Supplementary Material – Proofs of Statements

EC.1. More numerical experiments

EC.1.1. Illustrations of the Lovász extension

In this subsection, we show the Lovász extension of a two-dimensional function on $[3]^2 = \{1,2,3\}^2$. We consider the quadratic function

$$f(x) := x^T \begin{bmatrix} 0.101 & -0.068 \\ -0.068 & 0.146 \end{bmatrix} x, \quad \forall x \in \mathbb{R}^2.$$

By the results in Murota (2003, Section 7.3), we know the function $f(\cdot)$ is a $L^\natural$-convex function. We compare the landscapes of the original objective and the Lovász extension in Figure EC.1. We can see that the Lovász extension is a piecewise linear and convex function, which is consistent with the results in Section 4 and Murota (2003).

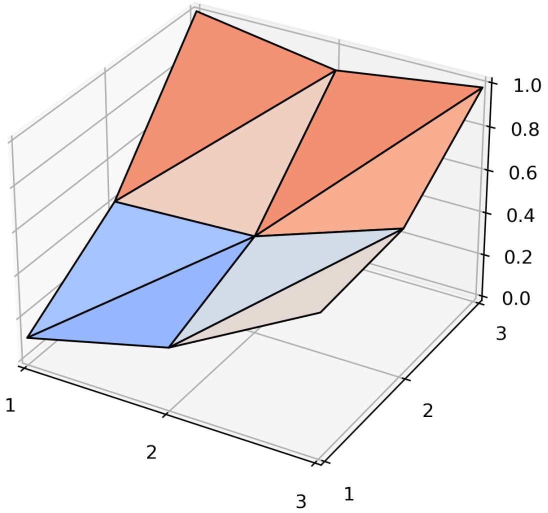

Figure EC.1 The Lovász extension of the objective function.

EC.2. Proofs in Section 3

EC.2.1. Proof of Theorem 1

Proof of Theorem 1. We denote the optimal value of $f(x)$ as f^*. Since point $\bar{x}$ satisfies the $(\epsilon/2, \delta/2)$-PGS guarantee, we have

$$\tilde{f}(\bar{x}) - f^* \le \epsilon/2$$

holds with probability at least $1-\delta/2$. We assume this event happens in the following of this proof. Let $S^0,S^1,\dots,S^d$ be the neighboring points of $\bar{x}$. Using the expression of the Lovász extension in (6), we know there exists an $\epsilon/2$-optimal solution among $S^0,S^1,\dots,S^d$. We denote the $\epsilon/2$-optimal solution and the solution returned by Algorithm 1 as S^* and $\hat{S}$, respectively. By the definition of confidence intervals, we know

$$\left|\hat{F}_n(S_i)-f(S_i)\right|\le\epsilon/4,\quad\forall i\in\{0,\dots,d\},\quad\left|\hat{F}_n(\hat{S})-f(\hat{S})\right|\le\epsilon/4$$

holds uniformly with probability at least $1-\delta/2$. Under this event, we know

$$f(\hat{S})-f^*\le\hat{F}_n(\hat{S})-f^*+\epsilon/4\le\hat{F}_n(S^*)-f^*+\epsilon/4\le f(S^*)-f^*+\epsilon/2\le\epsilon,$$

which implies that $x^*\in\mathcal{X}$ is an ϵ-optimal solution and the probability is at least $1-\delta/2-\delta/2=1-\delta$. Hence, we know x^* is an (ϵ,δ)-PGS solution to problem (1).

Now, we estimate the simulation cost of Algorithm 1. By Hoeffding bound, simulating

$$\frac{32}{\epsilon^2}\log\left(\frac{8d}{\delta}\right)$$

times on each neighboring point is enough to achieve $1-\delta/(4d)$ confidence half-width $\epsilon/4$. Hence, the simulation cost of Algorithm 1 is at most

$$\frac{32(d+1)}{\epsilon^2}\log\left(\frac{8d}{\delta}\right)=O\left[\frac{d}{\epsilon^2}\log\left(\frac{d}{\delta}\right)\right]=\tilde{O}\left[\frac{d}{\epsilon^2}\log\left(\frac{1}{\delta}\right)\right].$$

□

EC.2.2. Proof of Theorem 2

The following Azuma's inequality for martingales with sub-Gaussian tails plays as a major role for deriving high-probability bounds, i.e., the number of required samples to ensure the algorithms succeed with high probability.

LEMMA EC.1 **(Azuma's inequality for sub-Gaussian tails (Shamir 2011))**. *Let $X_0,\dots,X_{T-1}$ be a martingale difference sequence. Suppose there exist constants $b_1\ge1,b_2>0$ such that, for any $t\in\{0,\dots,T-1\}$,*

$$\mathbb{P}(|X_t|\ge a\mid X_1,\dots,X_{t-1})\le2b_1\exp(-b_2a^2),\quad\forall a\ge0.\tag{EC.1}$$

Then for any $\delta>0$, it holds with probability at least $1-\delta$ that

$$\frac{1}{T}\sum_{t=0}^{T-1}X_t\le\sqrt{\frac{28b_1}{b_2T}\log\left(\frac{1}{\delta}\right)}.$$

Since the stochastic subgradient $\hat{g}^t$ is truncated, the stochastic subgradient used for updating, namely $\tilde{g}^t$, is not unbiased. We define the bias at each step as

$$b_t := \mathbb{E}\left[\tilde{g}^t \mid x^0, x^1, \dots, x^t\right] - g^t, \quad \forall t \in \{0, 1, \dots, T-1\}.$$

First, we bound the ℓ_1-norm of the bias.

Lemma EC.2. *Suppose that Assumptions 1-4 hold. If we have*

$$M \geq 2\sigma \cdot \sqrt{\log\left(\frac{4\sigma d T}{\epsilon}\right)} = \Theta\left[\sqrt{\log\left(\frac{dT}{\epsilon}\right)}\right], \quad T \geq \frac{2\epsilon}{\sigma},$$

then it holds

$$\|b^t\|_1 \leq \frac{\epsilon}{2T}, \quad \forall t \in \{0, 1, \dots, T-1\}.$$

Proof. Let α_t be a consistent permutation of x^t and $S^{t,i}$ be the corresponding i-th neighboring points. We only need to prove

$$\left|b^t_{\alpha_t(i)}\right| \leq \frac{\epsilon}{2dT}, \quad \forall i \in [d].$$

We define two random variables

$$Y_1 := F\left(S^{t,i}, \xi_i^1\right) - f\left(S^{t,i}\right), \quad Y_2 := F\left(S^{t,i-1}, \xi_{i-1}^2\right) - f\left(S^{t,i-1}\right).$$

By Assumption 1, both Y_1 and Y_2 are independent and sub-Gaussian with parameter σ^2. Hence, we know

$$\begin{aligned} b^t_{\alpha_t(i)} &= \mathbb{E}\left[\tilde{g}^t_{\alpha_t(i)} - g^t_{\alpha_t(i)}\right] = \mathbb{E}\left[(Y_1 + Y_2)\cdot \mathbf{1}_{-M \leq Y_1 + Y_2 \leq M}\right] + \mathbb{E}\left[M \cdot \mathbf{1}_{Y_1+Y_2 > M}\right] + \mathbb{E}\left[-M \cdot \mathbf{1}_{Y_1 + Y_2 < -M}\right] \\ &= \mathbb{E}\left[(M - Y_1 - Y_2)\cdot \mathbf{1}_{Y_1+Y_2>M}\right] + \mathbb{E}\left[-(M + Y_1 + Y_2)\cdot \mathbf{1}_{Y_1+Y_2<-M}\right], \end{aligned}$$

where the second step is from $\mathbb{E}[Y_1] = \mathbb{E}[Y_2] = 0$. Taking the absolute value on both sides, we get

$$\begin{aligned} \left|b^t_{\alpha_t(i)}\right| &\leq \mathbb{E}\left[(Y_1 + Y_2 - M)\cdot \mathbf{1}_{Y_1+Y_2>M}\right] + \mathbb{E}\left[-(M + Y_1 + Y_2)\cdot \mathbf{1}_{Y_1+Y_2<-M}\right] \\ &= \mathbb{E}\left[(Y - M)\cdot \mathbf{1}_{Y>M}\right] + \mathbb{E}\left[-(Y + M)\cdot \mathbf{1}_{Y<-M}\right], \end{aligned} \tag{EC.2}$$

where we define the random variable $Y := Y_1 + Y_2$. Since Y_1, Y_2 are independent, random variable Y is sub-Gaussian with parameter $2\sigma^2$. Let $F(y) := \mathbb{P}[Y \leq y]$ be the distribution function of Y. Then, we have

$$\mathbb{E}\left[(Y - M)\cdot \mathbf{1}_{Y>M}\right] = \int_M^\infty (y - M)\, dF(y) = \int_M^\infty (1 - F(y))\, dy. \tag{EC.3}$$

By the Hoeffding bound, we know

$$1 - F(y) = \mathbb{P}[Y > y] \leq \exp\left(-y^2/4\sigma^2\right), \quad \forall y \geq 0.$$

Using the upper bound for Q-function in Borjesson and Sundberg (1979), it holds that

$$\int_M^\infty 1 - F(y)\ dy \le \int_M^\infty \exp\left(-y^2/4\sigma^2\right)\ dy \le \frac{2\sigma^2}{M}\exp\left(-\frac{M^2}{4\sigma^2}\right).$$

By the choice of M, we know

$$M \ge 2\sigma\sqrt{\log(8d)} \ge 2\sigma \quad \text{and} \quad \sigma\exp(-M^2/4\sigma^2) \le \frac{\epsilon}{4dT}.$$

which implies that

$$\int_M^\infty 1 - F(y)\ dy \le \frac{2\sigma^2}{M}\exp(-M^2/4\sigma^2) \le \frac{\epsilon}{4dT}.$$

Substituting the above inequality into (EC.3), we have

$$\mathbb{E}\left[(Y - M)\cdot \mathbf{1}_{Y>M}\right] \le \frac{\epsilon}{4dT}.$$

Considering $-Y$ in the same way, we can prove

$$\mathbb{E}\left[-(Y + M)\cdot \mathbf{1}_{Y<-M}\right] \le \frac{\epsilon}{4dT}.$$

Substituting the last two estimates into inequality (EC.2), we know

$$\left|b^t_{\alpha_t(i)}\right| \le \frac{\epsilon}{2dT}.$$

□

Next, we show that $\langle g^t + b^t - \tilde{g}^t, x^t - x^*\rangle$ forms a martingale sequence and use Azuma's inequality to bound the deviation, where x^* is a minimizer of $f(x)$.

Lemma EC.3. *Suppose that Assumptions 1-4 hold and let x^* be a minimizer of $f(x)$. The sequence*

$$X_t := \left\langle g^t + b^t - \tilde{g}^t, x^t - x^*\right\rangle \quad t = 0, 1, \ldots, T-1$$

forms a martingale difference sequence. Furthermore, if we have

$$M = \max\left\{L, 2\sigma\cdot\sqrt{\log\left(\frac{4\sigma dT}{\epsilon}\right)}\right\} = \tilde{\Theta}\left[\sqrt{\log\left(\frac{dT}{\epsilon}\right)}\right], \quad T \ge \frac{2\epsilon}{\sigma},$$

then it holds

$$\frac{1}{T}\sum_{t=0}^{T-1} X_t \le \sqrt{\frac{224 d\sigma^2}{T}\log\left(\frac{1}{\delta}\right)}$$

with probability at least $1-\delta$.

