Electronic Companion to "Tight Guarantees for Multi-unit Prophet Inequalities and Online Stochastic Knapsack"

EC.1. Final Algorithms

In this section, we present our final algorithms, which combine the random routing approach described in Section 2.2 and the pre-processing algorithms for the corresponding OCRS problem under the multi-unit and the knapsack settings, as well as the unit density special case of the knapsack setting.

The final algorithm for the multi-unit setting is presented below in Algorithm 4. We adopt the multi-resource formulation, where we have m resources and each resource j can serve up to k_j queries. The final algorithm for the knapsack setting is presented below in Algorithm 5. We again adopt the multi-resource formulation and the size of each query over each resource can be an arbitrary fraction of the initial capacity of that resource. Finally, we present our final algorithm for the unit-density special case of the knapsack setting below in Algorithm 6. The algorithm is presented under the multi-resource formulation where for each query over each resource, the corresponding reward and size are assumed to be equivalent to each other which can take an arbitrary fraction of the initial capacity of the resource.

Algorithm 4 Algorithm for the multi-unit setting

1: Solve the LP relaxation (3) and and obtain an optimal solution $\{x^*_{tj}(\mathbf{r}_t, \mathbf{d}_t), \forall t, \forall j, \forall(\mathbf{r}_t, \mathbf{d}_t)\}$.

2: Define $p_{tj} = \mathbb{E}_{(\tilde{\mathbf{r}}_t, \tilde{\mathbf{d}}_t)\sim F_t}[x^*_{tj}(\tilde{\mathbf{r}}_t, \tilde{\mathbf{d}}_t)]$ for each j and each t.

3: For each j, compute the value of $\gamma^*_{k_j}$ following the procedure described in Section 3.3.

4: For each j, construct a solution $\{x_{l,t,j}(\gamma^*_{k_j})\}_{\forall l, \forall t}$ as in Algorithm 1 with the input $\theta = \gamma^*_{k_j}$ and $\boldsymbol{p}_j = (p_{1j}, \ldots, p_{Tj})$.

5: **for** $t = 1, \ldots, T$ **do**

6: Observe the reward and size realization $(\tilde{\mathbf{r}}_t, \tilde{\mathbf{d}}_t)$ of query t.

7: Randomly route query t to a resource j with probability $x^*_{tj}(\tilde{\mathbf{r}}_t, \tilde{\mathbf{d}}_t)$.

8: Denote by l_j the quantity such that $l_j - 1$ queries have already been served by resource j.

9: **If** $l_j = 1$, **then**,

10: Serve query t with probability $\frac{x_{1,t,j}(\gamma^*_{k_j})}{p_{tj}\cdot(1-\sum_{\tau<t} x_{1,\tau,j}(\gamma^*_{k_j}))}$.

11: **Else**,

12: Serve query t with probability $\frac{x_{l_j,t,j}(\gamma^*_{k_j})}{p_{tj}\cdot\sum_{\tau<t}(x_{l_j-1,\tau,j}(\gamma^*_{k_j})-x_{l_j,\tau}(\gamma^*_{k_j}))}$.

13: **end for**

Algorithm 5 Algorithm for the knapsack setting

Solve the LP relaxation (3) and obtain an optimal solution $\{x^*_{tj}(\mathbf{r}_t, \mathbf{d}_t), \forall t, \forall j, \forall(\mathbf{r}_t, \mathbf{d}_t)\}$.
Define $p_{tj}(d_t) = \mathbb{E}_{(\tilde{\mathbf{r}}_t, \tilde{\mathbf{d}}_t)\sim F_t}[\mathbb{1}(\tilde{d}_{tj} = d_t) \cdot x^*_{tj}(\tilde{\mathbf{r}}_t, \tilde{\mathbf{d}}_t)]$ for each j, each t and each d_t.
For each j, obtain the distributions $\{\tilde{X}_{t,j,\gamma}, \forall t\}$ and the thresholds $\{\eta_{t,j,\gamma}(d_t), \forall t, \forall d_t\}$ from Algorithm 2 with input $\gamma = 1/(3+e^{-2})$ and $\{p_{tj}(d_t), \forall t, \forall d_t\}$.
for $t = 1, \ldots, T,$ **do**
    Observe the reward and size realization $(\tilde{\mathbf{r}}_t, \tilde{\mathbf{d}}_t)$ of query t.
    Randomly route query t to a resource j with probability $x^*_{tj}(\tilde{\mathbf{r}}_t, \tilde{\mathbf{d}}_t)$.
    Observe the capacity utilization $X_{t-1,j}$ of the routed resource j.
    If $\eta_{t,j,\gamma}(\tilde{d}_{tj}) < X_{t-1,j} \leq 1 - \tilde{d}_{tj}$, **then**,
        Serve query t with probability 1 using resource j.
    Else if $X_{t-1,j} = \eta_{t,j,\gamma}(\tilde{d}_{tj})$, **then**
        serve query t with probability $\frac{\gamma - P(\eta_{t,j,\gamma}(\tilde{d}_{tj}) < X_{t-1,j} \leq 1-\tilde{d}_{tj})}{P(\eta_{t,j,\gamma}(\tilde{d}_{tj}) = X_{t-1,j})}$ using resource j.
    Else,
        reject query t.
end for

Algorithm 6 Algorithm for the unit-density special case of the knapsack setting

Solve the LP relaxation (3) and obtain an optimal solution $\{x^*_{tj}(\mathbf{r}_t, \mathbf{d}_t), \forall t, \forall j, \forall(\mathbf{r}_t, \mathbf{d}_t)\}$.
Define $p_{tj}(d_t) = \mathbb{E}_{(\tilde{\mathbf{r}}_t, \tilde{\mathbf{d}}_t)\sim F_t}[\mathbb{1}(\tilde{d}_{tj} = d_t) \cdot x^*_{tj}(\tilde{\mathbf{r}}_t, \tilde{\mathbf{d}}_t)]$ for each j, each t and each d_t.
For each j, obtain the sequence $\boldsymbol{\gamma}_j = (\gamma_{1j}, \ldots, \gamma_{Tj})$ from (18).
For each j, obtain the distributions $\{\tilde{X}_{t,j,\boldsymbol{\gamma}_j}, \forall t\}$ and the thresholds $\{\eta_{t,j,\boldsymbol{\gamma}_j}(d_t), \forall t, \forall d_t\}$ from Algorithm 3 with input $\boldsymbol{\gamma}_j$ and $\{p_{tj}(d_t), \forall t, \forall d_t\}$.
for $t = 1, 2, \ldots, T$ **do**
    Observe the reward and size realization $(\tilde{\mathbf{r}}_t, \tilde{\mathbf{d}}_t)$ of query t.
    Randomly route query t to a resource j with probability $x^*_{tj}(\tilde{\mathbf{r}}_t, \tilde{\mathbf{d}}_t)$.
    Observe the capacity utilization $X_{t-1,j}$ of the routed resource j.
    If $\eta_{t,j,\boldsymbol{\gamma}_j}(\tilde{d}_{tj}) < X_{t-1,j} \leq 1 - \tilde{d}_{tj}$, **then**,
        Serve query t with probability 1 using resource j.
    Else if $X_{t-1,j} = \eta_{t,j,\boldsymbol{\gamma}_j}(\tilde{d}_{tj})$, **then**
        serve query t with probability $\frac{\gamma_{tj} - P(\eta_{t,j,\boldsymbol{\gamma}_j}(\tilde{d}_{tj}) < X_{t-1,j} \leq 1-\tilde{d}_{tj})}{P(\eta_{t,j,\boldsymbol{\gamma}_j}(\tilde{d}_{tj}) = X_{t-1,j})}$ using resource j.
    Else,
        reject query t.
end for

EC.2. Proofs in Section 3

EC.2.1. Proof of Lemma 1

We first present the following lemma, which shows that instead of checking whether all the constraints of $\mathsf{LP}_k^{\mathsf{OCRS}}$ are satisfied, it is enough to consider only one constraint.

LEMMA EC.1. *For any $\theta \in [0,1]$, $\{\theta, x_{l,t}(\theta)\}$ is a feasible solution to $\mathsf{LP}_k^{\mathsf{OCRS}}(\boldsymbol{p})$ if and only if $\sum_{\tau=1}^{T-1} x_{k,\tau}(\theta) \leq 1-\theta$.*

The proof is relegated to Section EC.2.2. We now prove the condition on θ such that $\sum_{\tau=1}^{T-1} x_{k,\tau}(\theta) \leq 1-\theta$. Due to Lemma EC.1, this condition implies the feasibility condition of $\{\theta, x_{l,t}(\theta)\}$. Specifically, we will first show that the term $\sum_{t=1}^{T-1} x_{k,t}(\theta)$ is continuously monotone increasing with θ in the next lemma, where the formal proof is in Section EC.2.3.

LEMMA EC.2. *For any $1 \leq l \leq k$ and any $1 \leq t \leq T$, define $y_{l,t}(\theta) = \sum_{\tau=1}^{t} x_{l,t}(\theta)$. Then $y_{l,t}(\theta)$ is monotone increasing with θ and $y_{l,t}(\theta)$ is also Lipschitz continuous with θ.*

We are now ready to prove Lemma 1.

Proof of Lemma 1: Note that when $\theta = 0$, $y_{k,T-1}(0) = \sum_{\tau=1}^{T-1} x_{k,\tau}(0) = 0 < 1-\theta = 1$, and when $\theta = 1$, $y_{k,T-1}(1) = \sum_{\tau=1}^{T-1} x_{k,\tau}(1) > 0 = 1-\theta$. Further note that $1-\theta$ is continuously strictly decreasing with θ while Lemma EC.2 shows that $y_{k,T-1}(\theta) = \sum_{\tau=1}^{T-1} x_{k,\tau}(\theta)$ is continuously increasing with θ, there must exist a unique $\theta^* \in [0,1]$ such that $\sum_{\tau=1}^{T-1} x_{k,\tau}(\theta^*) = 1-\theta^*$ and for any $\theta \in [0,\theta^*]$, it holds that $\sum_{\tau=1}^{T-1} x_{k,\tau}(\theta) \leq 1-\theta$. Combining the above arguments with Lemma EC.1, we complete our proof. $\square$

EC.2.2. Proof of Lemma EC.1

We first prove that for any $\theta \in [0,1]$, $\{x_{l,t}(\theta)\}$ are non-negative.

LEMMA EC.3. *For any $\theta \in [0,1]$, we have $x_{l,t}(\theta) \geq 0$ for any $l = 1,\dots,k$ and $t = 1,\dots,T$.*

Proof: We now use induction on l to show that for any l, we have that $x_{l,t}(\theta) \geq 0$ and $\sum_{v=1}^{l} x_{v,t}(\theta) \leq \theta \cdot p_t$ for any t. Since we focus on a fixed θ, we abbreviate θ in the expression $x_{l,t}(\theta)$ and substitute $x_{l,t}$ for $x_{l,t}(\theta)$ in the proof.

For $l = 1$, from definition, we have that for $1 \leq t \leq t_2$, it holds that $x_{1,t} \geq 0$ and $\sum_{v=1}^{1} x_{v,t} \leq \theta \cdot p_t$. We now use induction on t to show that for $t_2+1 \leq t \leq T$, we have that $0 \leq x_{1,t} \leq \theta \cdot p_t$. Note that from definition, we have that

$$x_{1,t_2+1} = p_{t_2+1} \cdot (1 - \sum_{\tau=1}^{t_2} \theta \cdot p_\tau) < p_{t_2+1} \cdot \theta$$

Also, note that $1 - \sum_{\tau=1}^{t_2-1} \theta \cdot p_\tau \geq \theta$ and $p_{t_2} \leq 1$, we have that

$$1 - \sum_{\tau=1}^{t_2} \theta \cdot p_\tau \geq 1 - \sum_{\tau=1}^{t_2-1} \theta \cdot p_\tau - \theta \geq 0$$

Thus, it holds that

$$0 \le x_{1,t_2+1} = p_{t_2+1} \cdot (1 - \sum_{\tau=1}^{t_2} \theta \cdot p_\tau) < p_{t_2+1} \cdot \theta$$

Now, suppose for a t such that $t_2+1 \le t \le T$, we have that $0 \le x_{1,\tau} \le \theta \cdot p_\tau$ for any $t_2+1 \le \tau \le t$. Then we have that

$$x_{1,t+1} \le p_{t+1} \cdot (1 - \sum_{\tau=1}^{t} x_{1,\tau}) \le p_{t+1} \cdot (1 - \sum_{\tau=1}^{t_2} x_{1,\tau}) = p_{t+1} \cdot (1 - \sum_{\tau=1}^{t_2} \theta \cdot p_\tau) < p_{t+1} \cdot \theta$$

Also, note that $x_{1,t} \ge 0$ implies that $1 - \sum_{\tau=1}^{t-1} x_{1,\tau} \ge 0$, we have that

$$x_{1,t+1}/p_{t+1} = 1 - \sum_{\tau=1}^{t-1} x_{1,\tau} - x_{1,t} = (1-p_t) \cdot (1 - \sum_{\tau=1}^{t-1} x_{1,\tau}) \ge 0$$

It holds that $0 \le x_{1,t+1} \le p_{t+1} \cdot \theta$. Thus, from induction, for any t, we have proved that $0 \le x_{1,t} \le p_t \cdot \theta$.

Suppose that for a l such that $1 \le l \le k$, we have that $x_{l,t} \ge 0$ and $\sum_{v=1}^{l} x_{v,t} \le \theta \cdot p_t$ for any t. We now consider the case for $l+1$. From definition, $x_{l+1,t} = 0$ when $1 \le t \le t_{l+1}$ and when $t_{l+1}+1 \le t \le t_{l+2}$, $x_{l+1,t} = \theta \cdot p_t - \sum_{v=1}^{l} x_{v,t}$. Thus, for $1 \le t \le t_{l+2}$, we have proved that $x_{l+1,t} \ge 0$ and $\sum_{v=1}^{l+1} x_{v,t} \le \theta \cdot p_t$. We now use induction on t for $t > t_{l+2}$. When $t = t_{l+2}+1$, from definition, we have that

$$x_{l+1,t_{l+2}+1} = p_{t_{l+2}+1} \cdot \sum_{\tau=1}^{t_{l+2}} (x_{l,\tau} - x_{l+1,\tau}) \le \theta \cdot p_{t_{l+2}+1} - \sum_{v=1}^{l} x_{v,t_{l+2}+1} \Rightarrow \sum_{v=1}^{l+1} x_{v,t_{l+2}+1} \le \theta \cdot p_{t_{l+2}+1}$$

Also, note that

$$0 \le x_{l+1,t_{l+2}} = \theta \cdot p_{t_{l+2}} - \sum_{v=1}^{l} x_{v,t_{l+2}} \le p_{t_{l+2}} \cdot \sum_{\tau=1}^{t_{l+2}-1} (x_{l,\tau} - x_{l+1,\tau})$$

we get that

$$x_{l+1,t_{l+2}+1}/p_{t_{l+2}+1} = \sum_{\tau=1}^{t_{l+2}} (x_{l,\tau} - x_{l+1,\tau}) \ge \sum_{\tau=1}^{t_{l+2}-1} (x_{l,\tau} - x_{l+1,\tau}) - x_{l+1,t_{l+2}} \ge (1 - p_{t_{l+2}}) \cdot \sum_{\tau=1}^{t_{l+2}-1} (x_{l,\tau} - x_{l+1,\tau})$$

Thus, we proved that $0 \le x_{l+1,t_{l+2}+1}$ and $\sum_{v=1}^{l+1} x_{v,t_{l+2}+1} \le \theta \cdot p_{t_{l+2}+1}$. Now suppose that for a t such that $t_{l+2}+1 \le t \le T$, it holds that $0 \le x_{l+1,t}$ and $\sum_{v=1}^{l+1} x_{v,t} \le \theta \cdot p_t$. Then we have that

$$\sum_{v=1}^{l+1} x_{v,t+1}/p_{t+1} = 1 - \sum_{\tau=1}^{t} x_{l+1,\tau} \le 1 - \sum_{\tau=1}^{t-1} x_{l+1,\tau} = \sum_{v=1}^{l+1} x_{v,t}/p_t \le \theta$$

Also, note that $0 \le x_{l+1,t} = p_t \cdot \sum_{\tau=1}^{t-1} (x_{l,\tau} - x_{l+1,\tau})$, we have that

$$x_{l+1,t+1}/p_{t+1} = \sum_{\tau=1}^{t} (x_{l,\tau} - x_{l+1,\tau}) \ge \sum_{\tau=1}^{t-1} (x_{l,\tau} - x_{l+1,\tau}) - x_{l+1,\tau} = (1-p_t) \cdot \sum_{\tau=1}^{t-1} (x_{l,\tau} - x_{l+1,\tau}) \ge 0$$

Thus, we have proved that $0 \le x_{l+1,t+1}$ and $\sum_{v=1}^{l+1} x_{v,t+1} \le \theta \cdot p_{t+1}$. From the induction on t, we can conclude that for any $1 \le t \le T$, it holds that $0 \le x_{l+1,t}$ and $\sum_{v=1}^{l+1} x_{v,t} \le \theta \cdot p_t$. Again, from the induction on l, we can conclude that for any $1 \le l \le k$ and any $1 \le t \le T$, it holds that $0 \le x_{l,t}$ and $\sum_{v=1}^{l} x_{v,t} \le \theta \cdot p_t$, which completes our proof. $\square$

Now we are ready to prove Lemma EC.1.

