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EC.1. Missing proofs in Section 4.2

In this section, we first provide additional intuition on the AA. Then, we prove Lemmas 1, 2, 3, 4, 5 and Claim 3 in Section 4.2, and the loss factor of $(1-O(M/C))$ in the CR.

EC.1.1. Additional Intuition on AA

In this section, we further illustrate with an example how the AA assigns the dual variables, and discuss some intuition on the design of the AA using the example.

EXAMPLE EC.1. Consider a two-class setting with $C=2$, $D^{(1)}=D^{(2)}=10$, $r^{(1)}D^{(1)}=l$, $r^{(2)}D^{(2)}=h$, where l/h is small, e.g., 0.001. For this setting, the optimal booking limits $C^{(0)*}$ and $C^{(1)*}$ as in (5) are close to 1, i.e., $2/(2-0.001)$ and $(1-0.001)2/(2-0.001)$ respectively. For the sake of exposition, we treat both booking limits as 1. Consider an instance I with $T=20$, $(r_t,D_t)=(r^{(1)},D^{(1)})$ for all $t\in[20]\backslash\{4\}$, and $(r_4,D_4)=(r^{(2)},D^{(2)})$.

The primal constraints and the dual constraints (11) without $\bar{z}_t$ for this example are shown in Figures EC.1 and EC.2 respectively. To elaborate, each row in Figure EC.1 is a primal constraint corresponding to (2), and each column in Figure EC.2 is a dual constraint corresponding to (11) but without $\bar{z}_t$. We omit $\bar{z}_t$ here since we aim to show that the inequality (15) holds. Indeed, if property (15) holds, $\bar{z}_t$ will not affect the CR since it is the revenue surplus which comes together with the primal increment (revenue). In addition, we show dual constraints vertically as columns, since it is visually convenient to keep the shape of the constraints consistent in both the primal and dual.

Despite the same shape of the constraints, it is still a formidable task to analyze these constraints. To further visually simplify the dual constraints, we use a vertical line to represent each dual constraint (without $\bar{z}_t$ at each t) in Figure EC.3, and we show the rejection price R_t at the bottom of each line.

The assignment of the dual variables can now be seen clearly. With such an assignment, for every line (i.e. dual constraint), the sum of $\bar{y}_\tau$ along the line is at least the corresponding rejection price

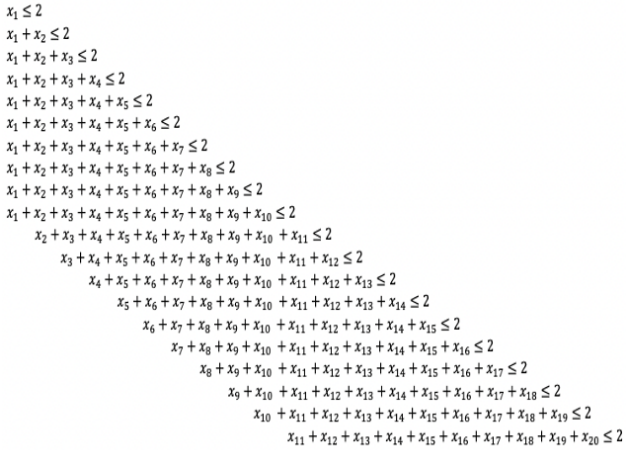


Figure EC.1 Primal constraints of Example EC.1

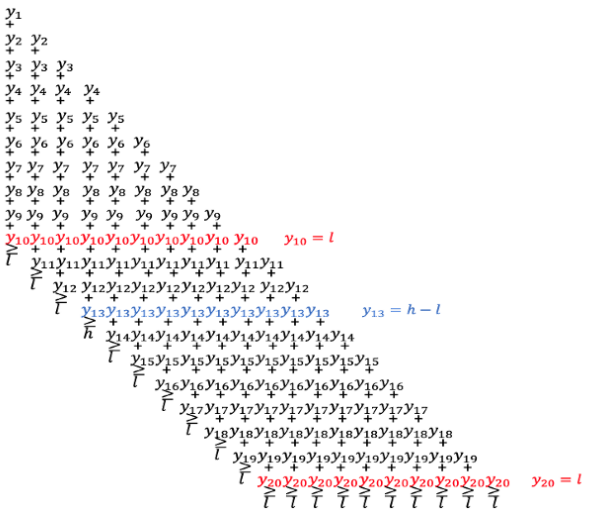


Figure EC.2 Dual constraints (without $\bar{z}_t$) of Example EC.1

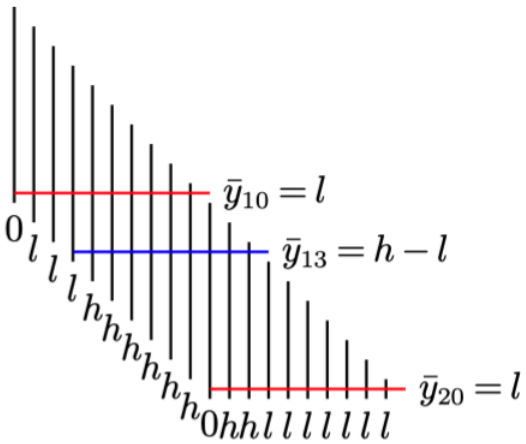


Figure EC.3 Simplified dual constraints of Example EC.1

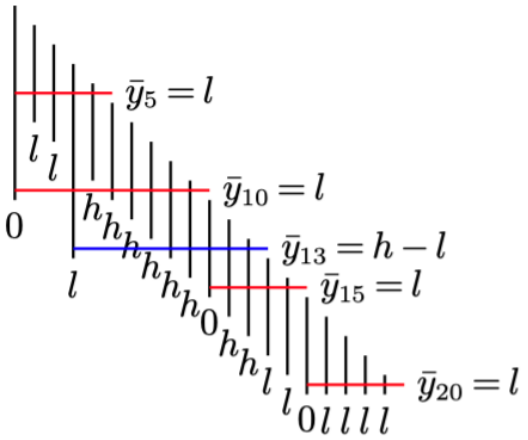


Figure EC.4 Simplified dual constraints of the modified example

R_t, so property (15) holds. If we alter the setting slightly so that $D^{(1)} = 5$ and $(r_1, D_1) = (r^{(2)}, D^{(2)})$ while all other parameters remain unchanged, which we refer to as the modified example, the dual constraints look like Figure EC.4. It can be seen that property (15) is still satisfied, since the sum of $\bar{y}_t$ along each vertical line is at least $\min\{R_t, r_t D_t\}$.

Exact dual optimum. The surprising fact is that, when the monotonicity condition holds and the durations are divisible, this simple way of assigning the dual variables achieves the *exact dual optimum* for instances in which at least C class i customers arrive for each $i \in [m]$, in low-then-high order, for any $m \in [M]$. Indeed, in Example EC.1, the optimal revenue is at least $C(h+l)$, while the revenue collected is $2l+h = C(\frac{2l}{2-l/h} + \frac{(1-l/h)h}{2-l/h})$, which equals the best possible CR $1/\Gamma = 1/(2-l/h)$

times $C(h+l)$. Consequently, the dual value of any feasible dual solution is at least $C(h+l)$, which is exactly what we achieve with our dual variables. This exact dual optimum achieved by the dual variables constructed in AA leads to the asymptotic optimality, when the monotone condition and duration divisibility hold.

Design to ensure primal-dual ratio bound. In fact, the exact dual optimum is achieved by design of the AA. In particular, a dual variable increment $r^{(m)}D^{(m)} - r^{(m-1)}D^{(m-1)}$ is only made when a revenue equal to a $1/\Gamma$ fraction of the resulting dual value increment is earned within a cycle of D. In Example EC.1, starting from $t=1$, when $C^{(0)*}$ units are allocated, a value of l is assigned to one of the $\bar{y}_t$ and the dual value increment is Cl. At this point, the primal-dual value ratio is exactly $C^{(0)*}/C = 1/\Gamma = 1/(2-l/h)$. At the next time step ($t=2$), the rejection price is raised to l. The intuitive explanation is that we concede that C class 1 customers could arrive, but we are willing to accept only $C^{(0)*}$ of them and risk losing the rest if there are no subsequent arrivals, in order to reserve capacity in anticipation of future arrivals of class 2 customers. One can easily verify that if some of the first $C^{(0)*}$ units (1 in Example EC.1) are allocated to class 2 customers, the primal-dual value ratio is still at least $1/\Gamma$, even though some $\bar{z}_t$ ($\bar{z}_1$ in Example EC.1) becomes positive.

When the remaining $C^{(1)*}$ units are all allocated, i.e., to class 2 customers, a value of $h-l$ is assigned to another $\bar{y}_t$, and the total dual value becomes Ch. The primal-dual increment ratio is again exactly $C^{(1)*}h/C(h-l) = 1/\Gamma = 1/(2-l/h)$. After the last unit is allocated, the rejection price is raised to h. The intuitive explanation is that we concede that the actual optimal revenue could be Ch; however, we can only accept $C^{(1)*}$ class 2 customers due to the previous allocation to class 1 customers. Lastly, since every primal-dual increment is bounded by $1/\Gamma$, naturally, the overall primal-dual value ratio is $(C^{(0)*}l + C^{(1)*}h)/Ch = 1/\Gamma = 1/(2-l/h)$.

Design to ensure dual feasibility. Such incremental assignment of $\bar{y}_t$ is feasible thanks to the monotone condition. In particular, since we assign the value $r^{(1)}D^{(1)}$ to $\bar{y}_\tau$ every $D^{(1)}$ time steps and $D^{(1)} \le D^{(2)}$, if $\min\{r_t D_t, R_t\} = r^{(2)}D^{(2)}$, then it must hold that $\sum_{\tau\in[T]} \mathbf{1}(t \le \tau \le t+D_t-1)\bar{y}_\tau \ge r^{(1)}D^{(1)}$. Consequently, the property (15) can be satisfied at t by just incrementing a $\bar{y}_\tau$ with a value of $r^{(2)}D^{(2)} - r^{(1)}D^{(1)}$, where $\tau \in [t, t+D_t-1]$. Indeed, in Figures EC.3 and EC.4, every vertical line (dual constraint) that passes through the blue horizontal line ($\bar{y}_t$ with value $h-l$) must also pass through at least one red horizontal line ($\bar{y}_t$ with value l). Therefore, the monotone condition allows a dual variable assignment that simultaneously achieves low dual value and dual feasibility.

EC.1.2. Proof of Lemma 1

Proof of Lemma 1 We demonstrate Lemma 1 as follows. For every $t \in \{t' \in [T] : R_{t'} \ge r^{(m)}D^{(m)}\}$, consider the smallest time index $\bar{t} \in \Phi^{(m)}$ that satisfies $t \le \bar{t}$. We assert that the inequalities $t \le \bar{t} \le t + D^{(m)} - 1$ hold. We demonstrate the assertion by a contradiction argument, where

we suppose that $\bar{t} > t + D^{(m)} - 1$, meaning $t < \bar{t} - D^{(m)} + 1$. The inequality $t < \bar{t} - D^{(m)} + 1$ implies the following statement: in the iteration of the **while** loop when $\bar{t}$ is identified in Line 7, the time index t is still in $A^{(m)}$ after the update in Line 12. However, by the definition of $\bar{t}$ and by the fact that the time indices identified in Line 7 form a decreasing sequence, we know that the $\bar{t}'$ identified in Line 7 in the next iteration of the **while** loop must satisfy $\bar{t}' < t$. The inequality $\bar{t}' < t$ leads to a contradiction to Line 7, since in Line 7 the Auxiliary Algorithm should have identified t instead of $\bar{t}'$. The contradiction means that the inequalities $t \le \bar{t} \le t + D^{(m)} - 1$ hold. ■

EC.1.3. Proof of Lemma 2

Proof of Lemma 2 $R_t = r^{(m)} D^{(m)}$ means that $t \in A^{(n)}$ for all $n \in \{1, \dots, m\}$. If $D_t \ge D^{(m)}$, by Lemma 1, we know that for every $n \in \{1, \dots, m\}$, there exists a $\bar{t}^{(n)} \in \Phi^{(n)}$ such that $t \le \bar{t}^{(n)} \le t + D^{(m)} - 1 \le t + D_t - 1$. However, if $D_t < D^{(m)}$, the same holds only for $n \in \{1, \dots, \text{index}(D_t)\}$, i.e., there exists a $\bar{t}^{(n)} \in \Phi^{(n)}$ such that $t \le \bar{t}^{(n)} \le t + D_t - 1$ only for such n, where $\text{index}(D_t) \in \{0, \dots, m\}$ denotes the class of the time t customer, identified by (r_t, D_t) (under **monotone** r **and** D, $D_t < D^{(m)}$ implies that this class is at most m). To summarize both scenarios, for every $n \in \{1, \dots, \min(m, \text{index}(D_t))\}$, there exists a $\bar{t}^{(n)} \in \Phi^{(n)}$ such that $t \le \bar{t}^{(n)} \le t + D_t - 1$. By Line 8 in Algorithm 2, we have $\bar{y}^{(n)}_{\bar{t}^{(n)}} = r^{(n)} D^{(n)} - r^{(n-1)} D^{(n-1)}$, which means that $\sum_{\tau=1}^{T} \mathbf{1}(t \le \tau \le t + D_t - 1) \bar{y}^{(n)}_{\tau} \ge r^{(n)} D^{(n)} - r^{(n-1)} D^{(n-1)}$ for all $n \in \{1, \dots, \min(m, \text{index}(D_t))\}$. Thus, by the definition of $\bar{y}_\tau$ (see Line 16), we have

$$\begin{aligned}
\sum_{\tau=1}^{T} \mathbf{1}(t \le \tau \le t + D_t - 1) \bar{y}_\tau &\ge \sum_{n=1}^{\min(m, \text{index}(D_t))} \sum_{\tau=1}^{T} \mathbf{1}(t \le \tau \le t + D_t - 1) \bar{y}^{(n)}_{\tau} \\
&\ge \sum_{n=1}^{\min(m, \text{index}(D_t))} \left(r^{(n)} D^{(n)} - r^{(n-1)} D^{(n-1)} \right) \\
&= \min\{ r^{(m)} D^{(m)}, r_t D_t \},
\end{aligned}$$

which proves the lemma. ■

EC.1.4. Proof of Lemma 3

Proof of Lemma 3 To see Lemma 3, note that whenever the Auxiliary Algorithm includes a time index $\bar{t}$ into $\Phi^{(m)}$ (Line 9), the time indices in $A^{(m)} \cap \{\bar{t} - D^{(m)} + 1, \dots, \bar{t}\}$ are removed from $A^{(m)}$ (Line 12). Consequently, the next time step $\bar{t}'$ to be included must satisfy $\bar{t}' \le \bar{t} - D^{(m)}$. ■

EC.1.5. Proof of Lemma 4

Proof of Lemma 4 Let $n = \sum_{\bar{t} \in \Phi^{(m)}} \mathbf{1}(t \in \text{Charge}^{(m)}_{\bar{t}})$. By the definition of the sets $\{\text{Charge}^{(m)}_t\}_{m \in \{1, \dots, M\}, t \in [T]}$ in Line 10, we have $\sum_{\bar{t}=1}^{T} \mathbf{1}\{t \le \bar{t} \le t + D_t - 1\} \mathbf{1}\{\bar{t} \in \Phi^{(m)}\} \ge n$. Due to

Lemma 3, we also know that these n values of $\bar{t}$ must be at least $D^{(m)}$ time steps apart from each other. This means that these n values of $\bar{t}$ must lie in n disjoint intervals of length $D^{(m)}$. By the fact that $D^{(m')} \le D^{(m)}\ \forall m' \in \{1,\dots,m\}$, and the way $\bar{t} \in \Phi^{(m')}$ are identified and sets $A^{(m')}$ are updated in Lines 7 and 12 for all $m' \in \{1,\dots,m\}$, we know that each newly identified $\bar{t} \in \Phi^{(m')}$ removes at most $D^{(m')} \le D^{(m)}$ time steps from $A^{(m')}$ in an iteration. As a consequence, the number of $\bar{t} \in \Phi^{(m')}$ within the period from t to $t + D_t - 1$ must be at least n as well, for every class $m' \in \{1,\dots,m\}$. That is,

$$\sum_{\bar{t}'=1}^{T} \mathbf{1}\{t \le \bar{t}' \le t + D_t - 1\}\mathbf{1}\{\bar{t}' \in \Phi^{(m')}\} \ge \sum_{\bar{t} \in \Phi^{(m)}} \mathbf{1}(t \in \text{Charge}_{\bar{t}}^{(m)}) \quad \forall m' \in \{1,\dots,m\}. \tag{EC.1}$$

Therefore, we have

$$\begin{aligned}
\sum_{\tau=1}^{T} \mathbf{1}(t \le \tau \le t + D_t - 1)\bar{y}_\tau &\ge \sum_{\bar{t}'=1}^{T} \mathbf{1}(t \le \bar{t}' \le t + D_t - 1)\mathbf{1}(\bar{t}' \in \bigcup_{m' \in \{1,\dots,m\}} \Phi^{(m')})\bar{y}_{\bar{t}'} \\
&\ge \sum_{m'=1}^{m} \sum_{\bar{t}'=1}^{T} \mathbf{1}(t \le \bar{t}' \le t + D_t - 1)\mathbf{1}(\bar{t}' \in \Phi^{(m')})\bar{y}_{\bar{t}'}^{(m')} \\
&= \sum_{m'=1}^{m} \left(r^{(m')}D^{(m')} - r^{(m'-1)}D^{(m'-1)}\right) \sum_{\bar{t}'=1}^{T} \mathbf{1}(t \le \bar{t}' \le t + D_t - 1)\mathbf{1}(\bar{t}' \in \Phi^{(m')}) \\
&\ge \sum_{m'=1}^{m} \sum_{\bar{t} \in \Phi^{(m)}} \mathbf{1}(t \in \text{Charge}_{\bar{t}}^{(m)}) \left(r^{(m')}D^{(m')} - r^{(m'-1)}D^{(m'-1)}\right),
\end{aligned} \tag{EC.2}$$

where the equality uses the fact that $\bar{y}_{\bar{t}'}^{(m')} = r^{(m')}D^{(m')} - r^{(m'-1)}D^{(m'-1)}$ for all $\bar{t}' \in \Phi^{(m')}$. Therefore,

$$\begin{aligned}
&\sum_{m'=1}^{m} \sum_{\bar{t} \in \Phi^{(m)}} \mathbf{1}(t \in \text{Charge}_{\bar{t}}^{(m)})(r^{(m')}D^{(m')} - r^{(m'-1)}D^{(m'-1)}) \\
&\qquad = r^{(m)}D^{(m)} \sum_{\bar{t} \in \Phi^{(m)}} \mathbf{1}(t \in \text{Charge}_{\bar{t}}^{(m)}),
\end{aligned} \tag{EC.3}$$

which proves the lemma. ■

EC.1.6. Proof of Lemma 5

Proof of Lemma 5 According to the way each Charge set is constructed in Line 10, all sets $\{\text{Charge}_{\tau}^{(m)}\}_{\tau \in \Phi^{(m)}, m \in \{1,\dots,M\}}$ that t belongs to have the same m, which equals $\text{index}(R_t) + 1$, where $\text{index}(R_t)$ is the type index of R_t. Therefore, $R_t = r^{(m-1)}D^{(m-1)}$. Since all such τ belong to $\Phi^{(m)}$ for the same m, they are at least $D^{(m)}$ time steps apart from each other. Consequently, at most $\left\lceil \frac{D_t}{D^{(m)}} \right\rceil$ such τ can lie within the range from t to $t + D_t - 1$, which proves the lemma. ■

EC.1.7. Proof of Claim 3

Proof of Claim 3 First of all, it is true that if $t \in \text{Charge}_{\bar{t}}^{(m)}$ for an $m \in \{1,\dots,M\}, \bar{t} \in \Phi^{(m)}$, then $r_t D_t > R_t = r^{(m-1)} D^{(m-1)}$. By the **monotone** r **and** D assumption, $D_t \geq D^{(m)}$.