Proof. Let $\mathcal{F}_t$ be the filtration generated by $x_0, x_1, \ldots, x_t$. By the definition of b^t, we know

$$\mathbb{E}\left[g^t + b^t - \tilde{g}^t \mid \mathcal{F}_t\right] = 0,$$

which implies that

$$\mathbb{E}\left[X_t \mid \mathcal{F}_t\right] = \left\langle \mathbb{E}\left[g^t + b^t - \tilde{g}^t \mid \mathcal{F}_t\right], x^t - x^* \right\rangle = 0.$$

Hence, the sequence $\{X_t\}$ is a martingale difference sequence. Next, we estimate the probability $\mathbb{P}[|X_t| \geq a \mid \mathcal{F}_t]$. We have the bound

$$|X_t| = \left|\left\langle g^t + b^t - \tilde{g}^t, x^t - x^* \right\rangle\right| \leq \left\|g^t + b^t - \tilde{g}^t\right\|_1 \left\|x^t - x^*\right\|_\infty \leq \left\|g^t + b^t - \tilde{g}^t\right\|_1 \leq \left\|g^t - \tilde{g}^t\right\|_1 + \left\|b^t\right\|_1.$$

Since M satisfies the condition in Lemma EC.2, we know $\|b^t\|_1 \leq \epsilon/2T$. Recalling Assumption 4, we get $|g_i^t| \leq L$ for all $i \in [d]$. By the truncation rule and the assumption $M \geq L$, we have

$$\left|\tilde{g}_i^t - g_i^t\right| = \left|(\hat{g}_i^t \wedge M) \vee (-M) - g_i^t\right| \leq \left|\hat{g}^t - g^t\right|, \quad \forall i \in [d].$$

Hence, we get

$$|X_t| \leq \frac{\epsilon}{2T} + \left\|\hat{g}^t - g^t\right\|_1. \tag{EC.4}$$

Define random variables $Y_i := |\hat{g}_i^t - g_i^t|$ for all $i \in [d]$. By Assumption 1, Y_i is sub-Gaussian with parameter σ^2. Hence, we have

$$Y := \left\|\hat{g}^t - g^t\right\|_1 = \sum_{i=1}^{d} Y_i$$

is sub-Gaussian with parameter $d\sigma^2$. First, we consider the case when $a \geq \epsilon/T$. Using inequality (EC.4), it follows that

$$\mathbb{P}\left[|X_t| \geq a \mid \mathcal{F}_\sigma\right] \leq \mathbb{P}\left[\frac{\epsilon}{2T} + Y \geq a\right] \leq \mathbb{P}\left[Y \geq a - \frac{\epsilon}{2T}\right] \leq \mathbb{P}\left[Y \geq \frac{a}{2}\right] \leq 2\exp\left(-\frac{a^2}{8d\sigma^2}\right), \tag{EC.5}$$

where the last inequality is from Hoeffding bound. In this case, we know condition (EC.1) holds with

$$b_1 = 1, \quad b_2 = \frac{1}{8d\sigma^2}.$$

Now, we consider the case when $a < \epsilon/T$. In this case, by the assumption that $T \geq 2\epsilon/\sigma$, we have

$$2b_1 \exp\left(-b_2 a^2\right) > 2\exp\left(-\frac{1}{8d\sigma^2} \cdot \frac{\epsilon^2}{T^2}\right) \geq 2\exp\left(-\frac{1}{32d}\right) \geq 2\exp\left(-\frac{1}{32}\right) > 1.$$

Hence, it holds

$$\mathbb{P}\left[|X_t| \geq a \mid \mathcal{F}_\sigma\right] \leq 1 < 2b_1 \exp\left(-b_2 a^2\right).$$

Combining with inequality (EC.5), we know condition (EC.1) holds with b and c defined above. Using Lemma EC.1, we know

$$\frac{1}{T}\sum_{t=0}^{T-1} X_t \leq \sqrt{\frac{224 d\sigma^2}{T} \log\left(\frac{1}{\delta}\right)}$$

holds with probability at least $1 - \delta$. □

Then, we prove a lemma similar to the Lemma in Zinkevich (2003).

Lemma EC.4. *Suppose that Assumptions 1-4 hold and let x^* be a minimizer of $f(x)$. If we choose*

$$\eta = \frac{1}{M\sqrt{T}},$$

then we have

$$\frac{1}{T}\sum_{t=0}^{T-1} \langle \tilde{g}^t, x^t - x^* \rangle \le \frac{dM}{\sqrt{T}}.$$

Proof. We define $\tilde{x}^{t+1} := x^t - \eta\tilde{g}^t$ as the next point before the projection onto $[0,1]^d$. Recalling the non-expansion property of orthogonal projection, we get

$$\begin{aligned}\|x^{t+1} - x^*\|_2^2 = \|\mathcal{P}_{\mathcal{X}}\left(\tilde{x}^{t+1} - x^*\right)\|_2^2 \le \|\tilde{x}^{t+1} - x^*\|_2^2 = \|x^t - x^* - \eta\tilde{g}^t\|_2^2 \\ = \|x^t - x^*\|_2^2 + \eta^2\|\tilde{g}^t\|_2^2 - 2\eta\langle\tilde{g}^t, x^t - x^*\rangle,\end{aligned}$$

and equivalently,

$$\langle \tilde{g}^t, x^t - x^* \rangle = \frac{1}{2\eta}\left[\left\|x^t - x^*\right\|_2^2 - \left\|x^{t+1} - x^*\right\|_2^2\right] + \frac{\eta}{2}\cdot\left\|\tilde{g}^t\right\|_2^2.$$

Summing over $t = 0, 1, \ldots, T-1$, we have

$$\begin{aligned}\sum_{t=0}^{T-1} \langle \tilde{g}^t, x^t - x^* \rangle &= \frac{\left\|x^0 - x^*\right\|_2^2 - \left\|x^T - x^*\right\|_2^2}{2\eta} + \frac{\eta}{2}\sum_{t=0}^{T-1}\left\|\tilde{g}^t\right\|_2^2 \\ &\le \frac{d\left\|x^0 - x\right\|_\infty^2}{2\eta} + \frac{\eta}{2}\sum_{t=0}^{T-1}\left\|\tilde{g}^t\right\|_2^2 \le \frac{d}{2\eta} + \frac{\eta}{2}\sum_{t=0}^{T-1}\left\|\tilde{g}^t\right\|_2^2.\end{aligned}$$

By the definition of truncation, it follows that $\|\tilde{g}^t\|_2^2 \le dM^2$. Choosing

$$\eta := \frac{1}{M\sqrt{T}},$$

it follows that

$$\sum_{t=0}^{T-1} \langle \tilde{g}^t, x^t - x^* \rangle \le \frac{d}{2\eta} + \frac{\eta}{2}\sum_{t=0}^{T-1}\left\|\tilde{g}^t\right\|_2^2 \le \frac{d}{2\eta} + \frac{\eta T d M^2}{2} = dM\sqrt{T}.$$

□

Finally, using Lemmas EC.2, EC.3 and EC.4, we can finish the proof of Theorem 2.

Proof of Theorem 2. Denote f^* as the optimal value of $\tilde{f}(x)$. Using the convexity of $\tilde{f}(x)$, we know

$$\begin{aligned}\tilde{f}(\bar{x}) - f^* &\le \frac{1}{T}\sum_{t=0}^{T-1}\left[\tilde{f}(x^t) - f^*\right] \le \frac{1}{T}\sum_{t=0}^{T-1} \left\langle g^t, x^t - x^*\right\rangle \qquad \text{(EC.6)} \\ &= \frac{1}{T}\sum_{t=0}^{T-1}\left[\left\langle g^t + b^t - \tilde{g}^t, x^t - x^*\right\rangle + \left\langle \tilde{g}^t, x^t - x^*\right\rangle - \left\langle b^t, x^t - x^*\right\rangle\right].\end{aligned}$$

We choose

$$T := \frac{3584 d\sigma^2}{\epsilon^2} \log\left(\frac{2}{\delta}\right) = \Theta\left[\frac{d}{\epsilon^2}\log\left(\frac{1}{\delta}\right)\right].$$

Recalling Assumption 1, we know δ is small enough and therefore we have the following estimates:

$$L^2 \le M^2 = \tilde{\Theta}\left[\log\left(\frac{dT}{\epsilon}\right)\right] = \tilde{O}\left[\log\left(\frac{d^2}{\epsilon^3}\right) + \log\log\left(\frac{1}{\delta}\right)\right] \le \frac{\epsilon^2 T}{64 d^2}, \quad T \ge \max\left\{\frac{2\epsilon}{\sigma}, 4\right\}.$$

Hence, the conditions in Lemmas EC.2 and EC.3 are satisfied. By Lemma EC.2, we know

$$-\frac{1}{T}\sum_{t=0}^{T-1} \left\langle b^t, x^t - x^* \right\rangle \le \frac{1}{T}\sum_{t=0}^{T-1} \left\|b^t\right\|_1 \left\|x^t - x^*\right\|_\infty \le \frac{\epsilon}{2T} \le \frac{\epsilon}{8}. \tag{EC.7}$$

By Lemma EC.3, it holds

$$\frac{1}{T}\sum_{t=0}^{T-1} \left\langle g^t + b^t - \tilde{g}^t, x^t - x^* \right\rangle \le \sqrt{\frac{224 d\sigma^2}{T}\log\left(\frac{2}{\delta}\right)} \le \frac{\epsilon}{4} \tag{EC.8}$$

with probability at least $1-\delta$, where the last inequality is from our choice of T. By Lemma EC.4, we know

$$\frac{1}{T}\sum_{t=0}^{T-1} \left\langle \tilde{g}^t, x^t - x^* \right\rangle \le \frac{dM}{\sqrt{T}} \le \frac{\epsilon}{8}. \tag{EC.9}$$

Substituting inequalities (EC.7), (EC.8) and (EC.9) into inequality (EC.6), we get

$$\tilde{f}(\bar{x}) - f^* \le \frac{\epsilon}{2}$$

holds with probability at least $1-\delta/2$. By the results of Theorem 1, we know Algorithm 2 returns an (ϵ,δ)-PGS solution.

Finally, we estimate the simulation cost of Algorithm 2. For each iteration, we need to generate a stochastic subgradient using (4) and the simulation cost is $2d$. Hence, the total simulation cost of all iterations is

$$2d \cdot T = \tilde{\Theta}\left[\frac{d^2}{\epsilon^2}\log\left(\frac{1}{\delta}\right)\right].$$

By Theorem 1, the simulation cost of rounding process is at most

$$\tilde{O}\left[\frac{d}{\epsilon^2}\log\left(\frac{1}{\delta}\right)\right].$$

Thus, we know the total simulation cost of Algorithm 2 is at most

$$\tilde{O}\left[\frac{d^2}{\epsilon^2}\log\left(\frac{1}{\delta}\right)\right].$$

□

EC.2.3. Analysis of the bounded stochastic subgradient case

In this subsection, we consider the special case when the stochastic subgradient is assumed to have a bounded ℓ_1-norm.

Assumption EC.1. *There exist a constant G and an unbiased subgradient estimator $\hat{g}$ such that*

$$\mathbb{P}\left[\|\hat{g}\|_1 \le G\right] = 1.$$

Moreover, the simulation cost of generating each $\hat{g}$ is at most β simulations.

We note that G and β may depend on d and N. In the field of stochastic optimization, this assumption is common when analyzing the high-probability convergence of stochastic subgradient methods (Hazan and Kale 2011, Xu et al. 2016). We first give examples where Assumption EC.1 holds.

Example EC.1. We consider the case when the randomness of each choice of decision variables shares the same measure space, i.e., there exists a measure space $(\mathrm{Z}, \mathcal{B}_{\mathrm{Z}})$ such that ξ_x can be any element in the measure space for all $x \in \mathcal{X}$. Moreover, for any fixed $\xi \in \mathcal{B}$, the function $F(\cdot, \xi)$ is also $L^\natural$-convex (or submodular when $N = 2$) and has ℓ_∞-Lipschitz constant $\tilde{L}$. Then, we consider the subgradient estimator

$$\hat{g}_{\alpha_x(i)} := F\left(S^{x,i}, \xi\right) - F\left(S^{x,i-1}, \xi\right), \quad \forall i \in [d]. \tag{EC.10}$$

The simulation cost of estimator (EC.10) is $d+1$. In addition, property (v) of Lemma 2 gives

$$\|\hat{g}\|_1 \le 3\tilde{L}/2.$$

Therefore, in this situation, the Assumption EC.1 holds with $G = 3\tilde{L}/2$ and $\beta = d+1$.

When the distribution at each choice of decision variables is the Bernoulli, we show that Assumption EC.1 also holds.

Example EC.2. We consider the case when the distribution at each point $x \in \mathcal{X}$ is Bernoulli, namely, we have

$$\mathbb{P}[F(x, \xi_x) = 1] = 1 - \mathbb{P}[F(x, \xi_x) = 0] = f(x) \in [0, 1], \quad \forall x \in \mathcal{X}.$$

We note that the Bernoulli distribution is a special case of sub-Gaussian distributions. In this case, the ℓ_∞-Lipschitz constant is 1 and property (v) in Lemma 2 gives $\|g\|_1 \le 3/2$ for any subgradient g. We consider the subgradient estimator (4). At point x, if index i is chosen, then we know that

$$\|\hat{g}\|_1 = d \cdot \left|F\left(S^{x,i}, \xi_i^1\right) - F\left(S^{x,i-1}, \xi_{i-1}^2\right)\right| \le d.$$

Hence, Assumption EC.1 holds with $G = d$ and $\beta = 2$.

Next, we estimate the expected simulation cost of Algorithm 2 under Assumption EC.1. Since the stochastic subgradient is bounded, the truncation step is unnecessary in Algorithm 2. The simulation cost of Algorithm 2 is estimated in the following theorem. The proof is similar to Lemma 10 in Hazan and Kale (2011) and, since the feasible set is the hypercube $[0,1]^d$, we use ℓ_∞-norm instead of ℓ_2-norm to bound distances between points.