Proof of Lemma EC.1: When $\{x_{l,t}(\theta)\}$ is feasible to $\mathsf{LP}_k^{\mathsf{OCRS}}(\boldsymbol{p})$ in (4), we get from constraint (4b) and (4c) that

$$x_{1,T}(\theta) \le p_T \cdot (1 - \sum_{t=1}^{T-1} x_{1,t}(\theta)) \text{ and } x_{l,T}(\theta) \le p_T \cdot \sum_{t=1}^{T-1} (x_{l-1,t}(\theta) - x_{l,t}(\theta)) \;\; \forall l = 2, \dots, k$$

Summing up the above inequalities, we get

$$\sum_{l=1}^{k} x_{l,T}(\theta) \le p_T \cdot (1 - \sum_{t=1}^{T-1} x_{k,t}(\theta))$$

Further note that by definition, we have $\sum_{l=1}^{k} x_{l,T}(\theta) = \theta \cdot p_T$. Thus, we show that $\{x_{l,t}(\theta)\}$ is feasible implies that $\sum_{t=1}^{T-1} x_{k,t}(\theta) \le 1 - \theta$.

Now we prove the reverse direction. Note that from the definition of $\{x_{l,t}(\theta)\}$, we have that $x_{l,t}(\theta) \le p_t \cdot \sum_{\tau=1}^{t-1}(x_{l-1,\tau}(\theta) - x_{l,\tau}(\theta))$ holds for any $1 \le l \le k-1$ and any $1 \le t \le T$, where we set $\sum_{\tau=1}^{t-1} x_{0,\tau}(\theta) = 1$ for any t for simplicity. Also, $\{x_{l,t}(\theta)\}$ are nonnegative as shown by Lemma EC.3. Thus, we have that

$$\{x_{l,t}(\theta)\} \text{ is feasible } \Leftrightarrow x_{k,t}(\theta) \le p_t \cdot \sum_{\tau=1}^{t-1} (x_{k-1,\tau}(\theta) - x_{k,\tau}(\theta)) \text{ holds for any } t_k + 1 \le t \le T$$

Moreover, note that from definition, for $t_k + 1 \le t \le T$, we have that $x_{l,t}(\theta) = p_t \cdot \sum_{\tau=1}^{t-1}(x_{l-1,\tau}(\theta) - x_{l,\tau}(\theta))$ when $1 \le l \le k-1$. Thus, for $t_k + 1 \le t \le T$, we have that

$$x_{k,t}(\theta) \le p_t \cdot \sum_{\tau=1}^{t-1} (x_{k-1,\tau}(\theta) - x_{k,\tau}(\theta)) \Leftrightarrow \theta = \sum_{v=1}^{k} x_{v,t}/p_t \le 1 - \sum_{\tau=1}^{t-1} x_{k,\tau}(\theta)$$

From the nonnegativity of $\{x_{l,t}(\theta)\}$, we know that $\sum_{\tau=1}^{t-1} x_{k,\tau}(\theta)$ is monotone increasing with t. Thus, it holds that

$$\{x_{l,t}(\theta)\} \text{ is feasible } \Leftrightarrow \theta \le 1 - \sum_{t=1}^{T-1} x_{k,t}(\theta)$$

which completes our proof. □

EC.2.3. Proof of Lemma EC.2

Proof: For any fixed $\theta \in [0,1]$ and any fixed $\Delta \ge 0$ such that $\theta + \Delta \in [0,1]$, we compare between $\{x_{l,t}(\theta)\}$ and $\{x_{l,t}(\theta + \Delta)\}$. Since we consider for a fixed θ and Δ, for notation brevity, we will omit θ and Δ by substituting $\{x_{l,t}\}$ for $\{x_{l,t}(\theta)\}$ and substituting $\{x'_{l,t}\}$ for $\{x_{l,t}(\theta+\Delta)\}$. Respectively, we denote $y_{l,t} = \sum_{\tau=1}^{t-1} x_{l,\tau}$ and $y'_{l,t} = \sum_{\tau=1}^{t-1} x'_{l,\tau}$. Also, we denote $\{t_l\}$ to be the time indexes associated with $\{x_{l,t}\}$ in the definition of $\{x_{l,t}\}$ and $\{t'_l\}$ to be the time indexes associated with $\{x'_{l,t}\}$. We will use induction to show that for each l, we have that $y_{l,t} \le y'_{l,t}$ and $\sum_{v=1}^{l} y'_{v,t} \le \sum_{v=1}^{l} y_{v,t} + \Delta \cdot \sum_{\tau=1}^{t} p_\tau$ hold for each t.

For the case $l=1$, obviously we have that $t_2'\le t_2$. When $1\le t\le t_2'$, from definition, it holds that $y_{1,t}\le y_{1,t}'\le y_{1,t}+\Delta\cdot\sum_{\tau=1}^{t}p_\tau$. We now use induction on t for $t_2'+1\le t\le t_2$. When $t=t_2'+1$, note that

$$x'_{1,t_2'+1}=p_{t_2'+1}\cdot(1-y'_{1,t_2'})\le(\theta+\Delta)\cdot p_{t_2'+1}\ \text{ and }\ x_{1,t_2'+1}=\theta\cdot p_{t_2'+1}\le p_{t_2'+1}\cdot(1-y_{1,t_2'})$$

we have that

$$y_{1,t_2'+1}=y_{1,t_2'}+x_{1,t_2'+1}\le p_{t_2'+1}+(1-p_{t_2'+1})\cdot y_{1,t_2'}\le p_{t_2'+1}+(1-p_{t_2'+1})\cdot y'_{1,t_2'}=y'_{1,t_2'+1}$$

and

$$y'_{1,t_2'+1}=y'_{1,t_2'}+x'_{1,t_2'+1}\le y_{1,t_2'}+\Delta\cdot\sum_{t=1}^{t_2'}p_t+(\theta+\Delta)\cdot p_{t_2'+1}=y_{1,t_2'+1}+\Delta\cdot\sum_{t=1}^{t_2'+1}p_t$$

Now suppose for a fixed t satisfying $t_2'+1\le t\le t_2-1$, it holds $y_{1,t}\le y'_{1,t}\le y_{1,t}+\Delta\cdot\sum_{\tau=1}^{t}p_\tau$. From definition, note that

$$x'_{1,t+1}=p_{t+1}\cdot(1-y'_{1,t})\le(\theta+\Delta)\cdot p_{t+1}\ \text{ and }\ x_{1,t+1}=\theta\cdot p_{t+1}\le p_{t+1}\cdot(1-y_{1,t})$$

we have

$$y_{1,t+1}=y_{1,t}+x_{1,t+1}\le p_{t+1}+(1-p_{t+1})\cdot y_{1,t}\le p_{t+1}+(1-p_{t+1})\cdot y'_{1,t}=y'_{1,t+1}$$

and

$$y'_{1,t+1}=y'_{1,t}+x'_{1,t+1}\le y_{1,t}+\Delta\cdot\sum_{\tau=1}^{t}p_\tau+(\theta+\Delta)\cdot p_{t+1}=y_{1,t+1}+\Delta\cdot\sum_{\tau=1}^{t+1}p_\tau$$

Thus, from induction on t, we conclude that $y_{1,t}\le y'_{1,t}\le y_{1,t}+\Delta\cdot\sum_{\tau=1}^{t}p_\tau$ holds for any $t_2'+1\le t\le t_2$. Finally, when $t\ge t_2+1$, note that

$$y_{1,t}=p_t+(1-p_t)\cdot y_{1,t-1}\ \text{ and }\ y'_{1,t}=p_t+(1-p_t)\cdot y'_{1,t-1}$$

which implies that

$$y'_{1,t}-y_{1,t}=(1-p_t)\cdot(y'_{1,t-1}-y_{1,t-1})=\cdots=(y'_{1,t_2}-y_{1,t_2})\cdot\prod_{\tau=t_2+1}^{t}(1-p_\tau)$$

Thus, we prove that for any $1\le t\le T$, it holds that $y_{1,t}\le y'_{1,t}\le y_{1,t}+\Delta\cdot\sum_{\tau=1}^{t}p_\tau$.

Suppose that for a fixed $1\le l\le k$, $y_{l,t}\le y'_{l,t}$ and $\sum_{v=1}^{l}y'_{v,t}\le\sum_{v=1}^{l}y_{v,t}+\Delta\cdot\sum_{\tau=1}^{t}p_\tau$ hold for each t. We now consider the case for $l+1$. When $t\le\min\{t_{l+2},t'_{l+2}\}$, from definition, we have that

$$\sum_{v=1}^{l+1}y'_{v,t}=(\theta+\Delta)\cdot\sum_{\tau=1}^{t}p_\tau\ \text{ and }\ \sum_{v=1}^{l+1}y_{v,t}=\theta\cdot\sum_{\tau=1}^{t}p_\tau$$

which implies that $\sum_{v=1}^{l+1} y'_{v,t} \le \sum_{v=1}^{l+1} y_{v,t} + \Delta \cdot \sum_{\tau=1}^{t} p_\tau$. Also, we have

$$y'_{l+1,t} - y_{l+1,t} = \Delta \cdot \sum_{\tau=1}^{t} p_\tau - \left(\sum_{v=1}^{l} y'_{v,t} - \sum_{v=1}^{l} y_{v,t} \right) \ge 0$$

where the last inequality holds from induction condition. Thus, we prove that $y_{l+1,t} \le y'_{l+1,t}$ and $\sum_{v=1}^{l+1} y'_{v,t} \le \sum_{v=1}^{l+1} y_{v,t} + \Delta \cdot \sum_{\tau=1}^{t} p_\tau$ hold for each $1 \le t \le \min\{t_{l+2}, t'_{l+2}\}$. Moreover, note that t_{l+2} is defined as the first time that $\theta > 1 - y_{l+1,t_{l+2}}$ while t'_{l+2} is defined as the first time that $\theta + \Delta > 1 - y'_{l+1,t'_{l+2}}$. Since $y'_{l+1,t} \ge y_{l+1,t}$ when $t \le \min\{t_{l+2}, t'_{l+2}\}$, we must have $t'_{l+2} \le t_{l+2}$. Then we use induction on t for $t'_{l+2} + 1 \le t \le t_{l+2}$. When $t = t'_{l+2} + 1 \le t_{l+2}$, from definition, we have

$$x'_{l+1,t'_{l+2}+1} = p_{t'_{l+2}+1} \cdot (y'_{l,t'_{l+2}} - y'_{l+1,t'_{l+2}}) \Rightarrow y'_{l+1,t'_{l+2}+1} = p_{t'_{l+2}+1} \cdot y'_{l,t'_{l+2}} + (1 - p_{t'_{l+2}+1}) \cdot y'_{l+1,t'_{l+2}}$$

and

$$x_{l+1,t'_{l+2}+1} \le p_{t'_{l+2}+1} \cdot (y_{l,t'_{l+2}} - y_{l+1,t'_{l+2}}) \Rightarrow y_{l+1,t'_{l+2}+1} \le p_{t'_{l+2}+1} \cdot y_{l,t'_{l+2}} + (1 - p_{t'_{l+2}+1}) \cdot y_{l+1,t'_{l+2}}$$

Note that $y'_{l,t'_{l+2}} \ge y_{l,t'_{l+2}}$ and $y'_{l+1,t'_{l+2}} \ge y_{l+1,t'_{l+2}}$, we get $y'_{l+1,t'_{l+2}+1} \ge y_{l+1,t'_{l+2}+1}$. Moreover, note that from the definition of t'_{l+2}, we have

$$\sum_{v=1}^{l+1} x'_{v,t'_{l+2}+1} \le p_{t'_{l+2}+1} \cdot (\theta + \Delta) = \sum_{v=1}^{l+1} x_{v,t'_{l+2}+1} + \Delta \cdot p_{t'_{l+2}+1}$$

which implies that

$$\begin{aligned}
\sum_{v=1}^{l+1} y'_{v,t'_{l+2}+1} &= \sum_{v=1}^{l+1} y'_{v,t'_{l+2}} + \sum_{v=1}^{l+1} x'_{v,t'_{l+2}+1} \le \sum_{v=1}^{l+1} y_{v,t'_{l+2}} + \Delta \cdot \sum_{j=1}^{t'_{l+2}} p_j + \sum_{v=1}^{l+1} x_{v,t'_{l+2}+1} + \Delta \cdot p_{t'_{l+2}+1} \\
&= \sum_{v=1}^{l+1} y_{v,t'_{l+2}+1} + \Delta \cdot \sum_{j=1}^{t'_{l+2}+1} p_j
\end{aligned}$$

Then suppose for a fixed t satisfying $t'_{l+2} + 1 \le t \le t_{l+2} - 1$, it holds that $y_{l+1,t} \le y'_{l+1,t}$ and $\sum_{v=1}^{l+1} y'_{v,t} \le \sum_{v=1}^{l+1} y_{v,t} + \Delta \cdot \sum_{\tau=1}^{t} p_\tau$. From definition, we have

$$x'_{l+1,t+1} = p_{t+1} \cdot (y'_{l,t} - y'_{l+1,t}) \Rightarrow y'_{l+1,t+1} = p_{t+1} \cdot y'_{l,t} + (1 - p_{t+1}) \cdot y'_{l+1,t}$$

and

$$x_{l+1,t+1} \le p_{t+1} \cdot (y_{l,t} - y_{l+1,t}) \Rightarrow y_{l+1,t+1} \le p_{t+1} \cdot y_{l,t} + (1 - p_{t+1}) \cdot y_{l+1,t}$$

Note that $y'_{l,t} \ge y_{l,t}$ and $y'_{l+1,t} \ge y_{l+1,t}$, we have $y'_{l+1,t+1} \ge y_{l+1,t+1}$. Also, from the definition of t'_{l+2}, we have

$$\sum_{v=1}^{l+1} x'_{v,t+1} \le p_{t+1} \cdot (\theta + \Delta) = \sum_{v=1}^{l+1} x_{v,t+1} + \Delta \cdot p_{t+1}$$

which implies that

$$\begin{aligned}\sum_{v=1}^{l+1} y'_{v,t+1} &= \sum_{v=1}^{l+1} y'_{v,t} + \sum_{v=1}^{l+1} x'_{v,t+1} \le \sum_{v=1}^{l+1} y_{v,t} + \Delta \cdot \sum_{\tau=1}^{t} p_\tau + \sum_{v=1}^{l+1} x_{v,t+1} + \Delta \cdot p_{t+1} \\ &= \sum_{v=1}^{l+1} y_{v,t+1} + \Delta \cdot \sum_{\tau=1}^{t+1} p_\tau\end{aligned}$$

Thus, from induction on t, we prove that $y_{l+1,t} \le y'_{l+1,t}$ and $\sum_{v=1}^{l+1} y'_{v,t} \le \sum_{v=1}^{l+1} y_{v,t} + \Delta \cdot \sum_{\tau=1}^{t} p_\tau$ hold for any $t'_{l+2}+1 \le t \le t_{l+2}$. Finally, when $t \ge t_{l+2}+1$, note that

$$y_{l+1,t} = p_t \cdot y_{l,t-1} + (1-p_t) \cdot y_{l+1,t-1} \quad \text{and} \quad y'_{l+1,t} = p_t \cdot y'_{l,t-1} + (1-p_t) \cdot y'_{l+1,t-1}$$

which implies that

$$y'_{l+1,t} - y_{l+1,t} = p_t \cdot (y'_{l,t-1} - y_{l,t-1}) + (1-p_t) \cdot (y'_{l+1,t-1} - y_{l+1,t-1})$$

It is direct to show inductively on t such that $y_{l+1,t} \le y'_{l+1,t}$ and $\sum_{v=1}^{l+1} y'_{v,t} \le \sum_{v=1}^{l+1} y_{v,t} + \Delta \cdot \sum_{\tau=1}^{t} p_\tau$ hold for any $t \ge t_{l+2}+1$.