Claim 3 follows from Lemmas 4 and 5. From Lemma 4, we know that

$$\begin{aligned}\sum_{\tau=1}^{T} \mathbf{1}(t \leq \tau \leq t + D_t - 1)\bar{y}_\tau &\geq r^{(m)} D^{(m)} \sum_{\bar{t} \in \Phi^{(m)}} \mathbf{1}(t \in \text{Charge}_{\bar{t}}^{(m)}) \\ &\geq r^{(m)} D^{(m)} \sum_{\bar{t} \in \Phi^{(m)}} \mathbf{1}(t \in \text{Charge}_{\bar{t}}^{(m)}) \frac{\frac{D_t}{D^{(m)}}}{\left\lceil \frac{D_t}{D^{(m)}} \right\rceil}. \end{aligned} \tag{EC.4}$$

Lemma 5 states that

$$\sum_{\bar{t} \in \Phi^{(m)}} \mathbf{1}(t \in \text{Charge}_{\bar{t}}^{(m)}) \leq \left\lceil \frac{D_t}{D^{(m)}} \right\rceil,$$

which implies that

$$\sum_{\bar{t} \in \Phi^{(m)}} \mathbf{1}(t \in \text{Charge}_{\bar{t}}^{(m)}) \frac{\frac{D_t}{D^{(m)}}}{\left\lceil \frac{D_t}{D^{(m)}} \right\rceil} \leq \frac{D_t}{D^{(m)}}.$$

Consequently,

$$r^{(m)} D^{(m)} \sum_{\bar{t} \in \Phi^{(m)}} \mathbf{1}(t \in \text{Charge}_{\bar{t}}^{(m)}) \frac{\frac{D_t}{D^{(m)}}}{\left\lceil \frac{D_t}{D^{(m)}} \right\rceil} \leq r^{(m)} D_t \leq r_t D_t. \tag{EC.5}$$

Now, according to how $\bar{z}_t$ is set up in line 19 of the Auxiliary Algorithm, we can consider the following 2 situations:

- if $\bar{z}_t > 0$,

$$\begin{aligned}\bar{z}_t &= r_t D_t - \sum_{\tau=1}^{T} \mathbf{1}(t \leq \tau \leq t + D_t - 1)\bar{y}_\tau \\ &\leq r_t D_t - r^{(m)} D^{(m)} \sum_{\bar{t} \in \Phi^{(m)}} \mathbf{1}(t \in \text{Charge}_{\bar{t}}^{(m)}) \frac{\frac{D_t}{D^{(m)}}}{\left\lceil \frac{D_t}{D^{(m)}} \right\rceil}, \end{aligned} \tag{EC.6}$$

  where (EC.6) follows from (EC.4).

- if $\bar{z}_t = 0$,

$$\bar{z}_t = 0 \leq r_t D_t - r^{(m)} D^{(m)} \sum_{\bar{t} \in \Phi^{(m)}} \mathbf{1}(t \in \text{Charge}_{\bar{t}}^{(m)}) \frac{\frac{D_t}{D^{(m)}}}{\left\lceil \frac{D_t}{D^{(m)}} \right\rceil}, \tag{EC.7}$$

  where (EC.7) follows from (EC.5).

Therefore, we always have that

$$\bar{z}_t \leq r_t D_t - r^{(m)} D^{(m)} \sum_{\bar{t} \in \Phi^{(m)}} \mathbf{1}(t \in \text{Charge}_{\bar{t}}^{(m)}) \frac{\frac{D_t}{D^{(m)}}}{\left\lceil \frac{D_t}{D^{(m)}} \right\rceil},$$

which proves the claim. ■

EC.1.8. Proof of the loss factor of $(1-O(M/C))$ due to rounding of the booking limits

In rounding the booking limits $\{C^{(m)*}\}_{m=0}^{M-1}$ as in (5) to $\{C_z^{(m)*}\}_{m=0}^{M-1}$ as in (7), the booking limit of each class is reduced by at most 1. Therefore, we have

$$\begin{aligned}
&\text{CR of PLP-RR with } \{C_z^{(i)*}\}_{i\in\{0,\dots,M-1\}} \geq \min_{m\in\{1,\dots,M\}} \frac{\Psi(C^{(m-1)*}-1)r^{(m)}D^{(m)}}{C(r^{(m)}D^{(m)}-r^{(m-1)}D^{(m-1)})}\\
&=\min_{m\in\{1,\dots,M\}} \frac{\Psi}{\Gamma}\left(1-\frac{1}{C^{(m-1)*}}\right)=\min_{m\in\{1,\dots,M\}} \frac{\Psi}{\Gamma}\left(1-\frac{M-\sum_{k=1}^{M-1}\frac{r^{(k)}D^{(k)}}{r^{(k+1)}D^{(k+1)}}}{C\left(1-\frac{r^{(m-1)}D^{(m-1)}}{r^{(m)}D^{(m)}}\right)}\right)\\
&=\min_{m\in\{1,\dots,M\}} \frac{\Psi}{\Gamma}\left(1-\frac{M}{C}\cdot\frac{1-\frac{\sum_{k=1}^{M-1}\frac{r^{(k)}D^{(k)}}{r^{(k+1)}D^{(k+1)}}}{M}}{1-\frac{r^{(m-1)}D^{(m-1)}}{r^{(m)}D^{(m)}}}\right)\geq\left(1-O\left(\frac{M}{C}\right)\right)\cdot\frac{\Psi}{\Gamma}.
\end{aligned}$$

EC.2. Proof of the CR of PLP-RR in the general case

In this section, we prove the CRs of PLP-RR in the general non-monotone case in Theorems 4 and 5.

EC.2.1. Proof of Theorems 4 and 5

In this section, we consider the general case with no monotone assumptions on either the prices per unit time or the durations, and we prove Theorems 4 and 5. When the monotone assumptions do not hold, we could have $D^{(m)}<D^{(m-1)}$ for some class m. According to the Auxiliary Algorithm, the interval between adjacent positive $\bar{y}_t^{(m-1)}$ could be longer than that of a higher class, $\bar{y}_t^{(m)}$. Therefore, for a time step $t\in A^{(m)}$, though the dual constraint at time t contains an element in $\Phi^{(m)}$, it does not necessarily contain elements in $\Phi^{(m')}$ for all $m'<m$. This means Lemma 2 no longer holds.

To solve this issue, the Auxiliary Algorithm needs to be modified for the general case. We change Line 12 to

$$\text{Set } A^{(m)}\leftarrow A^{(m)}\setminus\{\bar{t}-D_{min}+1,\dots,\bar{t}\}.$$

To explain this change, we need to make sure that for any class m, the interval between adjacent positive $\bar{y}_t^{(m)}$ is at most that of a higher class. Since the highest class M might have the shortest duration, we assign positive $\bar{y}_t^{(m)}$ every D_{min} time steps for all classes m. As a result of this modification, Lemma 5 will be replaced with Lemma EC.1 as follows.

LEMMA EC.1. *If $t\in Charge^{(m)}$, then $t\in Charge_\tau^{(m)}$ for at most $\left\lceil\frac{D_t}{D_{min}}\right\rceil$ values of $\tau\in[T]$. In addition, $R_t=r^{(m-1)}D^{(m-1)}$.*

Proof of Lemma EC.1 This holds since the τ's are at least D_{min} time steps apart from each other. Consequently, at most $\left\lceil\frac{D_t}{D_{min}}\right\rceil$ such τ can lie within the range from t to $t+D_t-1$, which proves the lemma. ■

Consequently, Claim 3 also needs to be replaced with Claim EC.1 as follows for the non-monotone case.

CLAIM EC.1. *If $t \in Charge^{(m)}$, then* $r_t D_t \geq \bar{z}_t + r^{(m)} D^{(m)} \sum_{\bar{t}\in\Phi^{(m)}} \mathbf{1}(t \in Charge_{\bar{t}}^{(m)}) \frac{1}{\left\lceil \frac{D_t}{D_{min}} \right\rceil}$.

The proof of Claim EC.1 is provided in Section EC.2.2. Next, we establish the CR of PLP-RR for the general case.

$$\begin{aligned}
\sum_{t=1}^{T} r_t D_t \bar{x}_t &= \sum_{t\in \text{NoCharge}} r_t D_t + \sum_{m=1}^{M} \sum_{t\in \text{Charge}^{(m)}} r_t D_t \\
&\geq \sum_{t\in \text{NoCharge}} \bar{z}_t + \sum_{m=1}^{M} \sum_{t\in \text{Charge}^{(m)}} \bar{z}_t + \sum_{m=1}^{M} \sum_{t\in \text{Charge}^{(m)}} r^{(m)} D^{(m)} \sum_{\bar{t}\in\Phi^{(m)}} \mathbf{1}(t \in \text{Charge}_{\bar{t}}^{(m)}) \frac{1}{\left\lceil \frac{D_t}{D_{min}} \right\rceil} \\
&\geq \Lambda \cdot \min_{m\in\{1,\dots,M\}} \frac{1}{\left\lceil \frac{D^{(m)}}{D_{min}} \right\rceil} \cdot \Big(\sum_{t\in[T]} \bar{z}_t + C \sum_{t=1}^{T} \bar{y}_t\Big) \qquad \text{(EC.8)} \\
&\geq \frac{\Lambda}{\kappa} \cdot \Big(\sum_{t\in[T]} \bar{z}_t + C \sum_{t=1}^{T} \bar{y}_t\Big). \qquad \text{(EC.9)}
\end{aligned}$$

Inequality (EC.8) is derived following a procedure similar to that for the case of monotone r and D. Inequality (EC.9) is because

$$\max_{m\in[M]} \left\lceil \frac{D^{(m)}}{D_{\min}} \right\rceil \leq \left\lceil \frac{D_{\max}}{D_{\min}} \right\rceil = \kappa.$$

Therefore Theorem 4 is proved. By setting $\{C^{(m-1)}\}_{m=1}^{M}$ according to equation (5), Λ becomes $1/\Gamma$. Therefore, we have

$$\sum_{t=1}^{T} r_t D_t \bar{x}_t \geq \sum_{t\in[T]} \bar{z}_t + \frac{1}{\kappa\Gamma} \cdot C \sum_{t=1}^{T} \bar{y}_t \geq \frac{1}{\kappa\Gamma} \cdot \text{opt(LP-D)}.$$

The rounding of the booking limits from (5) to (7) gives rise to the same loss factor of $1 - O(\frac{M}{C})$, which can be proved in the same way as in the case of monotone r and D. Altogether, Theorem 5 is proved.

EC.2.2. Proof of Claim EC.1

Proof of Claim EC.1 Similar to Claim 3, Claim EC.1 follows from Lemmas 4 and EC.1. From Lemma 4, we know that

$$\begin{aligned}
\sum_{\tau=1}^{T} \mathbf{1}(t \leq \tau \leq t + D_t - 1) \bar{y}_\tau &\geq r^{(m)} D^{(m)} \sum_{\bar{t}\in\Phi^{(m)}} \mathbf{1}(t \in \text{Charge}_{\bar{t}}^{(m)}) \\
&\geq r^{(m)} D^{(m)} \sum_{\bar{t}\in\Phi^{(m)}} \mathbf{1}(t \in \text{Charge}_{\bar{t}}^{(m)}) \frac{1}{\left\lceil \frac{D_t}{D_{min}} \right\rceil}. \qquad \text{(EC.10)}
\end{aligned}$$

Lemma EC.1 states that

$$\sum_{\bar{t}\in\Phi^{(m)}} \mathbf{1}(t\in \text{Charge}_{\bar{t}}^{(m)}) \le \left\lceil \frac{D_t}{D_{min}} \right\rceil,$$

which implies that

$$\sum_{\bar{t}\in\Phi^{(m)}} \mathbf{1}(t\in \text{Charge}_{\bar{t}}^{(m)}) \frac{1}{\left\lceil \frac{D_t}{D_{min}} \right\rceil} \le 1.$$

Consequently,

$$r^{(m)}D^{(m)} \sum_{\bar{t}\in\Phi^{(m)}} \mathbf{1}(t\in \text{Charge}_{\bar{t}}^{(m)}) \frac{1}{\left\lceil \frac{D_t}{D_{min}} \right\rceil} \le r^{(m)}D^{(m)} \le r_t D_t. \tag{EC.11}$$

Now, according to how $\bar{z}_t$ is set up in line 19 of the Auxiliary Algorithm, we can consider the following 2 situations:

- if $\bar{z}_t > 0$,

$$\begin{aligned}\bar{z}_t &= r_t D_t - \sum_{\tau=1}^{T} \mathbf{1}(t\le\tau\le t+D_t-1)\bar{y}_\tau \\ &\le r_t D_t - r^{(m)}D^{(m)} \sum_{\bar{t}\in\Phi^{(m)}} \mathbf{1}(t\in \text{Charge}_{\bar{t}}^{(m)}) \frac{1}{\left\lceil \frac{D_t}{D_{min}} \right\rceil}, \end{aligned} \tag{EC.12}$$

  where (EC.12) follows from (EC.10).

- if $\bar{z}_t = 0$,

$$\bar{z}_t = 0 \le r_t D_t - r^{(m)}D^{(m)} \sum_{\bar{t}\in\Phi^{(m)}} \mathbf{1}(t\in \text{Charge}_{\bar{t}}^{(m)}) \frac{1}{\left\lceil \frac{D_t}{D_{min}} \right\rceil}, \tag{EC.13}$$

  where (EC.13) follows from (EC.11).

Therefore, we always have that

$$\bar{z}_t \le r_t D_t - r^{(m)}D^{(m)} \sum_{\bar{t}\in\Phi^{(m)}} \mathbf{1}(t\in \text{Charge}_{\bar{t}}^{(m)}) \frac{1}{\left\lceil \frac{D_t}{D_{min}} \right\rceil},$$

which proves the claim. ■

EC.3. Missing details and proofs for the FCFS policy

In this section, we show the FCFS policy, and prove its CR in the monotone setting in Theorem 3, and that in the general non-monotone setting in Theorem 6.

EC.3.1. First-Come-First-Served (FCFS) policy, phrased as a special case of PLP-RR

In this section, we present the FCFS policy as a special case of PLP-RR as follows.

Algorithm 4 FCFS policy, phrased as a special case of PLP-RR

for time step $t = 1, \dots, T$ **do**

Set the rejection price R_t, based on the protection levels $(C^{(m)})_{m=0}^{M-1}$ and the occupancy measures $I_t^{(0)}$, where $I_t^{(0)} = \sum_{\tau=1}^{t-1} \bar{x}_\tau \cdot \mathbf{1}(t \le \tau + D_\tau - 1)$:

$$R_t = \begin{cases} 0 & \text{if } I_t^{(0)} < C \\ r^{(M)} D^{(M)} & \text{if } I_t^{(0)} = C \end{cases}.$$

Set $\bar{x}_t = \mathbf{1}(r_t D_t > R_t)$.

end for

Algorithm 5 Auxiliary Algorithm for the FCFS policy (AA-FCFS)

Initialize $A = \{t \in [T] : R_t = r^{(M)} D^{(M)}\}$, $\Phi = \emptyset$, Charge $= \emptyset$.

For $t \in [T]$, initialize $\text{Charge}_t = \emptyset, \bar{y}_t = 0$.

while $A \neq \emptyset$ **do**

Identify $\bar{t} = \max\{t : t \in A\}$.