THEOREM EC.1. *Suppose that Assumptions 1-4 and EC.1 hold. If we skip the truncation step in Algorithm 2 (i.e., set $M=\infty$) and choose*

$$T=\tilde{\Theta}\left[\frac{(L+G)^2}{\epsilon^2}\log\left(\frac{1}{\delta}\right)\right],\quad \eta=\sqrt{\frac{d}{TG^2}},$$

then Algorithm 2 returns an (ϵ,δ)-PGS solution. Furthermore, we have

$$T(\epsilon,\delta,\mathcal{MC})=O\left[\frac{\beta(L+G)^2+d}{\epsilon^2}\log\left(\frac{1}{\delta}\right)+\frac{d^2G^2}{\epsilon^2}\right]=\tilde{O}\left[\frac{\beta(L+G)^2+d}{\epsilon^2}\log\left(\frac{1}{\delta}\right)\right].$$

Proof of Theorem EC.1. The proof of Theorem EC.1 is given in EC.2.4. ☐

In the case of Example EC.1, we have $\beta=d+1, G=3\tilde{L}/2$ and then the asymptotic simulation cost of Algorithm 2 is at most

$$\tilde{O}\left[\frac{d(L+\tilde{L})^2}{\epsilon^2}\log\left(\frac{1}{\delta}\right)\right].$$

If both Lipschitz constants are independent of d and N, the asymptotic simulation cost becomes

$$\tilde{O}\left[\frac{d}{\epsilon^2}\log\left(\frac{1}{\delta}\right)\right],$$

which is $O(d)$ better than the general case without Assumption EC.1. In addition, in the case of Example EC.2, we have $G=d$ and $\beta=2$. Hence, the asymptotic simulation cost is at most

$$\tilde{O}\left[\frac{d^2}{\epsilon^2}\log\left(\frac{1}{\delta}\right)\right].$$

Finally, we note that if we substitute ϵ with $c/2$, all upper bounds of simulation cost under Assumption EC.1 also hold for the PCS-IZ guarantee.

EC.2.4. Proof of Theorem EC.1

In this subsection, we provide a proof to Theorem EC.2.4. Since the stochastic gradient is bounded, we apply the Azuma's inequality for martingale difference sequences with bounded tails.

LEMMA EC.5 **(Azuma's inequality with bounded tails)**. *Let $X_0,\dots,X_{T-1}$ be a martingale difference sequence. Suppose there exists a constant b such that for any $t\in\{0,\dots,T-1\}$,*

$$\mathbb{P}(|X_t|\le b)=1.$$

Then for any $\delta>0$, it holds with probability at least $1-\delta$ that

$$\frac{1}{T}\sum_{t=0}^{T-1}X_t\le b\sqrt{\frac{2}{T}\log\left(\frac{1}{\delta}\right)}. \tag{EC.11}$$

The proof of Theorem EC.1 follows a similar way as Theorem 2. We first bound the noise term by Azuma's inequality.

LEMMA EC.6. *Suppose that Assumptions 1-EC.1 hold and let x^* be a minimizer of $f(x)$. Then, it holds*

$$\frac{1}{T}\sum_{t=0}^{T-1} \left\langle g^t - \hat{g}^t, x^t - x^* \right\rangle \leq \left(\frac{3L}{2} + G\right)\sqrt{\frac{2}{T}\log\left(\frac{1}{\delta}\right)}$$

with probability at least $1-\delta$.

Proof. Same as the proof of Lemma EC.3, the fact that $\hat{g}^t$ is unbiased implies that

$$X_t := \left\langle g^t - \hat{g}^t, x^t - x^* \right\rangle \quad t = 0, 1, \ldots, T-1$$

is a martingale difference sequence. By Assumption EC.1 and property (v) in Lemma 2, we know

$$|X_t| = \left|\left\langle g^t - \hat{g}^t, x^t - x^* \right\rangle\right| \leq \left\|g^t - \hat{g}^t\right\|_1 \left\|x^t - x^*\right\|_\infty \leq \left\|g^t - \hat{g}^t\right\|_1 \leq 3L/2 + G,$$

which implies that the condition (EC.11) holds with $b = 3L/2 + G$. Using Lemma EC.5, we get the conclusion of this lemma. □

The following lemma bounds the error of the algorithm and is similar to Theorem 3.2.2 in Nesterov (2018).

LEMMA EC.7. *Suppose that Assumptions 1-EC.1 hold and let x^* be a minimizer of $f(x)$. If we choose*

$$\eta = \sqrt{\frac{d}{TG^2}},$$

then we have

$$\frac{1}{T}\sum_{t=0}^{T-1} \left\langle \hat{g}^t, x^t - x^* \right\rangle \leq \sqrt{\frac{dG^2}{T}}.$$

Proof. We define $\tilde{x}^{t+1} := x^t - \eta\hat{g}^t$ as the next point before the projection onto $[0,1]^d$. Recalling the non-expansion property of orthogonal projection, we get

$$\begin{aligned}\|x^{t+1} - x^*\|_2^2 &= \|\mathcal{P}_{\mathcal{X}}\left(\tilde{x}^{t+1} - x^*\right)\|_2^2 \leq \|\tilde{x}^{t+1} - x^*\|_2^2 = \|x^t - x^* - \eta\hat{g}^t\|_2^2 \\ &= \|x^t - x^*\|_2^2 + \eta^2\|\tilde{g}^t\|_2^2 - 2\eta\langle\hat{g}^t, x^t - x^*\rangle,\end{aligned}$$

and equivalently,

$$\langle\hat{g}^t, x^t - x^*\rangle = \frac{1}{2\eta}\left[\left\|x^t - x^*\right\|_2^2 - \left\|x^{t+1} - x^*\right\|_2^2\right] + \frac{\eta}{2}\cdot\left\|\hat{g}^t\right\|_2^2.$$

Using Assumption EC.1, we know $\|\hat{g}^t\|_2^2 \leq \|\hat{g}^t\|_1^2 \leq G^2$ and therefore

$$\langle\hat{g}^t, x^t - x^*\rangle = \frac{1}{2\eta}\left[\left\|x^t - x^*\right\|_2^2 - \left\|x^{t+1} - x^*\right\|_2^2\right] + \frac{\eta G^2}{2}.$$

Summing over $t = 0, 1, \ldots, T-1$, we have

$$\sum_{t=0}^{T-1} \langle \hat{g}^t, x^t - x^* \rangle = \frac{\|x^0 - x^*\|_2^2 - \|x^T - x^*\|_2^2}{2\eta} + T \cdot \frac{\eta G^2}{2} \leq \frac{d\|x^0 - x\|_\infty^2}{2\eta} + \frac{\eta T G^2}{2} \leq \frac{d}{2\eta} + \frac{\eta T G^2}{2}.$$

Choosing

$$\eta := \sqrt{\frac{d}{TG^2}},$$

it follows that

$$\sum_{t=0}^{T-1} \langle \tilde{g}^t, x^t - x^* \rangle \leq G\sqrt{dT}.$$

□

Now, we prove Theorem EC.1 using Lemmas EC.6 and EC.7.

Proof of Theorem EC.1. According to to the proof of Theorem 2, we have

$$\begin{aligned} \tilde{f}(\bar{x}) - f^* &\leq \frac{1}{T}\sum_{t=0}^{T-1} \left[\tilde{f}(x^t) - f^*\right] \leq \frac{1}{T}\sum_{t=0}^{T-1} \left\langle g^t, x^t - x^* \right\rangle \\ &= \frac{1}{T}\sum_{t=0}^{T-1} \left\langle \hat{g}^t, x^t - x^* \right\rangle + \frac{1}{T}\sum_{t=0}^{T-1} \left\langle g^t - \hat{g}^t, x^t - x^* \right\rangle. \end{aligned} \tag{EC.12}$$

By Lemmas EC.6 and EC.7, it holds

$$\frac{1}{T}\sum_{t=0}^{T-1} \left\langle \hat{g}^t, x^t - x^* \right\rangle \leq \left(\frac{3L}{2} + G\right)\sqrt{\frac{2}{T}\log\left(\frac{2}{\delta}\right)}, \quad \frac{1}{T}\sum_{t=0}^{T-1} \left\langle g^t - \hat{g}^t, x^t - x^* \right\rangle \leq \sqrt{\frac{dG^2}{T}}$$

with probability at least $1 - \delta/2$. Choosing

$$T = \left(\frac{3L}{2} + G\right)^2 \cdot \frac{32}{\epsilon^2}\log\left(\frac{2}{\delta}\right) = \Theta\left[\frac{(L+G)^2}{\epsilon^2}\log\left(\frac{1}{\delta}\right)\right],$$

we know

$$T \geq \frac{16dG^2}{\epsilon^2}$$

when δ is small enough. Hence, we have

$$\frac{1}{T}\sum_{t=0}^{T-1} \left\langle \hat{g}^t, x^t - x^* \right\rangle \leq \frac{\epsilon}{4}, \quad \frac{1}{T}\sum_{t=0}^{T-1} \left\langle g^t - \hat{g}^t, x^t - x^* \right\rangle \leq \frac{\epsilon}{4}$$

holds with probability at least $1 - \delta/2$. Substituting into inequality (EC.12), we have

$$\tilde{f}(\bar{x}) - f^* \leq \frac{\epsilon}{2}$$

holds with probability at least $1 - \delta/2$. By the results of Theorem 1, we know Algorithm 2 returns an (ϵ, δ)-PGS solution.

Finally, we estimate the simulation cost of Algorithm 2. For each iteration, the simulation cost is decided by the generation of a stochastic subgradient, which is at most β by Assumption EC.1. Hence, the total simulation cost of all iterations is

$$O[\beta T] = \tilde{O}\left[\frac{\beta(L+G)^2}{\epsilon^2}\log\left(\frac{1}{\delta}\right)\right].$$

By Theorem 1, the simulation cost of rounding process is at most

$$\tilde{O}\left[\frac{d}{\epsilon^2}\log\left(\frac{1}{\delta}\right)\right].$$

Thus, we know the total simulation cost of Algorithm 2 is at most

$$\tilde{O}\left[\frac{\beta(L+G)^2+d}{\epsilon^2}\log\left(\frac{1}{\delta}\right)\right].$$

□

EC.3. Proofs in Section 4

EC.3.1. Proof of Theorem 3

Proof of Theorem 3. To prove the function is well-defined, we only need to show that for any two different points $y,z\in[N-1]^d$ such that $\mathcal{C}_y\cap\mathcal{C}_z\neq\emptyset$, we have $\tilde{f}_y(x)=\tilde{f}_z(x)$ for all $x\in\mathcal{C}_y\cap\mathcal{C}_z$. We first consider the case when $\|y-z\|_1=1$. Without loss of generality, we assume

$$y=(1,1\ldots,1),\quad z=(2,1,\ldots,1).$$

In this case, we know that

$$\mathcal{C}_y\cap\mathcal{C}_z=\{(2,x_2,\ldots,x_d):x_2,\ldots,x_d\in[0,1]\}.$$

Suppose that point $x\in\mathcal{C}_y\cap\mathcal{C}_z$. We first calculate $\tilde{f}_y(x)$. We can define the "local coordinate" of x in $\mathcal{C}_y$ as

$$x-y=(1,x_2-1,\ldots,x_d-1).$$

Let α_1 be a consistent permutation of x in $\mathcal{C}_y$ and and $S^{1,i}$ be the corresponding i-th neighbouring point. Since $(x-y)_1=1$ is not smaller than any other components, we can assume $\alpha_1(1)=1$ and calculate $\tilde{f}_y(x)$ as

$$\tilde{f}_y(x)=[1-(x-y)_{\alpha_1(1)}]f(S^{1,0})+\sum_{i=1}^{d-1}[(x-y)_{\alpha_1(i)}-(x-y)_{\alpha_1(i+1)}]f(S^{1,i})+(x-y)_{\alpha_1(d)}f(S^{1,d}) \tag{EC.13}$$

$$=\sum_{i=1}^{d-1}[(x-y)_{\alpha_1(i)}-(x-y)_{\alpha_1(i+1)}]f(S^{1,i})+(x-y)_{\alpha_1(d)}f(S^{1,d})$$

$$=\sum_{i=1}^{d-1}[x_{\alpha_1(i)}-x_{\alpha_1(i+1)}]f(S^{1,i})+\left[x_{\alpha_1(d)}-1\right]f(S^{1,d}).$$

Next, we consider $\tilde{f}_z(x)$ and define the "local coordinate" of x in $\mathcal{C}_z$ is

$$x - z = (0, x_2 - 1, \ldots, x_d - 1).$$

We define the permutation α_2 as

$$\alpha_2(i) = \alpha_1(i+1), \quad \forall i \in [d-1], \quad \alpha_2(d) = \alpha_1(1) = 1.$$

By the definition of α_1, we know

$$\begin{aligned}
&(x-z)_{\alpha_2(i)} = (x-y)_{\alpha_1(i+1)} \geq (x-y)_{\alpha_1(i+2)} = (x-z)_{\alpha_2(i+1)}, \quad \forall i \in [d-2], \\
&(x-z)_{\alpha_2(d-1)} \geq 0 = (x_z)_{\alpha_2(d)}.
\end{aligned}$$

