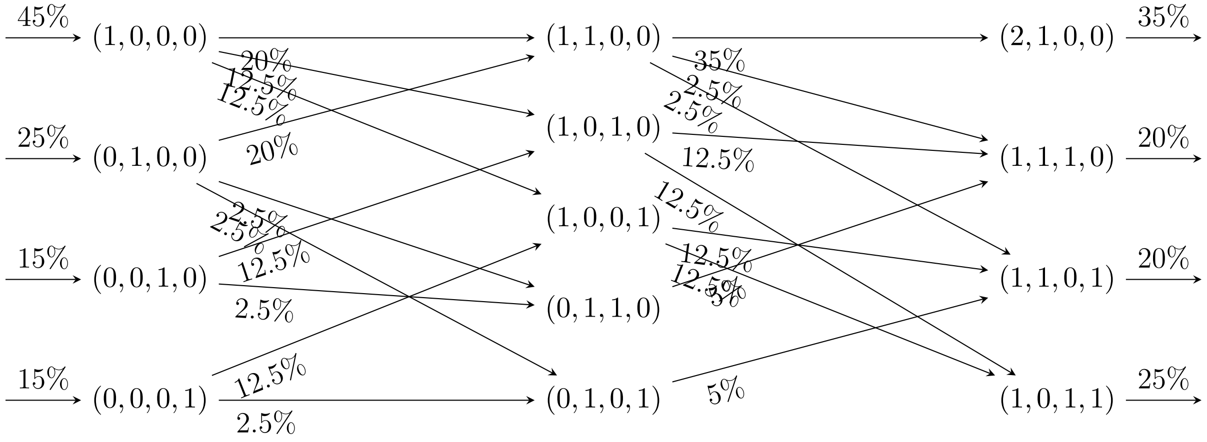
Appendix. Proofs of Statements

A. Pitfalls in the Development of House Monotone Methods

In Section 4.1, we claimed that for the population profile $\vec{p} = (45, 25, 15, 15)$ and house size $h = 3$ — thus, the standard quotas $(1.35, 0.75, 0.45, 0.45)$ — the following distribution over apportionments can be part of an apportionment method in which house monotonicity, quota, and ex ante proportionality hold across the inputs $\{(\vec{p}, h') \mid h' \leq 3\}$:

$$\vec{a} = \begin{cases} (2,1,0,0) \text{ with probability } 35\%, & (1,1,0,1) \text{ with probability } 20\%, \text{ and} \\ (1,1,1,0) \text{ with probability } 20\%, & (1,0,1,1) \text{ with probability } 25\%. \end{cases}$$

To obtain such an apportionment method, consider the following capacitated flow network:



One easily verifies that it is possible to send a total flow of 1 through this network, which necessarily uses all edges at their capacity. Consider any decomposition of this flow into paths. Our method will be defined as (1) choosing one of these paths $\vec{a}_1 \to \vec{a}_2 \to \vec{a}_3$ with probability proportional to its amount of flow and (2) returning a solution f such that $f(\vec{p}, j) = \vec{a}_j$ for $j = 1, 2, 3$, and some canonical apportionment for all other inputs.

Across the inputs $\{(\vec{p}, h') \mid h' \leq 3\}$, this method does not violate house monotonicity since edges in the flow network are such that no agent's seat number decreases along an edge. Since one verifies that all apportionments labeling the nodes of the flow network satisfy quota, the method satisfies quota on $\{(\vec{p}, h') \mid h' \leq 3\}$. On the same set of inputs, the method satisfies ex-ante proportionality, which follows from the fact that, for $h' = 1, 2, 3$, weighting the apportionments of the h'-th layer of the flow network by their internal flow, we obtain the vector of standard quotas for $\vec{p}$ and house size h'. The egress edges also ensure that, indeed, the method's distribution over apportionments on $\vec{p}, h$ is as given above.

Finally, we must prove that none of the apportionments in the last layer of the flow network are toxic. For this, observe that the quota solution by Balinski and Young (1975) (which is house monotone and satisfies quota), for one way of breaking ties in the definition, produces the following apportionments on $\vec{p}$: $(1,0,0,0) \to (1,1,0,0) \to (2,1,0,0) \to (2,1,1,0) \to (2,1,1,1), \ldots$. Since $(2,1,0,0)$ coincides with one of these values, it can be extended by the suffix of the quota solution and is therefore not toxic. Furthermore, we can extend $(1,1,1,0) \to (2,1,1,0)$, and then continue as the quota solution; $(1,1,0,1) \to (2,1,0,1) \to (2,1,1,1)$, and then as the quota solution; and $(1,0,1,1) \to (1,1,1,1) \to (2,1,1,1)$, and then as the quota solution. The claim follows by verifying that these extensions do not violate quota until merging with the quota solution.