Set $\Phi \leftarrow \Phi \cup \{\bar{t}\}$.

Set $\text{Charge}_{\bar{t}} = \{\tau : \bar{t} - D_\tau + 1 \le \tau \le \bar{t},\ \bar{x}_\tau = 1\}$.

Set $\text{Charge} \leftarrow \text{Charge} \cup \text{Charge}_{\bar{t}}$.

Set $\bar{y}_{\bar{t}} = r^{(M)} D^{(M)}$.

Set $A \leftarrow A \setminus \{\bar{t} - D_{\min} + 1, \dots, \bar{t}\}$.

end while

for $t = 1, \dots, T$ **do**

Set $\bar{z}_t = \max\{r_t D_t - \sum_{\tau=1}^{T} \mathbf{1}(t \le \tau \le t + D_t - 1)\bar{y}_\tau, 0\}$.

end for

Return $(\{\bar{y}_t\}_{t\in[T]}, \{\bar{z}_t\}_{t\in[T]})$.

EC.3.2. Proof of Theorem 3

To prove Theorem 3, we construct the AA for the FCFS policy as in Algorithm 5 to construct a set of feasible dual variables that satisfies $P \ge \tilde{\Psi} \cdot \frac{r^{(1)} D^{(1)}}{r^{(M)} D^{(M)}} \cdot D$, where P and D represent the primal objective value achieved by the FCFS policy and the dual objective value achieved by the dual variables constructed in Algorithm 5, respectively.

Similar to the AA for PLP-RR in the non-monotone case, the distance between any two time steps in Φ is at least $D_{\min} = D^{(1)}$, so Lemma EC.1 still holds here with the superscript (m) removed. Consequently, Claim EC.1 is replaced with Claim EC.2 as below.

Claim EC.2. *If $t \in$ Charge, then $\bar{z}_t \le r_t D_t - r^{(1)} D^{(1)} \sum_{\bar{t}\in\Phi} \mathbf{1}(t \in \text{Charge}_{\bar{t}}) \frac{\frac{D_t}{D^{(1)}}}{\left\lceil \frac{D_t}{D^{(1)}} \right\rceil}$.*

Proof of Claim EC.2 Due to the construction of AA for the FCFS policy in Algorithm 5, the following inequalities hold.

$$\sum_{\tau=1}^{T}\mathbf{1}(t\le\tau\le t+D_t-1)\bar{y}_\tau \ge r^{(M)}D^{(M)}\sum_{\bar{t}\in\Phi}\mathbf{1}(t\in\text{Charge}_{\bar{t}})$$
$$\ge r^{(1)}D^{(1)}\sum_{\bar{t}\in\Phi}\mathbf{1}(t\in\text{Charge}_{\bar{t}})\frac{\frac{D_t}{D^{(1)}}}{\left\lceil\frac{D_t}{D^{(1)}}\right\rceil}. \tag{EC.14}$$

Lemma EC.1 states that

$$\sum_{\bar{t}\in\Phi}\mathbf{1}(t\in\text{Charge}_{\bar{t}})\le\left\lceil\frac{D_t}{D^{(1)}}\right\rceil.$$

Consequently,

$$r^{(1)}D^{(1)}\sum_{\bar{t}\in\Phi}\mathbf{1}(t\in\text{Charge}_{\bar{t}})\frac{\frac{D_t}{D^{(1)}}}{\left\lceil\frac{D_t}{D^{(1)}}\right\rceil}\le r^{(1)}D_t\le r_tD_t. \tag{EC.15}$$

Now, according to how $\bar{z}_t$ is set up in line 12 of the AA for the FCFS policy, we can consider the following 2 situations:

- if $\bar{z}_t>0$,

$$\bar{z}_t = r_tD_t-\sum_{\tau=1}^{T}\mathbf{1}(t\le\tau\le t+D_t-1)\bar{y}_\tau$$
$$\le r_tD_t-r^{(1)}D^{(1)}\sum_{\bar{t}\in\Phi}\mathbf{1}(t\in\text{Charge}_{\bar{t}})\frac{\frac{D_t}{D^{(1)}}}{\left\lceil\frac{D_t}{D^{(1)}}\right\rceil}, \tag{EC.16}$$

  where (EC.16) follows from (EC.14).

- if $\bar{z}_t=0$,

$$\bar{z}_t=0\le r_tD_t-r^{(1)}D^{(1)}\sum_{\bar{t}\in\Phi}\mathbf{1}(t\in\text{Charge}_{\bar{t}})\frac{\frac{D_t}{D^{(1)}}}{\left\lceil\frac{D_t}{D^{(1)}}\right\rceil}, \tag{EC.17}$$

  where (EC.17) follows from (EC.15).

Therefore, we always have that

$$\bar{z}_t\le r_tD_t-r^{(1)}D^{(1)}\sum_{\bar{t}\in\Phi}\mathbf{1}(t\in\text{Charge}_{\bar{t}})\frac{\frac{D_t}{D^{(1)}}}{\left\lceil\frac{D_t}{D^{(1)}}\right\rceil},$$

which proves the claim. ■

Now, we are ready to establish the competitive ratio and prove Theorem 3:

Proof of Theorem 3

$$\sum_{t=1}^{T}r_tD_t\bar{x}_t=\sum_{t\in\text{NoCharge}}r_tD_t+\sum_{t\in\text{Charge}}r_tD_t$$

$$
\begin{aligned}
&\geq \sum_{t\in\text{NoCharge}} \bar{z}_t + \sum_{t\in\text{Charge}} \bar{z}_t + \sum_{t\in\text{Charge}} r^{(1)}D^{(1)} \sum_{\bar{t}\in\Phi} \mathbf{1}(t\in\text{Charge}_{\bar{t}}) \frac{\frac{D_t}{D^{(1)}}}{\left\lceil \frac{D_t}{D^{(1)}} \right\rceil} \\
&\geq \sum_{t\in[T]} \bar{z}_t + \tilde{\Psi} \sum_{\bar{t}\in\Phi} \sum_{t\in\text{Charge}_{\bar{t}}} r^{(1)}D^{(1)} \qquad \text{(EC.18)} \\
&\geq \sum_{t\in[T]} \bar{z}_t + \tilde{\Psi} \sum_{\bar{t}\in\Phi} C r^{(1)}D^{(1)} \\
&\geq \sum_{t\in[T]} \bar{z}_t + \tilde{\Psi} \cdot \frac{r^{(1)}D^{(1)}}{r^{(M)}D^{(M)}} \cdot C \sum_{\bar{t}\in\Phi} r^{(M)}D^{(M)} \\
&= \sum_{t\in[T]} \bar{z}_t + \tilde{\Psi} \cdot \frac{r^{(1)}D^{(1)}}{r^{(M)}D^{(M)}} \cdot C \sum_{t\in[T]} \bar{y}_t \geq \tilde{\Psi} \cdot \frac{r^{(1)}D^{(1)}}{r^{(M)}D^{(M)}} \cdot \Big(\sum_{t\in[T]} \bar{z}_t + C\sum_{t=1}^{T} \bar{y}_t\Big).
\end{aligned}
$$

With this, Theorem 3 is proved.

EC.3.3. Proof of Theorem 6

The proof for Theorem 6 largely follows the proof for Theorem 3 in Section EC.3.2, and the AA used in Section EC.3.2 is also used in this section. However, unlike the proof for Theorem 3 in Section EC.3.2, here we replace Claim EC.2 with Claim EC.3 as below.

Claim EC.3. *If $t \in$ Charge, then $\bar{z}_t \leq r_t D_t - r^{(1)}D^{(1)} \sum_{\bar{t}\in\Phi} \mathbf{1}(t\in \text{Charge}_{\bar{t}}) \frac{1}{\left\lceil \frac{D_t}{D_{min}} \right\rceil}$.*

Proof of Claim EC.3 Due to the construction of AA for the FCFS policy in Algorithm 5, the following inequalities hold.

$$
\begin{aligned}
\sum_{\tau=1}^{T} \mathbf{1}(t\leq\tau\leq t+D_t-1)\bar{y}_\tau &\geq r^{(M)}D^{(M)} \sum_{\bar{t}\in\Phi} \mathbf{1}(t\in\text{Charge}_{\bar{t}}) \\
&\geq r^{(1)}D^{(1)} \sum_{\bar{t}\in\Phi} \mathbf{1}(t\in\text{Charge}_{\bar{t}}) \frac{1}{\left\lceil \frac{D_t}{D_{min}} \right\rceil}. \qquad \text{(EC.19)}
\end{aligned}
$$

Lemma EC.1 states that

$$
\sum_{\bar{t}\in\Phi} \mathbf{1}(t\in\text{Charge}_{\bar{t}}) \leq \left\lceil \frac{D_t}{D_{min}} \right\rceil.
$$

Consequently,

$$
r^{(1)}D^{(1)} \sum_{\bar{t}\in\Phi} \mathbf{1}(t\in\text{Charge}_{\bar{t}}) \frac{1}{\left\lceil \frac{D_t}{D_{min}} \right\rceil} \leq r^{(1)}D^{(1)} \leq r_t D_t. \qquad \text{(EC.20)}
$$

Now, according to how $\bar{z}_t$ is set up in line 12 of the AA for the FCFS policy, we can consider the following 2 situations:

- if $\bar{z}_t > 0$,

$$
\begin{aligned}
\bar{z}_t &= r_t D_t - \sum_{\tau=1}^{T} \mathbf{1}(t\leq\tau\leq t+D_t-1)\bar{y}_\tau \\
&\leq r_t D_t - r^{(1)}D^{(1)} \sum_{\bar{t}\in\Phi} \mathbf{1}(t\in\text{Charge}_{\bar{t}}) \frac{1}{\left\lceil \frac{D_t}{D_{min}} \right\rceil}, \qquad \text{(EC.21)}
\end{aligned}
$$

where (EC.21) follows from (EC.19).

- if $\bar{z}_t = 0$,

$$\bar{z}_t = 0 \le r_t D_t - r^{(1)} D^{(1)} \sum_{\bar{t}\in\Phi} \mathbf{1}(t \in \text{Charge}_{\bar{t}}) \frac{1}{\left\lceil \frac{D_t}{D_{min}} \right\rceil}, \tag{EC.22}$$

where (EC.22) follows from (EC.20).

Therefore, we always have that

$$\bar{z}_t \le r_t D_t - r^{(1)} D^{(1)} \sum_{\bar{t}\in\Phi} \mathbf{1}(t \in \text{Charge}_{\bar{t}}) \frac{1}{\left\lceil \frac{D_t}{D_{min}} \right\rceil},$$

which proves the claim. ■

Now, we are ready to establish the competitive ratio.

Proof of Theorem 6

$$\begin{aligned}
\sum_{t=1}^{T} r_t D_t \bar{x}_t &= \sum_{t\in\text{NoCharge}} r_t D_t + \sum_{t\in\text{Charge}} r_t D_t \\
&\ge \sum_{t\in\text{NoCharge}} \bar{z}_t + \sum_{t\in\text{Charge}} \bar{z}_t + \sum_{t\in\text{Charge}} r^{(1)} D^{(1)} \sum_{\bar{t}\in\Phi} \mathbf{1}(t \in \text{Charge}_{\bar{t}}) \frac{1}{\left\lceil \frac{D_t}{D_{min}} \right\rceil} \\
&\ge \sum_{t\in[T]} \bar{z}_t + \frac{1}{\kappa} \sum_{\bar{t}\in\Phi} \sum_{t\in\text{Charge}_{\bar{t}}} r^{(1)} D^{(1)} \\
&\ge \sum_{t\in[T]} \bar{z}_t + \frac{1}{\kappa} \sum_{\bar{t}\in\Phi} C r^{(1)} D^{(1)} \\
&\ge \sum_{t\in[T]} \bar{z}_t + \frac{1}{\kappa} \cdot \frac{r^{(1)} D^{(1)}}{r^{(M)} D^{(M)}} \cdot C \sum_{\bar{t}\in\Phi} r^{(M)} D^{(M)} \\
&= \sum_{t\in[T]} \bar{z}_t + \frac{1}{\kappa} \cdot \frac{r^{(1)} D^{(1)}}{r^{(M)} D^{(M)}} \cdot C \sum_{t\in[T]} \bar{y}_t \ge \frac{1}{\kappa} \cdot \frac{r^{(1)} D^{(1)}}{r^{(M)} D^{(M)}} \cdot \Big(\sum_{t\in[T]} \bar{z}_t + C \sum_{t=1}^{T} \bar{y}_t\Big).
\end{aligned} \tag{EC.23}$$

With this, Theorem 6 is proved.

EC.4. Extension to advance booking

In this section, we study the case where advance booking on the resource units is allowed. We start with the advance booking model, then we propose a modified version of PLP-RR, PLP-AB, and show its performance guarantee and analysis.

EC.4.1. Model

We assume all customer classes have the same usage duration D. Unlike the admission control setting, for a customer arriving at time t, the usage starts at time $s_t \ge t$ instead of t, where s_t is revealed upon the arrival of customer t. The usage start times $\{s_t\}_{t\in[T]}$ can be chosen by an

adaptive adversary. We define $S = \max_{t\in[T]}\{s_t\}$ as the latest starting time of any usage period across all customers. All other settings on the online dynamics, model uncertainty, decision choices, objective and model assumptions are the same as those in the admission control setting.

The underlying IP of the advance booking problem is as follows, which is also revealed column by column in an online manner.

$$\begin{aligned}\text{(IP-AB):}\quad \max & \sum_{t=1}^{T} r_t D x_t \\ \text{s.t.} & \sum_{t=1}^{T} \mathbf{1}(s_t \le \tau \le s_t + D - 1) x_t \le C \qquad \forall\ \tau \in \{1,\dots,S\}, \\ & x_t \in \{0,1\} \qquad \forall\ t \in \{1,\dots,T\}.\end{aligned}$$

EC.4.2. Protection Level Policy for Advance Booking (PLP-AB)

In the following, we propose the Protection Level Policy for Advance Booking (PLP-AB), displayed in Algorithm 6. For each time step t, each $\tau \in \{1,2,\dots,S+D-1\}$ and $m \in \{0,1,\dots,M-1\}$, we

Algorithm 6 Protection Level Policy for Advance Booking (PLP-AB)

1: **Input:** Booking limit parameters $(C^{(m)})_{m=0}^{M-1}$.

2: **for** time step $t = 1,\dots,T$ **do**

3: Set the rejection price R_t, based on the protection levels $(C^{(m)})_{m=0}^{M-1}$ and the occupancy measures $\{I_{t,\tau}^{(m)}\}_{m\in\{0,\dots,M-1\},\tau\in\{s_t,\dots,s_t+D-1\}}$ (see equation (EC.24) for the definition of $I_{t,\tau}^{(m)}$):

$$R_t = \begin{cases} r^{(0)}D & \text{if } \forall \tau \in \{s_t,\dots,s_t+D-1\}, \quad I_{t,\tau}^{(0)} < C^{(0)} \\ r^{(1)}D & \text{if } \exists \tau \in \{s_t,\dots,s_t+D-1\}, \quad I_{t,\tau}^{(0)} = C^{(0)}, \\ & \quad \forall \tau \in \{s_t,\dots,s_t+D-1\}, \quad I_{t,\tau}^{(1)} < C^{(1)} \\ \vdots & \\ r^{(m)}D & \text{if } \forall \bar{m} \in \{0,\dots,m-1\}, \quad \exists \tau \in \{s_t,\dots,s_t+D-1\}, \quad I_{t,\tau}^{(\bar{m})} = C^{(\bar{m})}, \\ & \quad \forall \tau \in \{s_t,\dots,s_t+D-1\}, \quad I_{t,\tau}^{(m)} < C^{(m)} \\ \vdots & \\ r^{(M)}D & \text{if } \forall \bar{m} \in \{0,\dots,M-1\}, \quad \exists \tau \in \{s_t,\dots,s_t+D-1\}, \quad I_{t,\tau}^{(\bar{m})} = C^{(\bar{m})} \end{cases}.$$

4: Set $\bar{x}_t = \mathbf{1}(r_t D > R_t)$.

5: **end for**

define the quantity

$$I_{t,\tau}^{(m)} = \sum_{j=1}^{t-1} \bar{x}_j \cdot \mathbf{1}(s_j \le \tau \le s_j + D - 1) \cdot \mathbf{1}(R_j = r^{(m)}D) \tag{EC.24}$$

which is analogous to $I_t^{(m)}$ defined in (4) in the admission control setting. The quantity $I_{t,\tau}^{(m)}$ counts the number of accepted customers who arrive at time steps $1,\dots,t-1$ and are still occupying a unit of resource at time τ, with the additional property that their rejection price is equal to $r^{(m)}D$.

PLP-AB is based on a similar concept of booking limits as in PLP-RR. The difference here is that at each time step t, instead of only checking $\{I_t^{(m)}\}_{m\in\{0,\dots,M-1\}}$, now PLP-AB needs to check $\{I_{t,\tau}^{(m)}\}_{m\in\{0,\dots,M-1\},\tau\in\{s_t,\dots,s_t+D-1\}}$. That is, with the intuition of the nested protection level, PLP-RR only checks the number of units of each subset being occupied at time t, while PLP-AB checks the number of units of each subset being occupied for the *whole usage period* requested by the customer at time t, which is from s_t to s_t+D-1. As long as there exists a time step τ within the usage period from s_t to s_t+D-1 such that the booking limits of the 1st to m-th subsets are all reached at time τ, i.e. $I_{t,\tau}^{(k)}=C^{(k)}$ for all $k\in\{0,\dots,m-1\}$, the rejection price R_t needs to be raised to at least $r^{(m)}D$. The reason for such a difference between PLP-RR and PLP-AB is the generality of the usage start time in the advance booking setting. Next, we show the performance of PLP-AB.

