1-q_t^m)P_{t+} + q_t^m F - z_t \leq (1-q_t^m)\frac{h}{1-\gamma} + q_t^m F - z_t \text{ for } q_t^m > 0, \quad \text{and} \tag{C.13}$$

$$\left.\frac{dP_t}{dt}\right|_{P_t = \frac{h}{1-\gamma}} = \theta\frac{h}{1-\gamma} - q_t^n\left(F - \frac{h}{1-\gamma}\right) + z_t \leq 0 \text{ for } q_t^m = 0 \text{ and } q_t^n > 0. \tag{C.14}$$

Substituting (C.11) and (C.12) respectively into (C.13) and (C.14) yields

$$q_t^m\left[\frac{\beta}{\gamma}F + (1-\beta)P_{t+}^I - \frac{h}{1-\gamma}\right] \geq 0, \quad \text{and} \quad q_t^n\left[\frac{\beta}{\gamma}F + (1-\beta)P_{t+}^I - \frac{h}{1-\gamma}\right] \geq 0,$$

which gives the lower bound on P_{t+}^I in (C.5).

Conversely, given P_t satisfying (C.4)–(C.7) and nonnegative slack variable z_t specified by (C.8), it is straightforward to verify that P_t satisfies (9)–(10) and $U_t = \gamma P_t$ satisfies (7) and (8). Furthermore, if γ satisfies (C.3), then (P_t, U_t) also satisfies (6). □

By contrasting problem (C.2) with problem (B.1), we modify Lemma B.5 to obtain the following:

LEMMA C.2 **(Verification of Optimality)**. *The policy $\mathcal{P} := (P_t, Q_t)_{t\in[0,\infty)}$ solves (C.2) if the principal's cost-to-go function under $\mathcal{P}$, namely $C(\mathcal{I}_t, t)$ defined in (B.2), satisfies the following properties:*

Property 1: $C(\mathcal{I}_t, t)$ depends on $(\mathcal{I}_t, t)$ only through P_t and can hence be denoted as $C(\mathcal{I}_t, t) = C(P_t)$. That is, $C(\mathcal{I}_t, t) = C\left(\widehat{\mathcal{I}}_{\hat{t}}, \hat{t}\right)$ for any $(\mathcal{I}_t, t)$ and $\left(\widehat{\mathcal{I}}_{\hat{t}}, \hat{t}\right)$ such that $P_t(\mathcal{I}_t) = P_{\hat{t}}(\widehat{\mathcal{I}}_{\hat{t}})$.

Property 2: $C(p)$ is bounded, non-decreasing, and continuously differentiable in $p \in \left[0, \frac{h}{1-\gamma}\right]$.

Property 3: $C(p)$ satisfies

$$\lambda(r-p) - (\theta+\lambda)C(p) + \mathcal{N}C(p) \geq 0 \quad \text{and} \quad \mathcal{M}C(p) - C(p) \geq 0, \quad \text{for } p \in \left[0, \tfrac{h}{1-\gamma}\right], \tag{C.15}$$

where the functional operators $\mathcal{N}$ and $\mathcal{M}$ are defined as

$$\begin{aligned}\mathcal{N}C(p) := \min_{p_+^I, q^n, z \geq 0} \quad & q^n\left[k + C\left(p_+^I\right) - C(p)\right] + \{\theta p - q^n[F-p] + z\}\tfrac{dC(p)}{dp} \\ \text{subject to} \quad & \tfrac{1}{1-\beta}\left(\tfrac{h}{1-\gamma} - \tfrac{\beta}{\gamma}F\right)^+ \leq p_+^I \leq \min\left\{\tfrac{h}{1-\gamma}\tfrac{1-\beta/\gamma}{1-\beta}F, r\right\} \\ & z = q^n\left[(1-\beta/\gamma)F - (1-\beta)p_+^I\right],\end{aligned} \tag{C.16}$$

and

$$\begin{aligned}\mathcal{M}C(p) := \min_{p_+, p_+^I, q^m, z \geq 0} \quad & q^m \left(k + C\left(p_+^I\right)\right) + (1-q^m) C(p_+) \\ \text{subject to} \quad & \tfrac{1}{1-\beta}\left(\tfrac{h}{1-\gamma} - \tfrac{\beta}{\gamma}F\right)^+ \leq p_+^I \leq \min\left\{\tfrac{h}{1-\gamma}\tfrac{1-\beta/\gamma}{1-\beta}F, r\right\} \\ & z = q^m\left[(1-\beta/\gamma)F - (1-\beta)p_+^I\right] \\ & p = (1-q^m)p_+ + q^m F - z.\end{aligned} \tag{C.17}$$

Proof of Lemma C.2. The proof follows the same argument as that of Lemma B.5 by replacing the bound B with $\frac{h}{1-\gamma}$ and the constraint $p_+^I \leq B$ with the constraint implied by (C.5) and (C.8). $\square$

LEMMA C.3. *For any given γ satisfying (C.3), the solution to the following static constrained optimization problem, denoted as x_γ, exists and is unique:*

$$K_\gamma := \min_{\frac{1}{1-\beta}\left[1-\beta\frac{F(1-\gamma)}{h\gamma}\right]^+ \leq x \leq 1 \wedge \frac{F(\gamma-\beta)(1-\gamma)}{h(1-\beta)\gamma} \wedge \frac{r(1-\gamma)}{h}} \left(\frac{1-\gamma}{h}\right)^{\frac{\theta+\lambda}{\theta}} \frac{k + \beta\left[\frac{F}{\gamma} - \frac{h}{1-\gamma}x\right]}{\frac{\theta+\lambda}{\theta}\frac{1-\gamma}{h}\left[\beta\frac{F}{\gamma} + (1-\beta)\frac{h}{1-\gamma}x\right] - \frac{\lambda}{\theta} - x^{\frac{\theta+\lambda}{\theta}}}, \tag{C.18}$$

with $x_\gamma \in (0,1)$ and $K_\gamma > \frac{\theta}{\theta+\lambda}\left(\frac{1-\gamma}{h}\right)^{\frac{\lambda}{\theta}}$. Furthermore, let $\gamma^\star = \underset{\beta \leq \gamma \leq 1-h/\bar{h}}{\arg\min} K_\gamma$. Then, $(\gamma^\star, x_{\gamma^\star})$ is the solution to (19) with $K^\star = K_{\gamma^\star}$ and, in particular, $\frac{1}{1-\beta}\left[1-\beta\frac{F(1-\gamma^\star)}{h\gamma^\star}\right]^+ < x_{\gamma^\star} < 1$.