B. Deferred Proofs for House Monotone Apportionment

Lemma 1. An infinite seat sequence α satisfies quota iff it is the concatenation of infinitely many finite seat sequences $\beta^1, \beta^2, \beta^3, \dots$ of length p each satisfying quota, i.e.,

$$\alpha = \beta^1_1, \beta^1_2, \dots, \beta^1_p, \beta^2_1, \beta^2_2, \dots, \beta^2_p, \beta^3_1, \dots.$$

Proof. "$\Rightarrow$": Fix α and some $k \in \mathbb{Z}_{\geq 0}$. We must show that the finite seat sequence $\beta^{k+1} := \alpha_{kp+1}, \alpha_{kp+2} \dots, \alpha_{(k+1)p}$ satisfies quota. Indeed, for any $1 \leq r \leq p$, the number of seats allocated by β^{k+1} to state i at house size r is

$$\begin{aligned} \left|\{1 \leq h' \leq r \mid \beta^{k+1}_{h'} = i\}\right| &= |\{1 \leq h' \leq r \mid \alpha_{kp+h'} = i\}| \\ &= a_i(kp+r) - a_i(kp) \\ &\in \{\lfloor (kp+r)\, p_i/p \rfloor, \lceil (kp+r)\, p_i/p \rceil\} - k\, p_i \qquad \text{(since } \alpha \text{ satisfies quota)} \\ &= \{\lfloor r\, p_i/p \rfloor + k\, p_i, \lceil r\, p_i/p \rceil + k\, p_i\} - k\, p_i \\ &= \{\lfloor r\, p_i/p \rfloor, \lceil r\, p_i/p \rceil\}, \end{aligned}$$

which shows that α can be decomposed into finite seat sequences $\{\beta^k\}_{k\in\mathbb{N}}$ satisfying quota.

"$\Leftarrow$": Fix some h and choose $k := \lfloor (h-1)/p \rfloor + 1$ and $r := ((h-1) \bmod p) + 1$ such that $h = (k-1)\, p + r$, $k \geq 1$ and $1 \leq r \leq p$. We will show that α's allocation $a(h)$ on h satisfies quota. Denoting β^k's allocation for a house size h' by $b^k(h')$, it holds for all states i that $a_i(h) = \sum_{k'=1}^{k-1} b^{k'}(p) + b^k(r)$. By quota, $b^{k'}(p) = p_i$ for all k, and $b^k(r) \in \{\lfloor r\, p_i/p \rfloor, \lceil r\, p_i/p \rceil\}$. Thus, $a_i(h) \in \{(k-1)\, p_i + \lfloor r\, p_i/p \rfloor, (k-1)\, p_i + \lceil r\, p_i/p \rceil\}$. The conclusion follows since $h\, p_i/p = ((k-1)\, p + r)\, p_i/p = (k-1)\, p_i + r\, p_i/p$. ☐

Lemma 5. For any population profile $\vec{p}$, there is a probability distribution $\mathcal{D}$ over finite seat sequences such that one can sample a finite seat sequence $\alpha \sim \mathcal{D}$ in $\mathcal{O}(p^2\, n^2)$ randomized time, such that all finite seat sequences in the support of $\mathcal{D}$ satisfy quota, and such that, for all states i and $1 \leq h \leq p$,

$$\mathbb{P}[\alpha_h = i] = p_i/p.$$

Proof. As sketched in Section 4.2, we define $\mathcal{D}$ by invoking Theorem 4 on a star graph with $A = \{a\}, B = \{b_i \mid i \in N\}$, and $E = \big\{\{a, b_i\} \mid i \in N\big\}$. We set $T := p$, and, for each $1 \leq t \leq T$ and state i, set $w^t_{\{a,b_i\}} := p_i/p$.