THEOREM EC.1. *For the advance booking problem with identical usage durations, PLP-AB with* $\{C^{(m)}\}_{m\in\{0,\dots,M-1\}}\in\Delta_C$ *has a competitive ratio of* $\frac{1}{2}\cdot\Lambda$.

THEOREM EC.2. *For the advance booking problem with identical usage durations, PLP-AB with* $\{C^{(m)}\}_{m\in\{0,\dots,M-1\}}=\{C_z^{(m)*}\}_{m\in\{0,\dots,M-1\}}$ *has a competitive ratio of* $(1-O(\frac{M}{C}))\frac{1}{2\Gamma}$.

Theorem EC.2 says that with large capacity C, PLP-AB has a CR that is at least half of the CR upper bound. In addition, the loss factor in the CR, $1-O(M/C)$, is again due to the rounding from the optimally chosen set of booking limits, $\{C^{(m)}\}_{m\in\{0,\dots,M-1\}}$, to their closest integers, $\{C_z^{(m)*}\}_{m\in\{0,\dots,M-1\}}$.

EC.4.3. Analysis

Our analysis involves the following linear programming relaxation (LP-P-AB) of the IP:

$$\begin{aligned}
\text{(LP-P-AB):}\quad \max\ & \sum_{t=1}^{T} r_t D x_t \\
\text{s.t.}\ & \sum_{t=1}^{T} \mathbf{1}(s_t\le\tau\le s_t+D-1)x_t\le C \qquad \forall\ \tau\in\{1,\dots,S\},\\
& x_t\le 1 \qquad \forall\ t\in\{1,\dots,T\},\\
& x_t\ge 0 \qquad \forall\ t\in\{1,\dots,T\}.
\end{aligned}$$

$$\text{(LP-D-AB):}\quad \min \sum_{t=1}^{T} z_t + C\sum_{\tau=1}^{S} y_\tau$$

$$\begin{aligned} \text{s.t.} z_t + \sum_{\tau=s_t}^{s_t+D-1} y_\tau \geq r_t D & \qquad \forall\, t \in \{1,\dots,T\}, \\ y_t \geq 0 & \qquad \forall\, t \in \{1,\dots,S\}, \\ z_t \geq 0 & \qquad \forall\, t \in \{1,\dots,T\}. \end{aligned}$$

Similar to the admission control setting, to establish Theorem EC.1, we show that there exists $(\{\bar{y}_t\}_{t=1}^T, \{\bar{z}_t\}_{t=1}^T)$ such that

1. $(\{\bar{y}_t\}_{t=1}^T, \{\bar{z}_t\}_{t=1}^T)$ is feasible for (LP-D-AB).
2. the inequality $\sum_{t=1}^T r_t D \bar{x}_t \geq \frac{1}{2} \cdot \Lambda \cdot (\sum_{t=1}^T \bar{z}_t + C \sum_{\tau=1}^S \bar{y}_\tau)$ holds.

Again, we do so by constructing a dual solution $(\{\bar{y}_t\}_{t=1}^T, \{\bar{z}_t\}_{t=1}^T)$ with the Auxiliary Algorithm, as shown in Algorithm 7, and we show that such a dual solution is indeed feasible for (LP-D-AB) and makes the inequality in property 2 hold. We first need to define some notation to be used in the auxiliary algorithm. Similar to before, for $m \in \{1,\dots,M\}$, define $A^{(m)} = \{t \in [T] : R_t \geq r^{(m)} D\}$. By definition of R_t and $A^{(m)}$, for each $t \in A^{(m)}$, we can identify a $\tau \in \{s_t, \dots, s_t + D - 1\}$ such that

$$I_{t,\tau}^{(m-1)} = \sum_{j=1}^{t-1} \bar{x}_j \cdot \mathbf{1}(s_j \leq \tau \leq s_j + D - 1) \cdot \mathbf{1}(R_j = r^{(m-1)} D) = C^{(m-1)}.$$

For each $t \in A^{(m)}$, we define

$$\rho_t^{(m-1)} = \min\{\tau : \tau \in \{s_t, \dots, s_t + D - 1\},\ I_{t,\tau}^{(m-1)} = C^{(m-1)}\}. \tag{EC.25}$$

Now we are ready to show the Auxiliary Algorithm for the advance booking model.

To briefly explain the Auxiliary Algorithm for advance booking, the main difference from the one for admission control is that a pruning process is added. The key idea is that we need to upper bound the number of $\rho \in \Phi^{(m)}$ within each interval of D for each class $m \in \{1,\dots,M\}$, such that we can constrain the value of $\sum_{m\in\{1,\dots,M\}} \sum_{\rho\in\Phi^{(m)}} \bar{y}_\rho^{(m)}$, which is related to the dual objective value. The problem without the pruning process is that there could be an excessive number of ρ in $\Phi^{(m)}$. Consider the following case. Suppose a ρ is identified as $\bar{\rho}$ in line 7 and included in set $\Phi^{(m)}$. According to line 9, time steps with usage starting time from $\rho - D + 1$ to ρ will be removed from $A^{(m)}$. Therefore, the time step corresponding to ρ is removed from $A^{(m)}$. Suppose there is a time step t such that $s_t = \rho - D$, then t must still be in $A^{(m)}$. Now, the definition of $\rho_t^{(m-1)}$ states that every $\rho_t^{(m-1)}$ is within the range from s_t to $s_t + D - 1$, which means that $\rho_t^{(m-1)}$ could be as large as $\rho - 1$. Then $\rho - 1$ will be identified as the new $\bar{\rho}$ in the next round, therefore there is no distance guarantee between the ρ's in $\Phi^{(m)}$ for each class m.

To remove the redundant ρ from $\Phi^{(m)}$, we add the pruning process such that we can still ensure that there exists a $\rho \in \Phi^{(m)}$ for all $t \in A^{(m)}$ such that $s_t \leq \rho \leq s_t + D - 1$ (formalized in Lemma

Algorithm 7 Auxiliary Algorithm - Reusable Resource Advance Booking

for $m=1,\dots,M$ **do**
  Initialize $A^{(m)}=\{t\in[T]: R_t\geq r^{(m)}D\}$, $\Phi^{(m)}=\emptyset$, $\text{Charge}^{(m)}=\emptyset$.
  For $t\in[S]$, initialize $\text{Charge}_t^{(m)}=\emptyset, \bar{y}_t^{(m)}=0$.
end for
for $m=1,\dots,M$ **do**
  while $A^{(m)}\neq\emptyset$ **do**
    Identify $\bar{\rho}=\max_{t\in A^{(m)}}\{\rho_t^{(m-1)}\}$, where $\rho_t^{(m-1)}$ is defined in (EC.25).
    Set $\Phi^{(m)}\leftarrow\Phi^{(m)}\cup\{\bar{\rho}\}$.
    Set $A^{(m)}\leftarrow A^{(m)}\setminus\{t\in A^{(m)}: s_t\leq\bar{\rho}\leq s_t+D-1\}$.
  end while
  Set $\tilde{\rho}=\max\{\rho:\rho\in\Phi^{(m)}\}$ (if $\Phi^{(m)}=\emptyset$, skip the pruning process below).
  while $\{\rho:\rho\in\Phi^{(m)},\rho<\tilde{\rho}\}\neq\emptyset$ **do** ▷ Start of pruning process
    if $\{\rho:\rho\in\Phi^{(m)},\tilde{\rho}-D+1\leq\rho<\tilde{\rho}\}\neq\emptyset$ **then**
      Identify $\rho_{\min}=\min\{\rho:\rho\in\Phi^{(m)},\ \tilde{\rho}-D+1\leq\rho<\tilde{\rho}\}$.
      Set $\Phi^{(m)}=\Phi^{(m)}\setminus\{\rho_{\min}+1,\dots,\tilde{\rho}-1\}$.
      Set $\tilde{\rho}\leftarrow\rho_{\min}$.
    else
      Set $\tilde{\rho}\leftarrow\max\{\rho:\rho\in\Phi^{(m)},\ \rho<\tilde{\rho}\}$.
    end if
  end while ▷ End of pruning process
  for $\rho\in\Phi^{(m)}$ **do**
    Set $\text{Charge}_\rho^{(m)}=\{j:s_j\leq\rho\leq s_j+D-1,\ \bar{x}_j=1,\ R_j=r^{(m-1)}D\}$.
    Set $\text{Charge}^{(m)}\leftarrow\text{Charge}^{(m)}\cup\text{Charge}_\rho^{(m)}$.
    Set $\bar{y}_\rho^{(m)}=(r^{(m)}-r^{(m-1)})D$.
  end for
end for
for $s=1,\dots,S$ **do**
  Set $\bar{y}_s=\sum_{m=1}^M\bar{y}_s^{(m)}$.
end for
for $t=1,\dots,T$ **do**
  Set $\bar{z}_t=\max\{r_tD-\sum_{\tau=s_t}^{s_t+D-1}\bar{y}_\tau,0\}$.
end for
Return $(\{\bar{y}_s\}_{s=1}^S,\{\bar{z}_t\}_{t=1}^T)$.

EC.2), which is critical in ensuring $\sum_{\tau=s_t}^{s_t+D-1}\bar{y}_\tau\geq R_t$, and at the same time, we can constrain the number of $\rho\in\Phi^{(m)}$ within each interval of D to 2 for all $m\in\{1,\dots,M\}$ (formalized in Lemma EC.4). In fact, this is where the factor of $1/2$ in the CR originates. Indeed, with such a construction,

the overall dual value could potentially be doubled as compared to that with AA for admission control, halving the CR.

Next, we proceed to establish property 1. We establish property 2 in Section EC.4.4. Again, the output $(\{\bar{y}_s\}_{s=1}^S, \{\bar{z}_t\}_{t=1}^T)$ is indeed feasible for (LP-D-AB) due to the way they are set up in Lines 28 and 31 respectively. The establishment of property 2 for the advance booking problem largely resembles that of the admission control setting, and we show it in Section EC.4.4. Since both properties 1 and 2 hold, Theorem EC.1 is proved. A similar analysis to that of admission control in Section 4 can be used to prove Theorem EC.2, given that Theorem EC.1 holds.

EC.4.4. Establishment of Property 2 for advance booking

To establish property 2, we first define the set NoCharge the same way as in the admission control setting. Again, we need some lemmas and claims to facilitate the proof of property 2, and we state them below.

LEMMA EC.2. *For each $t \in A^{(m)}$, there exists one $\rho \in \Phi^{(m)}$ such that $s_t \le \rho \le s_t + D - 1$.*

The proof of Lemma EC.2 is provided in Section EC.4.5.1. Note that Lemma EC.2 corresponds to Lemma 1 in admission control in Section 4.2. Lemma EC.2 is used to prove the following lemma.

LEMMA EC.3. *If $R_t = r^{(m)} D$, then it holds that $\sum_{\tau=s_t}^{s_t+D-1} \bar{y}_\tau \ge r^{(m)} D$.*

The proof of Lemma EC.3 is provided in Section EC.4.5.2. Note that Lemma EC.3 corresponds to Lemma 2 in admission control in Section 4.2.

LEMMA EC.4. *For all $s \in [S]$, it holds that $\sum_{\rho=s}^{s+D-1} \mathbf{1}\{\rho \in \Phi^{(m)}\} \le 2$.*

Proof of Lemma EC.4 Lemma EC.4 can be proved using a contradiction argument. If Lemma EC.4 is not true, then there exists an $s \in [S]$ such that $\sum_{\rho=s}^{s+D-1} \mathbf{1}\{\rho \in \Phi^{(m)}\} = n > 2$. Denote these ρ's as $\rho_1, \rho_2, \ldots, \rho_n$. Then Line 15 prunes the ρ's in the middle, namely $\rho_2, \ldots, \rho_{n-1}$, and leaves only ρ_1 and ρ_n in $\Phi^{(m)}$ within the range from s to $s + D - 1$, which contradicts the assumption that $\sum_{\rho=s}^{s+D-1} \mathbf{1}\{\rho \in \Phi^{(m)}\} = n > 2$. This shows that Lemma EC.4 holds. ■

We remark that Lemma EC.4 pinpoints the biggest *difference* between the properties of AA in the admission control and advance booking settings. In particular, in the admission control setting, all elements ($\bar{t}$) in the set $\Phi^{(m)}$ are at least $D^{(m)}$ time steps apart from each other. However, in the advance booking setting, we can only guarantee that there are at most 2 elements (ρ) within each interval of length D time steps, which is due to the complexity caused by the arbitrary starting time of customers. This is essentially where the 1/2 in the CR of PLP-AB originates. Lemma EC.4 is useful in proving the following lemmas.

LEMMA EC.5. *If $t \in Charge^{(m)}$, then $t \in Charge_\rho^{(m)}$ for at most 2 such $\rho \in [S]$. In addition, $R_t = r^{(m-1)}D$.*

Proof of Lemma EC.5 According to the way each $\text{Charge}_\rho^{(m)}$ set is constructed in Line 22, all such sets $\{\text{Charge}_\rho^{(m)}\}_{\rho\in\Phi^{(m)},m\in\{1,\dots,M\}}$ that t belongs to have the same m, and $R_t = r^{(m-1)}D$ by definition of $\text{Charge}_\rho^{(m)}$. By Lemma EC.4, there are at most 2 such ρ in every interval of length D. Consequently, at most 2 such ρ can lie within the range from s_t to $s_t + D - 1$, which proves the lemma. ■

LEMMA EC.6. *If $t \in Charge^{(m)}$, then it holds that*

$$\sum_{\tau=s_t}^{s_t+D-1} \bar{y}_\tau \geq \frac{1}{2} r^{(m)} D \sum_{\rho\in\Phi^{(m)}} \mathbf{1}(t \in Charge_\rho^{(m)}).$$

The proof of Lemma EC.6 is provided in Section EC.4.5.3. Note that Lemmas EC.5 and EC.6 correspond to Lemmas 5 and 4 in admission control in Section 4.2 respectively.

Now we are ready to establish property 2. Again, we break down the primal objective into the sum of the revenue earned at time steps in $\text{Charge}^{(m)}$ for all $m \in \{1,\dots,M\}$, and the revenue earned at time steps in NoCharge.

$$\sum_{t=1}^{T} r_t D \bar{x}_t = \sum_{t\in\text{NoCharge}} r_t D + \sum_{m=1}^{M} \sum_{t\in\text{Charge}^{(m)}} r_t D.$$

Next, we lower bound the revenue in NoCharge and revenue in $\text{Charge}^{(m)}$ respectively using the following claims. They correspond to Claims 2 and 3 in admission control in Section 4.2 respectively.

CLAIM EC.4. *If $t \in NoCharge$, then $r_t D \geq \bar{z}_t$.*

Proof of Claim EC.4 This is true because of the way $\bar{z}_t$ is set for all t in line 31 of the Auxiliary Algorithm. ■

CLAIM EC.5. *If $t \in Charge^{(m)}$, then $r_t D \geq \bar{z}_t + \frac{1}{2} r^{(m)} D \sum_{\rho\in\Phi^{(m)}} \mathbf{1}(t \in Charge_\rho^{(m)})$.*

The proof of Claim EC.5 is provided in Section EC.4.5.4. Now, with Claims EC.4 and EC.5, we can proceed with the inequality as follows:

$$\begin{aligned}\sum_{t=1}^{T} r_t D \bar{x}_t &= \sum_{t\in\text{NoCharge}} r_t D + \sum_{m=1}^{M} \sum_{t\in\text{Charge}^{(m)}} r_t D \\ &\geq \sum_{t\in\text{NoCharge}} \bar{z}_t + \sum_{m=1}^{M} \sum_{t\in\text{Charge}^{(m)}} \bar{z}_t + \sum_{m=1}^{M} \sum_{t\in\text{Charge}^{(m)}} \frac{1}{2} r^{(m)} D \sum_{\rho\in\Phi^{(m)}} \mathbf{1}(t \in \text{Charge}_\rho^{(m)}).\end{aligned}$$

Similar to the admission control setting, the first two terms can be combined because of the following claim, which corresponds to Claim 1 in admission control in Section 4.2.