Proof of Lemma C.3. Direct calculation reveals that the sign of the derivative of the objective function in (C.18) with respect to x is given by

$$\begin{aligned}&\text{sign}\left\{\frac{\partial}{\partial x}\left(\frac{1-\gamma}{h}\right)^{\frac{\theta+\lambda}{\theta}} \frac{k + \beta\left[\frac{F}{\gamma} - \frac{h}{1-\gamma}x\right]}{\frac{\theta+\lambda}{\theta}\frac{1-\gamma}{h}\left[\beta\frac{F}{\gamma} + (1-\beta)\frac{h}{1-\gamma}x\right] - \frac{\lambda}{\theta} - x^{\frac{\theta+\lambda}{\theta}}}\right\} \\ =& -\frac{\beta h}{1-\gamma}\left\{\frac{\theta+\lambda}{\theta}\frac{1-\gamma}{h}\left[\beta\frac{F}{\gamma} + (1-\beta)\frac{h}{1-\gamma}x\right] - \frac{\lambda}{\theta} - x^{\frac{\theta+\lambda}{\theta}}\right\} - \frac{\theta+\lambda}{\theta}\left\{k + \beta\left[\frac{F}{\gamma} - \frac{h}{1-\gamma}x\right]\right\}\left\{1-\beta - x^{\frac{\lambda}{\theta}}\right\} \\ =& \frac{\theta+\lambda}{\theta}\left\{\beta\left(\frac{\lambda}{\lambda+\theta}\frac{h}{1-\gamma} - \frac{F}{\gamma}\right) - (1-\beta)k + \left(k + \beta\frac{F}{\gamma}\right)x^{\frac{\lambda}{\theta}} - \frac{\lambda}{\lambda+\theta}\beta\frac{h}{1-\gamma}x^{\frac{\lambda+\theta}{\theta}}\right\} := \Xi_\gamma(x).\end{aligned}$$

We then note that function $\Xi_\gamma(x)$ satisfies the following properties:

- $\Xi_\gamma'(x) = \lambda/\theta x^{\lambda/\theta-1}\left(k + \beta F/\gamma - \beta h x/(1-\gamma)\right) > 0$ for $0 \leq x \leq 1 \wedge \frac{F(\gamma-\beta)(1-\gamma)}{h(1-\beta)\gamma} \wedge \frac{r(1-\gamma)}{h}$;
- $\Xi_\gamma(0) = \beta\left(\frac{\lambda}{\lambda+\theta}\frac{h}{1-\gamma} - F/\gamma\right) - (1-\beta)k < 0$ for γ satisfying (C.3);
- and $\Xi_\gamma(1) = \beta k > 0$.

Thus, the objective function in (C.18) is strictly quasi-convex in x with positive derivative at $x = 0$. Thus, solution $x_\gamma \in (0,1)$ must exist and is unique, which immediately implies that $x_{\gamma^\star} \in (0,1)$.

To show $K_\gamma > \frac{\theta}{\theta+\lambda}\left(\frac{1-\gamma}{h}\right)^{\frac{\lambda}{\theta}}$, we note that it suffices to show

$$\frac{k + \beta\left[\frac{F}{\gamma} - \frac{h}{1-\gamma}x\right]}{\left[\beta\frac{F}{\gamma} + (1-\beta)\frac{h}{1-\gamma}x\right] - \frac{\lambda}{\theta+\lambda}\frac{h}{1-\gamma} - \frac{\theta}{\theta+\lambda}\frac{h}{1-\gamma}x^{\frac{\theta+\lambda}{\theta}}} > 1 \quad \Leftrightarrow \quad \Gamma(x) := k + \frac{\lambda}{\theta+\lambda}\frac{h}{1-\gamma} + \frac{\theta}{\theta+\lambda}\frac{h}{1-\gamma}x^{\frac{\theta+\lambda}{\theta}} - \frac{h}{1-\gamma}x > 0,$$

which holds for all $x \in [0,1]$, because function $\Gamma(x)$ is non-increasing in $x \in [0,1]$ (as $\Gamma'(x) = \frac{h}{1-\gamma}\left(x^{\frac{\lambda}{\theta}} - 1\right) \leq 0$) with $\Gamma(1) = k > 0$.

It is straightforward to see that $(\gamma^\star, x_{\gamma^\star})$ is the solution to (19), which can be solved via a two-stage optimization by first optimizing over x given γ as in (C.18) and then optimizing over γ. To show $x_{\gamma^\star} >$

$\frac{1}{1-\beta}\left[1-\beta\frac{F(1-\gamma^\star)}{h\gamma^\star}\right]$, we take the variable transformation, $\bar{p} := \frac{h}{1-\gamma}$ and $\underline{p} := \frac{h}{1-\gamma}x$, under which the objective function in (C.18) can be rewritten as

$$\left(\frac{1-\gamma}{h}\right)^{\frac{\theta+\lambda}{\theta}} \frac{k+\beta\left[\frac{F}{\gamma}-\frac{h}{1-\gamma}x\right]}{\frac{\theta+\lambda}{\theta}\frac{1-\gamma}{h}\left[\beta\frac{F}{\gamma}+(1-\beta)\frac{h}{1-\gamma}x\right]-\frac{\lambda}{\theta}-x^{\frac{\theta+\lambda}{\theta}}} = \frac{k+\phi(\underline{p},\bar{p})-\underline{p}}{\frac{\theta+\lambda}{\theta}\phi(\underline{p},\bar{p})\bar{p}^{\frac{\lambda}{\theta}}-\frac{\lambda}{\theta}\bar{p}^{\frac{\lambda+\theta}{\theta}}-\underline{p}^{\frac{\lambda+\theta}{\theta}}} := \kappa(\underline{p},\bar{p}), \quad \text{(C.19)}$$

with $\phi(\underline{p},\bar{p}) := \beta\frac{F}{1-h/\bar{p}}+(1-\beta)\underline{p}$, and the constraint $x \ge \frac{1}{1-\beta}\left[1-\beta\frac{F(1-\gamma)}{h\gamma}\right]$ can be rewritten as

$$\bar{p} \le \phi(\underline{p},\bar{p}) \quad \text{(C.20)}$$

with "=" holding at the same time. We further note that (C.20) is equivalent to

$$\underline{p} \ge \frac{1}{1-\beta}\left[\bar{p}-\beta\frac{F}{1-h/\bar{p}}\right],$$

whose right-hand side is monotonically increasing in $\bar{p}$. Hence, there exists a monotonically increasing function $\psi(\cdot)$ such that (C.20) is equivalent to $\bar{p} \le \psi(\underline{p})$.