Theorem 4 now defines a joint distribution over variables X^t_e satisfying marginal distribution, degree preservation, and cumulative degree preservation (as well as negative correlation, which we will not use). We will describe how each joint realization of the X^t_e can be mapped to a finite seat sequence and that the distribution $\mathcal{D}$ that arises from applying this mapping to the dependent-rounding distribution has the properties claim in the statement. The running time follows from the running time of applying dependent rounding, and the fact that the transformation for translating the outcome into a finite seat sequence requires only $\mathcal{O}(p\, n)$ time.

For a given joint realization of the X^t_e, let α be the finite seat sequence that maps each $h \in \{1, \dots, p\}$ to the state i such that $X^h_{\{a,b_i\}} = 1$. This definition presupposes that there is exactly one such i for each h, which follows from the degree-preservation guarantee for vertex a at time step h and from the fact that $d^h_a = \sum_{i\in N} p_i/p = 1$. This seat sequence α furthermore satisfies quota, which directly follows from cumulative degree preservation and from $\sum_{t'=1}^{h} d^{t'}_{b_i} = h\, p_i/p$. It only remains to show that α_h has value i with a probability of $p_i/p = w^h_{\{a,b_i\}}$ for all $i \in N$ and $1 \leq h \leq p$, but this immediately follows from the marginal distribution guarantee of Theorem 4. ☐

Theorem 5. There exists an apportionment method F that satisfies house monotonicity, quota, and ex ante proportionality.

Proof. For each population profile $\vec{p}$, Lemma 5 provides a probability distribution over finite seat sequences for that population profile. We define the outcomes such that ω contains, for each $\vec{p}$, a separate, independent $\alpha^{\vec{p}}$ following the distribution from Lemma 5.

From now on, we fix an $\omega \in \Omega$, which determines the values of all $\alpha^{\vec{p}}$. For this ω, we must construct an apportionment solution $f = F^{\omega}$. For a given input $\vec{p}, h$, let α be the concatenation of infinitely many copies of $\alpha^{\vec{p}}$ as in Lemma 1. Then, we define $f(\vec{p}, h)$ as the apportionment giving $a_i(h) = |\{1 \leq h' \leq h \mid \alpha(h') = i\}|$ seats to each state i.

By Lemma 5, $\alpha^{\vec{p}}$ satisfies quota, and, thus, α satisfies quota by Lemma 1, from which it follows immediately that f satisfies quota. Since f was constructed from a seat sequence allocating one seat at a time, it clearly satisfies house monotonicity.

It remains to argue that F satisfies ex ante proportionality. Fix any $\vec{p}$ and h. By construction, $F_i(\vec{p}, h) - F_i(\vec{p}, h-1) = \mathbb{1}\{\alpha^{\vec{p}}_{1+(h-1 \bmod p)} = i\}$, setting $F_i(\vec{p}, 0) = 0$. Hence,

$$\mathbb{E}[F_i(\vec{p}, h) - F_i(\vec{p}, h-1)] = \mathbb{P}[\alpha^{\vec{p}}_{1+(h-1 \bmod p)} = i] = p_i/p,$$

where the last equality follows from Lemma 5. By linearity of expectation, it follows that,

$$\mathbb{E}[F_i(\vec{p}, h)] = \sum_{h'=1}^{h} \mathbb{E}[F_i(\vec{p}, h) - F_i(\vec{p}, h-1)] = h\, p_i/p,$$

which shows ex ante proportionality. $\square$

Theorem 6. *For any population profile $\vec{p}$, we can construct a bipartite graph whose perfect matchings are in one-to-one correspondence with the finite seat sequences satisfying quota.*

Proof. We will first show a variant of the theorem, in which the finite seat sequences correspond not to perfect matchings but to perfect b-matchings, i.e., where each node is labeled with a target degree in $\mathbb{Z}_{\geq 0}$, and where a subset of edges is a perfect b-matching when each node has its target degree in the induced subgraph. We will then show how to modify the graph to obtain the claimed result for perfect matchings.

Then, the bipartite graph is the one to which we applied cumulative rounding in Lemma 5 (without the weights), with the following (technically necessary) modifications:

1. We set each node's target degree to its fractional degree as in Lemma 4. This is possible since all nodes have integer weight, including the nodes $v^{T:T+1}$ which have weight 1 given that $\sum_{t'=1}^{T} d_v^{t'}$ is an integer for all nodes v in the underlying graph for the chosen $T = p$.
2. Then, we delete all edges with zero weight (to ensure that they are never part of the b-matching).
3. Finally, for each edge with weight 1, we delete the edge and decrement the target degree of both adjacent nodes (simulating the constraint that these edges must be present in any b-matching).