CLAIM EC.6. *If $\bar{x}_t = 0$, then $r_t D \geq \bar{z}_t = 0$.*

Proof of Claim EC.6 $\bar{x}_t = 0$ implies that $R_t \geq r_t D$. Now, Lemma EC.3 asserts that $\sum_{\tau=s_t}^{s_t+D-1} \bar{y}_\tau \geq R_t \geq r_t D$, hence $\bar{z}_t = 0$ by line 31 of the Auxiliary Algorithm. ■

With Claim EC.6, we have:

$$\begin{aligned}
\sum_{t=1}^{T} r_t D \bar{x}_t &= \sum_{t\in \text{NoCharge}} r_t D + \sum_{m=1}^{M} \sum_{t\in \text{Charge}^{(m)}} r_t D \\
&\geq \sum_{t\in \text{NoCharge}} \bar{z}_t + \sum_{m=1}^{M} \sum_{t\in \text{Charge}^{(m)}} \bar{z}_t + \sum_{m=1}^{M} \sum_{t\in \text{Charge}^{(m)}} \frac{1}{2} r^{(m)} D \sum_{\rho\in\Phi^{(m)}} \mathbf{1}(t \in \text{Charge}_\rho^{(m)}) \\
&= \sum_{t\in[T]} \bar{z}_t + \frac{1}{2} \sum_{m=1}^{M} \sum_{\rho\in\Phi^{(m)}} \sum_{t\in \text{Charge}_\rho^{(m)}} r^{(m)} D \\
&\geq \sum_{t\in[T]} \bar{z}_t + \frac{1}{2} \sum_{m=1}^{M} \sum_{\rho\in\Phi^{(m)}} C^{(m-1)} r^{(m)} D \\
&= \sum_{t\in[T]} \bar{z}_t + \frac{1}{2} \sum_{m=1}^{M} \frac{C^{(m-1)} r^{(m)} D}{C(r^{(m)} D - r^{(m-1)} D)} \cdot C \sum_{\rho\in\Phi^{(m)}} (r^{(m)} D - r^{(m-1)} D) \\
&= \sum_{t\in[T]} \bar{z}_t + \frac{1}{2} \sum_{m=1}^{M} \frac{C^{(m-1)} r^{(m)} D}{C(r^{(m)} D - r^{(m-1)} D)} \cdot C \sum_{\tau=1}^{S} \bar{y}_\tau^{(m)} \\
&\geq \sum_{t\in[T]} \bar{z}_t + \frac{1}{2} \min_{m:1\leq m\leq M} \left\{ \frac{C^{(m-1)} r^{(m)} D}{C(r^{(m)} D - r^{(m-1)} D)} \right\} \cdot C \sum_{\tau=1}^{S} \bar{y}_\tau \\
&= \sum_{t\in[T]} \bar{z}_t + \frac{1}{2} \cdot \Lambda \cdot C \sum_{\tau=1}^{S} \bar{y}_\tau \\
&\geq \frac{1}{2} \cdot \Lambda \cdot \Big(\sum_{t\in[T]} \bar{z}_t + C \sum_{\tau=1}^{S} \bar{y}_\tau \Big).
\end{aligned}$$

Therefore, property 2 holds, which completes the proof of Theorem EC.1. When the booking limits take the values of $\{C_z^{(m)*}\}_{m\in\{0,\ldots,M-1\}}$ as in (7), Theorem EC.2 can be proved with the same analysis as in the admission control setting.

EC.4.5. Proofs of claims and lemmas for advance booking

EC.4.5.1. Proof of Lemma EC.2

Proof of Lemma EC.2 First, we refer to Lines 12 to 20 as the pruning process. We demonstrate Lemma EC.2 in two parts. In the first part, we prove that the property holds before the pruning process. In the second part, we prove that the property still holds after the pruning process by arguing that the pruning will not leave any $t \in A^{(m)}$ with no $\rho \in \Phi^{(m)}$ such that $s_t \leq \rho \leq s_t + D - 1$.

The first part follows a structure similar to that of the contradiction argument in the admission control setting. For every $t \in A^{(m)}$, consider the time index $\bar{t} \in A^{(m)}$ such that $\rho_{\bar{t}}^{(m-1)}$ is the smallest $\rho \in \Phi^{(m)}$ that satisfies $\rho_t^{(m-1)} \le \rho_{\bar{t}}^{(m-1)}$. We assert that the inequalities $s_t \le \rho_{\bar{t}}^{(m-1)} \le s_t + D - 1$ hold. Suppose that $\rho_{\bar{t}}^{(m-1)} > s_t + D - 1$, meaning $s_t < \rho_{\bar{t}}^{(m-1)} - D + 1$. The inequality $s_t < \rho_{\bar{t}}^{(m-1)} - D + 1$ implies the following statement: in the iteration of the **while** loop when $\rho_{\bar{t}}^{(m-1)}$ is identified in Line 7, the time index t is still in $A^{(m)}$ after the update in Line 9. However, by the definition of $\rho_{\bar{t}}^{(m-1)}$ and by the fact that the ρ's identified in Line 7 form a decreasing sequence, we know that the $\rho_{\bar{t}'}^{(m-1)}$ identified in Line 7 in the next iteration of the **while** loop must satisfy $\rho_{\bar{t}'}^{(m-1)} < \rho_t^{(m-1)}$. The inequality $\rho_{\bar{t}'}^{(m-1)} < \rho_t^{(m-1)}$ leads to a contradiction to Line 7, since in Line 7 the Auxiliary Algorithm should have identified $\rho_t^{(m-1)}$ instead of $\rho_{\bar{t}'}^{(m-1)}$. The contradiction means that the inequalities $s_t \le \rho_{\bar{t}}^{(m-1)} \le s_t + D - 1$ hold, which proves that Lemma EC.2 holds before the pruning process.

Now we show that after the pruning process Lemma EC.2 still holds. Consider every ρ pruned in Line 15, from Line 13 we know that there must be a $\rho_l \in \Phi^{(m)}$ and $\rho_h \in \Phi^{(m)}$ such that $\rho_l < \rho < \rho_h$ and $\rho_h - \rho_l \le D$. Therefore, after the pruning of ρ, for all $t \in A^{(m)}$ such that $\rho_l - D + 1 \le s_t \le \rho_h$, there exists $\rho' \in \Phi^{(m)}$ such that $s_t \le \rho' \le s_t + D - 1$ due to the existence of ρ_l and ρ_h in $\Phi^{(m)}$. Since $\{t : \rho - D + 1 \le s_t \le \rho\} \subseteq \{t : \rho_l - D + 1 \le s_t \le \rho_h\}$, the pruning does not affect the property. With this, we have proved that Lemma EC.2 holds. ■

EC.4.5.2. Proof of Lemma EC.3

Proof of Lemma EC.3 $R_t = r^{(m)} D$ means that $t \in A^{(n)}$ for all $n \in \{1, \dots, m\}$. By Lemma EC.2, we know that for every $n \in \{1, \dots, m\}$, there exists a $\rho \in \Phi^{(n)}$ such that $s_t \le \rho \le s_t + D - 1$. By Line 24 in Algorithm 7, we have $\bar{y}_\rho^{(n)} = r^{(n)} D - r^{(n-1)} D$, which means that $\sum_{\tau=s_t}^{s_t+D-1} \bar{y}_\tau^{(n)} \ge r^{(n)} D - r^{(n-1)} D$. Thus, by the definition of $\bar{y}_\tau$ (see Line 28), we have

$$\begin{aligned}
\sum_{\tau=s_t}^{s_t+D-1} \bar{y}_\tau &\ge \sum_{n=1}^{m} \sum_{\tau=s_t}^{s_t+D-1} \bar{y}_\tau^{(n)} \\
&\ge \sum_{n=1}^{m} \left(r^{(n)} D - r^{(n-1)} D \right) \\
&= r^{(m)} D,
\end{aligned}$$

which proves the lemma. ■

EC.4.5.3. Proof of Lemma EC.6

Proof of Lemma EC.6 $t \in \text{Charge}_\rho^{(m)}$ means that $s_t \le \rho \le s_t + D - 1$ by the way $\text{Charge}_\rho^{(m)}$ is set in Line 22 of AA. $t \in \text{Charge}_\rho^{(m)}$ also means that $R_t = r^{(m-1)} D$, and therefore $t \in A^{(m')}$ for all $m' \in \{1, \dots, m-1\}$. By Lemma EC.2, we know that there exists a $\rho^{(m')} \in \Phi^{(m')}$ that satisfies

$s_t \le \rho^{(m')} \le s_t + D - 1$ for each $m' \in \{1, \dots, m-1\}$. We also know that for all $\rho^{(m')} \in \Phi^{(m')}$, $\bar{y}^{(m')}_{\rho^{(m')}} = r^{(m')}D - r^{(m'-1)}D$. Therefore, we have

$$\sum_{\tau=s_t}^{s_t+D-1} \bar{y}_\tau \ge \sum_{m'=1}^{m} \left(r^{(m')}D - r^{(m'-1)}D \right) = r^{(m)}D.$$

Now, by Lemma EC.5, we know that $\sum_{\rho\in\Phi^{(m)}} \mathbf{1}(t \in \text{Charge}^{(m)}_\rho) \le 2$, which means $\frac{1}{2}\sum_{\rho\in\Phi^{(m)}} \mathbf{1}(t \in \text{Charge}^{(m)}_\rho) \le 1$. Therefore, we have

$$\sum_{\tau=s_t}^{s_t+D-1} \bar{y}_\tau \ge r^{(m)}D \ge \frac{1}{2} r^{(m)}D \sum_{\rho\in\Phi^{(m)}} \mathbf{1}(t \in \text{Charge}^{(m)}_\rho),$$

which proves the lemma. ■

EC.4.5.4. Proof of Claim EC.5

Proof of Claim EC.5 This claim follows from Lemmas EC.6 and EC.5. From Lemma EC.6, we know that

$$\sum_{\tau=s_t}^{s_t+D-1} \bar{y}_\tau \ge \frac{1}{2} r^{(m)}D \sum_{\rho\in\Phi^{(m)}} \mathbf{1}(t \in \text{Charge}^{(m)}_\rho). \tag{EC.26}$$

Lemma EC.5 states that

$$\sum_{\rho\in\Phi^{(m)}} \mathbf{1}(t \in \text{Charge}^{(m)}_\rho) \le 2.$$

Consequently,

$$\frac{1}{2} r^{(m)}D \sum_{\rho\in\Phi^{(m)}} \mathbf{1}(t \in \text{Charge}^{(m)}_\rho) \le r^{(m)}D \le r_t D. \tag{EC.27}$$

Now, according to how $\bar{z}_t$ is set up in Line 31, we can consider the following 2 situations:

- if $\bar{z}_t > 0$,

$$\begin{aligned} \bar{z}_t &= r_t D - \sum_{\tau=s_t}^{s_t+D-1} \bar{y}_\tau \\ &\le r_t D - \frac{1}{2} r^{(m)}D \sum_{\rho\in\Phi^{(m)}} \mathbf{1}(t \in \text{Charge}^{(m)}_\rho), \end{aligned} \tag{EC.28}$$

  where (EC.28) follows from (EC.26).

- if $\bar{z}_t = 0$,

$$\bar{z}_t = 0 \le r_t D - \frac{1}{2} r^{(m)}D \sum_{\rho\in\Phi^{(m)}} \mathbf{1}(t \in \text{Charge}^{(m)}_\rho), \tag{EC.29}$$

  where (EC.29) follows from (EC.27).

Therefore, we always have that

$$\bar{z}_t \le r_t D - \frac{1}{2} r^{(m)}D \sum_{\rho\in\Phi^{(m)}} \mathbf{1}(t \in \text{Charge}^{(m)}_\rho),$$

which proves the claim. ■

EC.5. Main results for PLP-AP on Assortment Planning

We first define the parameters Γ_i and Γ', where Γ_i corresponds to Γ in (6), but concerns the type i resource in the assortment setting.

$$\Gamma_i = M - \sum_{m=1}^{M-1} \frac{D_i^{(m)}}{D_i^{(m+1)}}, \qquad \Gamma' = \max_{i\in[n]} \Gamma_i. \tag{EC.30}$$

We claim that the CR upper bound in Proposition 1, with Γ replaced by Γ', still applies to the assortment setting, as shown in Proposition EC.1. Indeed, one can easily show this by considering the allocation of each resource type in the same low-then-high instances as in the admission control setting. Proposition EC.1 is proved in Section EC.7.3.

Proposition EC.1. *For the assortment planning problem, no online policy has a competitive ratio larger than* $1/\Gamma'$ *if* $D_i^{(1)} > M \cdot C_i$ *for all* $i \in [n]$.

We highlight that the upper bound in Proposition EC.1 applies to all online policies, not only to protection level policies. Before we show the performance of PLP-AP, we define the following notation. The parameter Ψ_i corresponds to Ψ in (8) but is specific to type i resource, and Λ' corresponds to Λ in (9).

$$\Psi_i = \min_{m'>m, m\in[M]} \frac{D_i^{(m')}/D_i^{(m)}}{\lceil D_i^{(m')}/D_i^{(m)} \rceil}, \qquad \Psi' = \min_{i\in[n]} \Psi_i. \tag{EC.31}$$

$$\Lambda' = \min_{m\in[M], i\in[n]} \left\{ \frac{C_i^{(m-1)} D_i^{(m)}}{C_i (D_i^{(m)} - D_i^{(m-1)})} \right\}. \tag{EC.32}$$

Theorem EC.3. *For the assortment planning problem, PLP-AP with* $\{C_i^{(m)}\}_{m\in\{0,\dots,M-1\}, i\in[n]} \in \Delta'_C$ *has a competitive ratio of* $\frac{1}{2}\Psi' \cdot \Lambda' \geq \frac{\Lambda'}{4}$.

For Theorem EC.3 to hold, we require $C_i \geq M$ for all $i \in [n]$, so that such a choice of protection levels $\{C_i^{(m)}\}_{m\in\{0,\dots,M-1\}, i\in[n]} \in \Delta'_C$ exists. To compare the CR in Theorem EC.3 with the CR upper bound in Proposition EC.1, we provide the following theorem. We first define the following notation. In particular, $\{C_i^{(m)*}\}_{m\in\{0,\dots,M-1\}}$ corresponds to $\{C^{(m)*}\}_{m\in\{0,\dots,M-1\}}$ in (5), and $\{C_{i,z}^{(m)*}\}_{m\in\{0,\dots,M-1\}}$ corresponds to $\{C_z^{(m)*}\}_{m\in\{0,\dots,M-1\}}$ in (7).

$$\{C_i^{(m)*}\}_{m\in\{0,\dots,M-1\}} = \left\{ \frac{C_i}{\Gamma_i} \left(1 - \frac{D_i^{(m)}}{D_i^{(m+1)}} \right) \right\}_{m\in\{0,\dots,M-1\}}. \tag{EC.33}$$

$$\{C_{i,z}^{(m)*}\}_{m\in\{0,\dots,M-1\}} = \left\{ \left\lfloor \sum_{k=0}^{m} C_i^{(k)*} + \frac{1}{2} \right\rfloor - \left\lfloor \sum_{k=0}^{m-1} C_i^{(k)*} + \frac{1}{2} \right\rfloor \right\}_{m\in\{0,\dots,M-1\}}. \tag{EC.34}$$

Theorem EC.4. *For the assortment planning problem, PLP-AP with protection levels* $\{C_{i,z}^{(m)*}\}_{m\in\{0,\dots,M-1\}, i\in[n]}$ *achieves a CR of* $(1 - O(\frac{M}{\min_{i\in[n]} C_i}))\frac{\Psi'}{2\Gamma'} \geq (1 - O(\frac{M}{\min_{i\in[n]} C_i})) \cdot \frac{1}{4\Gamma'}$.

Theorems EC.3 and EC.4 are proved in Appendix EC.6.1. The proofs largely follow those of Theorems 1 and 2 in the admission control setting, but with a more involved establishment of dual feasibility. Similar to the admission control setting, the factor $O(M/\min_{i\in[n]} C_i)$ in Theorem EC.4 is a loss factor due to the rounding of the booking limits $\{C_i^{(m)*}\}_{m\in\{0,\ldots,M-1\}}$ to their integer counterparts $\{C_{i,z}^{(m)*}\}_{m\in\{0,\ldots,M-1\}}$.

Recovery of the CR of 1/2 for the greedy policy in the identical-duration setting. In the special case when $D_i^{(1)} = \ldots = D_i^{(M)}$ for all resources i (which relaxes the strict ordering in Assumption (i)), our choice of $\{C_i^{(m)*}\}_{m\in\{0,\ldots,M-1\}}$ in (EC.33) gives $C_i^{(0)*} = C_i$ and $C_i^{(1)*} = \ldots = C_i^{(M-1)*} = 0$, so that PLP-AP coincides with the greedy policy. In this case, the CR of PLP-AP becomes 1/2, which recovers the CR of the greedy policy in Gong et al. (2022). Indeed, in this case we have $\Gamma' = 1$, $\Psi' = 1$, and the loss factor due to rounding becomes 0 since all booking limits are integers.

Comparison with the greedy policy in the general-duration setting. We next show, via the following results, that PLP-AP outperforms the greedy policy in general in the setting of assortment planning with customer-dependent durations. The greedy algorithm in this setting is shown in Section EC.6.2. We define the following notation, where $\tilde{\Psi}_i'$ corresponds to $\tilde{\Psi}$ in (10).

$$\tilde{\Psi}_i' = \min_{m\in[M]} \frac{D_i^{(m)}/D_i^{(1)}}{\lceil D_i^{(m)}/D_i^{(1)} \rceil}, \qquad \tilde{\Psi}' = \min_{i\in[n]} \tilde{\Psi}_i'. \tag{EC.35}$$

Corollary EC.1. *For the assortment planning problem, PLP-AP with protection levels* $\{C_{i,z}^{(m)*}\}_{m\in\{0,\ldots,M-1\},i\in[n]}$ *achieves a competitive ratio of* $(1-O(\frac{M}{\min_{i\in[n]} C_i}))\frac{\Psi'}{2} \cdot \frac{1}{1-ln(\min_{i\in[n]}\{D_i^{(1)}/D_i^{(M)}\})}$.

Theorem EC.5. *For the assortment planning problem, the greedy policy has a CR of* $(\tilde{\Psi}'/2) \cdot \min_{i\in[n]}\{D_i^{(1)}/D_i^{(M)}\}$.