Hence, we know α_2 is a consistent permutation of x in $\mathcal{C}_z$ and let $S^{2,i}$ be the corresponding i-th neighbouring point of x in $\mathcal{C}_z$. Similar to the first case, the Lovász extension $\tilde{f}_z(x)$ can be calculated as

$$\begin{aligned}
\tilde{f}_y(x) &= [1 - (x-z)_{\alpha_2(1)}] f(S^{2,0}) + \sum_{i=1}^{d-1} [(x-z)_{\alpha_2(i)} - (x-z)_{\alpha_2(i+1)}] f(S^{2,i}) + (x-z)_{\alpha_2(d)} f(S^{2,d}) \qquad \text{(EC.14)} \\
&= [1 - (x-z)_{\alpha_2(1)}] f(S^{2,0}) + \sum_{i=1}^{d-1} [(x-z)_{\alpha_2(i)} - (x-z)_{\alpha_2(i+1)}] f(S^{2,i}) \\
&= [2 - x_{\alpha_2(1)}] f(S^{2,0}) + \sum_{i=1}^{d-1} [x_{\alpha_2(i)} - x_{\alpha_2(i+1)}] f(S^{2,i}) + f(S^{2,d-1}).
\end{aligned}$$

Recalling the fact that $z = y + e_1$, for any $i \in [d-1]$, we have

$$S^{2,i} = z + \sum_{j=1}^{i} e_{\alpha_2(j)} = y + e_1 + \sum_{j=1}^{i} e_{\alpha_1(j+1)} = y + \sum_{j=1}^{i+1} e_{\alpha_1(i)} = S^{1,i+1}.$$

Substituting into equation (EC.14), we know

$$\begin{aligned}
\tilde{f}_y(x) &= [2 - x_{\alpha_2(1)}] f(S^{2,0}) + \sum_{i=1}^{d-1} [x_{\alpha_2(i)} - x_{\alpha_2(i+1)}] f(S^{2,i}) + f(S^{2,d-1}) \\
&= [2 - x_{\alpha_2(1)}] f(S^{1,1}) + \sum_{i=1}^{d-2} [x_{\alpha_2(i)} - x_{\alpha_2(i+1)}] f(S^{1,i+1}) + [x_{\alpha_2(d-1)} - x_{\alpha_2(d)}] f(S^{1,d}) + f(S^{1,d}) \\
&= [x_{\alpha_1(1)} - x_{\alpha_1(2)}] f(S^{1,1}) + \sum_{i=1}^{d-2} [x_{\alpha_1(i+1)} - x_{\alpha_1(i+2)}] f(S^{1,i+1}) + \left[x_{\alpha_1(d)} - 2\right] f(S^{1,d}) + f(S^{1,d}) \\
&= \sum_{i=1}^{d-1} [x_{\alpha_1(i)} - x_{\alpha_1(i+1)}] f(S^{1,i}) + \left[x_{\alpha_1(d)} - 1\right] f(S^{1,d}),
\end{aligned}$$

which is equal to $\tilde{f}_y(x)$ by equation (EC.13).

Then, we consider the case when $\|y-z\|_1 > 1$. Since $\mathcal{C}_y \cap \mathcal{C}_z \neq \emptyset$, we know $\|y-z\|_\infty = 1$. Without loss of generality, we consider the case when

$$y = (1,1,\dots,1), \quad z = y + \sum_{j=1}^{k} e_j,$$

where constant $k \in [d]$. In this case, we know

$$\mathcal{C}_y \cap \mathcal{C}_z = \left\{ x \in \mathbb{R}^d : x_j = 2,\ \forall j \le k,\ x_j \in [0,1],\ \forall j \ge k+1 \right\}.$$

We define

$$y_i := y + \sum_{j=1}^{i} e_j, \quad \forall i \in \{0,1,\dots,k\}.$$

Then, it follows that

$$\|y_i - y_{i-1}\|_1 = 1, \quad \forall i \in [k], \quad y_0 = y, \quad y_k = z$$

and

$$x \in \mathcal{C}_y \cap \mathcal{C}_z \subset \mathcal{C}_{y_i} \cap \mathcal{C}_{y_{i-1}} = \left\{ x \in \mathbb{R}^d : x_i = 2,\ x_j \in [0,1],\ \forall j \in [d] \backslash \{i\} \right\}, \quad \forall i \in [k].$$

Hence, by the results for the case when $\|y-z\|_1 = 1$, we know

$$\tilde{f}_y(x) = \tilde{f}_{y_0}(x) = \tilde{f}_{y_1}(x) = \cdots = \tilde{f}_{y_k}(x) = \tilde{f}_z(x),$$

which means $\tilde{f}(x)$ is well-defined.

Finally, we prove the convexity of $\tilde{f}(x)$. Since the Lovász extension is the support function of submodular functions (Fujishige 2005, section 6.3), the function $\tilde{f}_y(x)$ is the support function of $f(x)$ inside hypercube $\mathcal{C}_y$. In addition, Theorem 7.20 in Murota (2003) implies that the $L^\natural$-convex function $f(x)$ is integrally convex. Hence, we know that the support function of $f(x)$ on $\mathcal{X}$ is equal to $\tilde{f}_y(x)$ in each hypercube $\mathcal{C}_y$. By the definition of $\tilde{f}(x)$ in (7), the function $\tilde{f}(x)$ is the support function of $f(x)$ on $\mathcal{X}$. Since support functions are convex, we know $\tilde{f}(x)$ is convex. □

EC.3.2. Proof of Theorem 4

Proof of Theorem 4. The proof can be done in the same way as Theorem 2 and we only give a sketch of the proof. We use the same notation as the proof of Theorem EC.2.

- If we have

$$M \ge 2\sigma \cdot \sqrt{\log\left(\frac{8\sigma d T}{\epsilon}\right)} = \tilde{\Theta}\left[\sqrt{\log\left(\frac{dT}{\epsilon}\right)}\right], \quad T \ge \frac{2\epsilon}{\sigma},$$

then the proof of Lemma EC.2 implies that

$$\left\|b^t\right\|_1 \le \frac{\epsilon}{2T}, \quad \forall t \in \{0,1,\dots,T-1\}.$$

- If we have

$$M = \max\left\{L, 2\sigma\cdot\sqrt{\log\left(\frac{8\sigma dT}{\epsilon}\right)}\right\} = \tilde{\Theta}\left[\sqrt{\log\left(\frac{dNT}{\epsilon}\right)}\right], \quad T \geq \frac{2N\epsilon}{\sigma},$$

then the proof of Lemma EC.3 shows that

$$\frac{1}{T}\sum_{t=0}^{T-1} X_t \leq \sqrt{\frac{224dN^2\sigma^2}{T}\log\left(\frac{1}{\delta}\right)}$$

holds with probability at least $1-\delta$.

- If we choose

$$\eta = \frac{N}{M\sqrt{T}},$$

then the proof of Lemma EC.4 implies that

$$\frac{1}{T}\sum_{t=0}^{T-1}\langle \tilde{g}^t, x^t - x^*\rangle \leq \frac{dNM}{\sqrt{T}}.$$

Hence, choosing

$$T = \tilde{\Theta}\left[\frac{dN^2}{\epsilon^2}\log\left(\frac{1}{\delta}\right)\right], \quad M = \tilde{\Theta}\left[\sqrt{\log\left(\frac{dNT}{\epsilon}\right)}\right], \quad \eta = \frac{N}{M\sqrt{T}}$$

and using the inequality (EC.6), we know the averaging point $\bar{x}$ is an $(\epsilon/2, \delta/2)$-PGS solution. Combining with Theorem 1, Algorithm 2 returns an (ϵ,δ)-PGS solution. Since the simulation cost of each iteration is $2d$, the total simulation cost of Algorithm 2 is at most

$$\tilde{O}\left[\frac{d^2N^2}{\epsilon^2}\log\left(\frac{1}{\delta}\right)\right] + \tilde{O}\left[\frac{d}{\epsilon^2}\log\left(\frac{1}{\delta}\right)\right] = \tilde{O}\left[\frac{d^2N^2}{\epsilon^2}\log\left(\frac{1}{\delta}\right)\right].$$

□

Similarly, we can estimate the asymptotic simulation cost under Assumption EC.1.

THEOREM EC.2. *Suppose that Assumptions 1-4 and EC.1 hold. If we skip the truncation step in Algorithm 2 (or equivalently set $M=\infty$) and choose*

$$T = \tilde{\Theta}\left[\frac{(L+G)^2N^2}{\epsilon^2}\log\left(\frac{1}{\delta}\right)\right], \quad \eta = \sqrt{\frac{dN^2}{TG^2}},$$

then Algorithm 2 returns an (ϵ,δ)-PGS solution. Furthermore, we have

$$T(\epsilon,\delta,\mathcal{MC}) = O\left[\frac{\beta(L+G)^2N^2+d}{\epsilon^2}\log\left(\frac{1}{\delta}\right) + \frac{G^2d^2N^2}{\epsilon^2}\right] = \tilde{O}\left[\frac{\beta(L+G)^2N^2+d}{\epsilon^2}\log\left(\frac{1}{\delta}\right)\right].$$

The above theorem can be proved in the same way as Theorem EC.1 and we omit the proof. We note that the step size η does not depend on N in this case.

EC.3.3. Algorithms for the PCS-IZ case

We first prove that the existence of indifference zone is equivalent to the so-called weak sharp minima condition of the convex extension. Moreover, we use the ℓ_∞ norm in place of the ℓ_2 norm since the feasible set is a hypercube.

DEFINITION EC.1. We say a function $f(x) : \mathcal{X} \mapsto \mathbb{R}$ satisfies the **Weak Sharp Minimum (WSM) condition**, if the function $f(x)$ has a unique minimizer x^* and there exists a constant $\kappa > 0$ such that

$$\|x - x^*\|_\infty \le \kappa \left(f(x) - f^*\right), \quad \forall x \in \mathcal{X},$$

where $f^* := f(x^*)$.

The WSM condition was first defined in Burke and Ferris (1993), and is also called the polyhedral error bound condition in recent literature (Yang and Lin 2018). In addition, the WSM condition is a special case of the global growth condition in Xu et al. (2016) with $\theta = 1$. The WSM condition can be used to leverage the distance between intermediate solutions and (c,δ)-PCS-IZ solutions. The next theorem verifies that the WSM condition is equivalent to the existence of indifference zone.

THEOREM EC.3. *Suppose that function* $f(x) : \mathcal{X} \mapsto \mathbb{R}$ *is a* $L^\natural$*-convex function and* $\tilde{f}(x)$ *is the convex extension on* $[1,N]^d$*. Given a constant* $c > 0$*, function* $f(x) \in \mathcal{MC}_c$ *if and only if* $\tilde{f}(x)$ *satisfies the WSM condition with* $\kappa = c^{-1}$*.*

Proof of Theorem EC.3. We first prove the sufficiency part and then consider the necessity part.

Sufficiency. Suppose there exists a constant $\kappa > 0$ such that the function $\tilde{f}(x)$ satisfies the WSM condition with κ. Considering any point $x \in \mathcal{X} \backslash \{x^*\}$, we know $\|x - x^*\|_\infty \ge 1$ and, by the WSM condition,

$$f(x) - f^* = \tilde{f}(x) - f^* \ge \kappa^{-1} \|x - x^*\|_\infty \ge \kappa^{-1}.$$

Thus, we know the indifference zone parameter for $f(x)$ is at least κ^{-1} and $f(x) \in \mathcal{MC}_{\kappa^{-1}}$.

Necessity. Suppose there exists a constant $c > 0$ such that

$$f(x) - f^* \ge c, \quad \forall x \in \mathcal{X} \backslash \{x^*\}.$$

We first consider point $x \in [1,N]^d$ such that $\|x - x^*\|_\infty \le 1$. In this case, we know there exists a hypercube $\mathcal{C}_y$ containing both x and x^*. By the definition of Lovász extension, we know that

$$\tilde{f}(x) = [1 - x_{\alpha_x(1)}] f\left(S^{x,0}\right) + \sum_{i=1}^{d-1} [x_{\alpha_x(i)} - x_{\alpha_x(i+1)}] f\left(S^{x,i}\right) + x_{\alpha_x(d)} f\left(S^{x,d}\right) = \sum_{i=0}^{d} \lambda_i f\left(S^{x,i}\right),$$

where we define

$$\lambda_i := x_{\alpha_x(i)} - x_{\alpha_x(i+1)}, \quad \forall i \in [d-1], \quad \lambda_0 := 1 - x_{\alpha_x(1)}, \quad \lambda_d := x_{\alpha_x(d)}.$$

Recalling the definition of consistent permutation, we get

$$\sum_{i=0}^{d} \lambda_i = 1, \quad \lambda_i \geq 0, \quad \forall i \in \{0, \ldots, d\}$$

and $\tilde{f}(x)$ is a convex combination of $f\left(S^{x,0}\right), \ldots, f\left(S^{x,d}\right)$. In addition, we can calculate that

$$\left(\sum_{i=0}^{d} \lambda_i S^{x,i}\right)_{\alpha_x(k)} = \sum_{i=0}^{d} \lambda_i \cdot S^{x,i}_{\alpha_x(k)} = \sum_{i=0}^{d} \lambda_i \cdot \mathbf{1}(i \geq k) = \sum_{i=k}^{d} \lambda_i = x_{\alpha_x(k)},$$

which implies that

$$x = \sum_{i=0}^{d} \lambda_i S^{x,i}.$$

If $x^* \notin \left\{S^{x,0}, \ldots, S^{x,d}\right\}$, the assumption that indifference zone parameter is c gives

$$\tilde{f}(x) - f^* = \sum_{i=0}^{d} \lambda_i \left[f\left(S^{x,i}\right) - f^*\right] \geq \sum_{i=0}^{d} \lambda_i \cdot c = c.$$