If $x_{\gamma^\star} = \frac{1}{1-\beta}\left[1-\beta\frac{F(1-\gamma^\star)}{h\gamma^\star}\right]$, then $\bar{p}^\star := \frac{h}{1-\gamma^\star}$ and $\underline{p}^\star := \frac{h}{1-\gamma^\star}x^\star$ must satisfy $\bar{p}^\star = \phi(\underline{p}^\star,\bar{p}^\star)$, or equivalently $\bar{p}^\star = \psi(\underline{p}^\star)$. Then, direct calculation reveals

$$\text{sign}\left\{\frac{\partial\kappa(\underline{p}^\star,\bar{p}^\star)}{\partial\bar{p}}\right\} = \text{sign}\left\{\underbrace{\frac{\partial\phi(\underline{p}^\star,\bar{p}^\star)}{\partial\bar{p}}}_{<0}\underbrace{\left[1-\frac{\theta+\lambda}{\theta}\underbrace{\kappa(\underline{p}^\star,\bar{p}^\star)}_{=K_{\gamma^\star}}(\bar{p}^\star)^{\frac{\lambda}{\theta}}\right]}_{<0}\right\} > 0,$$

where we have shown above that $K_{\gamma^\star} > \frac{\theta}{\theta+\lambda}\left(\frac{1-\gamma^\star}{h}\right)^{\frac{\lambda}{\theta}}$. That is, there exists $\bar{p} < \bar{p}^\star = \psi(\underline{p}^\star)$ such that $\kappa(\underline{p},\bar{p}^\star) < \kappa(\underline{p}^\star,\bar{p}^\star) = K_{\gamma^\star}$, contradicting to the optimality of $K_{\gamma^\star}$. □

Lemma C.4. *For any given γ satisfying (C.3), let x_γ be the solution to (C.18) given in Lemma C.3. Then, the optimal policy that solves (C.2) is given as follows: for an initial period of length $t^\circ_\gamma = -\frac{1}{\theta}\ln\frac{(1-\gamma)p_\gamma}{h}$ with $p_\gamma = \left(\frac{\theta+\lambda}{\theta}K_\gamma\right)^{-\theta/\lambda} < \frac{h}{1-\gamma}$, the principal applies no audits (i.e., $q^{m\star}_t = q^{n\star}_t := 0$ for $t \in [0, t^\circ_\gamma]$) and charges the agent a payment according to*

$$P^\star_t = p_\gamma e^{\theta t}, \quad \textit{for } t \in [0, t^\circ_\gamma]. \quad \text{(C.21)}$$

Starting from t°_γ, the principal applies intensive audits at constant rate while maintaining the constant payment level, respectively given by

$$q^{n\star}_t := \frac{\theta}{\beta\frac{F(1-\gamma)}{h\gamma}+(1-\beta)x_\gamma-1}, \quad \textit{and} \quad P^\star_t := \frac{h}{1-\gamma}, \quad \textit{for } t \in (t^\circ_\gamma, \tau_1]\bigcup_{i=1}^{\infty}(\tau_i+t_\gamma, \tau_{i+1}], \quad \text{(C.22)}$$

where $t_\gamma = -\frac{1}{\theta}\ln x_\gamma$ and τ_i is the i-th audit. (If $q^{n\star}_t$ in (C.22) becomes infinity, then the intensive audit is equivalent to an impulsive audit with probability $q^{m\star}_t = 1$.) The principal applies no audits (i.e., $q^{m\star}_t = q^{n\star}_t := 0$) for $t \in (\tau_i, \tau_i + t_\gamma]$ and charges the agent a payment according to

$$P^\star_t = \frac{h}{1-\gamma}e^{-\theta(\tau_i+t_\gamma-t)}, \quad \textit{for } t \in \bigcup_{i=1}^{\infty}(\tau_i, \tau_i+t_\gamma]. \quad \text{(C.23)}$$

Finally, the principal's optimal total expected discounted cost in (C.2) is given by $c^\star_\gamma := \frac{\lambda}{\lambda+\theta}(r-p_\gamma)$, out of which she spends $a^\star_\gamma = \frac{\theta}{\lambda+\theta}\frac{kp_\gamma}{k+\beta\left[\frac{F}{\gamma}-\frac{h}{1-\gamma}x_\gamma\right]}$ on audits. The corresponding agent's total expected discounted cost is given by $c^\star_{a\gamma} = \frac{\lambda}{\lambda+\theta}p_\gamma + a^\star_\gamma$.

Proof of Lemma C.4. It is clear that the policy in the lemma are prescribed purely as a function of $P_t^\star$, and hence its current-value cost-to-go function depends on $(\mathcal{I}_t, t)$ only through $P_t^\star$. Then, (C.21) and (C.23) imply that staring from any $P_0^\star = p \in \left[0, \frac{h}{1-\gamma}\right]$, $P_t^\star$ evolves deterministically according to $P_t^\star = pe^{\theta t}$ before reaching the threshold $\frac{h}{1-\gamma}$, which takes $\tau(p) := \frac{1}{\theta}\ln\frac{h}{p(1-\gamma)}$ amount of time, i.e., $P_{\tau(p)}^\star = \frac{h}{1-\gamma}$. No audit ($q_t^{m\star} = q_t^{n\star} := 0$) is conducted between $[0, \tau(p))$. Therefore, we compute the cost-to-go function as follows:

$$
\begin{aligned}
C(p) =& \mathbb{E}\left[\int_0^\infty e^{-(\lambda+\theta)t}\left(kq_t^{n\star} + \lambda(r - P_t^\star)\right)dt + \sum_{t\geq 0, q_t^{m\star}>0} e^{-(\lambda+\theta)t} kq_t^{m\star} \,\middle|\, P_0^\star = p\right] \\
=& \int_0^{\tau(p)} \lambda e^{-(\lambda+\theta)t}\left[r - pe^{\theta t}\right]dt + e^{-(\lambda+\theta)\tau(p)} C\left(\frac{h}{1-\gamma}\right) \\
=& \frac{\lambda}{\lambda+\theta} r\left(1 - e^{-(\lambda+\theta)\tau(p)}\right) - p\left(1 - e^{-\lambda\tau(p)}\right) + e^{-(\lambda+\theta)\tau(p)} C\left(\frac{h}{1-\gamma}\right) \\
=& \frac{\lambda r}{\lambda+\theta} + \left[C\left(\frac{h}{1-\gamma}\right) - \frac{\lambda r}{\lambda+\theta} + \frac{h}{1-\gamma}\right]\left[\frac{p(1-\gamma)}{h}\right]^{\frac{\lambda+\theta}{\theta}} - p.
\end{aligned} \tag{C.24}
$$

Under the policy prescribed in the lemma, once $P_t^\star$ reaches $\frac{h}{1-\gamma}$, an intensity audit with constant rate prescribed in (C.22) is used while maintaining $P_t^\star := \frac{h}{1-\gamma}$, which suggests that

$$
\begin{aligned}
C\left(\frac{h}{1-\gamma}\right) =& \int_0^\infty e^{-(\theta+\lambda+q_t^{n\star})t}\left\{\lambda\left(r - \frac{h}{1-\gamma}\right) + q_t^{n\star}\left[k + C(\underline{p}^\star)\right]\right\}dt \\
=& \frac{1}{\lambda+\theta+\frac{\theta}{\beta\frac{F(1-\gamma)}{h\gamma}+(1-\beta)x_\gamma-1}}\left\{\lambda\left(r - \frac{h}{1-\gamma}\right) + \frac{\theta}{\beta\frac{F(1-\gamma)}{h\gamma}+(1-\beta)x_\gamma - 1}\left[k + C\left(\frac{h}{1-\gamma}e^{-\theta t_\gamma}\right)\right]\right\} \\
=& \frac{1}{\frac{\lambda+\theta}{\theta}\left[\beta\frac{F(1-\gamma)}{h\gamma}+(1-\beta)x_\gamma-1\right]+1}\left\{\frac{\lambda}{\theta}\left[\beta\frac{F(1-\gamma)}{h\gamma}+(1-\beta)x_\gamma-1\right]\left(r - \frac{h}{1-\gamma}\right) + k + C\left(\frac{h}{1-\gamma}e^{-\theta t_\gamma}\right)\right\}.
\end{aligned} \tag{C.25}
$$

Note that if $q_t^{n\star} := \infty$, or equivalently $\beta\frac{F(1-\gamma)}{h\gamma} + (1-\beta)x_\gamma - 1 = 0$, then (C.25) also holds and can be verified to coincide with the cost-to-go function for an impulsive audit with probability $q_t^{m\star} = 1$.