Corollary EC.1 follows directly from Theorem EC.4 and the fact that $\Gamma_i = M - \sum_{m=1}^{M-1} \frac{D_i^{(m)}}{D_i^{(m+1)}} \leq 1 - \ln\frac{D_i^{(1)}}{D_i^{(M)}}$ for all i (since $x \geq 1 + \ln x$ for $x > 0$). Theorem EC.5 is proved in Section EC.6.3. By inspecting Corollary EC.1 and Theorem EC.5 side by side, we see that PLP-AP outperforms the greedy policy in general. Indeed, the CR of PLP-AP has a *logarithmic* dependence on $\min_{i\in[n]}\{D_i^{(1)}/D_i^{(M)}\}$, while the CR of the greedy policy has a *linear* dependence on the same model parameter. In particular, PLP-AP with $\{C_i^{(m)*}\}_{m\in\{0,\ldots,M-1\}}$ in (EC.33) outperforms the greedy algorithm when $D_i^{(1)}, \ldots, D_i^{(M)}$ are "far away", which is defined by the following condition (denote $\epsilon_M = O(M/\min_i\{C_i\})$):

$$\frac{\Psi'(1-\epsilon_M)}{2} \cdot \frac{1}{\max_{i\in[n]}\left\{M - \sum_{m=1}^{M-1} D_i^{(m)}/D_i^{(m+1)}\right\}} \geq \frac{\tilde{\Psi}'}{2} \cdot \min_{i\in[n]}\left\{\frac{D_i^{(1)}}{D_i^{(M)}}\right\}. \tag{EC.36}$$

To get more intuition, let us specialize to $M = 2$, and the above inequality holds true when $D_i^{(2)} \geq (\tilde{\Psi}'/\Psi'(1-\epsilon_2))D_i^{(1)}$ for all resources i, which can be interpreted as the case when the longer duration type is "much longer" than the shorter duration type, where "much longer" means the condition in (EC.36). Intuitively, PLP-AP performs better than the greedy policy, since the former reserves resources in anticipation of future customers with long durations. Hence the former is able to achieve a better CR than the greedy policy on "low-then-high" instances in particular.

EC.6. Proof of CR lower bounds for assortment planning

In this section, we prove the CRs of PLP-AP in Theorems EC.3 and EC.4 in Section EC.5. In addition, we show the greedy policy and prove its CR in Theorem EC.5 in Section EC.5.

EC.6.1. Proof of Theorems EC.3 and EC.4

Our analysis involves the consideration of the linear program (LP-P-assort) and its dual formulation (LP-D-assort):

$$\begin{aligned}
\text{(LP-D-assort):}\quad \min & \sum_{t=1}^{T} z_t + \sum_{i=1}^{n} C_i \sum_{t=1}^{T} y_{i,t} && \text{(EC.37)}\\
\text{s.t.} & z_t + \sum_{i=1}^{n}\sum_{\tau=1}^{T} \mathbf{1}(t \leq \tau \leq t + D_{i,t} - 1)\phi_t(S,i) y_{i,\tau} \geq \sum_{i=1}^{n} r_i D_{i,t} \phi_t(S,i) && \forall S \in \mathcal{S}, t \in [T],\\
& z_t \geq 0 && \forall\ t \in [T],\\
& y_{i,t} \geq 0 && \forall\ i \in [n], t \in [T].
\end{aligned}$$

To proceed, we first define $\{\bar{x}_{S,t}\}_{S\in\mathcal{S},t\in[T]}$, where $\bar{x}_{S,t} = \mathbf{1}(S = S_t)$ for each $S \in \mathcal{S}, t \in [T]$. The expected reward under PLP-AP is precisely $\mathbb{E}[\sum_{t=1}^{T}\sum_{S\in\mathcal{S}}\sum_{i=1}^{n} r_i D_{i,t}\phi_t(S,i)\bar{x}_{S,t}]$. To establish Theorem EC.3, we show that there exist random variables $(\{\bar{y}_{i,t}\}_{i\in[n],t\in[T]}, \{\bar{z}_t\}_{t\in[T]})$ such that

1. $(\{\mathbb{E}[\bar{y}_{i,t}]\}_{i\in[n],t\in[T]}, \{\mathbb{E}[\bar{z}_t]\}_{t\in[T]})$ is feasible for (LP-D-assort).
2. the inequality $\mathbb{E}[\sum_{t=1}^{T}\sum_{S\in\mathcal{S}}\sum_{i=1}^{n} r_i D_{i,t}\phi_t(S,i)\bar{x}_{S,t}] \geq \frac{1}{2}\Psi'\cdot\Lambda'\cdot\mathbb{E}[\sum_{t=1}^{T}\bar{z}_t + \sum_{i=1}^{n} C_i \sum_{t=1}^{T}\bar{y}_{i,t}]$ holds.

As shown in our forthcoming construction, $(\{\bar{y}_{i,t}\}_{i\in[n],t\in[T]}, \{\bar{z}_t\}_{t\in[T]})$ are random variables that correlate with the randomness in the online dynamics.

Given random variables $(\{\bar{y}_{i,t}\}_{i\in[n],t\in[T]}, \{\bar{z}_t\}_{t\in[T]})$ that satisfy properties 1 and 2, Theorem EC.3 is established as follows. To start with, we observe that opt(LP-P-assort) = opt(LP-D-assort). Denote the revenue achieved by PLP-AP as P. We have

$$\begin{aligned}
\mathbb{E}[\mathrm{P}] &= \mathbb{E}\left[\sum_{t=1}^{T}\sum_{S\in\mathcal{S}}\sum_{i=1}^{n} r_i D_{i,t}\phi_t(S,i)\bar{x}_{S,t}\right] \geq \frac{1}{2}\Psi'\cdot\Lambda'\cdot\mathbb{E}\left[\left(\sum_{t=1}^{T}\bar{z}_t + \sum_{i=1}^{n} C_i\sum_{t=1}^{T}\bar{y}_{i,t}\right)\right] && \text{(EC.38)}\\
&\geq \frac{1}{2}\Psi'\cdot\Lambda'\cdot\mathrm{opt}(\text{LP-D-assort}) \geq \frac{1}{2}\Psi'\cdot\Lambda'\cdot\mathrm{opt}(\text{LP-P-assort}), && \text{(EC.39)}
\end{aligned}$$

where the inequality in (EC.38) is by property 2, and the first inequality in (EC.39) is by property 1. Thus, the theorem is proved.

We construct such dual variables with an Auxiliary Algorithm (AA-AP) as in Algorithm 8. Similar to the AA in the admission control setting, the AA-AP takes the realized decisions $\{\bar{x}_{S,t}\}_{t\in[T],S\in\mathcal{S}}$ and rejection prices $\{R_{i,t}\}_{t\in[T],i\in[n]}$ as input, and AA-AP outputs a dual solution $(\{\bar{y}_{i,t}\}_{i\in[n],t\in[T]},\{\bar{z}_t\}_{t\in[T]})$ that satisfies properties 1 and 2.

Algorithm 8 Auxiliary Algorithm for Assortment Planning (AA-AP)

for $i=1,\dots,n$ **do**
  for $m=1,\dots,M$ **do**
    Initialize $A_i^{(m)}=\{t\in[T]: R_{i,t}\geq r_i D_i^{(m)}\}$, $\Phi_i^{(m)}=\emptyset$, $\text{Charge}_i^{(m)}=\emptyset$.
    For $t\in[T]$, initialize $\text{Charge}_{i,t}^{(m)}=\emptyset, \bar{y}_{i,t}^{(m)}=0$.
  end for
end for
for $i=1,\dots,n$ **do**
  for $m=1,\dots,M$ **do**
    while $A_i^{(m)}\neq\emptyset$ **do**
      Identify $\bar{t}=\max\{t: t\in A_i^{(m)}\}$.
      Set $\Phi_i^{(m)}\leftarrow\Phi_i^{(m)}\cup\{\bar{t}\}$.
      Set $\text{Charge}_{i,\bar{t}}^{(m)}=\{\tau:\bar{t}-D_{i,\tau}+1\leq\tau\leq\bar{t},\ \mathbf{1}(i \text{ is chosen at } \tau)=1,\ R_{i,\tau}=r_i D_i^{(m-1)}\}$.
      Set $\text{Charge}_i^{(m)}\leftarrow\text{Charge}_i^{(m)}\cup\text{Charge}_{i,\bar{t}}^{(m)}$.
      Set $\bar{y}_{i,\bar{t}}^{(m)}=r_i(D_i^{(m)}-D_i^{(m-1)})$.
      Set $A_i^{(m)}\leftarrow A_i^{(m)}\setminus\{\bar{t}-D_i^{(m)}+1,\dots,\bar{t}\}$.
    end while
  end for
  for $t=1,\dots,T$ **do**
    Set $\bar{y}_{i,t}=\sum_{m=1}^{M}\bar{y}_{i,t}^{(m)}$.
  end for
end for
for $t=1,\dots,T$ **do**
  Set $\bar{z}_t=\sum_{i=1}^{n}(r_i D_{i,t}-R_{i,t})\mathbf{1}(i \text{ is chosen at } t)$.
end for
Return $(\{\bar{y}_{i,t}\}_{t\in[T],i\in[n]},\{\bar{z}_t\}_{t\in[T]})$.

Next, we proceed to establish properties 1 and 2. We start by stating the following lemmas which are essential for proving the feasibility.

LEMMA EC.7. *For each* $t \in \{t' \in [T] : R_{i,t'} \geq r_i D_i^{(m)}\}$*, there exists a time index* $\bar{t} \in \Phi_i^{(m)}$ *such that* $t \leq \bar{t} \leq t + D_i^{(m)} - 1$.

LEMMA EC.8. *If* $R_{i,t} = r_i D_i^{(m)}$*, then it holds that* $\sum_{\tau=1}^{T} \mathbf{1}(t \leq \tau \leq t + D_{i,t} - 1)\bar{y}_{i,\tau} \geq \min\{r_i D_i^{(m)}, r_i D_{i,t}\}$.

Note that Lemmas EC.7 and EC.8 are analogous to Lemmas 1 and 2 in admission control (in Section 4.2) respectively. Lemma EC.7 is useful to show Lemma EC.8, since the former ensures the existence of a time index τ such that $\bar{y}_{i,\tau} > 0$ and $\mathbf{1}(t \leq \tau \leq t + D_i^{(m)} - 1) = 1$ for any $t \in A_i^{(m)}$. The proofs of Lemmas EC.7 and EC.8 follow those of Lemmas 1 and 2 in the admission control setting, with the addition of the subscript i to the relevant quantities, since the lemmas here concern resource type i. Hence, we omit the proofs of Lemmas EC.7 and EC.8. Now we are ready to establish the dual feasibility. To show that the expectations of the dual variables are feasible, we take the expectation of $\bar{z}_t$ conditioned on (i) $\mathcal{F}_{t-1}$, the history up to the end of time step $t-1$, and (ii) the duration type and the choice model of the customer at time t, $(\{D_{i,t}\}_{i\in[n]}, \phi_t)$:

$$\begin{aligned}\mathbb{E}[\bar{z}_t | \mathcal{F}_{t-1}, (\{D_{i,t}\}_{i\in[n]}, \phi_t)] &= \mathbb{E}\left[\sum_{i=1}^{n} (r_i D_{i,t} - R_{i,t}) \mathbf{1}\{i \text{ is chosen at } t\} | \mathcal{F}_{t-1}, (\{D_{i,t}\}_{i\in[n]}, \phi_t)\right] \\ &= \sum_{i=1}^{n} (r_i D_{i,t} - R_{i,t}) \phi_t(S_t, i) = \max_{S\in\mathcal{S}} \left\{\sum_{i=1}^{n} (r_i D_{i,t} - R_{i,t}) \phi_t(S, i)\right\},\end{aligned}$$

where the last equality is because $S_t = \text{argmax}_{S\in\mathcal{S}} \sum_{i=1}^{n} (r_i D_{i,t} - R_{i,t}) \phi_t(S, i)$. The weak substitutability assumption ensures that a product i with $r_i D_{i,t} - R_{i,t} \leq 0$ is not included in S_t. Therefore, we have

$$\begin{aligned}\mathbb{E}[\bar{z}_t | \mathcal{F}_{t-1}, (\{D_{i,t}\}_{i\in[n]}, \phi_t)] &= \max_{S\in\mathcal{S}} \left\{\sum_{i=1}^{n} (r_i D_{i,t} - R_{i,t}) \phi_t(S, i)\right\} \\ &\geq \max_{S\in\mathcal{S}} \left\{\sum_{i=1}^{n} \left(r_i D_{i,t} - \sum_{\tau=1}^{T} \mathbf{1}\{t \leq \tau \leq t + D_{i,t} - 1\} \bar{y}_{i,\tau}\right) \phi_t(S, i)\right\}, \qquad \text{(EC.40)}\end{aligned}$$

where (EC.40) is because $\sum_{\tau=1}^{T} \mathbf{1}\{t \leq \tau \leq t + D_{i,t} - 1\} \bar{y}_{i,\tau} \geq \min\{R_{i,t}, r_i D_{i,t}\}$ for all $i \in [n]$ and $t \in [T]$ according to Lemma EC.8. Therefore, the feasibility of the expected values of the dual variables is ensured.

Next, we prove that property 2 holds. To proceed, we lower bound the expectation of the primal objective value in terms of the expectation of the dual objective value. More precisely, we show that

$$\mathbb{E}[P] \geq \mathbb{E}\left[\sum_{t=1}^{T} \bar{z}_t\right], \quad \mathbb{E}[P] \geq \Lambda' \cdot \Psi' \cdot \mathbb{E}\left[\sum_{i=1}^{n} C_i \sum_{t=1}^{T} \bar{y}_{i,t}\right]. \qquad \text{(EC.41)}$$

We first establish the first inequality in (EC.41):

$$\begin{aligned}\mathbb{E}[P] &= \mathbb{E}\left[\sum_{t=1}^{T}\sum_{S\in\mathcal{S}}\sum_{i=1}^{n} r_i D_{i,t}\phi_t(S,i)\cdot \bar{x}_{S,t}\right] = \mathbb{E}\left[\sum_{t=1}^{T}\sum_{i=1}^{n} r_i D_{i,t}\phi_t(S_t,i)\right] \\ &= \mathbb{E}\left[\sum_{t=1}^{T}\mathbb{E}\left[\sum_{i=1}^{n} r_i D_{i,t}\mathbf{1}\{i \text{ is chosen at } t\}|(\phi_t,\{D_{i,t}\}_{i=1}^n), S_t\right]\right] \\ &= \mathbb{E}\left[\sum_{t=1}^{T}\sum_{i=1}^{n} r_i D_{i,t}\mathbf{1}\{i \text{ is chosen at } t\}\right] \geq \mathbb{E}\left[\sum_{t=1}^{T}\bar{z}_t\right].\end{aligned}$$

In the second-to-last equality, the inner conditional expectation is purely over the random choice of resource in $S_t\cup\{0\}$, conditioned on the adversarial choice of customer type $(\phi_t,\{D_{i,t}\}_{i=1}^n)$ which specifies the choice model ϕ_t and the subsequent choice of assortment S_t by the DM. The last inequality is because for all $t\in[T]$,

$$\sum_{i=1}^{n} r_i D_{i,t}\mathbf{1}\{i \text{ is chosen at } t\} = \bar{z}_t + \sum_{i=1}^{n} R_{i,t}\mathbf{1}\{i \text{ is chosen at } t\} \geq \bar{z}_t.$$

Before we establish the second inequality in (EC.41), we provide the following essential lemmas:

Lemma EC.9. *For distinct $\bar{t},\bar{t}'\in\Phi_i^{(m)}$, it holds that $|\bar{t}-\bar{t}'|\geq D_i^{(m)}$.*

Lemma EC.10. *If $t\in Charge_i^{(m)}$, then $t\in Charge_{i,\tau}^{(m)}$ for at most $\lceil D_{i,t}/D_i^{(m)}\rceil$ values of $\tau\in[T]$. In addition, $R_{i,t}=r_i D_i^{(m-1)}$.*

Note that Lemmas EC.9 and EC.10 are analogous to Lemmas 3 and 5 in admission control (in Section 4.2) respectively. Lemma EC.9 is useful for proving Lemma EC.10, since the former establishes a lower bound on the distance between consecutive time steps belonging to $\Phi_i^{(m)}$, which leads to an upper bound on the number of charge sets that a time step can belong to. The proofs of Lemmas EC.9 and EC.10 follow those of Lemmas 3 and 5 in the admission control setting, with the addition of the subscript i to the relevant quantities, since the lemmas here concern resource type i. Hence, we omit the proofs of Lemmas EC.9 and EC.10. Now we are ready to lower bound the primal objective value in terms of the second term of the dual objective value.