Combining with $\|x - x^*\|_\infty \leq 1$, we have

$$\|x - x^*\|_\infty \leq c^{-1} \cdot \left[\tilde{f}(x) - f^*\right].$$

Otherwise if $x^* = S^{x,i}$ for some $i \in \{0, \ldots, d\}$. Then, we know

$$\tilde{f}(x) - f^* = \sum_{i=0}^{d} \lambda_i \left[f\left(S^{x,i}\right) - f^*\right] \geq \sum_{i \neq k} \lambda_i \cdot c = (1 - \lambda_k)c$$

and

$$\begin{aligned}\|x - x^*\|_\infty &= \left\|\sum_{i=0}^{d} \lambda_i S^{x,i} - x^*\right\|_\infty = \left\|\sum_{i=0}^{d} \lambda_i \left(S^{x,i} - x^*\right)\right\|_\infty = \left\|\sum_{i \neq k} \lambda_i \left(S^{x,i} - x^*\right)\right\|_\infty \\ &\leq \sum_{i \neq k} \lambda_i \left\|S^{x,i} - x^*\right\|_\infty \leq \sum_{i \neq k} \lambda_i = 1 - \lambda_k,\end{aligned}$$

where the last inequality is because $S^{x,i}$ and x^* are in the same hypercube $\mathcal{C}_y$. Combining the above two inequalities, it follows that

$$\|x - x^*\|_2 \leq c^{-1} \cdot \left[\tilde{f}(x) - f^*\right],$$

which means that the WSM condition holds with $\kappa = c^{-1}$. Now we consider point $x \in [1, N]^d$ such that $\|x - x^*\|_\infty \geq 1$. We define

$$\tilde{x} := x^* + \frac{x - x^*}{\|x - x^*\|_\infty}$$

to be the point on the segment $\overline{xx^*}$ such that $\|\tilde{x}-x^*\|_\infty = 1$. By the convexity of $\tilde{f}(x)$ and the WSM condition for point $\tilde{x}$, we know

$$\tilde{f}(x)-f^* \geq \frac{\|x-x^*\|_\infty}{\|\tilde{x}-x^*\|_\infty}\left[\tilde{f}(\tilde{x})-f^*\right] = \frac{\tilde{f}(\tilde{x})-f^*}{\|\tilde{x}-x^*\|_\infty}\cdot\|x-x^*\|_\infty \geq c^{-1}\cdot\|x-x^*\|_\infty,$$

which shows that the WSM condition holds with $\kappa = c^{-1}$. Hence, the WSM condition holds for all points in $[1,N]^d$ with $\kappa = c^{-1}$. □

Using the WSM condition, we can accelerate Algorithm 2 by dynamically shrinking the search space. To describe the shrinkage of search space, we define the ℓ_∞-neighbourhood of point x as

$$\mathcal{N}(x,a) := \{y \in [1,N]^d : \|y-x\|_\infty \leq a\}$$

and the orthogonal projection onto $\mathcal{N}(x,a)$ as

$$\mathcal{P}_{x,a}(y) := (y \wedge (x+a)\mathbf{1}) \vee (x-a)\mathbf{1}, \quad \forall x \in \mathbb{R}^d.$$

Now we give the adaptive SSGD algorithm for the PCS-IZ guarantee.

Algorithm 4 Adaptive SSGD method for the PCS-IZ guarantee

Input: Model $\mathcal{X}, \mathcal{B}_\mathsf{Y}, F(x,\xi_x)$, optimality guarantee parameter δ, indifference zone parameter c.

Output: An (c,δ)-PCS-IZ solution x^* to problem (1).

```
Set the initial guarantee $\epsilon_0 \leftarrow cN/4$.
Set the number of epochs $E \leftarrow \lceil \log_2(N) \rceil + 1$.
Set the initial search space $\mathcal{Y}_0 \leftarrow [1,N]^d$.
for $e = 0,\dots,E-1$ do
    Use Algorithm 2 to get an $(\epsilon_e, \delta/(2E))$-PGS solution $x_e$ in $\mathcal{Y}_e$.
    Update guarantee $\epsilon_{e+1} \leftarrow \epsilon_e/2$.
    Update the search space $\mathcal{Y}_{e+1} \leftarrow \mathcal{N}(x_e, 2^{-e-2}N)$.
end for
Round $x_{E-1}$ to an integral point satisfying the $(c,\delta)$-PCS-IZ guarantee by Algorithm 1.
```

Basically, the algorithm finds a $(c/2,\delta)$-PGS solution and, with the assumption that the indifference zone parameter is c, the solution satisfies the (c,δ)-PCS-IZ guarantee. We prove that the expected simulation cost of Algorithm 4 has only $O(\log(N))$ dependence on N.

Theorem EC.4. *Suppose that Assumptions 1-4 hold. Then, Algorithm 4 returns a (c,δ)-PCS-IZ solution. Furthermore, we have*

$$T(\delta,\mathcal{MC}_c) = O\left[\frac{d^2\log(N)}{c^2}\log\left(\frac{1}{\delta}\right) + \frac{d^3\log(N)}{c^2}\log\left(\frac{d^2N}{\epsilon^3}\right) + \frac{d^3\log(N)L^2}{c^2}\right] = \tilde{O}\left[\frac{d^2\log(N)}{c^2}\log\left(\frac{1}{\delta}\right)\right].$$

Proof of Theorem EC.4. We first prove the correctness of Algorithm 4. Let x^* be the minimizer of $f(x)$ and $f^* := f(x^*)$. We use the induction method to prove that, for each epoch e, it holds

$$\tilde{f}(x_e) - f^* \leq \epsilon_e$$

with probability at least $1-(e+1)\delta/(2E)$. For epoch 0, the solution x_0 is $(\epsilon_0, \delta/(2E))$-PGS and we know

$$\tilde{f}(x_0) - f^* \leq \epsilon_0$$

holds with probability at least $1-\delta/(2E)$. We assume that the above event happens for the $(e-1)$-th epoch with probability at least $1-e\cdot\delta/(2E)$ and consider the case when this event happens. By Theorem EC.3, function $\tilde{f}(x)$ satisfies the WSM condition with $\kappa = c^{-1}$. Hence, the intermediate solution x_{e-1} satisfies

$$\|x_{e-1}-x^*\|_\infty \leq c^{-1}\left[\tilde{f}(x_{e-1}) - f^*\right] \leq c^{-1}\epsilon_{e-1} = c^{-1}\cdot 2^{-e+1}\epsilon_0 = 2^{-e-1}N,$$

which implies that $x^* \in \mathcal{N}(x_{e-1}, 2^{-e-1}N) = \mathcal{N}_e$ and therefore $x^* \in \mathcal{N}_e$. For the epoch e, it holds

$$\tilde{f}(x_e) - f^* = \tilde{f}(x_e) - \min_{x\in\mathcal{N}_e} \tilde{f}(x) \leq \epsilon_e$$

with probability at least $1-\delta/(2E)$. Hence, the above event happens with probability at least $1-\delta/(2E) - e\cdot\delta/(2E) = 1-(e+1)\delta/(2E)$ for epoch e. By the induction method, we know the claim holds for all epochs. Considering the last epoch, we know

$$\tilde{f}(x_{E-1}) - f^* \leq \epsilon_{E-1} = 2^{-E+1}\epsilon_0 = 2^{-\lceil \log_2(N)\rceil -2}\cdot cN \leq 2^{-\log_2(N)-2}\cdot cN = c/4$$

holds with probability at least $1-\delta/2$. Thus, we know x_{E-1} satisfies the $(c/4,\delta/2)$-PGS guarantee. By Theorem 1, the integral solution returned by Algorithm 4 satisfies the $(c/2,\delta)$-PGS guarantee. Since the indifference zone parameter is c, the solution satisfying the $(c/2,\delta)$-PGS guarantee must satisfies the (c,δ)-PCS-IZ guarantee.

Next, we estimate the asymptotic simulation cost of Algorithm 4. By Theorem 2, the simulation cost of epoch e is at most

$$\tilde{O}\left[\frac{d^2\left(2^{-e}N\right)^2}{\epsilon_e^2}\log\left(\frac{E}{\delta}\right)\right] = \tilde{O}\left[\frac{d^2\left(2^{-e}N\right)^2}{\left(2^{-e-2}\cdot cN\right)^2}\log\left(\frac{E}{\delta}\right)\right] = \tilde{O}\left[\frac{d^2}{c^2}\log\left(\frac{1}{\delta}\right)\right].$$

Summing over $e = 0,1,\dots,E-1$, we know the total simulation cost of E epochs is at most

$$\tilde{O}\left[E\cdot\frac{d^2}{c^2}\log\left(\frac{1}{\delta}\right)\right] = \tilde{O}\left[\frac{d^2\log(N)}{c^2}\log\left(\frac{1}{\delta}\right)\right].$$

By Theorem 1, the simulation cost of the rounding process is at most

$$\tilde{O}\left[\frac{d}{c^2}\log\left(\frac{1}{\delta}\right)\right].$$

Combining the two parts, we know the asymptotic simulation cost of Algorithm 4 is at most

$$\tilde{O}\left[\frac{d^2\log(N)}{c^2}\log\left(\frac{1}{\delta}\right)\right].$$

□

Similarly, we can estimate the asymptotic simulation cost under Assumption EC.1 and we omit the proof.

Theorem EC.5. *Suppose that Assumptions 1-4 and EC.1 hold. Then, Algorithm 4 returns a (c,δ)-PCS-IZ solution. Furthermore, we have*

$$T(\delta,\mathcal{MC}_c)=\tilde{O}\left[\frac{\beta(L+G)^2\log(N)+d}{c^2}\log\left(\frac{1}{\delta}\right)\right].$$

EC.4. Proofs in Section 5

EC.4.1. Proof of Theorem 5

Proof of Theorem 5. In this proof, we change the feasible set to $\mathcal{X}=\{0,1,\dots,N\}^d$, where $N\geq 1$. We split the proof into three steps.

Step 1. We first show that the construction of $L^\natural$-convex functions can be reduced to the construction of submodular functions. Equivalently, we show that any submodular function defined on $\{0,1\}^d$ can be extended to a $L^\natural$-convex function on $\mathcal{X}$ with the same convex extension after scaling. Let $g(x)$ be a submodular function defined on $\{0,1\}^d$ and $\tilde{g}(x)$ be the Lovász extension of $g(x)$. We first extend the domain of the Lovász extension to $[0,N]^d$ by scaling, i.e.,

$$\tilde{f}(x):=\tilde{g}(x/N),\quad\forall x\in[0,N]^d.$$

Then, we define the discretization of $\tilde{f}(x)$ by restricting to the integer lattice

$$f(x):=\tilde{f}(x),\quad\forall x\in\mathcal{X}.$$

We prove that $f(x)$ is a $L^\natural$-convex function. By Proposition 7.25 in Murota (2003), we know the Lovász extension $\tilde{g}(x)$ is a polyhedral L-convex function. Since the scaling operation does not change the L-convexity, we know $\tilde{f}(x)$ is also polyhedral L-convex. Hence, by Theorem 7.29 in Murota (2003), the function $\tilde{f}(x)$ satisfies the SBF$^\natural[\mathbb{R}]$ property, namely,

$$\tilde{f}(p)+\tilde{f}(q)\geq\tilde{f}[(p-\alpha\mathbf{1})\vee q]+\tilde{f}(p\wedge(q+\alpha\mathbf{1})),\quad\forall p,q\in[0,N]^d,\ \alpha\geq 0.$$

Restricting to the integer lattice, we know the SBF$^\natural[\mathbb{Z}]$ property holds for $f(x)$, namely,

$$f(p)+f(q)\geq f[(p-\alpha\mathbf{1})\vee q]+f(p\wedge(q+\alpha\mathbf{1})),\quad\forall p,q\in\{0,\dots,N\}^d,\ \alpha\in\mathbb{N}.$$

Finally, Theorem 7.7 in Murota (2003) shows that the $L^\natural$-convexity is equivalent to the SBF$^\natural[\mathbb{Z}]$ property and therefore we know that $f(x)$ is a $L^\natural$-convex function.