Meanwhile, (C.24) implies

$$
C\left(\frac{h}{1-\gamma}e^{-\theta t_\gamma}\right) = \frac{\lambda r}{\lambda+\theta} + \left[C\left(\frac{h}{1-\gamma}\right) - \frac{\lambda r}{\lambda+\theta} + \frac{h}{1-\gamma}\right]x_\gamma^{\frac{\lambda+\theta}{\theta}} - \frac{h}{1-\gamma}x_\gamma, \tag{C.26}
$$

where we use $t_\gamma = -\frac{1}{\theta}\ln x_\gamma$. Combining (C.25) and (C.26) yields

$$
C\left(\frac{h}{1-\gamma}\right) - \frac{\lambda r}{\lambda+\theta} + \frac{h}{1-\gamma} = \frac{k + \beta\left[\frac{F}{\gamma} - \frac{h}{1-\gamma}x_\gamma\right]}{\frac{\theta+\lambda}{\theta}\frac{1-\gamma}{h}\left[\beta\frac{F}{\gamma} + (1-\beta)\frac{h}{1-\gamma}x_\gamma\right] - \frac{\lambda}{\theta} - x_\gamma^{\frac{\theta+\lambda}{\theta}}}. \tag{C.27}
$$

Substituting (C.27) into (C.24) yields

$$
C(p) = \frac{\lambda r}{\lambda+\theta} + K_\gamma p^{\frac{\lambda+\theta}{\theta}} - p, \quad \text{for } p \in \left[0, \frac{h}{1-\gamma}\right], \tag{C.28}
$$

where K_γ is defined in (C.18) and it is straightforward to verify that $p_\gamma = \left(\frac{\theta+\lambda}{\theta}K_\gamma\right)^{-\theta/\lambda} < \frac{h}{1-\gamma}$ because $K_\gamma > \frac{\theta}{\theta+\lambda}\left(\frac{1-\gamma}{h}\right)^{\frac{\lambda}{\theta}}$ by Lemma C.3, and that p_γ satisfies

$$
\frac{dC(p_\gamma)}{dp} = \frac{\lambda+\theta}{\theta}K_\gamma p_\gamma^{\frac{\lambda}{\theta}} - 1 = 0 \quad \Rightarrow \quad C(p_\gamma) = \frac{\lambda}{\lambda+\theta}(r - p_\gamma). \tag{C.29}
$$

It is straightforward to verify that $C(p)$ is bounded, non-decreasing, convex, and continuously differentiable function on $\left[0, \frac{h}{1-\gamma}\right]$. Therefore, by Lemma C.2, it remains to show that $C(p)$ satisfies (C.15) to establish the optimality of the policy specified in this lemma.

We first show $\lambda(r-p)-(\lambda+\theta)C(p)+\mathcal{N}C(p)\geq 0$ for $p\in\left[0,\frac{h}{1-\gamma}\right]$. By definition (C.16) (after eliminating slack variable z via its constraint therein), we have

$$\mathcal{N}C(p)=\theta p\frac{dC(p)}{dp}+\min_{\substack{\frac{1}{1-\beta}\left(\frac{h}{1-\gamma}-\frac{\beta}{\gamma}F\right)^+\leq p_+^I\leq\min\left\{\frac{h}{1-\gamma},\frac{\gamma-\beta}{1-\beta}F\right\}\\ q^n\geq 0}} q^n\left[k+C\left(p_+^I\right)-C(p)-\left(\beta/\gamma F+(1-\beta)p_+^I-p\right)\frac{dC(p)}{dp}\right], \tag{C.30}$$

where we note that the coefficient of q^n satisfies

$$\frac{\partial}{\partial p}\left\{k+C\left(p_+^I\right)-C(p)-\left(\beta/\gamma F+(1-\beta)p_+^I-p\right)\frac{dC(p)}{dp}\right\}=-\left(\beta/\gamma F+(1-\beta)p_+^I-p\right)\frac{d^2C(p)}{dp^2}\leq 0,$$

for all $p\leq\frac{h}{1-\gamma}$, because the lower bound on $p_+^I\geq\frac{1}{1-\beta}\left(\frac{h}{1-\gamma}-\frac{\beta}{\gamma}F\right)^+$ and the convexity of $C(p)$. Therefore, the coefficient of q^n must be nonnegative:

$$\begin{aligned}
&k+C\left(p_+^I\right)-C(p)-\left(\beta/\gamma F+(1-\beta)p_+^I-p\right)\frac{dC(p)}{dp}\\
\text{(with “=” at } p=\tfrac{h}{1-\gamma}) \quad &\geq k+C\left(p_+^I\right)-C\left(\frac{h}{1-\gamma}\right)-\left(\beta/\gamma F+(1-\beta)p_+^I-\frac{h}{1-\gamma}\right)\frac{d}{dp}C\left(\frac{h}{1-\gamma}\right)\\
\text{(by (C.28))} \quad &=k+\beta\left(F/\gamma-p_+^I\right)-K_\gamma\left[\frac{\lambda+\theta}{\theta}\left(\beta\frac{F}{\gamma}+(1-\beta)p_+^I\right)\left(\frac{h}{1-\gamma}\right)^{\frac{\lambda}{\theta}}-\frac{\lambda}{\theta}\left(\frac{h}{1-\gamma}\right)^{\frac{\lambda+\theta}{\theta}}-\left(p_+^I\right)^{\frac{\lambda+\theta}{\theta}}\right]\\
&=\left(\frac{1-\gamma}{h}\right)^{\frac{\theta+\lambda}{\theta}}\frac{k+\beta\left[\frac{F}{\gamma}-\frac{h}{1-\gamma}x\right]}{\frac{\theta+\lambda}{\theta}\frac{1-\gamma}{h}\left[\beta\frac{F}{\gamma}+(1-\beta)\frac{h}{1-\gamma}x\right]-\frac{\lambda}{\theta}-x^{\frac{\theta+\lambda}{\theta}}}-K_\gamma\geq 0,
\end{aligned} \tag{C.31}$$