Thus, we have proved that for any $1 \le t \le T$, we have $y_{l+1,t} \le y'_{l+1,t}$ and $\sum_{v=1}^{l+1} y'_{v,t} \le \sum_{v=1}^{l} y_{v,t} + \Delta \cdot \sum_{\tau=1}^{t} p_\tau$. By the induction on l, we finally prove that for any $1 \le l \le k$, $y_{l,t} \le y'_{l,t}$ and $\sum_{v=1}^{l} y'_{v,t} \le \sum_{v=1}^{l} y_{v,t} + \Delta \cdot \sum_{\tau=1}^{t} p_\tau$ hold for any $1 \le t \le T$. In this way, we prove that $y_{l,t}(\theta)$ is monotone increasing with θ for any l,t. Moreover, note that since $\sum_{\tau=1}^{T} p_\tau \le k$, we have that $y_{l,t}(\theta+\Delta) \le y_{l,t}(\theta) + k \cdot \Delta$ hold for any θ, Δ and any l,t. Thus, $y_{l,t}(\theta)$ is a continuous function on θ, which completes our proof. □

EC.2.4. Proof of Theorem 1

Proof: Given Lemma 1, in order to prove Theorem 1, it is enough for us to construct a feasible solution $\{\beta^*_{l,t}, \xi^*_t\}$ to $\mathsf{Dual}_k^{\mathsf{OCRS}}(\boldsymbol{p})$ in (5) such that the primal-dual pair $\{\theta^*, x_{l,t}(\theta^*)\}$ and $\{\beta^*_{l,t}, \xi^*_t\}$ satisfies the complementary slackness conditions. Specifically, we will construct a feasible solution $\{\beta^*_{l,t}, \xi^*_t\}$ to $\mathsf{Dual}_k^{\mathsf{OCRS}}(\boldsymbol{p})$ satisfying the following conditions:

$$\beta^*_{1,t} \cdot \left(x_{1,t}(\theta^*) - p_t \cdot (1 - \sum_{\tau<t} x_{1,\tau}(\theta^*)) \right) = 0, \quad \forall t = 1, \dots, T \tag{EC.1}$$

$$\beta^*_{l,t} \cdot \left(x_{l,t}(\theta^*) - p_t \cdot \sum_{\tau<t} (x_{l-1,\tau}(\theta^*) - x_{l,\tau}(\theta^*)) \right) = 0, \quad \forall t = 1, \dots, T, \forall l = 2, \dots, k$$

$$x_{l,t}(\theta^*) \cdot \left(\beta^*_{l,t} + \sum_{\tau>t} p_\tau \cdot (\beta^*_{l,\tau} - \beta^*_{l+1,\tau}) - \xi^*_t \right) = 0, \quad \forall t = 1, \dots, T, \forall l = 2, \dots, k$$

$$x_{k,t}(\theta^*) \cdot \left(\beta^*_{k,t} + \sum_{\tau>t} p_\tau \cdot \beta^*_{k,\tau} - \xi^*_t \right) = 0, \quad \forall t = 1, \dots, T$$

Note that from definitions, $\{x_{l,t}(\theta^*)\}$ satisfies the following conditions:

$$x_{l,t}(\theta^*) = 0 \le p_t \cdot \sum_{\tau=1}^{t-1} (x_{l-1,\tau}(\theta^*) - x_{l,\tau}(\theta^*)), \ \ \forall t \le t_l$$

$$x_{l,t}(\theta^*) = \theta^* \cdot p_t - \sum_{v=1}^{l-1} x_{v,t}(\theta^*) \le p_t \cdot \sum_{\tau=1}^{t-1} (x_{l-1,\tau}(\theta^*) - x_{l,\tau}(\theta^*)) \quad \text{for } t_l + 1 \le t \le t_{l+1}$$

where $\{t_l\}$ are the time indexes associated with the definition of $\{x_{l,t}(\theta^*)\}$ and we define $t_1 = 0$, $t_{k+1} = T-1$. Note that we can set $t_{k+1} = T-1$ because we focus on the solution $\{x_{l,t}(\theta^*)\}$. If we consider other solution $\{x_{l,t}(\theta)\}$ with $\theta \ne \theta^*$, then we cannot have $t_{k+1} = T-1$. Having $\theta = \theta^*$ in the solution $\{x_{l,t}(\theta)\}$ is the only way to make $t_{k+1} = T-1$ consistent with Algorithm 1. Note that if we set $\theta < \theta^*$, then we have $t_{k+1} = T$, and then we need to set $\beta^*_{k,T} = 0$ to satisfy the complementary slackness. However, later in the construction, we would set $\beta^*_{k,T} = R > 0$. On the other hand, if $\theta > \theta^*$, the primal solution given by Algorithm 1 is not feasible.

For simplicity, we also denote $\sum_{\tau=1}^{t-1} x_{0,\tau}(\theta^*) = 1$ for any t. Thus, in order for $\{\beta^*_{l,t}, \xi^*_t\}$ to satisfy the conditions in (EC.1), it is enough for $\{\beta^*_{l,t}, \xi^*_t\}$ to be feasible to $\mathsf{Dual}^{\mathsf{OCRS}}_k(\boldsymbol{p})$ and satisfy the following conditions:

$$\beta^*_{l,t} = 0 \quad \text{for } t \le t_{l+1} \tag{EC.2}$$

$$\beta^*_{l,t} + \sum_{\tau=t+1}^{T} p_\tau \cdot (\beta^*_{l,\tau} - \beta^*_{l+1,\tau}) = \xi^*_t \quad \text{for } t \ge t_l + 1 \tag{EC.3}$$

where we denote $\beta^*_{k+1,t} = 0$ for notation simplicity. We now show the construction of the solution $\{\beta^*_{l,t}, \xi^*_t\}$ to $\mathsf{Dual}^{\mathsf{OCRS}}_k(\boldsymbol{p})$.

We first define $\xi^*_T = R$, for a constant $R > 0$ that will be specified later. We also define $\beta^*_{l,T} = R$ for any $l = 1, \dots, k$. Then, inductively for $t = T-1, T-2, \dots, 1$, we follow the two steps below to specify the value of ξ^*_t and $\beta^*_{j,t}$ for any $l = 1, \dots, k$.

1. We fix l' such that $t_{l'} + 1 \le t \le t_{l'+1}$, and we define

$$\xi^*_t = \sum_{\tau=t+1}^{T} p_\tau \cdot (\beta^*_{l',\tau} - \beta^*_{l'+1,\tau}). \tag{EC.4}$$

2. For each $l = 1, \dots, k$, we define

$$\beta^*_{l,t} = \max\left\{0, \xi^*_t - \sum_{\tau=t+1}^{T} p_\tau \cdot (\beta^*_{l,\tau} - \beta^*_{l+1,\tau})\right\}. \tag{EC.5}$$

Finally, the constant R is selected such that $\sum_{t=1}^{T} p_t \cdot \xi^*_t = 1$. In what follows, we show the construction of $\{\beta^*_{l,t}, \xi^*_t\}$ above is feasible to $\mathsf{Dual}^{\mathsf{OCRS}}_k(\boldsymbol{p})$ and satisfy the requirements (EC.2) and (EC.3). Our proof would rely on the following property of $\{\beta^*_{l,t}, \xi^*_t\}$, which we prove at the end of this proof.

CLAIM EC.1. *Let $\{\beta^*_{l,t},\xi^*_t\}$ be constructed in (EC.4) and (EC.5). Then, it holds that*

$$\sum_{\tau=t}^{T} p_\tau \cdot (\beta^*_{l,\tau} - \beta^*_{l+1,\tau}) \leq \sum_{\tau=t}^{T} p_\tau \cdot (\beta^*_{l+1,\tau} - \beta^*_{l+2,\tau}) \tag{EC.6}$$

for any $t=1,\dots,T$ and any $l=1,\dots,k-1$. Moreover, $\beta^*_{l,t}\geq\beta^*_{l+1,t}$ for any $t=1,\dots,T$ and any $l=1,\dots,k$.

From Claim EC.1 and the construction (EC.4), we know that ξ^*_t is non-negative for any $t=1,\dots,T$. Therefore, from the construction (EC.5), we know that $\{\beta^*_{l,t},\xi^*_t\}$ above is feasible to $\mathsf{Dual}_k^{\mathsf{OCRS}}(\boldsymbol{p})$. It only remains to show that $\{\beta^*_{l,t},\xi^*_t\}$ satisfy the requirements (EC.2) and (EC.3).

For any $t=1,\dots,T$, we fix the index l' such that $t_{l'}+1\leq t\leq t_{l'+1}$. Then for any index $l\geq l'$, which is equivalent to $t\leq t_{l+1}$, we have

$$\xi^*_t = \sum_{\tau=t+1}^{T} p_\tau \cdot (\beta^*_{l',\tau} - \beta^*_{l'+1,\tau}) \leq \sum_{\tau=t+1}^{T} p_\tau \cdot (\beta^*_{l,\tau} - \beta^*_{l+1,\tau})$$

which follows from (EC.6) in Claim EC.1. Thus, we know that $\beta^*_{l,t}=0$ for any $l\geq l'$ such that $t\leq t_{l+1}$, which shows $\{\beta^*_{l,t},\xi^*_t\}$ satisfy the requirements (EC.2).

For any $t=1,\dots,T$, we fix the index l' such that $t_{l'}+1\leq t\leq t_{l'+1}$. Then for any index $l\leq l'$, which is equivalent to $t\geq t_l+1$, we have

$$\xi^*_t = \sum_{\tau=t+1}^{T} p_\tau \cdot (\beta^*_{l',\tau} - \beta^*_{l'+1,\tau}) \geq \sum_{\tau=t+1}^{T} p_\tau \cdot (\beta^*_{l,\tau} - \beta^*_{l+1,\tau})$$

which follows from (EC.6) in Claim EC.1. Thus, we know that

$$\beta^*_{l,t} = \xi^*_t - \sum_{\tau=t+1}^{T} p_\tau \cdot (\beta^*_{l,\tau} - \beta^*_{l+1,\tau})$$

for $t\geq t_l+1$, which shows that $\{\beta^*_{l,t},\xi^*_t\}$ satisfy the requirements (EC.3). Our proof of the theorem is thus completed. $\square$

Proof of Claim EC.1: We prove (EC.6) by induction. Clearly, for $t=T$, since $\beta^*_{l,T}=R>0$ for any $l=1,\dots,k$ and $\beta^*_{k+1,T}=0$, (EC.6) holds. We now suppose (EC.6) holds for $t+1$ and we consider the situation for t.

We fix the index l' such that $t_{l'}+1\leq t\leq t_{l'+1}$. We then consider the following two scenarios.

Scenario (i) when $l\geq l'$. Then, from the induction hypothesis, we know that

$$\xi^*_t = \sum_{\tau=t+1}^{T} p_\tau \cdot (\beta^*_{l',\tau} - \beta^*_{l'+1,\tau}) \leq \sum_{\tau=t+1}^{T} p_\tau \cdot (\beta^*_{l'',\tau} - \beta^*_{l''+1,\tau})$$

for $l''=l,l+1,l+2$. We thus have $\beta^*_{l,t}=\beta^*_{l+1,t}=\beta^*_{l+2,t}=0$ and we directly prove (EC.6) from the induction hypothesis.

Scenario (ii) when $l < l'$. From the induction hypothesis, it is clear to see that

$$\xi_t^* = \sum_{\tau=t+1}^{T} p_\tau \cdot (\beta_{l',\tau}^* - \beta_{l'+1,\tau}^*) \geq \sum_{\tau=t+1}^{T} p_\tau \cdot (\beta_{l,\tau}^* - \beta_{l+1,\tau}^*).$$

Then, we have

$$\beta_{l,t}^* = \xi_t^* - \sum_{\tau=t+1}^{T} p_\tau \cdot (\beta_{l,\tau}^* - \beta_{l+1,\tau}^*).$$

On the other hand, from the construction (EC.5), we know

$$-\beta_{l+1,t}^* \leq -\xi_t^* + \sum_{\tau=t+1}^{T} p_\tau \cdot (\beta_{l+1,\tau}^* - \beta_{l+2,\tau}^*).$$

Therefore, it holds that

$$\begin{aligned}\sum_{\tau=t}^{T} p_\tau \cdot (\beta_{l,\tau}^* - \beta_{l+1,\tau}^*) &= \sum_{\tau=t+1}^{T} p_\tau \cdot (\beta_{l,\tau}^* - \beta_{l+1,\tau}^*) + p_t \cdot (\beta_{l,t}^* - \beta_{l+1,t}^*) \\ &\leq (1-p_t) \cdot \sum_{\tau=t+1}^{T} p_\tau \cdot (\beta_{l,\tau}^* - \beta_{l+1,\tau}^*) + p_t \cdot \sum_{\tau=t+1}^{T} p_\tau \cdot (\beta_{l+1,\tau}^* - \beta_{l+2,\tau}^*).\end{aligned}$$

From the above inequality, we have

$$\begin{aligned}&\sum_{\tau=t}^{T} p_\tau \cdot (\beta_{l+1,\tau}^* - \beta_{l+2,\tau}^*) - \sum_{\tau=t}^{T} p_\tau \cdot (\beta_{l,\tau}^* - \beta_{l+1,\tau}^*) \\ &\geq \sum_{\tau=t}^{T} p_\tau \cdot (\beta_{l+1,\tau}^* - \beta_{l+2,\tau}^*) - (1-p_t) \cdot \sum_{\tau=t+1}^{T} p_\tau \cdot (\beta_{l,\tau}^* - \beta_{l+1,\tau}^*) - p_t \cdot \sum_{\tau=t+1}^{T} p_\tau \cdot (\beta_{l+1,\tau}^* - \beta_{l+2,\tau}^*) \\ &= p_t \cdot (\beta_{l+1,t}^* - \beta_{l+2,t}^*) + (1-p_t) \cdot \left(\sum_{\tau=t+1}^{T} p_\tau \cdot (\beta_{l+1,\tau}^* - \beta_{l+2,\tau}^*) - \sum_{\tau=t+1}^{T} p_\tau \cdot (\beta_{l,\tau}^* - \beta_{l+1,\tau}^*) \right) \\ &\geq p_t \cdot (\beta_{l+1,t}^* - \beta_{l+2,t}^*)\end{aligned}$$

where the last inequality follows from $p_t \leq 1$ and the induction hypothesis. Therefore, it only remains to show that $\beta_{l+1,t}^* \geq \beta_{l+2,t}^*$ under the induction hypothesis, which would prove our whole claim. From the induction hypothesis, we clearly have

$$\xi_t^* - \sum_{\tau=t+1}^{T} p_\tau \cdot (\beta_{l+1,\tau}^* - \beta_{l+2,\tau}^*) \geq \xi_t^* - \sum_{\tau=t+1}^{T} p_\tau \cdot (\beta_{l+2,\tau}^* - \beta_{l+3,\tau}^*)$$

which implies that

$$\beta_{l+1,t}^* = \max\left\{0, \xi_t^* - \sum_{\tau=t+1}^{T} p_\tau \cdot (\beta_{l+1,\tau}^* - \beta_{l+2,\tau}^*)\right\} \geq \beta_{l+2,t}^* = \max\left\{0, \xi_t^* - \sum_{\tau=t+1}^{T} p_\tau \cdot (\beta_{l+2,\tau}^* - \beta_{l+3,\tau}^*)\right\}. \tag{EC.7}$$

Therefore, we know that

$$\sum_{\tau=t}^{T} p_\tau \cdot (\beta^*_{l+1,\tau} - \beta^*_{l+2,\tau}) - \sum_{\tau=t}^{T} p_\tau \cdot (\beta^*_{l,\tau} - \beta^*_{l+1,\tau}) \geq p_t \cdot (\beta^*_{l+1,t} - \beta^*_{l+2,t}) \geq 0$$

which completes our induction. Thus, we prove (EC.6) for any $t = 1, \ldots, T$ and any $l = 1, \ldots, k-1$. Note that following the step in (EC.7), we can directly verify that $\beta^*_{l,t} \geq \beta^*_{l+1,t}$ given (EC.6) has been proved, for any $t = 1, \ldots, T$ and any $l = 1, \ldots, k-1$. Our proof of the claim is thus completed. □

EC.2.5. Construction of $\{\beta^*_{l,t}, \xi^*_t\}$ and a Constructive Proof of Theorem 1

Given Lemma 1, in order to prove Theorem 1, it is enough for us to construct a feasible solution $\{\beta^*_{l,t}, \xi^*_t\}$ to $\mathsf{Dual}_k^{\mathsf{OCRS}}(\boldsymbol{p})$ in (5) such that the primal-dual pair $\{\theta^*, x_{l,t}(\theta^*)\}$ and $\{\beta^*_{l,t}, \xi^*_t\}$ satisfies the complementary slackness conditions. Specifically, we will construct a feasible solution $\{\beta^*_{l,t}, \xi^*_t\}$ to $\mathsf{Dual}_k^{\mathsf{OCRS}}(\boldsymbol{p})$ satisfying the following conditions:

$$\begin{aligned}
&\beta^*_{1,t} \cdot \left(x_{1,t}(\theta^*) - p_t \cdot (1 - \sum_{\tau<t} x_{1,\tau}(\theta^*)) \right) = 0, \ \ \forall t = 1, \ldots, T \qquad \text{(EC.8)}\\
&\beta^*_{l,t} \cdot \left(x_{l,t}(\theta^*) - p_t \cdot \sum_{\tau<t} (x_{l-1,\tau}(\theta^*) - x_{l,\tau}(\theta^*)) \right) = 0, \ \ \forall t = 1, \ldots, T, \forall l = 2, \ldots, k\\
&x_{l,t}(\theta^*) \cdot \left(\beta^*_{l,t} + \sum_{\tau>t} p_\tau \cdot (\beta^*_{l,\tau} - \beta^*_{l+1,\tau}) - \xi^*_t \right) = 0, \ \ \forall t = 1, \ldots, T, \forall l = 2, \ldots, k\\
&x_{k,t}(\theta^*) \cdot \left(\beta^*_{k,t} + \sum_{\tau>t} p_\tau \cdot \beta^*_{k,\tau} - \xi^*_t \right) = 0, \ \ \forall t = 1, \ldots, T
\end{aligned}$$