Step 2. Next, we construct $d+1$ submodular functions on $\{0,1\}^d$ and extend them to $\mathcal{X}$ by the process defined in Step 1. The construction is based on the family of submodular functions defined in Graur et al. (2020). We denote $\mathcal{I} := \{0\} \cup [d]$. For each $i \in \mathcal{I}$, we define point $x^i \in \{0,1\}^d$ as

$$x^i := \sum_{j=1}^{i} e_j,$$

where e_j is the j-th unit vector of $\mathbb{R}^d$. Index $j(x)$ is defined as the maximal index j such that

$$x_i = 1, \quad \forall i \in [j].$$

If $x_1 = 0$, then we define $j(x) = 0$. Given $c : \mathcal{I} \mapsto \mathbb{R}$, we define a function on $\{0,1\}^d$ as

$$g^c(x) := \begin{cases} -c(i) & \text{if } x = x^i \text{ for some } i \in \mathcal{I} \\ (\|x\|_1 - j(x)) \cdot (d + 2 - j(x)) & \text{otherwise.} \end{cases}$$

By Lemma 6 in Graur et al. (2020), the function $g^c(x)$ is submodular if $c(i) \in \{0,1\}$. Using the fact that convex combinations of submodular functions are still submodular, we know that $g^c(x)$ is submodular for any c such that $c(i) \in [0,1]$. Then, for each $i \in \mathcal{I}$, we construct

$$c^i(0) := \frac{1}{2}, \quad c^i(j) := \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases}, \quad \forall j \in [d].$$

We denote $g^i(x) := g^{c^i}(x)$ and let $f^i(x)$ be the extension of $6\epsilon \cdot g^i(x)$ on $\mathcal{X}$ by the process in Step 1. By the result in Step 1, we know that $f^i(x)$ is $L^\natural$-convex.

Next, we prove that $f^0(x)$ has disjoint set of ϵ-optimal solutions with $f^i(x)$ for any $i \in [d]$. For each $f^i(x)$, we define the set of ϵ-optimal solutions as

$$\mathcal{X}_\epsilon^i := \{x \in \mathcal{X} : f^i(x) - \min_y \ f^i(y) \le \epsilon\}.$$

We first consider $\mathcal{X}_\epsilon^0$. By the definition of $g^0(x)$, we know that

$$f^0(x^0) = g^0(x^0) = -3\epsilon, \quad f^0(x) = g^0(x/N) \ge 0, \quad \forall x \in \{0,N\}^d \backslash \{x^0\}, \tag{EC.15}$$

which implies that

$$\mathcal{X}_\epsilon^0 = \{x \in \mathcal{X} : f^0(x) \le -2\epsilon\}.$$

Since $f^0(x)$ is defined by the scaled Lovász extension of $g^0(x)$, we have

$$f^0(x) = N^{-1} \cdot \left[(N - x_{\alpha(1)}) f^0(S^0) + \sum_{i=1}^{d-1} (x_{\alpha(i)} - x_{\alpha(i+1)}) f^0(S^i) + x_{\alpha(d)} f^0(S^d) \right], \tag{EC.16}$$

where α is a consistent permutation of x/N and $S^i := N \cdot S^{x/N,i} \in \{0,N\}^d$ is the i-th neighbouring points of x in the hypercube $\{0,N\}^d$. Using the relation in (EC.15) and the fact $S^0 = x^0$, we get

$$f^0(x) \ge N^{-1} \cdot (N - x_{\alpha(1)}) f(S_0) = N^{-1} \cdot (N - x_{\alpha(1)}) f(x^0) = -3\epsilon N^{-1} \cdot (N - x_{\alpha(1)}).$$

Hence, for any point $x \in \mathcal{X}_\epsilon^0$, we have $N - x_{\alpha(1)} = N - \max_i \ x_i \geq 2N/3$ and therefore

$$\mathcal{X}_\epsilon^0 \subset \{x \in \mathcal{X} : N - \max_i \ x_i \geq 2N/3\} = \{x \in \mathcal{X} : \max_i \ x_i \leq N/3\}. \tag{EC.17}$$

Next, we consider $\mathcal{X}_\epsilon^i$ with $i \in [d]$. By the definition of $g^i(x)$, we have

$$f^i(x^0) = g^i(x^0) = -3\epsilon, \quad f^i(x) = g^i(x) \geq -6\epsilon, \quad \forall x \in \{0, N\}^d \backslash \{x^0\},$$

which implies that

$$\mathcal{X}_\epsilon^i = \{x \in \mathcal{X} : f^i(x) \leq -5\epsilon\}.$$

Since the consistent permutation and neighboring points only depend on the coordinate of x, we know

$$\begin{aligned} f^i(x) &= N^{-1} \cdot \left[(N - x_{\alpha(1)}) f^i(S^0) + \sum_{i=1}^{d-1} (x_{\alpha(i)} - x_{\alpha(i+1)}) f^i(S^i) + x_{\alpha(d)} f^i(S^d) \right] \\ &\geq N^{-1} \cdot \left[-3\epsilon (N - x_{\alpha(1)}) - 6\epsilon \sum_{i=1}^{d-1} (x_{\alpha(i)} - x_{\alpha(i+1)}) - 6\epsilon \cdot x_{\alpha(d)} \right] \\ &= N^{-1} \cdot \left[-3\epsilon (N - x_{\alpha(1)}) - 6\epsilon \cdot x_{\alpha(1)} \right] = -3\epsilon N^{-1} \cdot (N + x_{\alpha(1)}). \end{aligned} \tag{EC.18}$$

Hence, the set $\mathcal{X}_\epsilon^i$ satisfies

$$\mathcal{X}_\epsilon^i \subset \{x \in \mathcal{X} : N + \max_i \ x_i \geq 5N/3\} = \{x \in \mathcal{X} : \max_i \ x_i \geq 2N/3\}. \tag{EC.19}$$

Combining the relations (EC.17) and (EC.19), we know $\mathcal{X}_\epsilon^0 \cap \mathcal{X}_\epsilon^i = \emptyset$ for all $i \in [d]$.

Step 3. Finally, we give a lower bound of $T_0(\epsilon, \delta, \mathcal{MC})$. For each $i \in \mathcal{I}$, we define M_i as the model such that the objective function is $f^i(x)$ and the distribution at each point is Gaussian with variance σ^2. Same as the one-dimensional case, given a zeroth-order algorithm and a model M, we denote $N_x(\tau)$ as the number of times that $F(x, \xi_x)$ is simulated when the algorithm terminates. By definition, we have

$$\mathbb{E}_M[\tau] = \sum_{x \in \mathcal{X}} \ \mathbb{E}_M \left[N_x(\tau) \right],$$

where $\mathbb{E}_M$ is the expectation when the model M is given. Similarly, we can define $\mathbb{P}_M$ as the probability when the model M is given. Suppose $\mathcal{A}$ is an $[(\epsilon, \delta)\text{-PGS}, \mathcal{MC}]$-algorithm and let $\mathcal{E}$ be the event that the solution returned by $\mathcal{A}$ is in the set $\mathcal{X}_\epsilon^0$. Since $\mathcal{X}_\epsilon^0 \cap \mathcal{X}_\epsilon^i = \emptyset$ for all $i \in [d]$, we know

$$\mathbb{P}_{M_0}[\mathcal{E}] \geq 1 - \delta, \quad \mathbb{P}_{M_i}[\mathcal{E}] \leq \delta, \quad \forall i \in [d].$$

Using the information-theoretical inequality (9), it holds

$$\sum_{x \in \mathcal{X}} \ \mathbb{E}_{M_0} \left[N_x(\tau) \right] \mathrm{KL}(\nu_{0,x}, \nu_{i,x}) \geq d(\mathbb{P}_{M_0}(\mathcal{E}), \mathbb{P}_{M_i}(\mathcal{E})) \geq d(1 - \delta, \delta) \geq \log \left(\frac{1}{2.4\delta} \right), \tag{EC.20}$$

where $d(x,y) := x\log(x/y) + (1-x)\log((1-x)/(1-y))$, $\mathrm{KL}(\cdot,\cdot)$ is the KL divergence and $\nu_{i,x}$ is the distribution of $F^i(x,\xi_x)$. Since the distributions $\nu_{i,x}$ are Gaussian with variance σ^2, the KL divergence can be calculated as

$$\mathrm{KL}(\nu_{0,x},\nu_{i,x}) = 2\sigma^{-2}\left(f^0(x) - f^i(x)\right)^2.$$

Now we estimate $f^0(x) - f^i(x)$ for all $i \in [d]$. By equations (EC.16) and (EC.18), we get

$$\begin{aligned} f^0(x) - f^i(x) = N^{-1}\Bigg[&(N - x_{\alpha(1)})\left(f^0(S^0) - f^i(S^0)\right) \\ &+ \sum_{j=1}^{d-1}(x_{\alpha(j)} - x_{\alpha(j+1)})\left(f^0(S^j) - f^i(S^j)\right) + x_{\alpha(d)}\left(f^0(S^d) - f^i(S^d)\right)\Bigg], \end{aligned} \tag{EC.21}$$

where α is a consistent permutation of x/N and S^i is the i-th neighboring point of x in hypercube $\{0,N\}^d$. By the definition of $f^0(x)$ and $f^i(x)$, we have

$$f^0(x) - f^i(x) = \begin{cases} 6\epsilon & \text{if } x = x^i \\ 0 & \text{otherwise.} \end{cases}$$

Since $\|x^i\|_1 = i$ and $\|S^j\|_1 = j$ for all $j \in \mathcal{I}$, we know

$$f^0(S^i) - f^i(S^i) \le 6\epsilon, \quad f^0(S^j) - f^i(S^j) = 0, \quad \forall j \in \mathcal{I}\backslash\{i\}.$$

Substituting into equation (EC.21), it follows that

$$f^0(x) - f^i(x) \le \begin{cases} \left(6\epsilon\cdot(x_{\alpha(i)} - x_{\alpha(i+1)})\right)^2 & \text{if } i \in [d-1] \\ \left(6\epsilon\cdot x_{\alpha(d)}\right)^2 & \text{if } i = d. \end{cases}$$

Hence, the KL divergence is bounded by

$$\mathrm{KL}(\nu_{0,x},\nu_{i,x}) = 2\sigma^{-2}\left(f^0(x) - f^i(x)\right)^2 \le \begin{cases} 72\sigma^{-2}N^{-2}\epsilon^2\left((x_{\alpha(i)} - x_{\alpha(i+1)})\right)^2 & \text{if } i \in [d-1] \\ 72\sigma^{-2}N^{-2}\epsilon^2 x_{\alpha(d)}^2 & \text{if } i = d. \end{cases}$$

Substituting the KL divergence into inequality (EC.20) and summing over $i = 1,\dots,d$, we get

$$\sum_{x\in\mathcal{X}} \mathbb{E}_{M_0}\left[N_x(\tau)\right]\cdot 72\sigma^{-2}N^{-2}\epsilon^2\left[\sum_{i=1}^{d-1}(x_{\alpha(i)} - x_{\alpha(i+1)})^2 + x_{\alpha(d)}^2\right] \ge d\log\left(\frac{1}{2.4\delta}\right). \tag{EC.22}$$

Since α is the consistent permutation of x, we know

$$0 \le x_{\alpha(i)} - x_{\alpha(i+1)} \le N, \quad \forall i \in [d-1]$$

and therefore

$$\sum_{i=1}^{d-1}(x_{\alpha(i)} - x_{\alpha(i+1)})^2 + x_{\alpha(d)}^2 \le N\cdot\left(\sum_{i=1}^{d-1}(x_{\alpha(i)} - x_{\alpha(i+1)}) + x_{\alpha(d)}\right) = N\cdot x_{\alpha(1)} \le N^2.$$

Combining with inequality (EC.22), we get

$$\sum_{x\in\mathcal{X}} \mathbb{E}_{M_0}\left[N_x(\tau)\right]\cdot 72\epsilon^2\sigma^{-2} \geq d\log\left(\frac{1}{2.4\delta}\right),$$

which implies that

$$\mathbb{E}_{M_0}[\tau] = \sum_{x\in\mathcal{X}} \mathbb{E}_{M_0}\left[N_x(\tau)\right] \geq \frac{d\sigma^2}{72\epsilon^2}\log\left(\frac{1}{2.4\delta}\right).$$

□

EC.4.2. Proof of Theorem 6

Proof of Theorem 6. We consider the submodular functions $g^0(x),\ldots,g^d(x)$ constructed in the proof of Theorem 5. We want to construct objective functions $f^0(x),\ldots,f^d(x)$ on $\mathcal{X}=[N]^d$ such that

$$f^i(x) = \begin{cases} 6c\cdot g^i(x-1)+h(x) & \text{if } x\in[2]^d \\ h(x) & \text{if } x\in[N]^d\backslash[2]^d, \end{cases} \qquad \forall i\in\{0,\ldots,d\},$$

where $(x-1)_j := x_j-1$ for all $j\in[d]$ and $h(x)$ is a suitably designed function. Similar to the proof of Theorem 5, we apply the information-theoretical inequality (9) to pairs $f^0(x)$ and $f^i(x)$ for all $i\in[d]$. Since the objective function values for $f^0(x)$ and $f^i(x)$ are equal for all $x\in[N]^d\backslash[2]^d$, the terms with respect to those x will disappear and we only need to analyze the terms with $x\in[2]^d$. Now, using the same analysis and notations as Theorem 5, we get the desired lower bound

$$\mathbb{E}_{M_0}[\tau] = \sum_{x\in\mathcal{X}} \mathbb{E}_{M_0}\left[N_x(\tau)\right] \geq \frac{d\sigma^2}{72c^2}\log\left(\frac{1}{2.4\delta}\right).$$

Therefore, it remains to chose a suitable function $h(x)$ such that $f^i(x)$ are $L^\natural$-convex on the whole feasible set $\mathcal{X}$. We define

$$M := \max_{x\in\{0,1\}^d,\ i\in\{0,\ldots,d\}} 6c\cdot|g^i(x)|.$$

The extended function $f^i(x)$ is defined by

$$h(x) := 4M\sum_{j=1}^d (x_j-1)(x_j-2) + 2M\max_j x_j + 2M\sum_{j=1}^d \mathbf{1}(x_j=1), \quad x\in[N]^d,$$

where $\mathbf{1}(\cdot)$ is the indicator function. The function $h(x)$ is the sum of two $L^\natural$-convex functions (Murota 2003) and thus is a $L^\natural$-convex function. We prove that for each $i\in[d]$, the function $f^i(x)$ is $L^\natural$-convex, namely, it satisfies the discrete mid-point convexity. Suppose that $x,y\in[N]^d$ are two feasible points. We consider three different cases.