$$\mathbb{E}[P] = \mathbb{E}\left[\sum_{t=1}^{T}\sum_{i=1}^{n} r_i D_{i,t}\mathbf{1}\{i \text{ is chosen at } t\}\right] \geq \mathbb{E}\left[\sum_{i=1}^{n}\sum_{m=1}^{M}\sum_{t\in \text{Charge}_i^{(m)}} r_i D_{i,t}\right] \tag{EC.42}$$

$$\geq \mathbb{E}\left[\sum_{i=1}^{n}\sum_{m=1}^{M}\sum_{t\in \text{Charge}_i^{(m)}} r_i D_i^{(m)} \sum_{\bar{t}\in\Phi_i^{(m)}} \mathbf{1}\{t\in \text{Charge}_{i,\bar{t}}^{(m)}\}\cdot \frac{D_{i,t}/D_i^{(m)}}{\left\lceil D_{i,t}/D_i^{(m)}\right\rceil}\right] \tag{EC.43}$$

$$= \mathbb{E}\left[\sum_{i=1}^{n}\sum_{m=1}^{M}\sum_{\bar{t}\in\Phi_i^{(m)}}\sum_{t\in \text{Charge}_{i,\bar{t}}^{(m)}} r_i D_i^{(m)}\cdot \frac{D_{i,t}/D_i^{(m)}}{\left\lceil D_{i,t}/D_i^{(m)}\right\rceil}\right],$$

where the last inequality in (EC.42) follows from the definition of $\text{Charge}_i^{(m)}$ in Line 12 in AA-AP, and (EC.43) is because of Lemma EC.10. Recall the definitions of Ψ_i, Ψ' and Λ' as in (EC.31) and (EC.32). We have

$$\begin{aligned}
\mathbb{E}[P] &\geq \mathbb{E}\left[\sum_{i=1}^{n}\sum_{m=1}^{M}\sum_{\bar{t}\in\Phi_i^{(m)}} \Psi_i \cdot C_i^{(m-1)} r_i D_i^{(m)}\right] && \text{(EC.44)}\\
&= \mathbb{E}\left[\sum_{i=1}^{n}\sum_{m=1}^{M} \frac{C_i^{(m-1)} D_i^{(m)}}{C_i(D_i^{(m)} - D_i^{(m-1)})} \cdot \Psi_i \cdot C_i \sum_{\bar{t}\in\Phi_i^{(m)}} r_i(D_i^{(m)} - D_i^{(m-1)})\right]\\
&= \mathbb{E}\left[\sum_{i=1}^{n}\sum_{m=1}^{M} \frac{C_i^{(m-1)} D_i^{(m)}}{C_i(D_i^{(m)} - D_i^{(m-1)})} \cdot \Psi_i \cdot C_i \sum_{t=1}^{T} \bar{y}_{i,t}^{(m)}\right]\\
&\geq \mathbb{E}\left[\sum_{i=1}^{n} \min_{m\in[M]} \frac{C_i^{(m-1)} D_i^{(m)}}{C_i(D_i^{(m)} - D_i^{(m-1)})} \cdot \Psi_i \cdot C_i \sum_{t=1}^{T} \bar{y}_{i,t}\right]\\
&\geq \min_{m\in[M], i\in[n]} \left\{\frac{C_i^{(m-1)} D_i^{(m)}}{C_i(D_i^{(m)} - D_i^{(m-1)})}\right\} \cdot \min_{i\in[n]} \Psi_i \cdot \mathbb{E}\left[\sum_{i=1}^{n} C_i \sum_{t=1}^{T} \bar{y}_{i,t}\right]\\
&= \Lambda' \cdot \Psi' \cdot \mathbb{E}\left[\sum_{i=1}^{n} C_i \sum_{t=1}^{T} \bar{y}_{i,t}\right],
\end{aligned}$$

where (EC.44) is because each non-empty $\text{Charge}_{i,\bar{t}}^{(m)}$ set contains exactly $C_i^{(m-1)}$ time steps. This is true because of the definitions of $R_{i,t}$ and $I_{i,t}^{(m)}$ and the way $\text{Charge}_{i,t}^{(m)}$ is constructed in AA-AP. Since we also have $\mathbb{E}[P] \geq \mathbb{E}[\sum_{t=1}^{T} \bar{z}_t]$, we have

$$\mathbb{E}[P] \geq \frac{1}{2} \cdot \Lambda' \cdot \Psi' \cdot \mathbb{E}\left[\sum_{t=1}^{T} \bar{z}_t + \sum_{i=1}^{n} C_i \sum_{t=1}^{T} \bar{y}_{i,t}\right] = \frac{1}{2} \cdot \Lambda' \cdot \Psi' \cdot \mathbb{E}[D],$$

where D is the dual objective value achieved by the dual variables in AA-AP. Therefore, property 2 is established. Combining this result with the feasibility established in property 1, Theorem EC.3 is proved.

By setting $\{C_i^{(m-1)}\}_{m\in[M]}$ to be $\{C_i^{(m-1)*}\}_{m\in[M]}$ as in (EC.33) for all $i \in [n]$, Λ' becomes $1/\Gamma'$. Therefore, we have

$$\mathbb{E}[P] \geq \frac{\Psi'}{2\Gamma'} \cdot \mathbb{E}[D].$$

Similar to the admission control setting, the loss of CR due to the rounding of $\{C_i^{(m)*}\}_{m\in\{0,\ldots,M-1\}}$ as in (EC.33) to $\{C_{i,z}^{(m)*}\}_{m\in\{0,\ldots,M-1\}}$ as in (EC.34) can be upper bounded. In particular, by following the derivation of the corresponding loss term in Section EC.1.8, we can show that

$$\text{CR of PLP-AP with } \{C_{i,z}^{(m)*}\}_{m\in\{0,\ldots,M-1\}, i\in[n]} \geq \left(1 - O\left(\frac{M}{\min_{i\in[n]} C_i}\right)\right) \cdot \frac{\Psi'}{2\Gamma'}.$$

With this, Theorem EC.4 is proved.

EC.6.2. The greedy policy in assortment planning

We show the greedy algorithm in Algorithm 9. The greedy algorithm is cast as a special case of PLP-AP by setting $C_i^{(0)} = C_i$ and $C_i^{(1)} = \ldots = C_i^{(M-1)} = 0$ for each i (allowing zero booking limits). Consequently, in the specialization, it suffices to consider the occupancy measure

$$I_{i,t}^{(0)} = \sum_{\tau=1}^{t-1} \mathbf{1}(i \text{ is chosen at } \tau) \cdot \mathbf{1}(t \le \tau + D_{i,\tau} - 1), \tag{EC.45}$$

and it is not necessary to track $I_{i,t}^{(m)}$ for $m \in [M-1]$.

Algorithm 9 Greedy policy, phrased as a special case of PLP-AP

1: **for** $t = 1, \ldots, T$ **do**

2: **for** $i = 1, \ldots, n$ **do**

3: Set the rejection price $R_{i,t}$, based on the occupancy measure $I_{i,t}^{(0)}$ (see equation (EC.45) for the definition of $I_{i,t}^{(0)}$):

$$R_{i,t} = \begin{cases} 0 & \text{if } I_{i,t}^{(0)} < C_i \\ r_i D_i^{(M)} & \text{if } I_{i,t}^{(0)} = C_i \end{cases}$$

4: **end for**

5: Offer assortment $S_t = \operatorname{argmax}_{S \in \mathcal{S}} \sum_{i \in [n]} (r_i D_{i,t} - R_{i,t}) \phi_t(S, i)$. If multiple optimal assortments exist, we choose an optimal assortment of minimum cardinality.

6: **end for**

EC.6.3. Proof of Theorem EC.5

We prove the theorem in a manner similar to the analysis in Section EC.6.1. We show that there exist random variables $(\{\bar{y}_{i,t}\}_{i \in [n], t \in [T]}, \{\bar{z}_t\}_{t \in [T]})$ such that

1. $(\{\mathbb{E}[\bar{y}_{i,t}]\}_{i \in [n], t \in [T]}, \{\mathbb{E}[\bar{z}_t]\}_{t \in [T]})$ is feasible for (LP-D-assort).
2. the inequality $\mathbb{E}[\sum_{t=1}^{T} \sum_{S \in \mathcal{S}} \sum_{i=1}^{n} r_i D_{i,t} \phi_t(S, i) \bar{x}_{S,t}] \ge \frac{\tilde{\Psi}'}{2} \cdot \min_{i \in [n]} \left\{ \frac{D_i^{(1)}}{D_i^{(M)}} \right\} \cdot \mathbb{E}[\sum_{t=1}^{T} \bar{z}_t + \sum_{i=1}^{n} C_i \sum_{t=1}^{T} \bar{y}_{i,t}]$ holds.

$(\{\bar{y}_{i,t}\}_{i \in [n], t \in [T]}, \{\bar{z}_t\}_{t \in [T]})$ are random variables that correlate with the randomness of the online dynamics. In property 2, we recall that $\bar{x}_{S,t} \in \{0, 1\}$ is the online decision made by the greedy policy at time t for assortment S, where we recall that $\bar{x}_{S,t} = \mathbf{1}(\text{assortment } S \text{ is offered at time } t) = \mathbf{1}(S = S_t)$. We construct such a feasible dual solution with the following Auxiliary Algorithm for the greedy policy.

The feasibility of the dual variables can be proved in a manner similar to the proof of feasibility for AA-AP. However, Lemmas EC.7 and EC.8 need to be replaced by the following two lemmas, respectively.

Algorithm 10 Auxiliary Algorithm for Greedy policy (AA-GP)

1: **for** $i = 1, \ldots, n$ **do**
2: Initialize $A_i = \{t \in [T] : R_{i,t} = r_i D_i^{(M)}\}$, $\Phi_i = \emptyset$, $\text{Charge}_i = \emptyset$.
3: For $t \in [T]$, initialize $\text{Charge}_{i,t} = \emptyset, \bar{y}_{i,t} = 0$.
4: **end for**
5: **for** $i = 1, \ldots, n$ **do**
6: **while** $A_i \neq \emptyset$ **do**
7: Identify $\bar{t} = \max\{t : t \in A_i\}$.
8: Set $\Phi_i \leftarrow \Phi_i \cup \{\bar{t}\}$.
9: Set $\text{Charge}_{i,\bar{t}} = \{\tau : \bar{t} - D_{i,\tau} + 1 \leq \tau \leq \bar{t},\ \mathbf{1}(i \text{ is chosen at } \tau) = 1\}$.
10: Set $\text{Charge}_i \leftarrow \text{Charge}_i \cup \text{Charge}_{i,\bar{t}}$.
11: Set $\bar{y}_{i,\bar{t}} = r_i D_i^{(M)}$.
12: Set $A_i \leftarrow A_i \setminus \{\bar{t} - D_i^{(1)} + 1, \ldots, \bar{t}\}$.
13: **end while**
14: **end for**
15: **for** $t = 1, \ldots, T$ **do**
16: Set $\bar{z}_t = \sum_{i=1}^{n} (r_i D_{i,t} - R_{i,t}) \mathbf{1}(i \text{ is chosen at } t)$.
17: **end for**
18: Return $(\{\bar{y}_{i,t}\}_{t \in [T], i \in [n]}, \{\bar{z}_t\}_{t \in [T]})$.

LEMMA EC.11. *For each $t \in \{t' \in [T] : R_{i,t'} = r_i D_i^{(M)}\}$, there exists a time index $\bar{t} \in \Phi_i$ such that $t \leq \bar{t} \leq t + D_i^{(1)} - 1$.*

LEMMA EC.12. *If $R_{i,t} = r_i D_i^{(M)}$, then it holds that $\sum_{\tau=1}^{T} \mathbf{1}(t \leq \tau \leq t + D_{i,t} - 1) \bar{y}_{i,\tau} \geq r_i D_i^{(M)}$.*

Lemmas EC.11 and EC.12 resemble Lemmas EC.7 and EC.8 in the analysis of PLP-AP, respectively, and are proved based on the way $\bar{t}$ is identified in Line 7 of AA-GP. Hence, we omit the proofs.

To establish property 2, which lower bounds the primal objective value by the dual objective value times the CR, we first need to replace Lemmas EC.9 and EC.10 with the following two lemmas, respectively.

LEMMA EC.13. *For distinct $\bar{t}, \bar{t}' \in \Phi_i$, it holds that $|\bar{t} - \bar{t}'| \geq D_i^{(1)}$.*

LEMMA EC.14. *If $t \in \text{Charge}_i$, then $t \in \text{Charge}_{i,\tau}$ for at most $\lceil D_{i,t}/D_i^{(1)} \rceil$ values of $\tau \in [T]$.*

Again, Lemmas EC.13 and EC.14 follow Lemmas EC.9 and EC.10 in the analysis of PLP-AP, respectively, so we omit the proofs. Denote the revenue achieved by the greedy policy as P. We can show that $\mathbb{E}[\text{P}] \geq \mathbb{E}[\sum_{t \in [T]} \bar{z}_t]$ in a similar manner as in the proof of the CR of PLP-AP in Section EC.6.1. Now we bound $\mathbb{E}[\text{P}]$ in terms of the second term of the dual objective function.

$$\mathbb{E}[P] = \mathbb{E}\left[\sum_{t=1}^{T}\sum_{i=1}^{n} r_i D_{i,t}\mathbf{1}\{i \text{ is chosen at } t\}\right] \geq \mathbb{E}\left[\sum_{i=1}^{n}\sum_{t\in\text{Charge}_i} r_i D_{i,t}\right] \tag{EC.46}$$

$$\geq \mathbb{E}\left[\sum_{i=1}^{n}\sum_{t\in\text{Charge}_i} r_i D_i^{(1)} \sum_{\bar{t}\in\Phi_i}\mathbf{1}\{t\in\text{Charge}_{i,\bar{t}}\}\cdot\frac{\frac{D_{i,t}}{D_i^{(1)}}}{\left\lceil\frac{D_{i,t}}{D_i^{(1)}}\right\rceil}\right] \tag{EC.47}$$

$$= \mathbb{E}\left[\sum_{i=1}^{n}\sum_{\bar{t}\in\Phi_i}\sum_{t\in\text{Charge}_{i,\bar{t}}} r_i D_i^{(1)}\cdot\frac{\frac{D_{i,t}}{D_i^{(1)}}}{\left\lceil\frac{D_{i,t}}{D_i^{(1)}}\right\rceil}\right],$$

where the last inequality in (EC.46) follows from the definition of Charge_i in Line 9 in AA-GP, and (EC.47) is because of Lemma EC.14. Recall the definitions of $\tilde{\Psi}_i'$ and $\tilde{\Psi}'$ as in (EC.35). Therefore, we have

$$\mathbb{E}[P] \geq \mathbb{E}\left[\sum_{i=1}^{n}\sum_{\bar{t}\in\Phi_i}\tilde{\Psi}_i'\cdot C_i r_i D_i^{(1)}\right] \tag{EC.48}$$

$$= \mathbb{E}\left[\sum_{i=1}^{n}\frac{D_i^{(1)}}{D_i^{(M)}}\cdot\tilde{\Psi}_i'\cdot C_i\sum_{\bar{t}\in\Phi_i} r_i D_i^{(M)}\right]$$

$$= \mathbb{E}\left[\sum_{i=1}^{n}\frac{D_i^{(1)}}{D_i^{(M)}}\cdot\tilde{\Psi}_i'\cdot C_i\sum_{t=1}^{T}\bar{y}_{i,t}\right]$$

$$\geq \tilde{\Psi}'\cdot\min_{i\in[n]}\left\{\frac{D_i^{(1)}}{D_i^{(M)}}\right\}\cdot\mathbb{E}\left[\sum_{i=1}^{n} C_i\sum_{t=1}^{T}\bar{y}_{i,t}\right],$$

where (EC.48) is because each non-empty $\text{Charge}_{i,\bar{t}}$ set contains exactly C_i time steps. This is true because of the definitions of $R_{i,t}$ and $I_{i,t}^{(0)}$ and the way $\text{Charge}_{i,t}$ is constructed in AA-GP. Since we also have $\mathbb{E}[P]\geq\mathbb{E}[\sum_{t=1}^{T}\bar{z}_t]$, we have

$$2\mathbb{E}[P] \geq \mathbb{E}\left[\sum_{t=1}^{T}\bar{z}_t\right] + \tilde{\Psi}'\cdot\min_{i\in[n]}\left\{\frac{D_i^{(1)}}{D_i^{(M)}}\right\}\cdot\mathbb{E}\left[\sum_{i=1}^{n} C_i\sum_{t=1}^{T}\bar{y}_{i,t}\right],$$

$$\mathbb{E}[P] \geq \frac{\tilde{\Psi}'}{2}\cdot\min_{i\in[n]}\left\{\frac{D_i^{(1)}}{D_i^{(M)}}\right\}\cdot\mathbb{E}\left[\sum_{t=1}^{T}\bar{z}_t + \sum_{i=1}^{n} C_i\sum_{t=1}^{T}\bar{y}_{i,t}\right] = \frac{\tilde{\Psi}'}{2}\cdot\min_{i\in[n]}\left\{\frac{D_i^{(1)}}{D_i^{(M)}}\right\}\cdot\mathbb{E}[D],$$

where D is the dual objective value achieved by the dual variables in AA-GP. Therefore, property 2 is established, and combining with the feasibility in property 1, Theorem EC.5 is proved.

EC.7. Proof of competitive ratio upper bounds

In this section, we prove the CR upper bounds in Propositions 1, 2 in admission control in Section 3.2, and in Proposition EC.1 in assortment planning in Section EC.5.

EC.7.1. Proof of CR upper bound (Proposition 1)

Proof of Proposition 1 The proof for the upper bound of $1/\Gamma$ largely follows the proofs for Proposition 3 and Theorem 4 from Ball and Queyranne (2009), where Yao's minimax principle is used to show the best possible competitive ratio. In short, the bound of $1/\Gamma$ is due to difficult instances with a low-then-high structure, similar to the non-reusable setting. The set of extreme instances in the proof for Theorem 4 from Ball and Queyranne (2009) becomes I_i $(i=1,\dots,M)$, where in instance I_i a sequence of $i\cdot C$ requests, namely, C requests for each price class $1,\dots,i$, arrive in this order. This is to suit our model where 1 denotes the lowest price class, and M denotes the highest price class. Observe that the optimal offline revenue is still $R^*(I_i)=r^{(i)}D^{(i)}C$ for all $i\in\{1,\dots,M\}$, because the shortest duration among all classes is still strictly larger than $M\cdot C$, which is the horizon length of the longest extreme instance I_M. Consequently, no unit will return in any of the instances. As an adjustment of the notation, the random instance $\hat{I}$ realizes as instance I_M with probability $p_M=1/\Gamma$, and realizes as instance I_i with probability $p_i=(1-\frac{r^{(i)}D^{(i)}}{r^{(i+1)}D^{(i+1)}})/\Gamma$ for $i\in[M-1]$. By the definition of Γ, we have $\sum_{i=1}^{M}p_i=1$. Let x_i^A denote the number of requests in class i accepted by algorithm A when presented with instance I_M. Note that A has no way of knowing whether it is facing instance I_i or some I_j with $j>i$ (or some other instance) before seeing the first $i\cdot C+1$ requests in the stated sequence. Therefore, the revenue earned by applying algorithm A to these instances is $R(A,I_i)=\sum_{j=1}^{i}r^{(j)}D^{(j)}x_j^A$, for all $i\in\{1,\dots,M\}$.