Note that from definitions, $\{x_{l,t}(\theta^*)\}$ satisfies the following conditions:

$$\begin{aligned}
&x_{l,t}(\theta^*) = 0 \leq p_t \cdot \sum_{\tau=1}^{t-1} (x_{l-1,\tau}(\theta^*) - x_{l,\tau}(\theta^*)), \ \ \forall t \leq t_l\\
&x_{l,t}(\theta^*) = \theta^* \cdot p_t - \sum_{v=1}^{l-1} x_{v,t}(\theta^*) \leq p_t \cdot \sum_{\tau=1}^{t-1} (x_{l-1,\tau}(\theta^*) - x_{l,\tau}(\theta^*)) \quad \text{for } t_l + 1 \leq t \leq t_{l+1}
\end{aligned}$$

where $\{t_l\}$ are the time indexes associated with the definition of $\{x_{l,t}(\theta^*)\}$ and we define $t_1 = 0$, $t_{k+1} = T - 1$. For simplicity, we also denote $\sum_{\tau=1}^{t-1} x_{0,\tau}(\theta^*) = 1$ for any t. Thus, in order for $\{\beta^*_{l,t}, \xi^*_t\}$ to satisfy the conditions in (EC.8), it is enough for $\{\beta^*_{l,t}, \xi^*_t\}$ to be feasible to $\mathsf{Dual}_k^{\mathsf{OCRS}}(\boldsymbol{p})$ and satisfy the following conditions:

$$\beta^*_{l,t} = 0 \quad \text{for } t \leq t_{l+1} \qquad \text{(EC.9)}$$

$$\beta^*_{l,t} + \sum_{\tau=t+1}^{T} p_\tau \cdot (\beta^*_{l,\tau} - \beta^*_{l+1,\tau}) = \xi^*_t \quad \text{for } t \geq t_l + 1 \qquad \text{(EC.10)}$$

where we denote $\beta^*_{k+1,t}=0$ for notation simplicity. We now show the construction of the solution $\{\beta^*_{l,t},\xi^*_t\}$ to $\mathsf{Dual}_k^{\mathsf{OCRS}}(\boldsymbol{p})$. Define the following constants for each $l,q\in\{1,2,\dots,k\}$:

$$B_{l,q}=\sum_{t_l+1\le j_1<j_2<\dots<j_q\le t_{l+1}}\frac{p_{j_1}p_{j_2}\dots p_{j_q}}{(1-p_{j_1})(1-p_{j_2})\dots(1-p_{j_q})}\cdot\prod_{w=t_l+1}^{t_{l+1}}(1-p_w)$$

and we set $B_{l,0}=\prod_{w=t_l+1}^{t_{l+1}}(1-p_w)$. We also define the following terms for each $l,q\in\{1,2,\dots,k\}$ and each $t\in\{t_l+1,\dots,t_{l+1}\}$, where $\{t_l\}$ are the time indexes defined in the construction of $\{\theta^*,x_{l,t}(\theta^*)\}$ and we define $t_1=0$, $t_{k+1}=T-1$:

$$A_{l,q}(t)=\sum_{t+1\le j_1<j_2<\dots<j_q\le t_{l+1}}\frac{p_{j_1}p_{j_2}\dots p_{j_q}}{(1-p_{j_1})(1-p_{j_2})\dots(1-p_{j_q})}\cdot\prod_{w=t+1}^{t_{l+1}}(1-p_w)$$

and we set $A_{l,0}(t)=\prod_{w=t+1}^{t_{l+1}}(1-p_w)$. Then our construction of the solution $\{\beta^*_{l,t},\xi^*_t\}$ can be fully described as follows:

$$\begin{aligned}
&\xi^*_T=\beta^*_{l_1,T}=R,\quad\forall l_1=1,2,\dots,k\\
&\beta^*_{l_1,t}=0,\quad\forall l_1=1,2,\dots,k,\forall t\le t_{l_1+1}\\
&\xi^*_t=\phi_l\cdot p_TR,\quad\forall l=1,2,\dots,k,\forall t_l+1\le t\le t_{l+1}\\
&\beta^*_{l_1,t}=p_TR\cdot\sum_{w=l_1}^{l_2-1}\delta_{w,l_2}\cdot A_{l_2,w-l_1}(t),\quad\forall l_1=1,2,\dots,k,\forall l_2=l_1+1,\dots,k,\forall t_{l_2}+1\le t\le t_{l_2+1}
\end{aligned}\tag{EC.11}$$

where the parameters $\{\phi_l,\delta_{l_1,l_2},R\}$ are defined as:

$$\delta_{l,k}=1\qquad\forall l=1,2,\dots,k-1$$

$$\delta_{l,l}=0\qquad\forall l=1,2,\dots,k$$

$$\delta_{l_1,l_2}=\sum_{w_0=l_1+1}^{l_2}\sum_{w_1=w_0}^{l_2+1}\sum_{w_2=w_1}^{l_2+2}\cdots\sum_{w_{k-1-l_2}=w_{k-2-l_2}}^{k-1}B_{l_2+1,w_1-w_0}\cdot B_{l_2+2,w_2-w_1}\dots B_{k-1,w_{k-1-l_2}-w_{k-2-l_2}}\cdot B_{k,k-w_{k-1-l_2}},$$
$$\forall l_2=1,2,\dots,k-1\text{ and }l_1=1,2,\dots,l_2-1$$

$$\phi_k=1$$

$$\phi_l=\sum_{q=l+1}^{k}\sum_{w=l+1}^{q}(\delta_{w-1,q}-\delta_{w,q})\cdot(1-\sum_{v=0}^{w-l-1}B_{q,v})\quad\forall l=1,2,\dots,k-1$$

and R is a positive constant such that $\sum_{t=1}^T p_t\cdot\xi^*_t=1$. We then prove the feasibility of $\{\beta^*_{l,t},\xi^*_t\}$ and the conditions (EC.9), (EC.10) are satisfied. Obviously, from definition, $\beta^*_{l,t}$ is nonnegative for each l and each t. We first prove that ξ^*_t is also nonnegative for each t.

LEMMA EC.4. *For each $l_2=1,2,\dots,k$ and each $l_1=1,2,\dots,l_2-1$, we have that $\delta_{l_1,l_2}\ge\delta_{l_1+1,l_2}$.*

Proof: Note that when $l_2=k$, we have that $\delta_{l,k}=1$ for each $l=1,2,\dots,k-1$, thus it holds that $\delta_{l,k}\ge\delta_{l+1,k}$. When $l_2\le k-1$, from definitions, we have that for each $l_1=1,2,\dots,l_2-1$

$$\delta_{l_1,l_2}-\delta_{l_1+1,l_2}=\sum_{w_1=l_1+1}^{l_2+1}\sum_{w_2=w_1}^{l_2+2}\cdots\sum_{w_{k-1-l_2}=w_{k-2-l_2}}^{k-1}B_{l_2+1,w_1-l_1-1}\cdot B_{l_2+2,w_2-w_1}\dots B_{k-1,w_{k-1-l_2}-w_{k-2-l_2}}\cdot B_{k,k-w_{k-1-l_2}}$$

which completes our proof. $\square$

We then show that the term $1-\sum_{w=0}^{q} B_{l,w}$ is nonnegative for each l and each q. Note that the following lemma essentially implies that $\sum_{t=t_l+1}^{t_{l+1}} p_t \cdot A_{l,q}(t) = 1-\sum_{w=0}^{q} B_{l,w}$, by replacing i_1 with t_l and i_2 with t_{l+1} in (EC.12), which establishes the nonnegativity of the term $1-\sum_{w=0}^{q} B_{l,w}$.

LEMMA EC.5. *For each $q\in\{1,2,\dots,k\}$ and any $1\le i_1+1\le i_2\le T$, it holds that*

$$\begin{aligned}&\sum_{t=i_1+1}^{i_2} p_t\cdot \sum_{t+1\le j_1<j_2<\dots<j_q\le i_2} \frac{p_{j_1}p_{j_2}\dots p_{j_q}}{(1-p_{j_1})(1-p_{j_2})\dots(1-p_{j_q})}\cdot\prod_{v=t+1}^{i_2}(1-p_v)\\ &=1-\sum_{w=0}^{q}\sum_{i_1+1\le j_1<j_2<\dots<j_w\le i_2}\frac{p_{j_1}p_{j_2}\dots p_{j_w}}{(1-p_{j_1})(1-p_{j_2})\dots(1-p_{j_w})}\cdot\prod_{v=i_1+1}^{i_2}(1-p_v),\end{aligned}\tag{EC.12}$$

Proof: We will do induction on q from $q=0$ to $q=k$ to prove (EC.12). When $q=0$, we have that

$$\begin{aligned}\sum_{t=i_1+1}^{i_2} p_t\cdot\prod_{v=t+1}^{i_2}(1-p_v)&=\sum_{t=i_1+1}^{i_2}(1-(1-p_t))\cdot\prod_{v=t+1}^{i_2}(1-p_v)=\sum_{t=i_1+1}^{i_2}\left(\prod_{v=t+1}^{i_2}(1-p_v)-\prod_{v=t}^{i_2}(1-p_v)\right)\\ &=1-\prod_{v=i_1+1}^{i_2}(1-p_v)\end{aligned}$$

Thus, we have (EC.12) holds for $q=0$. Suppose (EC.12) holds for $1,2,\dots,q-1$, we consider the case for q. For any $1\le i_1+1\le i_2\le T$, we have that

$$\begin{aligned}&\sum_{t=i_1+1}^{i_2} p_t\cdot\sum_{t+1\le j_1<j_2<\dots<j_q\le i_2}\frac{p_{j_1}p_{j_2}\dots p_{j_q}}{(1-p_{j_1})(1-p_{j_2})\dots(1-p_{j_q})}\cdot\prod_{v=t+1}^{i_2}(1-p_v)\\ &=\sum_{t=i_1+1}^{i_2} p_t\cdot\sum_{j_1=t+1}^{i_2}\sum_{j_1<j_2<\dots<j_q\le i_2}\frac{p_{j_1}p_{j_2}\dots p_{j_q}}{(1-p_{j_1})(1-p_{j_2})\dots(1-p_{j_q})}\cdot\prod_{v=t+1}^{i_2}(1-p_v)\\ &=\sum_{j_1=i_1+2}^{i_2}\sum_{t=i_1+1}^{j_1-1} p_t\cdot\frac{p_{j_1}}{1-p_{j_1}}\cdot\sum_{j_1<j_2<\dots<j_q\le i_2}\frac{p_{j_2}\dots p_{j_q}}{(1-p_{j_2})\dots(1-p_{j_q})}\cdot\prod_{v=t+1}^{i_2}(1-p_v)\\ &=\sum_{j_1=i_1+2}^{i_2}\frac{p_{j_1}}{1-p_{j_1}}\cdot\sum_{j_1<j_2<\dots<j_q\le i_2}\frac{p_{j_2}\dots p_{j_q}}{(1-p_{j_2})\dots(1-p_{j_q})}\cdot\prod_{v=j_1}^{i_2}(1-p_v)\cdot\sum_{t=i_1+1}^{j_1-1}p_t\cdot\prod_{v=t+1}^{j_1-1}(1-p_v)\end{aligned}$$

where the second equality holds by exchanging the order of summation. Note that for induction purpose, we assume (EC.12) holds for $q=0$, which implies that $\sum_{t=i_1+1}^{j_1-1} p_t\cdot\prod_{v=t+1}^{j_1-1}(1-p_v)=1-\prod_{v=i_1+1}^{j_1-1}(1-p_v)$. Then we have

$$\begin{aligned}&\sum_{j_1=i_1+2}^{i_2}\frac{p_{j_1}}{1-p_{j_1}}\cdot\sum_{j_1<j_2<\dots<j_q\le i_2}\frac{p_{j_2}\dots p_{j_q}}{(1-p_{j_2})\dots(1-p_{j_q})}\cdot\prod_{v=j_1}^{i_2}(1-p_v)\cdot\sum_{t=i_1+1}^{j_1-1}p_t\cdot\prod_{v=t+1}^{j_1-1}(1-p_v)\\ &=\sum_{j_1=i_1+2}^{i_2}p_{j_1}\cdot\sum_{j_1<j_2<\dots<j_q\le i_2}\frac{p_{j_2}\dots p_{j_q}}{(1-p_{j_2})\dots(1-p_{j_q})}\cdot\prod_{v=j_1+1}^{i_2}(1-p_v)\cdot\left(1-\prod_{v=i_1+1}^{j_1-1}(1-p_v)\right)\\ &=\sum_{j_1=i_1+1}^{i_2}p_{j_1}\cdot\sum_{j_1<j_2<\dots<j_q\le i_2}\frac{p_{j_2}\dots p_{j_q}}{(1-p_{j_2})\dots(1-p_{j_q})}\cdot\prod_{v=j_1+1}^{i_2}(1-p_v)\cdot\left(1-\prod_{v=i_1+1}^{j_1-1}(1-p_v)\right)\end{aligned}$$

where the second equality holds by noting that when $j_1 = i_1 + 1$, we have $1 - \prod_{v=i_1+1}^{j_1-1}(1-p_v) = 0$. Thus, it holds that

$$\sum_{t=i_1+1}^{i_2} p_t \cdot \sum_{t+1\le j_1<j_2<\cdots<j_q\le i_2} \frac{p_{j_1}p_{j_2}\dots p_{j_q}}{(1-p_{j_1})(1-p_{j_2})\dots(1-p_{j_q})} \cdot \prod_{v=t+1}^{i_2}(1-p_v)$$
$$= \sum_{j_1=i_1+1}^{i_2} p_{j_1} \cdot \sum_{j_1<j_2<\cdots<j_q\le i_2} \frac{p_{j_2}\dots p_{j_q}}{(1-p_{j_2})\dots(1-p_{j_q})} \cdot \prod_{v=j_1+1}^{i_2}(1-p_v) \cdot \left(1 - \prod_{v=i_1+1}^{j_1-1}(1-p_v)\right)$$

Note that for the induction purpose, we assume that (EC.12) holds for $q-1$. Then, we have that

$$\sum_{j_1=i_1+1}^{i_2} p_{j_1} \cdot \sum_{j_1<j_2<\cdots<j_q\le i_2} \frac{p_{j_2}\dots p_{j_q}}{(1-p_{j_2})\dots(1-p_{j_q})} \cdot \prod_{v=j_1+1}^{i_2}(1-p_v)$$
$$= \sum_{t=i_1+1}^{i_2} p_t \cdot \sum_{t+1\le j_1<j_2<\cdots<j_{q-1}\le i_2} \frac{p_{j_1}p_{j_2}\dots p_{j_{q-1}}}{(1-p_{j_1})(1-p_{j_2})\dots(1-p_{j_{q-1}})} \cdot \prod_{v=t+1}^{i_2}(1-p_v)$$
$$= 1 - \sum_{w=0}^{q-1} \sum_{i_1+1\le j_1<j_2<\cdots<j_w\le i_2} \frac{p_{j_1}p_{j_2}\dots p_{j_w}}{(1-p_{j_1})(1-p_{j_2})\dots(1-p_{j_w})} \cdot \prod_{v=i_1+1}^{i_2}(1-p_v)$$

where the second equality holds from replacing the index j_{l+1} with j_l for $l = 2, \dots, q$ and replace the index j_1 with t. Also, note that

$$\sum_{j_1=i_1+1}^{i_2} p_{j_1} \cdot \sum_{j_1<j_2<\cdots<j_q\le i_2} \frac{p_{j_2}\dots p_{j_q}}{(1-p_{j_2})\dots(1-p_{j_q})} \cdot \prod_{v=j_1+1}^{i_2}(1-p_v) \cdot \prod_{v=i_1+1}^{j_1-1}(1-p_v)$$
$$= \sum_{i_1+1\le j_1<j_2<\cdots<j_q\le i_2} \frac{p_{j_1}p_{j_2}\dots p_{j_q}}{(1-p_{j_1})(1-p_{j_2})\dots(1-p_{j_q})} \cdot \prod_{v=i_1+1}^{i_2}(1-p_v)$$

Thus, we have that

$$\sum_{t=i_1+1}^{i_2} p_t \cdot \sum_{t+1\le j_1<j_2<\cdots<j_q\le i_2} \frac{p_{j_1}p_{j_2}\dots p_{j_q}}{(1-p_{j_1})(1-p_{j_2})\dots(1-p_{j_q})} \cdot \prod_{v=t+1}^{i_2}(1-p_v)$$
$$= \sum_{j_1=i_1+1}^{i_2} p_{j_1} \cdot \sum_{j_1<j_2<\cdots<j_q\le i_2} \frac{p_{j_2}\dots p_{j_q}}{(1-p_{j_2})\dots(1-p_{j_q})} \cdot \prod_{v=j_1+1}^{i_2}(1-p_v) \cdot \left(1 - \prod_{v=i_1+1}^{j_1-1}(1-p_v)\right)$$
$$= 1 - \sum_{w=0}^{q} \sum_{i_1+1\le j_1<j_2<\cdots<j_w\le i_2} \frac{p_{j_1}p_{j_2}\dots p_{j_w}}{(1-p_{j_1})(1-p_{j_2})\dots(1-p_{j_w})} \cdot \prod_{v=i_1+1}^{i_2}(1-p_v)$$

which completes our proof by induction on q. $\square$

Combining Lemma EC.4 and Lemma EC.5, we draw the following conclusion.

LEMMA EC.6. *For each $l = 1, 2, \dots, k$ and each $t = 1, 2, \dots, T$, we have that $\beta^*_{l,t} \ge 0$ and $\xi^*_t \ge 0$.*

Proof: Note that from definition, $\beta^*_{l,t} \ge 0$ for each l and t. We then show the non-negativity of ξ^*_t for each t. Note that Lemma EC.4 shows that $\delta_{l_1,l_2} \ge \delta_{l_1+1,l_2}$ for each $l_2 = 1, 2, \dots, k$ and each $l_1 = 1, 2, \dots, l_2 - 1$.

It only remains to show the non-negativity of the term $1-\sum_{w=0}^{q} B_{l,w}$, which can be directly established by Lemma EC.5. Specifically, by replacing i_1 with t_l and i_2 with t_{l+1} in (EC.12), we have $1-\sum_{w=0}^{q} B_{l,w} = \sum_{t=t_l+1}^{t_{l+1}} p_t \cdot A_{l,q}(t) \geq 0$. $\square$

From the definition of $\{\beta^*_{l,t}, \xi^*_t\}$, condition (EC.9) holds obviously. We then prove that condition (EC.10) is satisfied.

LEMMA EC.7. *For each $l = 1, 2, \ldots, k$ and each $t \geq t_l + 1$, it holds that*

$$\beta^*_{l,t} + \sum_{j=t+1}^{T} p_j \cdot (\beta^*_{l,j} - \beta^*_{l+1,j}) = \xi^*_t$$

where we denote $\beta^*_{k+1,t} = 0$ for notation simplicity.