Case I. We first consider the case when $x,y \in [2]^d$. In this case, the fact that $[2]^d$ is a $L^\natural$-convex set implies that

$$\left\lceil \frac{x+y}{2} \right\rceil, \left\lfloor \frac{x+y}{2} \right\rfloor \in [2]^d.$$

Since the function $6c \cdot g^i(x) + h(x)$ is $L^\natural$-convex, the discrete mid-point convexity holds for x and y.

Case II. We consider the case when $x,y \notin [2]^d$. Since the function $\sum_j \mathbf{1}(x_j = 1)$ is $L^\natural$-convex, it satisfies the discrete mid-point convexity and we can safely ignore its effect in this case. If $\lfloor (x+y)/2 \rfloor, \lceil (x+y)/2 \rceil \notin [2]^d$, then the $L^\natural$-convexity of $h(x)$ implies the discrete mid-point convexity of points x and y. Now, we consider the case when $\lfloor (x+y)/2 \rfloor, \lceil (x+y)/2 \rceil \in [2]^d$. Since at least one component of x and y is larger than 2, it holds that

$$f^i(x) \geq 4M \cdot (3-1)(3-2) + 3M = 11M, \quad f^i(y) \geq 11M.$$

Hence, we get

$$f^i(x) + f^i(y) \geq 22M \geq f^i\left(\left\lceil \frac{x+y}{2} \right\rceil \right) + f^i\left(\left\lfloor \frac{x+y}{2} \right\rfloor \right).$$

The only remaining case is when

$$\left\lceil \frac{x+y}{2} \right\rceil \notin [2]^d, \quad \left\lfloor \frac{x+y}{2} \right\rfloor \in [2]^d.$$

In this case, we have

$$\left\lfloor \frac{x_j+y_j}{2} \right\rfloor \leq 2, \quad \forall j \in [d], \quad \max_j \left\lceil \frac{x_j+y_j}{2} \right\rceil \geq 3,$$

which implies that

$$x_j + y_j \leq \max_j (x_j + y_j) = 5, \quad \forall j \in [d]$$

and

$$\max_j x_j \geq 3, \quad \max_j y_j \geq 3, \quad \max_j \left\lceil \frac{x_j+y_j}{2} \right\rceil = 3, \quad \max_j \left\lfloor \frac{x_j+y_j}{2} \right\rfloor = 2.$$

Let

$$\mathcal{J}_x := \{ j \in [d] : \ x_j \geq 3 \}, \quad \mathcal{J}_y := \{ j \in [d] : \ y_j \geq 3 \}, \quad \mathcal{J} := \{ j \in [d] : \ x_j + y_j = 5 \}. \tag{EC.23}$$

The analysis above gives

$$\mathcal{J} \subset \mathcal{J}_x \cup \mathcal{J}_y, \quad \mathcal{J}_x \cap \mathcal{J}_y = \emptyset.$$

Hence, we know

$$\sum_j (x_j - 1)(x_j - 2) + \sum_j (y_j - 1)(y_j - 2) \geq 2|\mathcal{J}_x| + 2|\mathcal{J}_y|,$$

$$\sum_j \left(\left\lceil \frac{x_j+y_j}{2} \right\rceil - 1 \right) \left(\left\lceil \frac{x_j+y_j}{2} \right\rceil - 2 \right) + \sum_j \left(\left\lfloor \frac{x_j+y_j}{2} \right\rfloor - 1 \right) \left(\left\lfloor \frac{x_j+y_j}{2} \right\rfloor - 2 \right) = 2|\mathcal{J}|.$$

Combining with inequality (EC.23), we get

$$h(x)+h(y)-h\left(\left\lceil\frac{x_j+y_j}{2}\right\rceil\right)-h\left(\left\lfloor\frac{x_j+y_j}{2}\right\rfloor\right)\geq 8M(|\mathcal{J}_x|+|\mathcal{J}_y|-|\mathcal{J}|)+2M\geq 2M.$$

Therefore, it holds that

$$\begin{aligned}f^i(x)+f^i(y)=h(x)+h(y)&\geq h\left(\left\lceil\frac{x_j+y_j}{2}\right\rceil\right)+h\left(\left\lfloor\frac{x_j+y_j}{2}\right\rfloor\right)+2M\\&\geq h\left(\left\lceil\frac{x_j+y_j}{2}\right\rceil\right)+h\left(\left\lfloor\frac{x_j+y_j}{2}\right\rfloor\right)+6c\cdot g^i(x-1)\\&=f^i\left(\left\lceil\frac{x_j+y_j}{2}\right\rceil\right)+f^i\left(\left\lfloor\frac{x_j+y_j}{2}\right\rfloor\right).\end{aligned}$$

Case III. Finally, we consider the case when $x\in[2]^d$ and $y\notin[2]^d$. If

$$\left\lceil\frac{x+y}{2}\right\rceil,\left\lfloor\frac{x+y}{2}\right\rfloor\in[2]^d,$$

we know

$$f^i(y)+f^i(x)\geq 11M-M>6M\geq f^i\left(\left\lceil\frac{x_j+y_j}{2}\right\rceil\right)+f^i\left(\left\lfloor\frac{x_j+y_j}{2}\right\rfloor\right).$$

Next, for the case where

$$\left\lceil\frac{x+y}{2}\right\rceil,\left\lfloor\frac{x+y}{2}\right\rfloor\notin[2]^d,$$

we get

$$\max_j\frac{x_j+y_j}{2}\geq 3,$$

which implies that

$$\max_j y_j\geq 4.$$

Considering the component j such that $y_j\geq 4$, it follows that

$$\begin{aligned}&\left(\left\lceil\frac{x_j+y_j}{2}\right\rceil-1\right)\left(\left\lceil\frac{x_j+y_j}{2}\right\rceil-2\right)+\left(\left\lfloor\frac{x_j+y_j}{2}\right\rfloor-1\right)\left(\left\lfloor\frac{x_j+y_j}{2}\right\rfloor-2\right)\\\leq&\left(\frac{x_j+y_j+1}{2}-1\right)\left(\frac{x_j+y_j+1}{2}-2\right)+\left(\frac{x_j+y_j}{2}-1\right)\left(\frac{x_j+y_j}{2}-2\right)\\\leq&\left(\frac{y_j+3}{2}-1\right)\left(\frac{y_j+3}{2}-2\right)+\left(\frac{y_j+2}{2}-1\right)\left(\frac{y_j+2}{2}-2\right)=\frac{1}{2}y_j^2-\frac{1}{2}y_j-\frac{1}{4}.\end{aligned}$$

Combining with the $L^\natural$-convexity of functions $(x_k-1)(y_k-2)$ for each $k\in[d]$ and $\max_k x_k$, we get

$$\begin{aligned}&h(x)+h(y)-h\left(\left\lceil\frac{x+y}{2}\right\rceil\right)-h\left(\left\lfloor\frac{x+y}{2}\right\rfloor\right)\geq h_j(x)+h_j(y)-h_j\left(\left\lceil\frac{x+y}{2}\right\rceil\right)-h_j\left(\left\lfloor\frac{x+y}{2}\right\rfloor\right)\\\geq&4M(x_j-1)(x_j-2)+4M(y_j-1)(y_j-2)\\&\qquad-4M\left(\left\lceil\frac{x_j+y_j}{2}\right\rceil-1\right)\left(\left\lceil\frac{x_j+y_j}{2}\right\rceil-2\right)-4M\left(\left\lfloor\frac{x_j+y_j}{2}\right\rfloor-1\right)\left(\left\lfloor\frac{x_j+y_j}{2}\right\rfloor-2\right)\\\geq&0+4M\left[y_j^2-3y_j+2-\left(\frac{1}{2}y_j^2-\frac{1}{2}y_j-\frac{1}{4}\right)\right]=M(2y_j^2-10y_j+9)\geq M.\end{aligned}$$

Therefore, we have

$$\begin{aligned}
f^i(x)+f^i(y) &= h(x)+h(y)+6c\cdot g^i(x-1)\geq h(x)+h(y)-M\\
&\geq h\left(\left\lceil\frac{x+y}{2}\right\rceil\right)+h\left(\left\lfloor\frac{x+y}{2}\right\rfloor\right)+M-M\\
&\geq h\left(\left\lceil\frac{x+y}{2}\right\rceil\right)+h\left(\left\lfloor\frac{x+y}{2}\right\rfloor\right)=f^i\left(\left\lceil\frac{x+y}{2}\right\rceil\right)+f^i\left(\left\lfloor\frac{x+y}{2}\right\rfloor\right).
\end{aligned}$$

Now, we consider the last case where

$$\left\lceil\frac{x+y}{2}\right\rceil\notin[2]^d,\quad \left\lfloor\frac{x+y}{2}\right\rfloor\in[2]^d.$$

Similar to Case II, we can prove that

$$x_j+y_j\leq 5,\quad \forall j\in[d].$$

If it holds that

$$h(y)>h\left(\left\lceil\frac{x+y}{2}\right\rceil\right),$$

we can utilize that fact that $y,\lceil\frac{x+y}{2}\rceil\in\mathbb{Z}^d$ to prove

$$h(y)\geq h\left(\left\lceil\frac{x+y}{2}\right\rceil\right)+2M,$$

which leads to

$$\begin{aligned}
f^i(x)+f^i(y) &\geq h(x)+h(y)-M\geq 0+h\left(\left\lceil\frac{x+y}{2}\right\rceil\right)+2M-M\\
&\geq h\left(\left\lceil\frac{x+y}{2}\right\rceil\right)+6c\cdot g^i\left(\left\lfloor\frac{x+y}{2}\right\rfloor\right)=h\left(\left\lceil\frac{x+y}{2}\right\rceil\right)+0+6c\cdot g^i\left(\left\lfloor\frac{x+y}{2}\right\rfloor\right)\\
&\geq h\left(\left\lceil\frac{x+y}{2}\right\rceil\right)+h\left(\left\lfloor\frac{x+y}{2}\right\rfloor\right)+6c\cdot g^i\left(\left\lfloor\frac{x+y}{2}\right\rfloor\right)\\
&=f^i\left(\left\lceil\frac{x+y}{2}\right\rceil\right)+f^i\left(\left\lfloor\frac{x+y}{2}\right\rfloor\right).
\end{aligned}$$

Therefore, we focus on the case when

$$h(y)\leq h\left(\left\lceil\frac{x+y}{2}\right\rceil\right). \tag{EC.24}$$

First, using the facts that $x\in[2]^d$ and $y\notin[2]^d$, it is easy to prove that

$$\max_j y_j\geq\max_j\left\lceil\frac{x_j+y_j}{2}\right\rceil=3,\quad \mathbf{1}(y_j=1)\geq\mathbf{1}\left(\left\lceil\frac{x_j+y_j}{2}\right\rceil=1\right),\quad \forall j\in[d]. \tag{EC.25}$$

Moreover, using the condition that $x_j\in[2]$, it holds that

$$\left|y_j-\frac{3}{2}\right|\geq\left|\left\lceil\frac{x_j+y_j}{2}\right\rceil-\frac{3}{2}\right|,\quad \forall j\in[d],$$

which implies that

$$\sum_j (y_j-1)(y_j-2) \geq \sum_j \left(\left\lceil \frac{x_j+y_j}{2}\right\rceil -1\right)\left(\left\lceil \frac{x_j+y_j}{2}\right\rceil -2\right).$$

Combining with inequalities in (EC.25), we get

$$h(y) \geq h\left(\left\lceil \frac{x+y}{2}\right\rceil\right).$$

In addition, the equality of the above inequality holds in combination with our assumption in (EC.24). The equality conditions imply that

$$\max_j y_j = 3, \quad \mathbf{1}(y_j=1) = \mathbf{1}\left(\left\lceil \frac{x_j+y_j}{2}\right\rceil = 1\right), \quad \left|y_j - \frac{3}{2}\right| = \left|\left\lceil \frac{x_j+y_j}{2}\right\rceil - \frac{3}{2}\right|, \quad \forall j \in [d].$$

The above three conditions imply that

$$y = \left\lceil \frac{x_j+y_j}{2}\right\rceil.$$

Utilizing the identity

$$x+y = \left\lceil \frac{x_j+y_j}{2}\right\rceil + \left\lfloor \frac{x_j+y_j}{2}\right\rfloor,$$

we know

$$x = \left\lfloor \frac{x_j+y_j}{2}\right\rfloor.$$

In this case, the discrete mid-point convexity holds evidently.