The proof of Lemma 5 indicated a way to map perfect b-matchings to finite seat sequences, and we have to show that this mapping is a bijection, i.e., that it is injective and surjective.

To show that the mapping is injective, observe that two perfect b-matchings that differ in whether a certain edge $\{v^t, (v')^t\}$ is included lead to different seat sequences. Furthermore, the characterization of the edges in Fig. 3 (whose correctness follows from the proof of Theorem 4 and does not rely on properties of the result of dependent rounding other than those required by our b-matchings) implies that the set of edges of shape $\{v^t, (v')^t\}$ in the matching uniquely determines which of the other edges are included in the perfect b-matching, which means that there are never multiple perfect b-matchings that would be mapped to the same finite seat sequence.

It is more involved to show that the mapping is surjective. For a given finite seat sequence α, we will construct a perfect b-matching which is mapped to α. Clearly, each edge $\{a^t, (b_i)^t\}$ is included in the matching iff $\alpha_t = i$ (none of these edges have weight zero or one since $n \geq 2$ and each state has a positive population). We label all other edges according to the edges' events described in Fig. 3. One verifies that, by quota, this step would not have taken any edges with zero weight in the cumulative-rounding graph and would have taken all edges with weight one in the cumulative-rounding graph,

value	(a)?	(b)?	(c)?	Impossibility proof
2	✓ $a_i(t)=a_i(t-1)+1$	✓ $a_i(t-1)=\lfloor q_i(t-1)\rfloor+1$	✓ $a_i(t)=\lfloor q_i(t)\rfloor$	$a_i(t-1)=\lfloor q_i(t-1)\rfloor+1=\lfloor q_i(t)\rfloor=a_i(t)=a_i(t-1)+1$
2	$a_i(t)=a_i(t-1)$	$a_i(t-1)=\lfloor q_i(t-1)\rfloor$	?	$\lfloor q_i(t)\rfloor=\lfloor q_i(t-1)\rfloor+1=a_i(t-1)+1=a_i(t)+1>a_i(t)$, violates quota
2	✓ $a_i(t)=a_i(t-1)+1$	$a_i(t-1)=\lfloor q_i(t-1)\rfloor$	$a_i(t)=\lfloor q_i(t)\rfloor+1$	$a_i(t-1)=\lfloor q_i(t-1)\rfloor=\lfloor q_i(t)\rfloor-1=a_i(t)-2=a_i(t-1)-1$
2	$a_i(t)=a_i(t-1)$	✓ $a_i(t-1)=\lfloor q_i(t-1)\rfloor+1$	$a_i(t)=\lfloor q_i(t)\rfloor+1$	$a_i(t-1)=\lfloor q_i(t-1)\rfloor+1=\lfloor q_i(t)\rfloor=a_i(t)-1=a_i(t-1)-1$
1	$a_i(t)=a_i(t-1)$	$a_i(t-1)=\lfloor q_i(t-1)\rfloor$	$a_i(t)=\lfloor q_i(t)\rfloor+1$	$a_i(t-1)=\lfloor q_i(t-1)\rfloor=\lfloor q_i(t)\rfloor=a_i(t)-1=a_i(t-1)-1$
1	✓ $a_i(t)=a_i(t-1)+1$	?	✓ $a_i(t)=\lfloor q_i(t)\rfloor$	$a_i(t-1)=a_i(t)-1=\lfloor q_i(t)\rfloor-1=\lfloor q_i(t-1)\rfloor-1$, violates quota
1	✓ $a_i(t)=a_i(t-1)+1$	✓ $a_i(t-1)=\lfloor q_i(t-1)\rfloor+1$	$a_i(t)=\lfloor q_i(t)\rfloor+1$	$a_i(t-1)=\lfloor q_i(t-1)\rfloor+1=\lfloor q_i(t)\rfloor+1=a_i(t)=a_i(t-1)+1$
1	$a_i(t)=a_i(t-1)$	✓ $a_i(t-1)=\lfloor q_i(t-1)\rfloor+1$	✓ $a_i(t)=\lfloor q_i(t)\rfloor$	$a_i(t-1)=\lfloor q_i(t-1)\rfloor+1=\lfloor q_i(t)\rfloor+1=a_i(t)+1=a_i(t-1)+1$

Table 2 Case distinction on the value of $\lfloor q_i(t)\rfloor-\lfloor q_i(t-1)\rfloor+1$ for proving that nodes of the shape $\overline{b}_i^t$ have the target degree in the proof of Theorem 6.

which allows us to pretend for ease of exposition that we are producing a b-matching on the labeled graph before the preprocessing steps (2) and (3). One easily verifies that the resulting edge set gives the target degree to all nodes of shape v^t, $\overline{\overline{v}}^t$, and $v^{t:t+1}$ (including the special cases $v^{0:1}$ and $v^{T:T+1}$).