Proof: When $l = k$, from definition, we have $\beta^*_{l,j} = 0$ for each $j \leq t_{l+1} = T-1$ and $\beta^*_{l,T} = R$, thus the lemma holds directly. When $t = T$, it is also direct to show from definition that the lemma holds. We then focus on the case where $l \leq k-1$ and $t \leq T-1$.

For a fixed $l \leq k-1$ and a fixed $t_l + 1 \leq t \leq T-1$, we denote an index $l_1 \geq l$ such that $t_{l_1} + 1 \leq t \leq t_{l_1+1}$. We then consider the following cases separately based on the value of l_1.

(i). When $l_1 \leq k-1$, we have that

$$\beta^*_{l,t} = p_T R \cdot \sum_{w=l}^{l_1-1} \delta_{w,l_1} \cdot A_{l_1,w-l}(t) \tag{EC.13}$$

also, for any $t+1 \leq j \leq t_{l_1+1}$, we have that

$$\beta^*_{l,j} - \beta^*_{l+1,j} = p_T R \cdot \sum_{w=l}^{l_1-1} (\delta_{w,l_1} - \delta_{w+1,l_1}) \cdot A_{l_1,w-l}(j)$$

which implies that

$$\sum_{j=t+1}^{t_{l_1+1}} p_j \cdot (\beta^*_{l,j} - \beta^*_{l+1,j}) = p_T R \cdot \sum_{w=l}^{l_1-1} (\delta_{w,l_1} - \delta_{w+1,l_1}) \cdot \sum_{j=t+1}^{t_{l_1+1}} p_j \cdot A_{l_1,w-l}(j)$$

Note that from (EC.12), it holds that $\sum_{j=t+1}^{t_{l_1+1}} p_j \cdot A_{l_1,w-l}(j) = 1 - \sum_{q=0}^{w-l} A_{l_1,q}(t)$. Thus, we have that

$$\begin{aligned}
\sum_{j=t+1}^{t_{l_1+1}} p_j \cdot (\beta^*_{l,j} - \beta^*_{l+1,j}) &= p_T R \cdot \sum_{w=l}^{l_1-1} (\delta_{w,l_1} - \delta_{w+1,l_1}) \cdot \left(1 - \sum_{q=0}^{w-l} A_{l_1,q}(t)\right) \\
&= p_T R \cdot \delta_{l,l_1} - p_T R \cdot \sum_{w=l}^{l_1-1} \delta_{w,l_1} \cdot A_{l_1,w-l}(t)
\end{aligned} \tag{EC.14}$$

where the last equality holds from $\delta_{l_1,l_1} = 0$. Similarly, for any $l_2 \geq l_1 + 1$ and any $t_{l_2} + 1 \leq j \leq t_{l_2+1}$, we have that

$$\beta^*_{l,j} - \beta^*_{l+1,j} = p_T R \cdot \sum_{w=l}^{l_2-1} (\delta_{w,l_2} - \delta_{w+1,l_2}) \cdot A_{l_2,w-l}(j)$$

which implies that

$$\sum_{j=t_{l_2}+1}^{t_{l_2+1}} p_j\cdot(\beta^*_{l,j}-\beta^*_{l+1,j})=p_TR\cdot\sum_{w=l}^{l_2-1}(\delta_{w,l_2}-\delta_{w+1,l_2})\cdot\sum_{j=t_{l_2}+1}^{t_{l_2+1}} p_j\cdot A_{l_2,w-l}(j)$$

Note that from Lemma EC.5, we have that $\sum_{j=t_{l_2}+1}^{t_{l_2+1}} p_j\cdot A_{l_2,w-l}(j)=1-\sum_{q=0}^{w-l}B_{l_2,q}$. Thus, we have that

$$\begin{aligned}\sum_{j=t_{l_1+1}+1}^{t_{k+1}} p_j\cdot(\beta^*_{l,j}-\beta^*_{l+1,j})&=\sum_{l_2=l_1+1}^{k}\sum_{j=t_{l_2}+1}^{t_{l_2+1}} p_j\cdot(\beta^*_{l,j}-\beta^*_{l+1,j})\\&=\sum_{l_2=l_1+1}^{k} p_TR\cdot\sum_{w=l}^{l_2-1}(\delta_{w,l_2}-\delta_{w+1,l_2})\cdot(1-\sum_{q=0}^{w-l}B_{l_2,q})\\&=p_TR\cdot\sum_{l_2=l_1+1}^{k}\sum_{w=l+1}^{l_2}(\delta_{w-1,l_2}-\delta_{w,l_2})\cdot(1-\sum_{q=0}^{w-l-1}B_{l_2,q})\end{aligned}\tag{EC.15}$$

Combining (EC.13), (EC.14) and (EC.15), we have that

$$\beta^*_{l,t}+\sum_{j=t+1}^{T}p_j\cdot(\beta^*_{l,j}-\beta^*_{l+1,j})=p_TR\cdot\delta_{l,l_1}+p_TR\cdot\sum_{l_2=l_1+1}^{k}\sum_{w=l+1}^{l_2}(\delta_{w-1,l_2}-\delta_{w,l_2})\cdot(1-\sum_{q=0}^{w-l-1}B_{l_2,q})\tag{EC.16}$$

Note that

$$\xi^*_t=p_TR\cdot\sum_{l_2=l_1+1}^{k}\sum_{w=l_1+1}^{l_2}(\delta_{w-1,l_2}-\delta_{w,l_2})\cdot(1-\sum_{q=0}^{w-l_1-1}B_{l_2,q})$$

in order to show $\beta^*_{l,t}+\sum_{j=t+1}^{T}p_j\cdot(\beta^*_{l,j}-\beta^*_{l+1,j})=\xi^*_t$, it is enough to prove that

$$\delta_{l,l_1}+\sum_{l_2=l_1+1}^{k}\sum_{w=l+1}^{l_2}(\delta_{w-1,l_2}-\delta_{w,l_2})\cdot(1-\sum_{q=0}^{w-l-1}B_{l_2,q})=\sum_{l_2=l_1+1}^{k}\sum_{w=l_1+1}^{l_2}(\delta_{w-1,l_2}-\delta_{w,l_2})\cdot(1-\sum_{q=0}^{w-l_1-1}B_{l_2,q})\tag{EC.17}$$

Further note that

$$\begin{aligned}&\sum_{l_2=l_1+1}^{k}\sum_{w=l+1}^{l_2}(\delta_{w-1,l_2}-\delta_{w,l_2})\cdot(1-\sum_{q=0}^{w-l-1}B_{l_2,q})\\=&\sum_{l_2=l_1+1}^{k}\sum_{w=l+1}^{l_2}(\delta_{w-1,l_2}-\delta_{w,l_2})-\sum_{l_2=l_1+1}^{k}\sum_{w=l+1}^{l_2}\sum_{q=0}^{w-l-1}B_{l_2,q}\cdot(\delta_{w-1,l_2}-\delta_{w,l_2})\\=&\sum_{l_2=l_1+1}^{k}\delta_{l,l_2}-\sum_{l_2=l_1+1}^{k}\sum_{q=0}^{l_2-l-1}B_{l_2,q}\cdot\delta_{q+l,l_2}\end{aligned}$$

and similarly, note that

$$\sum_{l_2=l_1+1}^{k}\sum_{w=l_1+1}^{l_2}(\delta_{w-1,l_2}-\delta_{w,l_2})\cdot(1-\sum_{q=0}^{w-l_1-1}B_{l_2,q})=\sum_{l_2=l_1+1}^{k}\delta_{l_1,l_2}-\sum_{l_2=l_1+1}^{k}\sum_{q=0}^{l_2-l_1-1}B_{l_2,q}\cdot\delta_{q+l_1,l_2}$$

in order to prove (EC.17), it is enough to show that

$$\sum_{l_2=l_1}^{k} \delta_{l,l_2} - \sum_{l_2=l_1+1}^{k} \sum_{q=0}^{l_2-l-1} B_{l_2,q} \cdot \delta_{q+l,l_2} = \sum_{l_2=l_1+1}^{k} \delta_{l_1,l_2} - \sum_{l_2=l_1+1}^{k} \sum_{q=0}^{l_2-l_1-1} B_{l_2,q} \cdot \delta_{q+l_1,l_2} \tag{EC.18}$$

When $l_1 = l$, it is direct to check that (EC.18) holds. The proof of (EC.18) when $l_1 \geq l+1$ is relegated to Lemma EC.8. Thus, we prove that when $l_1 \leq k-1$, it holds that $\beta^*_{l,t} + \sum_{j=t+1}^{T} p_j \cdot (\beta^*_{l,j} - \beta^*_{l+1,j}) = \xi^*_t$.

(ii). When $l_1 = k$, we have that

$$\beta^*_{l,t} = p_T R \cdot \sum_{w=l}^{k-1} A_{k,w-l}(t)$$

and for each $t+1 \leq j \leq T-1$, it holds that

$$\beta^*_{l,j} - \beta^*_{l+1,j} = p_T R \cdot \left(\sum_{w=l}^{k-1} A_{k,w-l}(j) - \sum_{w=l+1}^{k-1} A_{k,w-l-1}(j) \right) = p_T R \cdot A_{k,k-1-l}(j)$$

Note that $\beta^*_{l,T} = \beta^*_{l+1,T} = R$, we have

$$\beta^*_{l,t} + \sum_{j=t+1}^{T-1} p_j \cdot (\beta^*_{l,j} - \beta^*_{l+1,j}) = p_T R \cdot \left(\sum_{w=l}^{k-1} A_{k,w-l}(t) + \sum_{j=t+1}^{T-1} p_j \cdot A_{k,k-1-l}(j) \right)$$

Note that from (EC.12), it holds that $\sum_{j=t+1}^{T-1} p_j \cdot A_{k,k-1-l}(j) = 1 - \sum_{q=0}^{k-1-l} A_{k,q}(t)$. Thus, we have that

$$\beta^*_{l,t} + \sum_{j=t+1}^{T-1} p_j \cdot (\beta^*_{l,j} - \beta^*_{l+1,j}) = p_T R = \xi^*_t$$

which completes our proof. $\square$

LEMMA EC.8. *For each $l = 1, 2, \ldots, k-1$ and each $l_1 = l, l+1, \ldots, k-1$, it holds that*

$$\sum_{l_2=l_1}^{k} \delta_{l,l_2} - \sum_{l_2=l_1+1}^{k} \sum_{q=0}^{l_2-l-1} B_{l_2,q} \cdot \delta_{q+l,l_2} = \sum_{l_2=l_1+1}^{k} \delta_{l_1,l_2} - \sum_{l_2=l_1+1}^{k} \sum_{q=0}^{l_2-l_1-1} B_{l_2,q} \cdot \delta_{q+l_1,l_2} \tag{EC.19}$$

Proof: We now prove (EC.19) by induction on l from $l = k-1$ to $l = 1$. When $l = k-1$, we must have $l_1 = k-1 = l$, then (EC.19) holds obviously. Suppose that there exists a $1 \leq l' \leq k-2$ such that for any l satisfying $l'+1 \leq l \leq k-1$, (EC.19) holds for each l_1 such that $l \leq l_1 \leq k-1$, then we consider the case when $l = l'$. For this case, we again use induction on l_1 from $l_1 = k-1$ to $l_1 = l+1 = l'+1$. When $l_1 = k-1$, we have that

$$\sum_{l_2=l_1}^{k} \delta_{l,l_2} - \sum_{l_2=l_1+1}^{k} \sum_{q=0}^{l_2-l-1} B_{l_2,q} \cdot \delta_{q+l,l_2} = \delta_{l,k-1} + \delta_{l,k} - \sum_{q=0}^{k-l-1} B_{k,q} \cdot \delta_{q+l,k}$$

and

$$\sum_{l_2=l_1+1}^{k} \delta_{l_1,l_2} - \sum_{l_2=l_1+1}^{k} \sum_{q=0}^{l_2-l_1-1} B_{l_2,q} \cdot \delta_{q+l_1,l_2} = \delta_{k-1,k} - B_{k,0} \cdot \delta_{k-1,k}$$

Further note that from definition, $\delta_{v,k}=1$ for each $v\leq k-1$ and $\delta_{l,k-1}=\sum_{w_0=l+1}^{k-1}B_{k,k-w_0}=\sum_{q=1}^{k-l-1}B_{k,q}$, it is obvious that (EC.19) holds when $l_1=k-1$. Now suppose that (EC.19) holds for l_1+1 (we assume $l_1\geq l+1$ since when $l_1=l$, it is direct from definition that (EC.19) holds), we consider the case for l_1. Note that

$$\text{LHS of (EC.19)}=\delta_{l,l_1}-\sum_{q=0}^{l_1-l}B_{l_1+1,q}\cdot\delta_{q+l,l_1+1}+\sum_{l_2=l_1+1}^{k}\delta_{l,l_2}-\sum_{l_2=l_1+2}^{k}\sum_{q=0}^{l_2-l-1}B_{l_2,q}\cdot\delta_{q+l,l_2}$$

and

$$\text{RHS of (EC.19)}=\delta_{l_1,l_1+1}-\sum_{l_2=l_1+1}^{k}B_{l_2,l_2-l_1-1}\cdot\delta_{l_2-1,l_2}+\sum_{l_2=l_1+2}^{k}\delta_{l_1,l_2}-\sum_{l_2=l_1+2}^{k}\sum_{q=0}^{l_2-l_1-2}B_{l_2,q}\cdot\delta_{q+l_1,l_2}$$

Since we suppose for induction that (EC.19) holds for l_1+1, we have that

$$\text{(EC.19) holds for }(l,l_1)\Leftrightarrow\delta_{l,l_1}-\sum_{q=0}^{l_1-l}B_{l_1+1,q}\cdot\delta_{q+l,l_1+1}=\delta_{l_1,l_1+1}-\sum_{l_2=l_1+1}^{k}B_{l_2,l_2-l_1-1}\cdot\delta_{l_2-1,l_2}$$

Further note that we have supposed for induction that (EC.19) holds for $(l+1,l_1)$, which implies

$$\delta_{l+1,l_1}-\sum_{q=0}^{l_1-l-1}B_{l_1+1,q}\cdot\delta_{q+l+1,l_1+1}=\delta_{l_1,l_1+1}-\sum_{l_2=l_1+1}^{k}B_{l_2,l_2-l_1-1}\cdot\delta_{l_2-1,l_2}$$

Thus, it holds that

$$\text{(EC.19) holds for }(l,l_1)\Leftrightarrow\delta_{l,l_1}-\delta_{l+1,l_1}=\sum_{q=0}^{l_1-l}B_{l_1+1,q}\cdot(\delta_{q+l,l_1+1}-\delta_{q+l+1,l_1+1})$$

Finally, from definition, we have

$$\delta_{l,l_1}-\delta_{l+1,l_1}=\sum_{w_1=l+1}^{l_1+1}\sum_{w_2=w_1}^{l_1+2}\cdots\sum_{w_{k-1-l_1}=w_{k-2-l_1}}^{k-1}B_{l_1+1,w_1-l-1}\cdot B_{l_1+2,w_2-w_1}\dots B_{k-1,w_{k-1-l_1}-w_{k-2-l_1}}\cdot B_{k,k-w_{k-1-l_1}}$$

and

$$\delta_{q+l,l_1+1}-\delta_{q+l+1,l_1+1}=\sum_{w_2=q+l+1}^{l_1+2}\cdots\sum_{w_{k-1-l_1}=w_{k-2-l_1}}^{k-1}B_{l_1+2,w_2-q-l-1}\dots B_{k-1,w_{k-1-l_1}-w_{k-2-l_1}}\cdot B_{k,k-w_{k-1-l_1}}$$

which implies that

$$\delta_{l,l_1}-\delta_{l+1,l_1}=\sum_{q=0}^{l_1-l}B_{l_1+1,q}\cdot(\delta_{q+l,l_1+1}-\delta_{q+l+1,l_1+1})\tag{EC.20}$$

Thus, from induction, we prove that (EC.19) holds for each $l_1\geq l+1$. Note that (EC.19) holds obviously for $l_1=l$, (EC.19) holds for each $l_1\geq l$. From the induction on l, we know that (EC.19) holds for each $1\leq l\leq k-1$ and each $l\leq l_1\leq k-1$, which completes our proof. □

Finally, we only need to prove feasibility of $\{\beta^*_{l,t}, \xi^*_t\}$ in the following lemma.

LEMMA EC.9. *For each $l = 1, 2, \ldots, k$ and each $t = 1, 2, \ldots, t_l$, it holds that*

$$\beta^*_{l,t} + \sum_{j=t+1}^{T} p_j \cdot (\beta^*_{l,j} - \beta^*_{l+1,j}) \geq \xi^*_t$$

where we denote $\beta^*_{k+1,t} = 0$ for notation simplicity.