Let c^* denote the best possible CR over all online policies, $\mathcal{A}$ the set of deterministic online policies, and x_j^A the number of class j customers accepted by policy A (which is the same across the instances I_i with $i\geq j$, since the instances share the same prefix). Let $P=\{x\in\mathbb{R}^M_{\geq 0}:\sum_{j=1}^{M}x_j\leq C\}$ denote the set of feasible acceptance vectors, which contains x^A for every $A\in\mathcal{A}$ since no unit is released within the horizon of any I_i. After adapting the last inequality in Ball and Queyranne (2009) to our notation and definitions, we have the following as the last step in proving the proposition:

$$c^*\leq\sup_{A\in\mathcal{A}}\sum_{i=1}^{M}p_i\frac{\sum_{j=1}^{i}r^{(j)}D^{(j)}x_j^A}{r^{(i)}D^{(i)}C}\leq\sup_{x^A\in P}\sum_{j=1}^{M}\left(\sum_{i=j}^{M}p_i\frac{r^{(j)}D^{(j)}}{r^{(i)}D^{(i)}}\right)\frac{x_j^A}{C}=\sup_{x^A\in P}\sum_{j=1}^{M}\frac{1}{\Gamma}\frac{x_j^A}{C}=\frac{1}{\Gamma}.$$

This completes the proof. ■

EC.7.2. Proof of CR upper bound involving η and Ψ (Proposition 2)

Proof of Proposition 2 The $1/\Gamma$ inside the minimum is derived from the same set of low-then-high instances as in the proof for the monotone case in Section EC.7.1. Therefore, we focus on the upper bound of η here, which is derived from a difficult high-then-low instance.

Recall that $\eta = \min_{m'>m,m',m\in[M]} \frac{D^{(m')}/D^{(m)}}{\lceil D^{(m')}/D^{(m)}\rceil} \cdot \frac{r^{(m')}}{r^{(m)}}$. We also know that $(i',j') = \operatorname{argmin}_{\{(m,m')\}_{m'>m,m,m'\in[M]}} \frac{D^{(m')}/D^{(m)}}{\lceil D^{(m')}/D^{(m)}\rceil} \cdot \frac{r^{(m')}}{r^{(m)}}$, which means that $\eta = \frac{D^{(j')}/D^{(i')}}{\lceil D^{(j')}/D^{(i')}\rceil} \cdot \frac{r^{(j')}}{r^{(i')}}$. Now, consider a high-then-low instance I with C arrivals of class j' customers, followed by $D^{(j')} - C$ class i' customers. The arrival instance is shown below.

$$I = \underbrace{\{\underbrace{j', j', \ldots, j'}_{C \text{ customers}}, i', i', \ldots, i'\}}_{D^{(j')} \text{ customers}}.$$

Note that since $D^{(j')} - D^{(i')}\lfloor D^{(j')}/D^{(i')}\rfloor > 2C$, it holds that $\left\lceil (D^{(j')} - 2C)/D^{(i')}\right\rceil = \left\lceil D^{(j')}/D^{(i')}\right\rceil$. Therefore, $\lceil D^{(j')}/D^{(i')}\rceil$ rounds of allocations (each round contains allocations of C resource units) can be made to class i' customers during $[C+1, D^{(j')}]$, generating a total revenue of $\lceil D^{(j')}/D^{(i')}\rceil \cdot Cr^{(i')}D^{(i')}$. On the other hand, any protection level policy would allocate at most C units to class j' customers who arrive during $[1, C]$, and reject all class i' customers who arrive later. Indeed, since $j' > i'$, by the time the first class i' customer arrives, the rejection price is already $r^{(j')}D^{(j')} \geq r^{(i')}D^{(i')}$, and therefore by definition of online protection level policies, the booking limit for class i' customers would have been reached. Consequently, the revenue achieved by any protection level policy is at most $Cr^{(j')}D^{(j')}$. Suppose $(i,j) = \operatorname{argmin}_{\{(m,m')\}_{m'>m,m,m'\in[M]}} \frac{D^{(m')}/D^{(m)}}{\lceil D^{(m')}/D^{(m)}\rceil}$. That is, $\Psi = \frac{D^{(j)}/D^{(i)}}{\lceil D^{(j)}/D^{(i)}\rceil}$. Consequently, the CR of any online protection level policy is at most

$$\begin{aligned}
&\frac{Cr^{(j')}D^{(j')}}{C\lceil D^{(j')}/D^{(i')}\rceil r^{(i')}D^{(i')}} = \frac{D^{(j')}/D^{(i')}}{\lceil D^{(j')}/D^{(i')}\rceil} \cdot \frac{r^{(j')}}{r^{(i')}} = \eta \\
&= \min_{\{(m,m')\}_{m'>m,m,m'\in[M]}} \frac{D^{(m')}/D^{(m)}}{\lceil D^{(m')}/D^{(m)}\rceil} \cdot \frac{r^{(m')}}{r^{(m)}} \leq \frac{D^{(j)}/D^{(i)}}{\lceil D^{(j)}/D^{(i)}\rceil} \cdot \frac{r^{(j)}}{r^{(i)}} = \Psi \cdot \frac{r^{(j)}}{r^{(i)}}.
\end{aligned}$$

This completes the proof. ■

EC.7.3. Proof of CR upper bound for assortment planning (Proposition EC.1)

Proof of Proposition EC.1 Let $\mathcal{I}$ denote the set of instances of the assortment planning problem, and $\mathcal{Q}$ denote the set of all probability distributions on the instance set $\mathcal{I}$. Let $I(Q)$ denote a random instance chosen according to probability distribution $Q \in \mathcal{Q}$. Let $\mathcal{A}$ denote the set of all deterministic online algorithms. Let $R(A, I(Q))$ denote the revenue obtained by applying $A \in \mathcal{A}$ to instance $I(Q)$, and $R^*(I(Q))$ denote the maximum revenue achievable by an offline algorithm for this instance. According to Yao's minimax principle as in Seiden (2000), for any $Q \in \mathcal{Q}$, the best possible competitive ratio c^* satisfies

$$c^* \leq \sup_{A\in\mathcal{A}} \mathbb{E}_Q \left[\frac{R(A, I(Q))}{R^*(I(Q))}\right]. \tag{EC.49}$$

Consider M types of picky customers who are only interested in product type i, that is, $\phi^{(m)}(S,i)=1$ if $i\in S$ and $\phi^{(m)}(S,j)=0$ for all $j\in S\backslash\{i\}$. We call these customers the 'i-craving' customers. Note that though they are all interested in type i product, each customer duration type has its own requested usage duration. Consider a set of extreme instances $\{I_i^{(m)}\}_{m\in[M]}$ where in instance $I_i^{(m)}$ a sequence of $m\cdot C_i$ requests, namely, C_i 'i-craving' customers for each customer duration type $1,\dots,m$, arrive in this order. It is easy to see that the offline optimal revenue is $R^*(I_i^{(m)})=r_iD_i^{(m)}C_i$ for all $m\in[M]$, because even the shortest duration among all duration types is strictly larger than $M\cdot C_i$, which is the length of the longest extreme instance $I_i^{(M)}$. Consider a random instance $\hat{I}_i$ that equals instance $I_i^{(M)}$ with probability $p_i^{(M)}=1/\Gamma_i$ and each other instance $I_i^{(m)}$ with probability $p_i^{(m)}=\left(1-\frac{D_i^{(m)}}{D_i^{(m+1)}}\right)/\Gamma_i$. By definition of Γ_i we have $\sum_{m=1}^M p_i^{(m)}=1$. Let $x_i^{A,m}$ denote the number of units of type i product offered to and selected by type m customers under algorithm A when presented with instance $I_i^{(M)}$, and let $P_i=\{x\in\mathbb{R}_{\geq 0}^M:\sum_{m=1}^M x_m\leq C_i\}$ denote the set of feasible allocation vectors for product i, which contains $x_i^A=(x_i^{A,1},\dots,x_i^{A,M})$ since no unit is released within the horizon of the instance. Note that here, A has no way of knowing whether it is facing instance $I_i^{(m)}$ or instance $I_i^{(k)}$ with $k>m$ (or some other instance) before it has seen the first $m\cdot C_i+1$ requests in the stated sequence. Therefore, the revenue earned by applying algorithm A to these instances is $R(A,I_i^{(m)})=\sum_{j=1}^m r_iD_i^{(j)}x_i^{A,j}$, for all $m\in[M]$.

Therefore, according to Yao's minimax principle, we have

$$\begin{aligned}
c^* &\leq \sup_{A\in\mathcal{A}}\mathbb{E}_Q\left[\frac{R(A,\hat{I}_i)}{R^*(\hat{I}_i)}\right]\\
&\leq \sup_{A\in\mathcal{A}}\sum_{m=1}^M p_i^{(m)}\frac{\sum_{j=1}^m D_i^{(j)}x_i^{A,j}}{D_i^{(m)}C_i}\\
&\leq \sup_{x_i^A\in P_i}\sum_{j=1}^M\left(\sum_{m=j}^M p_i^{(m)}\frac{D_i^{(j)}}{D_i^{(m)}}\right)\frac{x_i^{A,j}}{C_i}\\
&= \sup_{x_i^A\in P_i}\sum_{j=1}^M\frac{1}{\Gamma_i}\frac{x_i^{A,j}}{C_i}\\
&= \frac{1}{\Gamma_i}.
\end{aligned}$$

Now consider random instances $\hat{I}_j$ for all $j\in[n]\backslash\{i\}$, which are formed in the same way as $\hat{I}_i$. That is, $\hat{I}_j$ equals instance $I_j^{(M)}$ with probability $p_j^{(M)}=1/\Gamma_j$ and each other instance $I_j^{(m)}$ with probability $p_j^{(m)}=\left(1-\frac{D_j^{(m)}}{D_j^{(m+1)}}\right)/\Gamma_j$, where $I_j^{(1)},\dots,I_j^{(M)}$ are the corresponding instances involving 'j-craving' customers. Therefore, we also have

$$c^*\leq\frac{1}{\Gamma_j},\ \forall j\in[n]\backslash\{i\}. \tag{EC.50}$$

As a result, we have $c^*\leq\frac{1}{\Gamma'}$ since $\frac{1}{\Gamma'}=\min_{i\in[n]}\frac{1}{\Gamma_i}$, which proves the proposition.

EC.8. Numerical experiments for advance booking and assortment planning

In this section, we complement Section 6 by examining the empirical performance of PLP-AB and PLP-AP in advance booking and assortment planning respectively.

EC.8.1. Advance booking

Patterns similar to those in admission control in the monotone setting can be observed in advance booking. In particular, our online policy has much better empirical performance than its competitive ratio, dominates FCFS under both types of arrival instances, and its advantage grows as the number of classes increases. Details are as follows.

The same setup as in Section 6.1 is used for the advance booking problem. The same adversarial arrival instance is carried over to advance booking, and with probability 0.8 the starting time s_t for each time t is the time step after the end of the cycle that t is in, and with probability 0.2 s_t is a random time between t and $2T$. That is, most of the arrivals in the same cycle have the same starting time, which is after the end of that cycle. For the stochastic arrival instance for advance booking, both the arrival class and the starting time of each arrival are random. For advance booking, the revenue ratios of our online policy and FCFS on adversarial arrival instances are shown in Table EC.1, and those on stochastic arrival instances are shown in Table EC.2.

	C = 20		C = 50		C = 100	
Price classes	PLP-AB	FCFS	PLP-AB	FCFS	PLP-AB	FCFS
M = 2	**0.82**	0.70	**0.82**	0.70	**0.81**	0.72
M = 5	**0.63**	0.41	**0.65**	0.41	**0.64**	0.39
M = 9	**0.53**	0.28	**0.54**	0.26	**0.52**	0.21

Table EC.1 Adversarial arrival - Advance booking

	C = 20		C = 50		C = 100	
Price classes	PLP-AB	FCFS	PLP-AB	FCFS	PLP-AB	FCFS
M = 2	**0.89**	0.87	**0.91**	0.88	**0.93**	0.89
M = 5	**0.66**	0.51	**0.68**	0.51	**0.71**	0.52
M = 9	**0.51**	0.30	**0.53**	0.30	**0.55**	0.31

Table EC.2 Stochastic arrival - Advance booking

EC.8.2. Assortment planning

In this section, we compare the empirical performance of PLP-AP with the Inventory Balancing (IB) policy of Feng et al. (2019) and the greedy policy, under both a completely stochastic arrival instance and an adversarial arrival instance with slight stochasticity. The experimental setups are as follows.

We construct the adversarial instance with slight stochasticity as follows. The instance comprises two cycles, each of which is a low-then-high arrival instance. In particular, at each time step within the i-th segment of length K, where $i \in [M]$, one customer of duration type i arrives with probability 0.95, and a customer with a random duration type arrives with probability 0.05. For the rest of the cycle, no customer arrives. The value K is set such that $K > \sum_{i=1}^{n} C_i$, where n is the number of product types, and C_i is the inventory for type i product. We set $C_i = 20$ for all $i \in [n]$. In particular, we run the experiments with $n = 2, 3, 4$ and $K = 50, 80, 100$ respectively. This ensures that IB and the greedy policy have the chance to exhaust the inventory on customers with the shortest usage duration, which exposes their failure to hedge against future customers with longer usage durations, thereby highlighting an advantage of PLP-AP. In addition, we run the experiments with different numbers of customer duration types $M = 2, 3, 4$. We set $D_i^{(m)} = D_j^{(m)} = D^{(m)}$ for all $i, j \in [n], i \neq j, m \in [M]$. The duration $D^{(m)}$ for customers with duration type m is set as mMK for all $m \in [M]$. This ensures that no resource is returned within each cycle, so that no unit is allocated twice within each cycle. We set the length of each cycle as M^2K and $T = 2M^2K$. For the stochastic arrival instance, at each time step a customer whose duration type is drawn uniformly at random from all duration types (including the dummy type for no arrival) arrives. In addition, the revenue per unit time for the type i product is $r_i = 5i$.

Under both types of arrival instances, we use the multinomial logit (MNL) choice model for the customers, and the weight of each product type is randomly generated at each time step such that the weights of all product types sum to 0.99, and the weight of the outside option is 0.01. Again, we use the revenue ratio, i.e., the revenue achieved by the algorithm divided by the optimal objective value of LP-P-assort, which upper bounds the optimal expected revenue achieved by an offline policy that knows the full instance from the beginning, as the performance measure for the algorithms.

The empirical performance of PLP-AP, the IB policy, and the greedy policy under the adversarial (with slight stochasticity) and completely stochastic arrival instances are shown in Tables EC.3 and EC.4, respectively. Each entry is the average over 10 simulation iterations, with the standard deviation over the 10 simulation iterations in parentheses.

As we can see from Table EC.4, when the arrival instance is stochastic, all algorithms perform at a similar level. However, when the arrival instance is adversarial, Table EC.3 shows that the revenue ratio of PLP-AP exceeds those of IB and the greedy policy by approximately 0.16–0.22. This echoes the intuition that IB and the greedy policy only look at the number of available inventory units and therefore hedge only against future choice models, but not durations. In other words, they do not reserve capacity in anticipation of future customers with longer durations. By contrast, PLP-AP restricts the number of resource units that can be allocated to customers with shorter

no. of duration types	M = 2			M = 3			M = 4		
no. of product types	PLP-AP	IB	Greedy	PLP-AP	IB	Greedy	PLP-AP	IB	Greedy
n = 2	**0.81** (±0.01)	0.64 (±0.01)	0.63 (±0.01)	**0.76** (±0.01)	0.56 (±0.01)	0.57 (±0.01)	**0.81** (±0.01)	0.60 (±0.01)	0.63 (±0.01)
n = 3	**0.75** (±0.00)	0.58 (±0.01)	0.58 (±0.01)	**0.77** (±0.01)	0.56 (±0.01)	0.58 (±0.01)	**0.81** (±0.01)	0.60 (±0.01)	0.62 (±0.02)
n = 4	**0.75** (±0.01)	0.58 (±0.01)	0.58 (±0.01)	**0.77** (±0.01)	0.57 (±0.01)	0.57 (±0.01)	**0.81** (±0.01)	0.62 (±0.01)	0.65 (±0.01)

Table EC.3 Adversarial arrival - Assortment Planning

no. of duration types	M = 2			M = 3			M = 4		
no. of product types	PLP-AP	IB	Greedy	PLP-AP	IB	Greedy	PLP-AP	IB	Greedy
n = 2	0.86 (±0.00)	0.87 (±0.00)	0.87 (±0.01)	0.85 (±0.01)	0.86 (±0.01)	0.87 (±0.00)	0.83 (±0.01)	0.81 (±0.01)	0.81 (±0.01)
n = 3	0.84 (±0.00)	0.85 (±0.00)	0.85 (±0.00)	0.82 (±0.00)	0.82 (±0.00)	0.81 (±0.00)	0.81 (±0.01)	0.81 (±0.01)	0.81 (±0.01)
n = 4	0.84 (±0.00)	0.87 (±0.00)	0.87 (±0.01)	0.82 (±0.00)	0.83 (±0.01)	0.83 (±0.00)	0.82 (±0.01)	0.81 (±0.00)	0.81 (±0.01)

Table EC.4 Stochastic arrival - Assortment planning

durations and reserves capacity for potential customers requesting longer durations in the future. In addition, we can see that the performance of PLP-AP under both types of arrival instances is significantly better than its theoretical CR lower bound.