where $x=\frac{(1-\gamma)p_+^I}{h}\in\left[\frac{1}{1-\beta}\left[1-\beta\frac{F(1-\gamma)}{h\gamma}\right]^+,1\wedge\frac{F(\gamma-\beta)(1-\gamma)}{h(1-\beta)\gamma}\right]$, and the nonnegativity in (C.31) follows from (C.18) with the “=” holding at $x=x_\gamma$ or equivalently $p_+^I=\frac{h}{1-\gamma}x_\gamma$. As such, (C.30) reduces to

$$\mathcal{N}C(p)=\theta p\frac{dC(p)}{dp}=(\lambda+\theta)K_\gamma p^{\frac{\lambda+\theta}{\theta}}-\theta p,$$

which immediately implies that

$$\lambda(r-p)-(\lambda+\theta)C(p)+\mathcal{N}C(p)=\lambda(r-p)-\lambda r-(\lambda+\theta)K_\gamma p^{\frac{\lambda+\theta}{\theta}}+(\lambda+\theta)p+(\lambda+\theta)K_\gamma p^{\frac{\lambda+\theta}{\theta}}-\theta p=0,$$

establishing the result.

We next show $\mathcal{M}C(p)-C(p)\geq 0$ for $p\in\left[0,\frac{h}{1-\gamma}\right]$. The functional operator $\mathcal{M}$ defined by (C.17) can be rewritten as

$$\mathcal{M}C(p):=\min_{\frac{1}{1-\beta}\left(\frac{h}{1-\gamma}-\frac{\beta}{\gamma}F\right)^+\leq p_+^I\leq\min\left\{\frac{h}{1-\gamma}\,\frac{1-\beta/\gamma}{1-\beta}F\right\}} k+C(p_+^I)-\left[\beta/\gamma F+(1-\beta)p_+^I-p\right]\frac{k+C(p_+^I)-C(p_+)}{\beta/\gamma F+(1-\beta)p_+^I-p_+}, \tag{C.32}$$

where we eliminate decision variable q^m and slack variable z via the two equality constraints in (C.17):

$$q^m=\frac{p-p_+}{\beta/\gamma F+(1-\beta)p_+^I-p_+}=1-\frac{\beta/\gamma F+(1-\beta)p_+^I-p}{\beta/\gamma F+(1-\beta)p_+^I-p_+}\in[0,1]. \tag{C.33}$$

By (C.31), we note that

$$\frac{\partial}{\partial p_+}\frac{k+C(p_+^I)-C(p_+)}{\beta/\gamma F+(1-\beta)p_+^I-p_+}=\frac{k+C\left(p_+^I\right)-C(p_+)-\left(\beta/\gamma F+(1-\beta)p_+^I-p_+\right)\frac{dC(p_+)}{dp}}{\left[\beta/\gamma F+(1-\beta)p_+^I-p_+\right]^2}\geq 0. \tag{C.34}$$

By (C.33), there are two cases for us to consider:

- For $p \geq p_+$ and $\beta/\gamma F + (1-\beta)p_+^I - p_+ \geq \beta/\gamma F + (1-\beta)p_+^I - p \geq 0$, we have, by (C.34),

$$\begin{aligned} &k + C(p_+^I) - \left[\beta/\gamma F + (1-\beta)p_+^I - p\right] \frac{k + C(p_+^I) - C(p_+)}{\beta/\gamma F + (1-\beta)p_+^I - p_+} \\ \geq & k + C(p_+^I) - \left[\beta/\gamma F + (1-\beta)p_+^I - p\right] \frac{k + C(p_+^I) - C(p)}{\beta/\gamma F + (1-\beta)p_+^I - p} = C(p), \end{aligned}$$

establishing $\mathcal{M}C(p) - C(p) \geq 0$.

- For $p \leq p_+$ and $\beta/\gamma F + (1-\beta)p_+^I - p_+ \leq \beta/\gamma F + (1-\beta)p_+^I - p \leq 0$, we again have, by (C.34),

$$\begin{aligned} &k + C(p_+^I) - \left[\beta/\gamma F + (1-\beta)p_+^I - p\right] \frac{k + C(p_+^I) - C(p_+)}{\beta/\gamma F + (1-\beta)p_+^I - p_+} \\ \geq & k + C(p_+^I) - \left[\beta/\gamma F + (1-\beta)p_+^I - p\right] \frac{k + C(p_+^I) - C(p)}{\beta/\gamma F + (1-\beta)p_+^I - p} = C(p), \end{aligned}$$

also establishing $\mathcal{M}C(p) - C(p) \geq 0$.

Finally, we derive different cost components. The principal's cost $c_\gamma^\star$ immediately follows from (C.29). To compute the agent's cost, we denote the agent's cost-to-go function as $C_a(p) := \mathbb{E}\left[e^{-\theta(T-t)} P_T^\star \,\middle|\, P_t^\star = p, t < T\right]$. Then, letting $r = k = 0$ in (C.28) yields $-C_a(p)$, namely

$$C_a(p) = p - \left(\frac{1-\gamma}{h}\right)^{\frac{\theta+\lambda}{\theta}} \frac{\beta\left[\frac{F}{\gamma} - \frac{h}{1-\gamma}x_\gamma\right]}{\frac{\theta+\lambda}{\theta}\frac{1-\gamma}{h}\left[\beta\frac{F}{\gamma} + (1-\beta)\frac{h}{1-\gamma}x_\gamma\right] - \frac{\lambda}{\theta} - x_\gamma^{\frac{\theta+\lambda}{\theta}}} p^{\frac{\lambda+\theta}{\theta}}. \tag{C.35}$$

Thus, we can easily obtain the agent's cost as follows:

$$\begin{aligned} c_{a\gamma}^\star = C_a(p_\gamma) = & p_\gamma - \left(\frac{1-\gamma}{h}\right)^{\frac{\theta+\lambda}{\theta}} \frac{\beta\left[\frac{F}{\gamma} - \frac{h}{1-\gamma}x_\gamma\right]}{\frac{\theta+\lambda}{\theta}\frac{1-\gamma}{h}\left[\beta\frac{F}{\gamma} + (1-\beta)\frac{h}{1-\gamma}x_\gamma\right] - \frac{\lambda}{\theta} - x_\gamma^{\frac{\theta+\lambda}{\theta}}} p_\gamma^{\frac{\lambda+\theta}{\theta}} \\ = & \left[1 - \frac{\theta}{\lambda+\theta}\frac{\beta\left[\frac{F}{\gamma} - \frac{h}{1-\gamma}x_\gamma\right]}{k + \beta\left[\frac{F}{\gamma} - \frac{h}{1-\gamma}x_\gamma\right]}\right] p_\gamma = \frac{\lambda}{\lambda+\theta}p_\gamma + \frac{\theta}{\lambda+\theta}\frac{kp_\gamma}{k + \beta\left[\frac{F}{\gamma} - \frac{h}{1-\gamma}x_\gamma\right]}, \end{aligned}$$

where the second equality follows from the first equation in (C.29). According to (A.14), we then have

$$\frac{\lambda}{\lambda+\theta}(r - p_\gamma) = c_\gamma^\star = \frac{\lambda}{\theta+\lambda}r + \underbrace{\mathbb{E}\left[k\int_0^T e^{-\theta t} dN_t \,\middle|\, Q^\star\right]}_{a_\gamma^\star} - \underbrace{\mathbb{E}\left[e^{-\theta T} P_T^\star\right]}_{c_{a\gamma}^\star}$$

which immediately implies that $a_\gamma^\star = c_{a\gamma}^\star - \frac{\lambda}{\lambda+\theta}p_\gamma = \frac{\theta}{\lambda+\theta}\frac{kp_\gamma}{k+\beta\left[\frac{F}{\gamma} - \frac{h}{1-\gamma}x_\gamma\right]}$. □