Proof: Note that from Lemma EC.4, we have $\delta_{w,l_2} \geq \delta_{w+1,l_2}$, which implies that $\beta^*_{l,j} \geq \beta^*_{l+1,j}$ for each l and j. Thus, we have that for each $t = 1, 2, \ldots, t_l$, it holds that

$$\beta^*_{l,t} + \sum_{j=t+1}^{T} p_j \cdot (\beta^*_{l,j} - \beta^*_{l+1,j}) \geq \beta^*_{l,t_l+1} + \sum_{j=t_l+2}^{T} p_j \cdot (\beta^*_{l,j} - \beta^*_{l+1,j})$$

Further note that Lemma EC.7 implies that

$$\beta^*_{l,t_l+1} + \sum_{j=t_l+2}^{n} p_j \cdot (\beta^*_{l,j} - \beta^*_{l+1,j}) \geq \xi^*_{t_l+1}$$

Thus, it is enough to show that $\xi^*_i \leq \xi^*_{t_l+1}$ for each $t = 1, 2, \ldots, t_l$. From the definition of ξ^*_i, it is enough to show that $\phi_l \leq \phi_{l+1}$. When $l = k-1$, we have $\phi_{l+1} = \phi_k = 1$ and $\phi_l = \phi_{k-1} = 1 - B_{k,0}$, which implies that $\phi_{k-1} \leq \phi_k$. When $l \leq k-2$, from definition, we have

$$\phi_l - \phi_{l+1} = \sum_{q=l+1}^{k} (\delta_{l,q} - \delta_{l+1,q}) \cdot (1 - B_{q,0}) - \sum_{q=l+2}^{k} \sum_{w=l+2}^{q} (\delta_{w-1,q} - \delta_{w,q}) \cdot B_{q,w-l-1}$$

Note that in the proof of Lemma EC.8, we proved (EC.20), then when $k-1 \geq q \geq l+1$, we have

$$\delta_{l,q} - \delta_{l+1,q} = \sum_{w=0}^{q-l} B_{q+1,w} \cdot (\delta_{w+l,q+1} - \delta_{w+l+1,q+1}) = \sum_{w=l+1}^{q+1} B_{q+1,w-l-1} \cdot (\delta_{w-1,q+1} - \delta_{w,q+1})$$

Thus, it holds that

$$\begin{aligned}
\phi_l - \phi_{l+1} &= -\sum_{q=l+1}^{k-1} (\delta_{l,q} - \delta_{l+1,q}) \cdot B_{q,0} + \sum_{q=l+1}^{k-1} \sum_{w=l+1}^{q+1} B_{q+1,w-l-1} \cdot (\delta_{w-1,q+1} - \delta_{w,q+1}) \\
&\quad - \sum_{q=l+2}^{k} \sum_{w=l+2}^{q} (\delta_{w-1,q} - \delta_{w,q}) \cdot B_{q,w-l-1} \\
&= -\sum_{q=l+1}^{k-1} (\delta_{l,q} - \delta_{l+1,q}) \cdot B_{q,0} + \sum_{q=l+1}^{k-1} B_{q+1,0} \cdot (\delta_{l,q+1} - \delta_{l+1,q+1}) \\
&= -B_{l+1,0} \cdot \delta_{l,l+1} \leq 0
\end{aligned}$$

which completes our proof. □

Together, Lemma EC.6, Lemma EC.7, and Lemma EC.9 establish the feasibility of $\{\beta^*_{l,t}, \xi^*_t\}$. Then, from the definition of $\{\beta^*_{l,t}, \xi^*_t\}$, obviously condition (EC.9) is satisfied and from Lemma EC.7, condition (EC.10) is satisfied. Thus, we finish the proof of Theorem 1.

EC.2.6. Proof of Lemma 2

Proof: Since we have $\mathsf{LP}_k^{\mathsf{OCRS}}(\boldsymbol{p}) = \mathsf{Dual}_k^{\mathsf{OCRS}}(\boldsymbol{p})$, it is enough to consider the dual LP $\mathsf{Dual}_k^{\mathsf{OCRS}}(\boldsymbol{p})$ in (5) and prove that $\mathsf{Dual}_k^{\mathsf{OCRS}}(\boldsymbol{p}) \geq \mathsf{Dual}_k^{\mathsf{OCRS}}(\tilde{\boldsymbol{p}})$. Suppose the optimal solution of $\mathsf{Dual}_k^{\mathsf{OCRS}}(\boldsymbol{p})$ is denoted as $\{\beta_{l,t}^*, \xi_t^*\}$, as constructed in (EC.11), we then construct a feasible solution $\{\tilde{\beta}_{l,t}, \tilde{\xi}_t\}$ to $\mathsf{Dual}_k^{\mathsf{OCRS}}(\tilde{\boldsymbol{p}})$ as follows:

$$\tilde{\xi}_t = \xi_t^* \quad \forall 1 \leq t < q, \quad \tilde{\xi}_q = \tilde{\xi}_{q+1} = \xi_q^*, \quad \tilde{\xi}_{t+1} = \xi_t^* \quad \forall q+1 \leq t \leq T$$

$$\tilde{\beta}_{l,t} = \beta_{l,t}^* \quad \forall l = 1, \ldots, k, \forall 1 \leq t < q$$

$$\tilde{\beta}_{l,q} = \tilde{\beta}_{l,q+1} = \beta_{l,q}^* \quad \forall l = 1, \ldots, k$$

$$\tilde{\beta}_{l,t+1} = \beta_{l,t}^* \quad \forall l = 1, \ldots, k, \forall q+1 \leq t \leq T$$

Note that we have

$$\mathsf{Dual}_k^{\mathsf{OCRS}}(\boldsymbol{p}) = \sum_{t=1}^{T} p_t \cdot \beta_{1,t}^* = \sum_{t=1}^{T+1} \tilde{p}_t \cdot \tilde{\beta}_{1,t}$$

it is enough to prove that $\{\tilde{\beta}_{l,t}, \tilde{\xi}_t\}$ is feasible to Primal$(\tilde{\boldsymbol{p}}, k)$. Obviously, we have $\{\tilde{\beta}_{l,t}, \tilde{\xi}_t\}$ are non-negative and $\sum_{t=1}^{T+1} \tilde{p}_t \cdot \tilde{\xi}_t = \sum_{t=1}^{T} p_t \cdot \xi_t^* = 1$, then we only need to check whether the following constraint is satisfied:

$$\tilde{\beta}_{l,t} + \sum_{\tau > t} \tilde{p}_\tau \cdot (\tilde{\beta}_{l,\tau} - \tilde{\beta}_{l+1,\tau}) - \tilde{\xi}_t \geq 0, \quad \forall l = 1, \ldots, k, \forall t = 1, \ldots, T+1 \tag{EC.21}$$

where we denote $\tilde{\beta}_{k+1,t} = 0$ for notation simplicity. Note that when $t \geq q+1$, we have that

$$\tilde{\beta}_{l,t} + \sum_{\tau > t} \tilde{p}_\tau \cdot (\tilde{\beta}_{l,\tau} - \tilde{\beta}_{l+1,\tau}) - \tilde{\xi}_t = \beta_{l,t}^* + \sum_{\tau > t} p_\tau \cdot (\beta_{l,\tau}^* - \beta_{l+1,\tau}^*) - \xi_t^* \geq 0, \ \ \forall l = 1, \ldots, k$$

and when $1 \leq t \leq q-1$, we also have

$$\tilde{\beta}_{l,t} + \sum_{\tau > t} \tilde{p}_\tau \cdot (\tilde{\beta}_{l,\tau} - \tilde{\beta}_{l+1,\tau}) - \tilde{\xi}_t = \beta_{l,t}^* + \sum_{\tau > t} p_\tau \cdot (\beta_{l,\tau}^* - \beta_{l+1,\tau}^*) - \xi_t^* \geq 0, \ \ \forall l = 1, \ldots, k$$

by noting $\tilde{p}_q + \tilde{p}_{q+1} = p_q$. Now we consider the case when $t = q$, then for each $l = 1, \ldots, k$, we have

$$\begin{aligned}
\tilde{\beta}_{l,q} + \sum_{j=q+1}^{T+1} \tilde{p}_j \cdot (\tilde{\beta}_{l-1,j} - \tilde{\beta}_{l,j}) - \tilde{\xi}_q &= \tilde{\beta}_{l,q} + \sum_{j=q+2}^{T+1} \tilde{p}_j \cdot (\tilde{\beta}_{l-1,j} - \tilde{\beta}_{l,j}) - \tilde{\xi}_q + \tilde{p}_{q+1} \cdot (\tilde{\beta}_{l-1,q+1} - \tilde{\beta}_{l,q+1}) \\
&= \beta_{l,q}^* + \sum_{j=q+1}^{T} p_j \cdot (\beta_{l-1,j}^* - \beta_{l,j}^*) - \xi_q^* + p_q \cdot (1-\sigma) \cdot (\beta_{l-1,q}^* - \beta_{l,q}^*) \\
&\geq p_q \cdot (1-\sigma) \cdot (\beta_{l-1,q}^* - \beta_{l,q}^*)
\end{aligned}$$

Thus, it is enough to show that $\beta_{l-1,q}^* \geq \beta_{l,q}^*$ to prove feasibility. Note that from Lemma EC.4, for each $l_2 = 1, 2, \ldots, k$ and each $l_1 = 1, 2, \ldots, l_2 - 1$, we have $\delta_{l_1,l_2} \geq \delta_{l_1+1,l_2}$, then, it is direct to show that $\beta_{l-1,q}^* \geq \beta_{l,q}^*$ from the construction (EC.11), which completes our proof. $\square$

EC.2.7. Proof of Proposition 1

Proof: We consider the following problem instance $\mathcal{H}$. At the beginning, there are two queries arriving deterministically with a reward 1. Then, over the time interval $[0,1]$, there are queries with reward $r_1 > 1$ arriving according to a Poisson process with rate λ. At last, there is one query with a reward $\frac{r_2}{\epsilon}$ arriving with a probability ϵ for some small $\epsilon > 0$.

Obviously, since $r_1 > 1$ and ϵ is set to be small, the prophet will first serve the last query as long as it arrives, and then serve the queries with a reward r_1 as much as possible, and at least serve the first two queries. Then, we have that

$$\mathbb{E}_{\boldsymbol{I}\sim\boldsymbol{F}}[V^{\mathrm{off}}(\boldsymbol{I})] = \hat{V} := r_2 + 2\cdot\exp(-\lambda) + (r_1+1)\cdot\lambda\cdot\exp(-\lambda) + 2r_1\cdot(1-(\lambda+1)\cdot\exp(-\lambda) + O(\epsilon)$$

Moreover, for any online algorithm π, we consider the following situations separately based on the number of the first two queries that π will serve.

(i). If π will always serve the first two queries, then it is obvious that $\mathbb{E}_{\pi,\boldsymbol{I}\sim\boldsymbol{F}}[V^{\pi}(\boldsymbol{I})] = 2$.

(ii). If π serves only one of the first two queries, then the optimal way for π to serve the second query will depend on the value of r_1 and r_2. To be more specific, if $r_1 \geq r_2$, then the optimal way is to serve the query with reward r_1 as long as it arrives, and if $r_1 < r_2$, then the optimal way is to reject all the arriving queries with reward r_1 and only serve the last query. Thus, it holds that

$$\mathbb{E}_{\pi,\boldsymbol{I}\sim\boldsymbol{F}}[V^{\pi}(\boldsymbol{I})] \leq V(1) := 1 + \exp(-\lambda)\cdot r_2 + (1-\exp(-\lambda))\cdot\max\{r_1, r_2\} + O(\epsilon)$$

(iii). If π rejects all the first two queries, then conditioning on there are more than one queries with reward r_1 arriving during the interval $[0,1]$, the optimal way for π is to serve both queries with reward r_1 if $r_1 \geq r_2$ and only serve one query with reward r_1 if $r_1 < r_2$. Then, it holds that

$$\mathbb{E}_{\pi,\boldsymbol{I}\sim\boldsymbol{F}}[V^{\pi}(\boldsymbol{I})] \leq V(2) := \exp(-\lambda)\cdot r_2 + \lambda\cdot\exp(-\lambda)\cdot(r_1+r_2) + (1-(\lambda+1)\cdot\exp(-\lambda)\cdot(r_1+\max\{r_1, r_2\})$$

Thus, we conclude that for any online algorithm π, it holds that

$$\frac{\mathbb{E}_{\pi,\boldsymbol{I}\sim\boldsymbol{F}}[V^{\pi}(\boldsymbol{I})]}{\mathbb{E}_{\boldsymbol{I}\sim\boldsymbol{F}}[V^{\mathrm{off}}(\boldsymbol{I})]} \leq g(r_1, r_2, \lambda) := \frac{\max\{V(1), V(2), 2\}}{\hat{V}}$$

where we can neglect the $O(\epsilon)$ term by letting $\epsilon \to 0$. In this way, we can focus on the following optimization problem

$$\inf_{r_1>1, r_2>1, \lambda} g(r_1, r_2, \lambda)$$

to obtain the upper bound of the guarantee of any online algorithm relative to the prophet's value. We can numerically solve the above problem and show that when $r_1 = r_2 = 1.4119$, $\lambda = 1.2319$, the value of $g(r_1, r_2, \lambda)$ reaches its minimum and equals 0.6269, which completes our proof. $\square$

EC.2.8. Proof of Theorem 2

Proof: For each $\boldsymbol{p} = (p_1, \dots, p_T)$ satisfying $\sum_{t=1}^{T} p_t = k$, since each irrational number can be arbitrarily approximated by a rational number, we assume without loss of generality that p_t is a rational number for each t, i.e., $p_t = \frac{n_t}{N}$ where n_t is an integer for each t and N is an integer to denote the common denominator. We first split p_1 into $\frac{1}{N}$ and $\frac{n_1-1}{N}$ to form a new sequence $\tilde{\boldsymbol{p}} = (\frac{1}{N}, \frac{n_1-1}{N}, \frac{n_2}{N}, \dots, \frac{n_T}{N})$. From Lemma 2, we know $\mathsf{LP}_k^{\mathsf{OCRS}}(\boldsymbol{p}) \geq \mathsf{LP}_k^{\mathsf{OCRS}}(\tilde{\boldsymbol{p}})$. We then split $\frac{n_1-1}{N}$ into $\frac{1}{N}$ and $\frac{n_1-2}{N}$ and so on. In this way, we split p_1 into n_1 copies of $\frac{1}{N}$ to form a new sequence $\tilde{\boldsymbol{p}} = (\frac{1}{N}, \dots, \frac{1}{N}, \frac{n_2}{N}, \dots, \frac{n_T}{N})$ and Lemma 2 guarantees that $\mathsf{LP}_k^{\mathsf{OCRS}}(\boldsymbol{p}) \geq \mathsf{LP}_k^{\mathsf{OCRS}}(\tilde{\boldsymbol{p}})$. We repeat the above operation for each t. Finally, we form a new sequence of arrival probabilities, denoted as $\boldsymbol{p}^N = (\frac{1}{N}, \dots, \frac{1}{N}) \in \mathbb{R}^{Nk}$, and we have $\mathsf{LP}_k^{\mathsf{OCRS}}(\boldsymbol{p}) \geq \mathsf{LP}_k^{\mathsf{OCRS}}(\boldsymbol{p}^N)$.

From the above argument, we know that for each $\boldsymbol{p} = (p_1, \dots, p_T)$ satisfying $\sum_{t=1}^{T} p_t = k$, there exists an integer N such that $\mathsf{LP}_k^{\mathsf{OCRS}}(\boldsymbol{p}) \geq \mathsf{LP}_k^{\mathsf{OCRS}}(\boldsymbol{p}^N)$, which implies that

$$\inf_{\boldsymbol{p}: \sum_t p_t = k} \mathsf{LP}_k^{\mathsf{OCRS}}(\boldsymbol{p}) = \liminf_{N \to \infty} \mathsf{LP}_k^{\mathsf{OCRS}}(\boldsymbol{p}^N)$$

Thus, it is enough to consider $\liminf_{N \to \infty} \mathsf{LP}_k^{\mathsf{OCRS}}(\boldsymbol{p}^N)$.