It only remains to show that the nodes $\overline{v}^t$ have their target degree, $\left\lfloor \sum_{t'=1}^{t} d_v^{t'} \right\rfloor - \left\lfloor \sum_{t'=1}^{t-1} d_v^{t'} \right\rfloor - \lfloor d_v^t \rfloor + 1$. For nodes $v=a^t$, it holds that $d_v^{t'}=D_v^{t'}=1$ for all t', which means that the target degree is one and indeed only one adjacent edge, namely, $\{\overline{v}^t, v^{t:t+1}\}$, is taken.

We will now consider the case of a node $v=b_i^t$. Observe that $d_v^{t'}=p_i/\sum_{j\in N} p_j<1$ for all t', which means that $\sum_{t''=1}^{t'} d_v^{t''}=t'\,p_i/\sum_{j\in N} p_j$, which is just i's standard quota for house size t', which we will write as $q_i(t')$. Furthermore, note that $\sum_{t''=1}^{t'} D_v^{t'}=a_i(t')$. With this, the target degree of v is just $\lfloor q_i(t)\rfloor-\lfloor q_i(t-1)\rfloor+1$, and the three edges incident to v are selected if

(a) $a_i(t)=a_i(t-1)+1$ (rather than $a_i(t)=a_i(t-1)$),
(b) $a_i(t-1)=\lfloor q_i(t-1)\rfloor+1$ (rather than $a_i(t-1)=\lfloor q_i(t-1)\rfloor$), and
(c) $a_i(t)=\lfloor q_i(t)\rfloor$ (rather than $a_i(t)=\lfloor q_i(t)\rfloor$),

respectively, where the values in parentheses are the only alternatives to the properties, by house monotonicity and quota. That is, we want to show that, for our house monotone and quota α, exactly $\lfloor q_i(t)\rfloor-\lfloor q_i(t-1)\rfloor+1\in\{1,2\}$ many out of the statements (a), (b), and (c), are true. In Table 2, we rule out all other cases via a case distinction, which shows that we indeed produced a perfect b-matching, and that the mapping is surjective.

The above establishes the one-to-one correspondence to the vertices on the polytope of perfect fractional b-matchings. Though this polytope is very nicely behaved already, we prefer to state the theorem for a classical perfect matching polytope, which is more widely known. Thus, we will adapt the bipartite graph above such that all nodes have target degree one, while keeping the perfect b-matchings in one-to-one correspondence. First, we remove all nodes with target degrees zero from the graph, which clearly does not change the set of perfect b-matchings. Looking at Lemma 4, only two kinds of nodes can have a target degree larger than one: nodes a^t and some nodes $\overline{b}_i^t$. In fact, the nodes a^t are no problem: While they have degree 2 in the cumulative-rounding construction, one of their adjacent edges, $\{a^t,\overline{\overline{a}}^t\}$ had weight 1, and thus the target degree of a^t was already lowered to one in step (3) of the preprocessing.

Thus, once more, the only issue are nodes of the form $\overline{b}_i^t$, specifically, when their target degree is 2 (it is never higher, as discussed above). If such a node only has two adjacent edges remaining in the graph these edges must be taken in any perfect b-matching, so we can eliminate $\overline{b}_i^t$ and its neighbors. Thus, say that the node still has all three adjacent edges, $\{\overline{b}_i^t, \overline{\overline{b}}_i^t\}$, $\{\overline{b}_i^t, b_i^{t-1:t}\}$, and $\{\overline{b}_i^t, b_i^{t:t+1}\}$. In this case, replace node $\overline{b}_i^t$ by two fresh nodes n_1 and n_2, both with target degree 1, and connect these nodes using four edges $(n_1, \overline{\overline{b}}_i^t), (n_1, b_i^{t-1:t}), (n_2, b_i^{t-1:t}), (n_2, b_i^{t:t+1})$. One verifies that, that in any perfect b-matching on the graph before replacement, one can replace the two edges incident to $\overline{b}_i^t$ by exactly one subset of the new edges to obtain a perfect b-matching on the new graph, and that the analogous step in the other direction also works in one unique way. Thus, after making these replacements, the finite seat sequences satisfying quota correspond one-to-one to the perfect matchings of the graph, which are the corner points of the polytope of fractional perfect matchings by the Birkhoff–von Neumann Theorem. ☐

C. Deferred Proofs for Cumulative Rounding

Lemma 2. The graph of Construction 7 is bipartite.