Proof of Theorem 4 and Corollary 3. The results in the theorem and corollary follow from Lemmas C.3 and C.4 by letting $p_0^\star = p_{\gamma^\star}$ (and hence $t_0^\star = t_{\gamma^\star}^\circ$) and $x^\star = x_{\gamma^\star}$ (and hence $t^\star = t_{\gamma^\star}$). □

Proof of Proposition 1. The "if" direction directly follows from Theorem 3. To show the "only if" part, we note that under a cyclic deterministic policy $(\bar{t}, \bar{p})$ with $q_{\tau_i}^m = 1$, $q_t^m = 0$ for $t \neq \tau_i$, and $q_t^n := 0$ for all t, (8) implies that

$$U_{\tau_i} = U_{\tau_{i-1}+}^I e^{\theta\bar{t}}, \quad \text{for } i = 1, 2, \dots \text{ with } \tau_0 = 0, \tag{C.36}$$

while (7) implies that

$$U_{\tau_i} = \beta F + (1-\beta)U_{\tau_i+}^I, \quad \text{for } i = 1, 2, \dots. \tag{C.37}$$

Combining (C.36) and (C.37) yields

$$U_{\tau_i+}^I = \rho U_{\tau_{i-1}+}^I - \frac{\beta}{1-\beta}F, \quad \text{for } i = 1, 2, \dots,$$

with $\rho := e^{\theta\bar{t}}/(1-\beta) > 1$, which implies

$$U^I_{\tau_i+} = \left(U_0 - \frac{\beta}{1-\beta}\frac{F}{\rho-1}\right)\rho^i + \frac{\beta}{1-\beta}\frac{F}{\rho-1}, \quad \text{for } i = 1, 2, \ldots.$$

Thus, if $U_0 \gtrless \frac{\beta}{1-\beta}\frac{F}{\rho-1}$, then $U^I_{\tau_i+} \to \pm\infty$ as $i \to \infty$. Thus, for the feasibility of U_t, we must have

$$U^I_{\tau_i+} = U_0 = \frac{\beta}{1-\beta}\frac{F}{\rho-1}, \quad \text{for } i = 1, 2, \ldots.$$

implying that

$$U_t = \frac{\beta}{1-\beta}\frac{F}{\rho-1}e^{-\theta(\tau_i - t)}, \quad \text{for } t \in (\tau_{i-1}, \tau_i] \text{ and } i = 1, 2, \ldots,$$

which is proportional to P_t. Thus, (P, Q) belongs to the class of proportional policies, among which Theorem 4 (resp., Theorem 3) has shown that the optimal audit policy cannot be periodic deterministic when $h < \hat{h}(\beta)$ (resp., $h \geq \hat{h}(\beta)$ and $r < F$). □

Appendix D: Proofs in Section 9

Proof of Theorem 5. Under any policy $\mathcal{P} = \left(F_t, \overline{F}_t, P_t, Q_t\right)$, the agent's and the principal's expected discounted costs onwards after having taken an evasive action at time t (i.e., $H_t = 1$) are now modified to

$$U_t := \mathbb{E}\left[-\int_t^\infty e^{-\theta(\zeta - t)}\overline{F}_\zeta dZ_\zeta \,\middle|\, T \leq t, Z_t = 1, H_t = 1, \mathcal{I}_t, \mathcal{P}\right], \text{ and} \tag{D.1}$$

$$V_t := \mathbb{E}\left[\int_t^\infty e^{-\theta(\zeta - t)}\left\{Z_\zeta\left(cd\zeta + kdN_\zeta\right) - (r - \overline{F}_\zeta)dZ_\zeta\right\} \,\middle|\, T \leq t, Z_t = 1, H_t = 1, \mathcal{I}_t, \mathcal{P}\right]. \tag{D.2}$$

If a self-correction is conducted, then the agent's expected continuation cost would be reduced to 0 and the principal's expected continuation cost is still given by (A.6).

For any given policy $\widehat{\mathcal{P}} = \left(\widehat{F}_t, \widehat{\overline{F}}_t, \widehat{P}_t, \widehat{Q}_t\right)$ and the agent's corresponding best response $\widehat{\sigma}^\star$, we now construct an alternative policy $\mathcal{P} := \left(F_t, \overline{F}_t, P_t, \overline{P}_t, Q_t\right)_{t \in [0,\infty)}$ by letting $F_t := \widehat{F}_t$, $\overline{F}_t := \widehat{\overline{F}}_t$, $Q_t := \widehat{Q}_t$, and,

$$P_t := \mathbb{E}\left[e^{-\theta(\widehat{\sigma}^\star(t) - t)}\min\left\{\widehat{P}_{\widehat{\sigma}^\star(t)}, h + \widehat{U}_{\widehat{\sigma}^\star(t)}, r\right\}Z_{\widehat{\sigma}^\star(t)} - \int_t^{\widehat{\sigma}^\star(t)} e^{-\theta(\zeta - t)}\widehat{F}_\zeta dZ_\zeta \,\middle|\, T \leq t, Z_t = 1, H_t = 0, \mathcal{I}_t, \widehat{\mathcal{P}}\right], \quad \forall \mathcal{I}_t. \tag{D.3}$$

Clearly, $\mathcal{P}$ is well defined and in particular, $F_t, \overline{F}_t$ and P_t are all bounded above by F. Because $\overline{F}_t := \widehat{\overline{F}}_t$ and $Q_t := \widehat{Q}_t$ by construction, we immediately have $\widehat{U}_t = U_t$ and $\widehat{W}_t = W_t$.

Now we demonstrate that the above-defined policy $\mathcal{P}$ satisfies the following properties.

Property 1: (6) holds, i.e., the agent prefers disclosure over evasion or self-correction. Indeed, the optimality of $\widehat{\sigma}^\star$ in (D.3) immediately implies $P_t \leq \min\left\{\widehat{P}_t, h + \widehat{U}_t, r\right\} \leq \min\left\{h + \widehat{U}_t, r\right\} = \min\left\{h + U_t, r\right\}$.