We denote $\tilde{\boldsymbol{y}}_\theta(t) = (\tilde{y}_{1,\theta}(t), \dots, \tilde{y}_{k,\theta}(t))$. We define a function $\boldsymbol{f}_\theta(\cdot) = (f_{1,\theta}(\cdot), \dots, f_{k,\theta}(\cdot))$, where we denote $\tilde{y}_{0,\theta}(t) = 1$ and for each $l = 1, \dots, k-1$

$$f_{l,\theta}(\tilde{y}_{1,\theta}, \dots, \tilde{y}_{k,\theta}, t) = \begin{cases} 0, & \text{if } \tilde{y}_{l-1,\theta}(t) \leq 1-\theta \\ \tilde{y}_{l-1,\theta}(t) - (1-\theta), & \text{if } \tilde{y}_{l,\theta}(t) \leq 1-\theta \leq \tilde{y}_{l-1,\theta}(t) \\ \tilde{y}_{l-1,\theta}(t) - \tilde{y}_{l,\theta}(t), & \text{if } \tilde{y}_{l,\theta}(t) \geq 1-\theta \end{cases}$$

and

$$f_{k,\theta}(\tilde{y}_{1,\theta}, \dots, \tilde{y}_{k,\theta}, t) = \begin{cases} 0, & \text{if } \tilde{y}_{k-1,\theta}(t) \leq 1-\theta \\ \tilde{y}_{k-1,\theta}(t) - (1-\theta), & \text{if } \tilde{y}_{k-1,\theta}(t) \geq 1-\theta \end{cases}$$

Moreover, variable $(\tilde{y}_1, \dots, \tilde{y}_k, t)$ belongs to the feasible set of the function $f_{l,\theta}(\cdot)$ if and only if $y_{v-1} \geq y_v$ for $v = 1, \dots, k-1$. Then, for each $\theta \in [0,1]$, the function $\tilde{\boldsymbol{y}}_\theta(t)$ in Definition 3 should be the solution to the following ordinary differential equation (ODE):

$$\frac{d\tilde{\boldsymbol{y}}_\theta(t)}{dt} = \boldsymbol{f}_\theta(\tilde{\boldsymbol{y}}_\theta, t) \text{ for } t \in [0,k] \text{ with starting point } \tilde{\boldsymbol{y}}_\theta(0) = (0, \dots, 0) \tag{EC.22}$$

For each integer N and $\boldsymbol{p}^N = (\frac{1}{N}, \dots, \frac{1}{N})$ where $\|\boldsymbol{p}^N\|_1 = k$, for any fixed $\theta \in [0,1]$, we denote $\{x_{l,t}(\theta, N)\}$ as the variables constructed in Definition 1 under the arrival probabilities $\boldsymbol{p}^N$, where $l = 1, \dots, k$ and $t = 1, \dots, Nk$. We further denote $y_{l,\theta,N}(\frac{t}{N}) = \sum_{\tau=1}^{t} x_{l,\tau}(\theta, N)$ and denote $\boldsymbol{y}_{\theta,N}(\cdot) = (y_{1,\theta,N}(\cdot), \dots, y_{k,\theta,N}(\cdot))$. It is direct to check that for each $t = 1, \dots, Nk$, it holds that

$$(\boldsymbol{y}_{\theta,N}(\frac{t}{N}) - \boldsymbol{y}_{\theta,N}(\frac{t-1}{N}))/(\frac{1}{N}) = \boldsymbol{f}_\theta(\boldsymbol{y}_{\theta,N}, \frac{t-1}{N})$$

Thus, $\{\boldsymbol{y}_{\theta,N}(t)\}_{\forall t\in[0,k]}$ can be viewed as the result obtained from applying Euler's method (Butcher and Goodwin, 2008) to solve ODE (EC.22), where there are Nk discrete points uniformly distributed within $[0,k]$. Note that for each $\theta\in[0,1]$, the function $\boldsymbol{f}_\theta(\cdot)$ is Lipschitz continuous with a Lipschitz constant 2 under infinity norm. Moreover, it is direct to note that for each $\theta\in[0,1]$ and each $t\in[0,k]$, it holds that $\|\boldsymbol{f}_\theta(\tilde{\boldsymbol{y}},t)\|_\infty\le 1$. Then, for each $\theta\in[0,1]$, each $t_1,t_2\in[0,k]$ and each $l=1,\dots,k$, we have

$$|\frac{d\tilde{y}_{l,\theta}(t_1)}{dt}-\frac{d\tilde{y}_{l,\theta}(t_2)}{dt}|\le 2\cdot\|\tilde{\boldsymbol{y}}_\theta(t_1)-\tilde{\boldsymbol{y}}_\theta(t_2)\|_\infty\le 2\cdot|t_1-t_2|$$

Thus, we know that

$$|\tilde{y}_{l,\theta}(t_1)-\tilde{y}_{l,\theta}(t_2)-\frac{d\tilde{y}_{l,\theta}(t_2)}{dt}\cdot(t_1-t_2)|\le 2\cdot(t_1-t_2)^2$$

We can apply the global truncation error of Euler's method (Theorem 212A (Butcher and Goodwin, 2008)) to show that $\boldsymbol{y}_{\theta,N}(k)$ converges to $\tilde{\boldsymbol{y}}_\theta(k)$ when $N\to\infty$. Specifically, we have

$$\|\boldsymbol{y}_{\theta,N}(k)-\tilde{\boldsymbol{y}}_\theta(k)\|_\infty\le(\exp(2k)-1)\cdot\frac{1}{N},\ \ \forall\theta\in[0,1] \tag{EC.23}$$

Now we define $Y(\theta)=\tilde{y}_{k,\theta}(k)$ as a function of $\theta\in[0,1]$ and for each N, we define $Y_N(\theta)=y_{k,\theta,N}(k)$ as a function of $\theta\in[0,1]$. (EC.23) implies that the function sequence $\{Y_N\}_{\forall N}$ converges uniformly to the function Y when $N\to\infty$. Note that for each N, the function $Y_N(\theta)$ is continuously monotone increasing with θ due to Lemma EC.2, then from uniform limit theorem, $Y(\theta)$ must be a continuously monotone increasing function over θ. Thus, the equation $Y(\theta)=1-\theta$ has a unique solution, denoted as γ_k^*. For each N, we denote θ_N^* as the unique solution to the equation $Y_N(\theta)=1-\theta$, where we have that $\theta_N^*=\mathsf{LP}_k^{\mathsf{OCRS}}(\boldsymbol{p}^N)$. Since $\{Y_N\}_{\forall N}$ converges uniformly to the function Y, it must hold that $\gamma_k^*=\lim_{N\to\infty}\theta_N^*$, which completes our proof. $\square$

EC.3. Proofs in Section 4

EC.3.1. Proof of Proposition 2

Proof: The proof is the same as the proof of Proposition 3.1 in Jiang et al. (2022).

Consider a problem setup $\mathcal{H}$ with 4 queries and

$$(r_1,p_1,d_1)=(r,1,\epsilon),\ \ (r_2,p_2,d_2)=(r_3,p_3,d_3)=(r,\frac{1-2\epsilon}{1+2\epsilon},\frac{1}{2}+\epsilon),\ \ (r_4,p_4,d_4)=(r/\epsilon,\epsilon,1)$$

for $r>0$ and some $\epsilon>0$. Obviously, if the policy π only serves queries with a size greater than 1/2, then the expected total reward is $V_L^\pi=r$. If the policy π only serves queries with a size no greater than 1/2, then the expected total reward is $V_S^\pi=r$. Thus, the expected total reward of the policy π is

$$V^\pi=\max\{V_L^\pi,V_S^\pi\}=r+O(\epsilon)$$

Moreover, it is direct to see that $\sum_{t=1}^4 p_t\cdot d_t=1$, then, we have $\mathrm{UP}(\mathcal{H})=4r$. Thus, the guarantee of π is upper bounded by $1/4+O(\epsilon)$, which converges to $1/4$ as $\epsilon\to 0$. $\square$

EC.3.2. Proof of Theorem 3

Proof: It is enough to prove that the Best-fit Magician policy π_γ in Algorithm 2 is feasible when $\gamma = \frac{1}{3+e^{-2}}$. In the remaining proof, we set $\gamma = \frac{1}{3+e^{-2}}$. For a fixed t, and any a and b, denote $\mu_{t,\gamma}(a,b] = P(a < \tilde{X}_{t,\gamma} \le b)$ assuming $\tilde{X}_{t,\gamma}$ is well-defined, it is enough to prove that $\mu_{t,\gamma}(0,1] \le 1-\gamma$ thus the random variable $\tilde{X}_{t+1,\gamma}$ is well-defined.

We define $U_t(s) = \mu_{t,\gamma}(0,s]$ for any $s \in (0,1]$. Note that by definition, we have $\mathbb{E}[\tilde{X}_{t,\gamma}] = \gamma \cdot \sum_{\tau=1}^{t} p_\tau \cdot d_\tau \le \gamma$. From integration by parts, we have that

$$\gamma \ge \mathbb{E}[\tilde{X}_{t,\gamma}] = \int_{s=0}^{1} s dU_t(s) = U_t(1) - \int_{s=0}^{1} U_t(s) ds \tag{EC.24}$$

We then bound the term $\int_{s=0}^{1} U_t(s)ds$. Now suppose $U_t(1) > \gamma$, otherwise $U_t(1) \le \gamma$ immediately implies that $U_t(1) \le 1-\gamma$, which proves our result. Then there must exist a constant $u^* \in (0,1)$ such that $\gamma \cdot u^* - \gamma \cdot \ln(u^*) = U_t(1)$. We further define

$$s^* = \begin{cases} \min\{s \in (0,1/2] : U_t(s) \ge \gamma \cdot u^*\}, & \text{if } U_t(\frac{1}{2}) \ge \gamma \cdot u^* \\ \frac{1}{2}, & \text{if } U_t(\frac{1}{2}) < \gamma \cdot u^* \end{cases}$$

Denote $U_t(s^*-) = \lim_{s \to s^*-} U_t(s)$, it holds that

$$\begin{aligned} \int_{s=0}^{1} U_t(s)ds &= \int_{s=0}^{s^*-} U_t(s)ds + \int_{s=s^*}^{1/2} U_t(s)ds + \int_{s=1/2}^{1-s^*} U_t(s)ds + \int_{s=1-s^*}^{1} U_t(s)ds \\ &\le s^* \cdot (U_t(s^*-) + U_t(1)) + \int_{s=s^*}^{1/2} U_t(s)ds + \int_{s=1/2}^{1-s^*} U_t(s)ds \\ &\le s^* \cdot (2\gamma u^* - \gamma \cdot \ln(u^*)) + \int_{s=s^*}^{1/2} (2U_t(s) - \gamma \cdot \ln(\frac{U_t(s)}{\gamma}))ds \end{aligned}$$

where the last inequality holds by noting that $U_t(s^*-) \le \gamma u^*$ and for any $s \in [s^*, 1/2]$, from Lemma 3, we have $\frac{U_t(s)}{\gamma} \le \exp(-\frac{U_t(1-s)-U_t(s)}{\gamma})$, which implies that $\frac{U_t(1-s)}{\gamma} \le \frac{U_t(s)}{\gamma} - \ln(\frac{U_t(s)}{\gamma})$. Note that for any $s \in [s^*, 1/2]$, we have that $\gamma \cdot u^* \le U_t(s^*) \le U_t(s) \le U_t(1/2) \le \gamma$, where $U_t(1/2) \le \gamma$ holds directly from Lemma 3. Further note that the function $2x - \gamma \cdot \ln(x/\gamma)$ is a convex function, thus is quasi convex. Then, for any $s \in [s^*, 1/2]$, it holds that

$$2U_t(s) - \gamma \cdot \ln(\frac{U_t(s)}{\gamma}) \le \max\{2\gamma u^* - \gamma \cdot \ln(u^*), 2\gamma\}$$

Thus, we have that

$$\int_{s=0}^{1} U_t(s)ds \le s^* \cdot (2\gamma u^* - \gamma \cdot \ln(u^*)) + (1/2 - s^*) \cdot \max\{2\gamma u^* - \gamma \cdot \ln(u^*), 2\gamma\}$$

If $2\gamma u^* - \gamma \cdot \ln(u^*) \le 2\gamma$, we have $\int_{s=0}^{1} U_t(s)ds \le 2s^*\gamma + \gamma - 2s^*\gamma = \gamma$. From (EC.24), we have that $U_t(1) \le 2\gamma < 1-\gamma$.

If $2\gamma u^* - \gamma \cdot \ln(u^*) > 2\gamma$, we have $\int_{s=0}^{1} U_t(s)ds \le \gamma u^* - \frac{\gamma}{2} \cdot \ln(u^*)$. From (EC.24) and the definition of u^*, we have that

$$U_t(1) = \gamma u^* - \gamma \cdot \ln(u^*) \le \gamma + \gamma u^* - \frac{\gamma}{2} \cdot \ln(u^*)$$

which implies that $u^* \ge \exp(-2)$. Note that the function $x - \ln(x)$ is non-increasing on $(0,1)$, we have $U_t(1) \le \gamma \cdot \exp(-2) + 2\gamma = 1 - \gamma$, which completes our proof. □

EC.3.3. Proof of Theorem 4

Proof: We denote by d_t the size of query t. Then, we have $\boldsymbol{p} = (p_1, \dots, p_t)$ and $\boldsymbol{D} = (d_1, \dots, d_T)$. For each $\epsilon > 0$, we consider the following $\boldsymbol{p}$ and $\boldsymbol{D}$:

$$(p_1, d_1) = (1, \epsilon),\ \ (p_t, d_t) = (\frac{1-2\epsilon}{(T-2)(\frac{1}{2}+\epsilon)}, \frac{1}{2} + \epsilon)\ \text{ for all } 2 \le t \le T-1 \ \text{ and }\ (p_T, d_T) = (\epsilon, 1)$$

It is direct to check that $\sum_{t=1}^{T} p_t \cdot d_t = 1$.

We denote by θ^* the maximum ratio in the knapsack OCRS problem. We now focus on the last query $(p_T, d_T) = (\epsilon, 1)$. Note that in order to accept this last query with probability θ^*, we must not accept any query during the period 1 to period $T-1$, with probability at least θ^*. Therefore, it holds that

$$\begin{aligned}\theta^* &\le 1 - P(\text{accept some query } t \le T-1) \\ &\le 1 - P(\text{accept query 1 and all queries } 2 \le t \le T-1 \text{ are inactive}) - P(\text{accept some query } 2 \le t \le T-1).\end{aligned}$$

Note that we can bound

$$\begin{aligned}P(\text{accept query 1 and all queries } 2 \le t \le T-1 \text{ are inactive}) &= \theta^* \cdot \left(1 - \frac{1-2\epsilon}{(T-2)(\frac{1}{2}+\epsilon)}\right)^{T-2} \\ &= \theta^* \cdot e^{-2} + O(\epsilon).\end{aligned}$$

On the other hand, we know that

$$P(\text{accept some query } 2 \le t \le T-1) = \theta^* \cdot \sum_{t=2}^{T-1} p_t = 2\theta^* + O(\epsilon).$$

Therefore, when $\epsilon \to 0$, the optimal value θ^* must satisfy the inequality

$$\theta^* \le 1 - \theta^* \cdot e^{-2} - 2\theta^*$$

which implies that $\theta^* \le \frac{1}{3+e^{-2}}$. Our proof is thus completed.

□

EC.4. Proofs in section 5

EC.4.1. Proof of Lemma 4

Proof: We prove (13) by induction on t. When $t=0$, since $\mu_{0,\gamma}(0,b]=0$ for any $0<b\le 1/2$, (13) holds trivially. Now suppose that (13) holds for $t-1$, we consider the case for t. Denote $\mathcal{F}_t$ as the support of $\tilde{d}_t$ and for each $d_t\in\mathcal{F}_t$, we denote $\eta_{t,\gamma}(d_t)$ as the threshold defined in (12). Then we define the following division of $\mathcal{F}_t$:

$$\mathcal{F}_{t,1}:=\{d_t\in\mathcal{F}_t:\eta_{t,\gamma}(d_t)=0 \text{ and } d_t\le b\}$$

$$\mathcal{F}_{t,2}=\{d_t\in\mathcal{F}_t:\eta_{t,\gamma}(d_t)=0 \text{ and } b<d_t\le 1-b\}$$

$$\mathcal{F}_{t,3}=\{d_t\in\mathcal{F}_t:\eta_{t,\gamma}(d_t)>0 \text{ and } d_t\le 1-b\}$$

$$\mathcal{F}_{t,4}=\{d_t\in\mathcal{F}_t:\eta_{t,\gamma}(d_t)=0 \text{ and } 1-b<d_t\}$$

$$\mathcal{F}_{t,5}=\{d_t\in\mathcal{F}_t:\eta_{t,\gamma}(d_t)>0 \text{ and } 1-b<d_t\}$$

Note that for each $d_t\in\mathcal{F}_t$, $\eta_{t,\gamma}(d_t)=0$ implies that a measure $p_t(d_t)\cdot(\gamma_t-\mu_{t-1,\gamma}(0,1-d_t])$ of empty sample paths will be moved to d_t due to the inclusion of realization d_t when defining $\tilde{X}_{t,\gamma}$. More specifically, the movement of sample paths due to the inclusion of each realization $d_t\in\mathcal{F}_t$ can be described as follows:
(i). For each $d_t\in\mathcal{F}_{t,1}$, obviously, $p_t(d_t)\cdot(\gamma_t-\mu_{t-1,\gamma}(0,1-d_t])$ measure of sample paths, which is upper bounded by $p_t(d_t)\cdot(\gamma_t-\mu_{t-1,\gamma}(0,1-b])$, will be moved from 0 to the range $(0,b]$, while a quantity $a_1(d_t)\le p_t(d_t)\cdot\mu_{t-1,\gamma}(0,b]$ measure of sample paths will be moved out of the range $(0,b]$. Moreover, at most $p_t(d_t)\cdot\mu_{t-1,\gamma}(0,b]$ measure of sample paths will be moved into the range $(b,1-b]$.
(ii). For each $d_t\in\mathcal{F}_{t,2}$, $p_t(d_t)\cdot\mu_{t-1,\gamma}(0,b]$ measure of sample paths will be moved out of the range $(0,b]$. Moreover, $p_t(d_t)\cdot(\gamma_t-\mu_{t-1,\gamma}(0,1-d_t])$ measure of sample paths, which is upper bounded by $p_t(d_t)\cdot(\gamma_t-\mu_{t-1,\gamma}(0,b])$, will be moved from 0 into the range $(b,1-b]$, while at most $p_t(d_t)\cdot\mu_{t-1,\gamma}(0,b]$ measure of sample paths will be moved from $(0,b]$ into $(b,1-b]$. Thus, the measure of new sample path that is moved into the range $(b,1-b]$ is upper bounded by $\gamma_t\cdot p_t(d_t)$.
(iii). For each $d_t\in\mathcal{F}_{t,3}$, then a quantity $a_3(d_t)\le p_t(d_t)\cdot\mu_{t-1,\gamma}(0,b]$ measure of sample paths is moved out of the range $(0,b]$, and at most $p_t(d_t)\cdot\mu_{t-1,\gamma}(0,b]$ measure of sample paths is moved into the range $(b,1-b]$.
(iv). For each $d_t\in\mathcal{F}_{t,4}$ or $d_t\in\mathcal{F}_{t,5}$, since $d_t>1-b$, obviously, no new sample path will be added to the range $(b,1-b]$ due to the inclusion of such realization d_t when defining $\tilde{X}_{t,\gamma}$, while the measure of the sample paths within the range $(0,b]$ can only become smaller.
To conclude, denoting

$$a_1=\sum_{d_t\in\mathcal{F}_{t,1}}a_1(d_t) \text{ and } \hat{p}_1=\sum_{d_t\in\mathcal{F}_{t,1}}p_t(d_t) \text{ and } \hat{p}_2=\sum_{d_t\in\mathcal{F}_{t,2}}p_t(d_t) \text{ and } a_3=\sum_{d_t\in\mathcal{F}_{t,3}}a_3(d_t) \text{ and } \hat{p}_3=\sum_{d_t\in\mathcal{F}_{t,3}}p_t(d_t)$$

we have that

$$\mu_{t,\gamma}(0,b] \leq \mu_{t-1,\gamma}(0,b] + (\gamma_t - \mu_{t-1,\gamma}(0,1-b]) \cdot \hat{p}_1 - a_1 - \mu_{t-1,\gamma}(0,b] \cdot \hat{p}_2 - a_3 \tag{EC.25}$$

and

$$\mu_{t,\gamma}(b,1-b] \leq \mu_{t-1,\gamma}(b,1-b] + a_1 + \gamma_t \cdot \hat{p}_2 + a_3 \tag{EC.26}$$

Moreover, it holds that $\hat{p}_1 + \hat{p}_2 + \hat{p}_3 \leq 1$. We now consider the following two cases separately.