Proof. Note that the set of nodes

$$\{a^t \mid a \in A, 1 \le t \le T\} \cup \{\overline{a}^t \mid a \in A, 1 \le t \le T\} \cup \{\overline{\overline{b}}^t \mid b \in B, 1 \le t \le T\} \cup \{b^{t:t+1} \mid b \in B, 0 \le t \le T\}$$

has no internal edges, and neither does the complement of this set. ☐

Lemma 3. All edge weights lie between 0 and 1.

Proof. If the edge has the shape $\{v^t, (v')^t\}$ for some $v, v' \in A \cup B$, then the edge weight is one of the w_e^t, which are in $[0,1]$ by assumption. All other edge weights either have the shape $x - \lfloor x \rfloor$ or the shape $1 - x + \lfloor x \rfloor = 1 - (x - \lfloor x \rfloor)$ for some $x \in \mathbb{R}$. The claim follows since $x - 1 < \lfloor x \rfloor \le x$. ☐

Lemma 4. Let $1 \le t \le T$. The node v^t has fractional degree $\lfloor d_v^t \rfloor + 1$, the node $\overline{v}^t$ has fractional degree $\lfloor \sum_{t'=1}^{t} d_v^{t'} \rfloor - \lfloor \sum_{t'=1}^{t-1} d_v^{t'} \rfloor - \lfloor d_v^t \rfloor + 1$, and $\overline{\overline{v}}^t$ has fractional degree 1.

Further, the node $v^{0:1}$ has fractional degree 0, and the nodes $v^{1:2}, \ldots, v^{t-1:t}$ have fractional degree 1.

Proof. Within this proof, denote by *frac*$(\cdot)$ the fractional degree of a vertex in the constructed graph.

$$\begin{aligned}
\mathit{frac}(v^t) &= \underbrace{\textstyle\sum_{v \in e \in E} w_e^t}_{=d_v^t} + \left(1 - d_v^t + \lfloor d_v^t \rfloor\right) = \lfloor d_v^t \rfloor + 1 \\
\mathit{frac}(\overline{v}^t) &= \left(d_v^t - \lfloor d_v^t \rfloor\right) + \Big(\textstyle\sum_{t'=1}^{t-1} d_v^{t'} - \lfloor \sum_{t'=1}^{t-1} d_v^{t'} \rfloor\Big) + \Big(1 - \sum_{t'=1}^{t} d_v^{t'} + \lfloor \sum_{t'=1}^{t} d_v^{t'} \rfloor\Big) \\
&= \lfloor \textstyle\sum_{t'=1}^{t} d_v^{t'} \rfloor - \lfloor d_v^t \rfloor - \lfloor \sum_{t'=1}^{t-1} d_v^{t'} \rfloor + 1 \\
\mathit{frac}(\overline{\overline{v}}^t) &= \left(1 - d_v^t + \lfloor d_v^t \rfloor\right) + \left(d_v^t - \lfloor d_v^t \rfloor\right) = 1 \\
\mathit{frac}(v^{t:t+1}) &= \Big(1 - \textstyle\sum_{t'=1}^{t} d_v^{t'} + \lfloor \sum_{t'=1}^{t} d_v^{t'} \rfloor\Big) + \Big(\sum_{t'=1}^{t} d_v^{t'} - \lfloor \sum_{t'=1}^{t} d_v^{t'} \rfloor\Big) = 1 \quad \text{(if } 1 \le t \le T-1\text{)} \\
\mathit{frac}(v^{0:1}) &= \textstyle\sum_{t'=1}^{0} d_v^{t'} - \lfloor \sum_{t'=1}^{0} d_v^{t'} \rfloor = 0 - \lfloor 0 \rfloor = 0 \quad \square
\end{aligned}$$