Property 2: Prompt disclosure is the agent's best response to $\mathcal{P}$. Indeed, we can follow the same argument as in (A.11) to show

$$P_t \leq \mathbb{E}\left[e^{-\theta(s-t)}Z_s P_s - \int_t^s e^{-\theta(\zeta - t)}F_\zeta dZ_\zeta \,\middle|\, T \leq t, Z_t = 1, H_t = 0, \mathcal{I}_t, \mathcal{P}\right], \quad \forall s \geq t. \tag{D.4}$$

That is, the agent's cost of immediate disclosure is always dominated by the agent's total expected discounted cost of delaying the disclosure to any stopping time $s \geq t$ under $\mathcal{P}$. As such, the agent always prefers to disclose without delay.

Property 3: The principal is not worse off under $\mathcal{P}$ than under $\widehat{\mathcal{P}}$. We first note that since $c \geq \theta r$ and $k \geq 0$, (D.2) (applied under $\widehat{\mathcal{P}}$) implies that

$$\begin{aligned}
\widehat{V}_t =& \mathbb{E}\left[\int_t^\infty e^{-\theta(\zeta-t)}\left\{Z_\zeta\left(cd\zeta + kdN_\zeta\right) - (r - \widehat{\overline{F}}_\zeta)dZ_\zeta\right\}\,\middle|\, T \leq t, Z_t = 1, H_t = 1, \mathcal{I}_t, \widehat{\mathcal{P}}\right] \\
\geq& r\mathbb{E}\left[\theta\int_t^\infty e^{-\theta(\zeta-t)}Z_\zeta d\zeta - \int_t^\infty e^{-\theta(\zeta-t)}dZ_\zeta\,\middle|\, T \leq t, Z_t = 1, H_t = 1, \mathcal{I}_t, \widehat{\mathcal{P}}\right] \\
&+ \mathbb{E}\left[\int_t^\infty e^{-\theta(\zeta-t)}\widehat{\overline{F}}_\zeta dZ_\zeta\,\middle|\, T \leq t, Z_t = 1, H_t = 1, \mathcal{I}_t, \widehat{\mathcal{P}}\right] \\
=& r - \widehat{U}_t,
\end{aligned} \tag{D.5}$$

where the last equality follows from (A.13) and the definition in (D.1). Then, similar to (A.14), we can use (D.5) to show that $C(\mathcal{P}, T) \leq C\left(\widehat{\mathcal{P}}, \widehat{\sigma}^\star\right)$, i.e., the principal's expected cost under policy $\mathcal{P}$ is no larger than that under $\widehat{\mathcal{P}}$. In particular, we note that the principal saves on the inspection cost k and damage cost c thanks to the agent's prompt disclosure under $\mathcal{P}$.

Property 4: It is optimal for the principal to set $F_t = \overline{F}_t := F$ for all $t \geq 0$, which immediately yields the recursive representation of (A.4) and (A.11) in (7), (8), (9) and (10) by following a similar derivation as in Lemma 1 of Wang et al. (2016). As shown by the previous properties, the agent will choose to disclose without delay nor evasion under $\mathcal{P}$, and hence the principal's expected payoff is given by $C(\mathcal{P}, T) = \mathbb{E}\left[k\int_0^T e^{-\theta t}dN_t + e^{-\theta T}\left(r - P_T\right)\,\middle|\,\mathcal{P}\right]$, in which the variables F_t and $\overline{F}_t$ are absent. Variable F_t only appears on the right-hand side of the constraint (D.4) and variable $\overline{F}_t$ only on the right-hand side of (6) through U_t in (D.1). Therefore, it is optimal for the principal to relax the constraints (D.4) and (6) by setting both F_t and $\overline{F}_t$ to the maximum F.

Property 5: Policy $\mathcal{P}$ and policy $\widehat{\mathcal{P}}$ are payoff-equivalent to the agent, i.e., $C_a(\mathcal{P}, T) = C_a(\widehat{\mathcal{P}}, \widehat{\sigma}^\star)$. This is because, by construction in (D.3), P_t is the agent's minimum expected discounted cost from t onwards under policy $\widehat{\mathcal{P}}$ by following $\widehat{\sigma}^\star$ as the response. On the other hand, by Properties 1 and 2, the agent will always promptly disclose at time T under policy $\mathcal{P}$ and hence incur the same expected cost P_t, leading to the conclusion. □

Proof of Proposition 2. With an exogenous detection at rate μ, the agent's disclosure incentive is strengthen and it is straightforward to show, by replicating the proof of Theorem 1, that it is optimal for the principal to impose maximal fine F upon exogenous detection and to restrict to policies that induce the agent's prompt disclosure (and hence the penalty F will never realize). That is, the principal can optimize within the class of policies $\widetilde{\mathcal{P}} = \left(\widetilde{P}_t, \widetilde{Q}_t\right)_{t\in[0,\infty)}$ satisfying

$$\widetilde{P}_t \leq \min\{r, h + \widetilde{U}_t\}, \quad \text{for all } t \geq 0, \tag{D.6}$$

with the dynamic evolution of $\widetilde{U}_t$ and $\widetilde{P}_t$ given as follows:

$$\widetilde{U}_t = (1 - \tilde{q}_t^m)\widetilde{U}_{t+} + \tilde{q}_t^m\left(\beta F + (1-\beta)\widetilde{U}_{t+}^I\right), \quad \text{for } \tilde{q}_t^m > 0, \tag{D.7}$$

$$\frac{d\widetilde{U}_t}{dt} = (\theta + \mu)\widetilde{U}_t - \mu F - \tilde{q}_t^n\left[\beta F + (1-\beta)\widetilde{U}_{t+}^I - \widetilde{U}_t\right], \quad \text{for } \tilde{q}_t^m = 0, \tag{D.8}$$

$$\widetilde{P}_t \leq (1 - \tilde{q}_t^m)\widetilde{P}_{t+} + \tilde{q}_t^m F, \quad \text{for } \tilde{q}_t^m > 0, \quad \text{and} \tag{D.9}$$

$$\widetilde{P}_t \le \widetilde{P}_{t+}, \quad \text{or} \quad \frac{d\widetilde{P}_t}{dt} \ge (\theta+\mu)\widetilde{P}_t - \mu F - \tilde{q}_t^n\left(F - \widetilde{P}_t\right), \quad \text{for} \quad \tilde{q}_t^m = 0, \tag{D.10}$$

where $\widetilde{U}_{t+}$ (resp., $\widetilde{U}_{t+}^I$) is the value of $\widetilde{U}_t$ right after time t in the absence (resp., presence) of an audit, and $\widetilde{P}_{t+}$ (resp., $\widetilde{P}_{t+}^I$) is the value of $\widetilde{P}_t$ right after time t in the absence (resp., presence) of an audit. Under such a policy satisfying (D.6)–(D.10), the principal's objective can be written as

$$\frac{\lambda}{\theta+\lambda} r + \min_{\widetilde{\mathcal{P}}} \ \mathbb{E}\left[k\int_0^T e^{-\theta t} dN_t - e^{-\theta T}\widetilde{P}_T \,\middle|\, \widetilde{\mathcal{P}}\right]. \tag{D.11}$$