Case 1: If $\hat{p}_1 > 0$, then we must have $\gamma_t \geq \mu_{t-1,\gamma}(0,1-b]$. Notice that $\hat{p}_1 \leq 1 - \hat{p}_2$, from (EC.25), we have

$$\begin{aligned}
\mu_{t,\gamma}(0,b] &\leq \mu_{t-1,\gamma}(0,b] + (\gamma_t - \mu_{t-1,\gamma}(0,1-b]) \cdot \hat{p}_1 - a_1 - \mu_{t-1,\gamma}(0,b] \cdot \hat{p}_2 - a_3 \\
&\leq \mu_{t-1,\gamma}(0,b] + (\gamma_t - \mu_{t-1,\gamma}(0,1-b]) \cdot (1-\hat{p}_2) - a_1 - \mu_{t-1,\gamma}(0,b] \cdot \hat{p}_2 - a_3 \\
&= (\gamma_t - \mu_{t-1,\gamma}(b,1-b]) \cdot (1-\hat{p}_2) - a_1 - a_3 \\
&\leq (\gamma_1 - \mu_{t-1,\gamma}(b,1-b]) \cdot (1-\hat{p}_2) - a_1 - a_3
\end{aligned} \tag{EC.27}$$

where the last inequality holds from $\gamma_1 \geq \gamma_t$. Moreover, from (EC.26), we have that

$$\begin{aligned}
\exp(-\frac{1}{\gamma_1} \cdot \mu_{t,\gamma}(b,1-b]) &\geq \exp(-\frac{1}{\gamma_1} \cdot \mu_{t-1,\gamma}(b,1-b] - \frac{\gamma_t \hat{p}_2}{\gamma_1}) \cdot \exp(-\frac{1}{\gamma_1} \cdot a_1 - \frac{1}{\gamma_1} \cdot a_3) \\
&\geq \exp(-\frac{1}{\gamma_1} \cdot \mu_{t-1,\gamma}(b,1-b] - \hat{p}_2) \cdot \exp(-\frac{1}{\gamma_1} \cdot a_1 - \frac{1}{\gamma_1} \cdot a_3) \\
&\geq \exp(-\frac{1}{\gamma_1} \cdot \mu_{t-1,\gamma}(b,1-b] - \hat{p}_2) \cdot (1 - \frac{1}{\gamma_1} \cdot a_1 - \frac{1}{\gamma_1} \cdot a_3) \\
&= \exp(-\frac{1}{\gamma_1} \cdot \mu_{t-1,\gamma}(b,1-b] - \hat{p}_2) \\
&\quad - \exp(-\frac{1}{\gamma_1} \cdot \mu_{t-1,\gamma}(b,1-b] - \hat{p}_2) \cdot (\frac{1}{\gamma_1} \cdot a_1 + \frac{1}{\gamma_1} \cdot a_3) \\
&\geq \exp(-\frac{1}{\gamma_1} \cdot \mu_{t-1,\gamma}(b,1-b] - \hat{p}_2) - \frac{1}{\gamma_1} \cdot a_1 - \frac{1}{\gamma_1} \cdot a_3
\end{aligned} \tag{EC.28}$$

where the second inequality holds from $\gamma_1 \geq \gamma_t$, the third inequality holds from $\exp(-x) \geq 1 - x$ for any $x \geq 0$ and the last inequality holds from $\exp(-x) \leq 1$ for any $x \geq 0$. Further note that

$$\exp(-\frac{1}{\gamma_1} \cdot \mu_{t-1,\gamma}(b,1-b] - \hat{p}_2) = \exp(-\frac{1}{\gamma_1} \cdot \mu_{t-1,\gamma}(b,1-b]) \cdot \exp(-\hat{p}_2) \geq (1 - \frac{1}{\gamma_1} \cdot \mu_{t-1,\gamma}(b,1-b]) \cdot (1-\hat{p}_2)$$

From (EC.27) and (EC.28), we have

$$\begin{aligned}
\exp(-\frac{1}{\gamma_1} \cdot \mu_{t,\gamma}(b,1-b]) &\geq (1 - \frac{1}{\gamma_1} \cdot \mu_{t-1,\gamma}(b,1-b]) \cdot (1-\hat{p}_2) - \frac{1}{\gamma_1} \cdot a_1 - \frac{1}{\gamma_1} \cdot a_3 \\
&\geq \frac{1}{\gamma_1} \cdot \mu_{t,\gamma}(0,b]
\end{aligned}$$

Case 2: If $\hat{p}_1 = 0$ which also implies $a_1 = 0$, then we have

$$\mu_{t,\gamma}(0,b] \leq \mu_{t-1,\gamma}(0,b] - \mu_{t-1,\gamma}(0,b] \cdot \hat{p}_2 - a_3 \tag{EC.29}$$

and

$$\mu_{t,\boldsymbol{\gamma}}(b,1-b] \le \mu_{t-1,\boldsymbol{\gamma}}(b,1-b] + \gamma_t \cdot \hat{p}_2 + a_3 \le \mu_{t-1,\boldsymbol{\gamma}}(b,1-b] + \gamma_1 \cdot \hat{p}_2 + a_3$$

Thus, it holds that

$$\begin{aligned}
\exp(-\frac{1}{\gamma_1}\cdot\mu_{t,\boldsymbol{\gamma}}(b,1-b]) &\ge \exp(-\frac{1}{\gamma_1}\cdot\mu_{t-1,\boldsymbol{\gamma}}(b,1-b]-\hat{p}_2)\cdot\exp(-\frac{1}{\gamma_1}\cdot a_3) \\
&\ge \exp(-\frac{1}{\gamma_1}\cdot\mu_{t-1,\boldsymbol{\gamma}}(b,1-b]-\hat{p}_2)\cdot(1-\frac{1}{\gamma_1}\cdot a_3) \\
&\ge \exp(-\frac{1}{\gamma_1}\cdot\mu_{t-1,\boldsymbol{\gamma}}(b,1-b]-\hat{p}_2)-\frac{1}{\gamma_1}\cdot a_3 \\
&\ge \exp(-\frac{1}{\gamma_1}\cdot\mu_{t-1,\boldsymbol{\gamma}}(b,1-b])\cdot(1-\hat{p}_2)-\frac{1}{\gamma_1}\cdot a_3 \\
&\ge \frac{1}{\gamma_1}\cdot\mu_{t-1,\boldsymbol{\gamma}}(0,b]\cdot(1-\hat{p}_2)-\frac{1}{\gamma_1}\cdot a_3
\end{aligned} \tag{EC.30}$$

where the third inequality holds from $\exp(-a)\le 1$ for any $a\ge 0$ and the last inequality holds from induction hypothesis. Our proof is completed immediately by combining (EC.29) and (EC.30). □

EC.4.2. Proof of Theorem 5

Proof: For each fixed t, we define $U_t(s)=\mu_{t,\boldsymbol{\gamma}}(0,s]=P(0<\tilde{X}_{t,\boldsymbol{\gamma}}\le s)$ for any $s\in(0,1]$. Note that by Algorithm 3, we have $\mathbb{E}[\tilde{X}_{t,\gamma}]=\sum_{\tau=1}^{t}\gamma_\tau\cdot\psi_\tau$. From integration by parts, we have that

$$\sum_{\tau=1}^{t}\gamma_\tau\cdot\psi_\tau=\mathbb{E}[\tilde{X}_{t,\gamma}]=\int_{s=0}^{1} s dU_t(s)=U_t(1)-\int_{s=0}^{1}U_t(s)ds \tag{EC.31}$$

We then bound the term $\int_{s=0}^{1}U_t(s)ds$. If $U_t(1)\le\gamma_1$, then we immediately have

$$P(\tilde{X}_{t,\boldsymbol{\gamma}}=0)\ge 1-\gamma_1\ge 1-\gamma_1-\sum_{\tau=1}^{t}\gamma_\tau\cdot\psi_\tau$$

which proves (14). Thus, in the remaining part of the proof, it is enough for us to only focus on the case $U_t(1)>\gamma_1$.

If $U_t(1)>\gamma_1$, then there must exists a constant $u^*\in(0,1)$ such that

$$\gamma_1\cdot u^*-\gamma_1\cdot\ln(u^*)=U_t(1).$$

We further define

$$s^*=\begin{cases}\min\{s\in(0,1/2]:U_t(s)\ge\gamma_1\cdot u^*\}, & \text{if } U_t(\frac{1}{2})\ge\gamma_1\cdot u^* \\ \frac{1}{2}, & \text{if } U_t(\frac{1}{2})<\gamma_1\cdot u^*\end{cases}$$

Following the proof of Theorem 3, we can show that

$$\int_{s=0}^{1}U_t(s)ds\le s^*\cdot(2\gamma_1\cdot u^*-\gamma_1\cdot\ln(u^*))+(1/2-s^*)\cdot\max\{2\gamma_1\cdot u^*-\gamma_1\cdot\ln(u^*),2\gamma_1\}$$

We further simplify the above expression separately by comparing the value of $2\gamma_1 \cdot u^* - \gamma_1 \cdot \ln(u^*)$ and $2\gamma_1$.

Case 1: If $2\gamma_1 \cdot u^* - \gamma_1 \cdot \ln(u^*) \le 2\gamma_1$, we have $\int_{s=0}^{1} U_t(s)ds \le 2s^*\gamma_1 + \gamma_1 - 2s^*\gamma_1 = \gamma_1$. From (EC.31), we have that

$$U_t(1) \le \gamma_1 + \sum_{\tau=1}^{t} \gamma_\tau \cdot \psi_\tau$$

Case 2: If $2\gamma_1 \cdot u^* - \gamma_1 \cdot \ln(u^*) > 2\gamma_1$, we have $\int_{s=0}^{1} U_t(s)ds \le \gamma_1 \cdot u^* - \frac{\gamma_1}{2} \cdot \ln(u^*)$. From (EC.31) and the definition of u^*, we have that

$$U_t(1) = \gamma_1 \cdot u^* - \gamma_1 \cdot \ln(u^*) \le \sum_{\tau=1}^{t} \gamma_\tau \cdot \psi_\tau + \gamma_1 \cdot u^* - \frac{\gamma_1}{2} \cdot \ln(u^*)$$

The above inequality implies that

$$u^* \ge \exp(-\frac{2}{\gamma_1} \cdot \sum_{\tau=1}^{t} \gamma_\tau \cdot \psi_\tau)$$

Note that the function $x - \ln(x)$ is non-increasing on $(0,1)$, hence we have

$$U_t(1) \le 2 \cdot \sum_{\tau=1}^{t} \gamma_\tau \cdot \psi_\tau + \gamma_1 \cdot \exp(-\frac{2}{\gamma_1} \cdot \sum_{\tau=1}^{t} \gamma_\tau \cdot \psi_\tau)$$

Combing the above two cases, we conclude that

$$U_t(1) \le \max\{\gamma_1 + \sum_{\tau=1}^{t} \gamma_\tau \cdot \psi_\tau, \quad 2 \cdot \sum_{\tau=1}^{t} \gamma_\tau \cdot \psi_\tau + \gamma_1 \cdot \exp(-\frac{2}{\gamma_1} \cdot \sum_{\tau=1}^{t} \gamma_\tau \cdot \psi_\tau)\}$$

Note that $P(\tilde{X}_{t,\boldsymbol{\gamma}} = 0) = 1 - U_t(1)$, we conclude that

$$P(\tilde{X}_{t,\boldsymbol{\gamma}} = 0) \ge \min\{1 - \gamma_1 - \sum_{\tau=1}^{t} \gamma_\tau \cdot \psi_\tau, \;\; 1 - 2 \cdot \sum_{\tau=1}^{t} \gamma_\tau \cdot \psi_\tau - \gamma_1 \cdot \exp(-\frac{2}{\gamma_1} \cdot \sum_{\tau=1}^{t} \gamma_\tau \cdot \psi_\tau)\}$$

which completes our proof. □

EC.4.3. Proof of Lemma 5

Proof: Since the function $h_{\gamma_0}(\cdot)$ is non-increasing and non-negative over $[0,1]$, it is direct to see that

$$1 \ge \hat{\gamma}_1 \ge \cdots \ge \hat{\gamma}_T \ge 0$$

Note that for each $t = 1, \ldots, T$, we have

$$\int_{\tau=0}^{k_t} h_{\gamma_0}(\tau) d\tau = \sum_{\tau=1}^{t} \hat{\gamma}_\tau \cdot \psi_\tau$$

and $\gamma_0 \geq \hat{\gamma}_1$. Then, for each $t = 1, \ldots, T-1$ and each $\tau \in [k_t, k_{t+1}]$, it holds that

$$h_{\gamma_0}(\tau) \leq 1 - \gamma_0 - \int_{\tau'=0}^{\tau} h_{\gamma_0}(\tau')d\tau' \leq 1 - \gamma_0 - \int_{\tau'=0}^{k_t} h_{\gamma_0}(\tau')d\tau' \leq 1 - \hat{\gamma}_1 - \sum_{\tau'=1}^{t} \hat{\gamma}_{\tau'} \cdot \psi_{\tau'}$$

which implies that

$$\hat{\gamma}_{t+1} \leq 1 - \hat{\gamma}_1 - \sum_{\tau'=1}^{t} \hat{\gamma}_{\tau'} \cdot \psi_{\tau'}$$

since $\hat{\gamma}_{t+1}$ is defined as the average of function $h_{\gamma_0}(\cdot)$ over $[k_t, k_{t+1}]$ in (18).

Similarly, note that the function $2x + \gamma_0 \cdot \exp(-\frac{2}{\gamma_0} \cdot x)$ is monotone increasing when $x \geq 0$. Then, for each $t = 1, \ldots, T-1$ and each $\tau \in [k_t, k_{t+1}]$, we have

$$\begin{aligned} h_{\gamma_0}(\tau) &\leq 1 - 2 \cdot \int_{\tau'=0}^{\tau} h_{\gamma_0}(\tau')d\tau' - \gamma_0 \cdot \exp(-\frac{2}{\gamma_0} \cdot \int_{\tau'=0}^{\tau} h_{\gamma_0}(\tau')d\tau') \\ &\leq 1 - 2 \cdot \int_{\tau'=0}^{k_t} h_{\gamma_0}(\tau')d\tau' - \gamma_0 \cdot \exp(-\frac{2}{\gamma_0} \cdot \int_{\tau'=0}^{k_t} h_{\gamma_0}(\tau')d\tau') \\ &= 1 - 2 \cdot \sum_{\tau'=1}^{t} \hat{\gamma}_{\tau'} \cdot \psi_{\tau'} - \gamma_0 \cdot \exp(-\frac{2}{\gamma_0} \cdot \sum_{\tau'=1}^{t} \hat{\gamma}_{\tau'} \cdot \psi_{\tau'}) \end{aligned}$$

which implies that

$$\hat{\gamma}_{t+1} \leq 1 - 2 \cdot \sum_{\tau'=1}^{t} \hat{\gamma}_{\tau'} \cdot \psi_{\tau'} - \gamma_0 \cdot \exp(-\frac{2}{\gamma_0} \cdot \sum_{\tau'=1}^{t} \hat{\gamma}_{\tau'} \cdot \psi_{\tau'})$$

since $\hat{\gamma}_{t+1}$ is defined as the average of function $h_{\gamma_0}(\cdot)$ over $[k_t, k_{t+1}]$ in (18). Thus, we conclude that $\{\hat{\gamma}_t\}_{t=1}^{T}$ is a feasible solution to OP($\boldsymbol{\psi}$). $\square$

EC.4.4. Proof of Proposition 3

It is enough for us to consider a problem setup $\mathcal{H}$ with T queries, where each query has a deterministic size $\frac{1}{2} + \frac{1}{T}$ and is active with probability $\frac{2}{T}$. It is clear that $\mathbf{UP}(\mathcal{H}) = 1$. However, any online algorithm π can serve at most one query, given at least one query has arrived. Then, the expected capacity utilization of any online algorithm π is upper bound by

$$(\frac{1}{2} + \frac{1}{T}) \cdot (1 - (1 - \frac{2}{T})^T) = \frac{1 - e^{-2}}{2} + O(\frac{1}{T})$$

This implies an upper bound $\frac{1-e^{-2}}{2}$ as $T \to \infty$.