Now, with slight abuse of notation, let $\widetilde{F} = \frac{\theta}{\theta+\mu}F$, $\tilde{r} = r - \frac{\mu}{\theta+\mu}F$ (which is positive because of the assumption $\mu < \bar{\mu} := \min\{\lambda, \frac{\theta r}{(F-r)^+}\}$) and policy $\mathcal{P}$ be

$$Q_t = (q_t^m, q_t^n) := \widetilde{Q}_t = (\tilde{q}_t^m, \tilde{q}_t^n), \quad P_t := \widetilde{P}_t - \frac{\mu}{\theta+\mu}F, \quad \text{and} \quad U_t := \widetilde{U}_t - \frac{\mu}{\theta+\mu}F, \quad \text{for all} t \ge 0. \tag{D.12}$$

Then, under this variable transformation, the constraints (D.6)–(D.10) can be rewritten as

$$P_t \le \min\{\tilde{r}, h+U_t\}, \quad \text{for all } t \ge 0, \tag{D.13}$$

$$U_t = (1-q_t^m)U_{t+} + q_t^m\left(\beta\widetilde{F} + (1-\beta)U_{t+}^I\right), \quad \text{for} \quad q_t^m > 0, \tag{D.14}$$

$$\frac{dU_t}{dt} = (\theta+\mu)U_t - q_t^n\left[\beta\widetilde{F} + (1-\beta)U_{t+}^I - U_t\right], \quad \text{for} \quad q_t^m = 0, \tag{D.15}$$

$$P_t \le (1-q_t^m)P_{t+} + q_t^m\widetilde{F}, \quad \text{for} \quad q_t^m > 0, \quad \text{and} \tag{D.16}$$

$$P_t \le P_{t+}, \quad \text{or} \quad \frac{dP_t}{dt} \ge (\theta+\mu)P_t - q_t^n\left(\widetilde{F} - P_t\right), \quad \text{for} \quad q_t^m = 0. \tag{D.17}$$

Similarly, the principal's objective can be rewritten as

$$\begin{aligned}
&\frac{\lambda}{\theta+\lambda}\tilde{r} + \min_{\mathcal{P}} \ \mathbb{E}\left[k\int_0^T e^{-\theta t} dN_t - e^{-\theta T}P_T \,\middle|\, \mathcal{P}\right] \\
=&\frac{\lambda}{\theta+\lambda}\tilde{r} + \min_{\mathcal{P}} \ \mathbb{E}\left[\int_0^\infty e^{-(\lambda+\theta)t}\left(k dN_t - \lambda P_t\right) \,\middle|\, \mathcal{P}\right] \\
=&\frac{\lambda}{\theta+\lambda}\tilde{r} + \min_{\mathcal{P}} \ \frac{\lambda}{\lambda-\mu}\mathbb{E}\left[\int_0^\infty e^{-(\theta+\mu+\lambda-\mu)t}\left(\frac{\lambda-\mu}{\lambda}k dN_t - (\lambda-\mu)P_t\right) \,\middle|\, \mathcal{P}\right]
\end{aligned} \tag{D.18}$$

Similar to Lemma A.1, the constraints (D.13)–(D.17) imply the feasible range of (P_t, U_t) is $\widetilde{\Omega}(h,\beta) := \left\{(p,u): 0 \le \beta p \le u \le p \le \tilde{r} \wedge \widetilde{F},\ p \le h+u\right\}$. Therefore, the principal's problem (D.18) subject to (D.13)–(D.17) is equivalent to the principal's problem in (11) with (i) the discounting rate θ replaced by $\theta+\mu$, (ii) the limited liability F by $\widetilde{F} = \frac{\theta}{\theta+\mu}F$, the remedial cost r by $\tilde{r} = r - \frac{\mu}{\theta+\mu}F$, (iv) the hazard rate λ by $\lambda-\mu$ (which is positive because of the assumption $\mu < \bar{\mu} := \min\{\lambda, \frac{\theta r}{(F-r)^+}\}$), and (v) the auditing cost k by $k(\lambda-\mu)/\lambda$, as shown by the last line of equation in (D.18). □

Proof of Proposition 3. First, we note that Theorem 1 still applies to (24), because Theorem 1(1)-(3) only depends on the agent's IC constraint in (24), which is the same as (5), and Theorem 1(4) also holds as the principal's cost is enlarged to account for the agent's portion. Thus, we can again restrict to the optimization among policies that induce prompt disclosure, under which the principal's problem is reformulated as (11) with the constraints unchanged but the objective function replaced by

$$\begin{aligned}
(1+\alpha)C(\mathcal{P},\sigma) + C_a(\mathcal{P},\sigma) =& \frac{\lambda}{\theta+\lambda}(1+\alpha)r + \mathbb{E}\left[(1+\alpha)k\int_0^T e^{-\theta t}dN_t - e^{-\theta T}\alpha P_T \,\middle|\, \mathcal{P}\right] \\
=&\alpha\left\{\frac{\lambda}{\theta+\lambda}(1/\alpha+1)r + \mathbb{E}\left[(1/\alpha+1)k\int_0^T e^{-\theta t}dN_t - e^{-\theta T}P_T \,\middle|\, \mathcal{P}\right]\right\},
\end{aligned}$$

which the same (up to a constant difference) as the objective function of (11) by replacing k with $(1/\alpha+1)k$.

□

Proof of Proposition 4. Straightforward calculation yields

$$\begin{aligned}
&\mathbb{E}\left[c\int_T^{\tau(T)} e^{-\theta t}dt + e^{-\theta\tau(T)}\left(k+r-\underline{F}\right)\middle|\,\mathcal{P}\right]\\
=&\mathbb{E}\left[e^{-\theta T}\left\{c\int_0^{\tau(T)-T} e^{-\theta s}ds + e^{-\theta(\tau(T)-T)}\left(k+r-\underline{F}\right)\right\}\middle|\,\mathcal{P}\right]\\
=&\mathbb{E}\left[e^{-\theta T}\left\{c/\theta + e^{-\theta(\tau(T)-T)}\left(k+r-\underline{F}-c/\theta\right)\right\}\,\middle|\,\mathcal{P}\right],
\end{aligned}$$

which reduces to a constant $\frac{\lambda}{\theta+\lambda}\frac{c}{\theta}$ if $\underline{F} = k+r-c/\theta$, rendering the objective function in (25) to

$$\frac{\lambda}{\theta+\lambda}\left[\delta r + (1-\delta)c/\theta\right] + \delta\mathbb{E}\left[k/\delta\int_0^T e^{-\theta t}dN_t - e^{-\theta T}P_T\,\middle|\,\mathcal{P}\right]. \tag{D.19}$$

Thus, the proposition follows immediately by contrasting (D.19) with (11). □

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