□

EC.5. Proofs in Section 6

EC.5.1. Proof of Theorem 7

First, the following lemma shows that the lower bound of $\mathbb{E}[H_x(y,\eta_y)]$ in $\mathcal{N}_x$ implies a global lower bound of $f(x)$.

Lemma EC.8. *Suppose that Assumptions 1-5 hold. If we have*

$$\mathbb{E}[H_x(y,\eta_y)] \geq -b, \quad \forall y \in \mathcal{N}_x$$

for some constant $b \geq 0$, then it holds

$$f(y) \geq f(x) - \frac{2N}{1-a}\cdot b, \quad \forall y \in \mathcal{X}.$$

Proof of Lemma EC.8. The proof follows the same framework as Theorem EC.3. We first consider points $y \in \mathcal{N}_x$. By the condition of this lemma and inequality (10), we have

$$f(y) - f(x) \geq (1-a)^{-1} \cdot \mathbb{E}[H_x(y, \eta_y)] \geq -(1-a)^{-1}(1-a)^{-1}b.$$

Next, we consider point $y \in \mathcal{X}$ such that $\|y - x\|_\infty \leq 1$. Then, there exists two disjoint sets $\mathcal{S}_1, \mathcal{S}_2 \subset [d]$ such that

$$y = x + e_{\mathcal{S}_1} - e_{\mathcal{S}_2},$$

where $e_{\mathcal{S}} := \sum_{i \in \mathcal{S}} e_i$ is the indicator vector of $\mathcal{S}$. Using the translation submodularity of $f(x)$, we have

$$f(y) \geq f(x + e_{\mathcal{S}_1}) - f(x) + f(x - e_{\mathcal{S}_2}) - f(x) \geq -2(1-a)^{-1}b.$$

Now, let $\tilde{f}(x)$ be the convex extension of $f(x)$ defined in (7) and consider $y \in [1, N]^d$ such that $\|y - x\|_\infty \leq 1$. We consider the hypercube $\mathcal{C}_z$ that contains both x and y and denote $S^{y,i}$ as the i-th neighboring point of y in $\mathcal{C}_z$. Recalling the expression (6), we know $f(y)$ is a convex combination of $f(S^{y,0}), \ldots, f(S^{y,d})$. Since the neighboring point $S^{y,i} \in \mathcal{X}$ satisfies $\|S^{y,i} - x\|_\infty \leq 1$, we know

$$\tilde{f}(y) \geq \min_{i \in \{0\} \cup [d]} f(S^{y,i}) \geq -2(1-a)^{-1}b.$$

Finally, we consider points $y \in [1, N]^d$. We define

$$\tilde{y} := x + \frac{y - x}{\|y - x\|_\infty}.$$

Then, we know $\|\tilde{y} - x\|_\infty = 1$ and $\tilde{f}\tilde{y}) \geq -2(1-a)^{-1}b$. By the convexity of $\tilde{f}(x)$,

$$\tilde{f}(y) - f(x) \geq \frac{\|y - x\|_\infty}{\|\tilde{y} - x\|_\infty} \left[\tilde{f}(\tilde{y}) - f(x)\right] \geq -N \cdot 2(1-a)^{-1}b = -\frac{2N}{1-a} \cdot b.$$

□

Hence, to find an (ϵ, δ)-PGS solution, it suffices to find point x such that

$$\mathbb{E}[H_x(y, \eta_y)] \geq -\frac{(1-a)\epsilon}{2N}, \quad \forall y \in \mathcal{N}_x$$

holds with probability at least $1 - \delta$.

Proof of Theorem 7. Let x^* be a minimizer of $f(x)$. We use the induction method to prove that

$$f(x^{e,0}) - f(x^*) \leq 2^{-e} \cdot NL, \quad \forall e \in \{0, 1, \ldots, E\} \tag{EC.26}$$

holds with probability at least $1 - e \cdot \delta / E$. Using Assumption 4, we have

$$f(x^{0,0}) - f(x^*) \leq L \cdot \|x^{0,0} - x^*\|_\infty \leq NL,$$

which means the induction assumption holds for epoch 0. Suppose the induction assumption is true for epochs $0, 1, \ldots, e-1$. Now we consider epoch e. We assume the event

$$f(x^{e-1,0}) - f(x^*) \leq 2^{-e+1} \cdot NL$$

happens in the following proof, which has probability at least $1 - (e-1)\delta/E$. We suppose epoch e terminates after T_e iterations and discuss by two different cases.

Case I. We first consider the case when $T_e \le T-1$. This event happens only if epoch $e-1$ is terminated by the condition in Line 13, i.e.,

$$\hat{H}_{x^{e-1,T_e-1}}(y) > -2h_{e-1}, \quad \forall y \in \mathcal{N}_{x^{e-1,T_e-1}}.$$

By the definition of confidence intervals, it follows that

$$\min_{y\in\mathcal{N}_{x^{e-1,T_e-1}}} \mathbb{E}[H_{x^{e-1,T_e-1}}(y,\eta_y)] \ge -3h_{e-1} = -3\cdot 2^{-e+1}h_0 = -2^{-e-1}\cdot(1-a)L$$

holds with probability at least $1-\delta/(ET)$. Then, considering the results of Lemma EC.8, we know

$$f(x^{e,0}) - f(x^*) = f(x^{e-1,T_e-1}) - f(x^*) \le \frac{2N}{1-a}\cdot 2^{-e-1}\cdot(1-a)L = 2^{-e}\cdot NL$$

happens with the same probability. Combining with the induction assumption for epoch $e-1$, the above event happens with probability at least $1-(e-1)\delta/E - \delta/(ET) \ge 1 - e\cdot\delta/E$ and the induction assumption holds for epoch e.

Case II. Next, we consider the case when $T_e = T$. We estimate the object function decrease for each iteration $t=0,1,\dots,T-1$. By the definition of confidence intervals, it holds

$$\mathbb{E}[H_{x^{e-1,t}}(y,\eta_y)] \le -h_{e-1}$$

with probability at least $1-\delta/(ET)$, where $y = x^{e-1,t+1}$ is the next iteration point. Recalling inequality (10), we know

$$f(x^{e-1,t+1}) - f(x^{e-1,t}) \le -(1+a)^{-1}h_{e-1}$$

happens with probability at least $1-\delta/(ET)$. We assume the above event happens for all $t = 1,2,\dots,T$, which has probability at least $1 - T\cdot\delta/(ET) = 1-\delta/E$. Then, we have

$$\begin{aligned} f(x^{e,0}) - f(x^{e-1,0}) = f(x^{e-1,T}) - f(x^{e-1,0}) &= \sum_{t=1}^{T} f(x^{e-1,t}) - f(x^{e-1,t-1}) \\ &\le -T\cdot(1+a)^{-1}h_{e-1} = -2^{-e}\cdot NL \end{aligned}$$

holds with the same probability. Combining with the induction assumption for epoch $e-1$, we know

$$f(x^{e,0}) - f(x^*) \le 2^{-e}\cdot NL$$

happens with probability at least $1-(e-1)\delta/E - \delta/E = 1 - e\cdot\delta/E$. This means the induction assumption holds for epoch e.

Combining the above two cases, we know the induction assumption is true for epoch e. By the induction method, we know inequality (EC.26) holds for epoch E, i.e.,

$$f(x^{E,0}) - f(x^*) \le 2^{-E}\cdot NL = 2^{-\lceil \log_2(NL/\epsilon)\rceil}\cdot NL \le 2^{-\log_2(NL/\epsilon)}\cdot NL = \epsilon$$

with probability at least $1-E\cdot\delta/E=1-\delta$. Hence, Algorithm 3 returns an (ϵ,δ)-PGS solution.

Next, we estimate the simulation cost of Algorithm 3. For each iteration in epoch e, Hoeffding bound implies that simulating $H_x(y,\eta_y)$ for

$$\frac{2\tilde{\sigma}^2}{h_e^2}\log\left(\frac{2ET}{\delta}\right)=2^{2e}\cdot\frac{288\tilde{\sigma}^2}{(1-a)^2L^2}\log\left(\frac{2ET}{\delta}\right)$$

times is sufficient to ensure that the $1-\delta/(ET)$ confidence half-width is at most T_e. Since the simulation cost of each evaluation of all $H_x(y,\eta_y)$ is γ, the simulation cost of epoch e is at most

$$\gamma\cdot T\cdot 2^{2e}\cdot\frac{288\tilde{\sigma}^2}{(1-a)^2L^2}\log\left(\frac{2ET}{\delta}\right)=2^{2e}\cdot\frac{1728(1+a)\tilde{\sigma}^2\gamma N}{(1-a)^3L^2}\log\left(\frac{2ET}{\delta}\right).$$

Summing over $e=0,1,\dots,E-1$, we get the bound of total simulation cost as

$$\begin{aligned}&\sum_{e=0}^{E-1}2^{2e}\cdot\frac{1728(1+a)\tilde{\sigma}^2\gamma N}{(1-a)^3L^2}\log\left(\frac{2ET}{\delta}\right)=\left(4^E-1\right)\cdot\frac{576(1+a)\tilde{\sigma}^2\gamma N}{(1-a)^3L^2}\log\left(\frac{2ET}{\delta}\right)\\ \le&4^{\lceil\log_2(NL/\epsilon)\rceil}\cdot\frac{576(1+a)\tilde{\sigma}^2\gamma N}{(1-a)^3L^2}\log\left(\frac{2ET}{\delta}\right)\le4^{\log_2(NL/\epsilon)+1}\cdot\frac{576(1+a)\tilde{\sigma}^2\gamma N}{(1-a)^3L^2}\log\left(\frac{2ET}{\delta}\right)\\ =&\frac{4N^2L^2}{\epsilon^2}\cdot\frac{576(1+a)\tilde{\sigma}^2\gamma N}{(1-a)^3L^2}\log\left(\frac{2ET}{\delta}\right)=\frac{2304(1+a)\tilde{\sigma}^2\gamma N^3}{(1-a)^3\epsilon^2}\log\left(\frac{2ET}{\delta}\right).\end{aligned}$$

When δ is small enough, the asymptotic simulation cost is at most

$$\frac{2304(1+a)\tilde{\sigma}^2\gamma N^3}{(1-a)^3\epsilon^2}\log\left(\frac{2ET}{\delta}\right)=\tilde{O}\left[\frac{\gamma N^3}{(1-a)^3\epsilon^2}\log\left(\frac{1}{\delta}\right)\right].$$

□

EC.5.2. First-order algorithms for the PCS-IZ case

We first give the stochastic steepest descent method for the PCS-IZ guarantee in Algorithm 5.

Algorithm 5 Adaptive stochastic steepest descent method for PCS-IZ guarantee

```
Input: Model $\mathcal{X},\mathcal{B}_{\mathsf{Y}},F(x,\xi_x)$, optimality guarantee parameter $\delta$, indifference zone parameter $c$,
    biased subgradient estimator $H_x(y,\eta_y)$, bias ratio $a$.
Output: A $(c,\delta)$-PCS-IZ solution $x^*$ to problem (1).
Set the initial confidence half-width threshold $h\leftarrow(1-a)c/12$.
Set maximal number of iterations $T\leftarrow(1+a)/(1-a)\cdot 12N$.
Use Algorithm 3 to find an $(Nc,\delta/2)$-PGS solution.
for $t=0,1,\dots,T-1$ do
    repeat simulate $H_{x^t}(y,\eta_y)$ for all $y\in\mathcal{N}_{x^t}$
        Compute the empirical mean $\hat{H}_{x^t}(y)$ using all simulated samples for all $y\in\mathcal{N}_{x^t}$.
```

```
 7:             Compute the $1-\delta/(2T)$ confidence interval
                    $$\left[\hat{H}_{x^t}(y)-h_y,\hat{H}_{x^t}(y)+h_y\right],\quad \forall y\in\mathcal{N}_{x^t}.$$
 8:         until the confidence half-width $h_y \le h$ for all $y\in\mathcal{N}_{x^t}$
 9:         if $\hat{H}_{x^t}(y)\le -2h$ for some $y\in\mathcal{N}_{x^t}$ then    ▷ This step takes $2^{d+1}$ arithmetic operations.
10:             Update $x^{t+1}\leftarrow y$.
11:         else if $\hat{H}_{x^t}(y) > -2h$ for some $y\in\mathcal{N}_{x^t}$ then
12:             break
13:         end if
14:     end for
15:     Return $x^t$.
```

The following theorem verifies the correctness of Algorithm 5 and estimates its asymptotic simulation cost.

THEOREM EC.6. *Suppose that Assumptions 1-4, 5 hold. Algorithm 5 returns an (c,δ)-PCS-IZ solution and we have*

$$T(\delta,\mathcal{MC}_c)=O\left[\frac{\gamma N}{(1-a)^3c^2}\log\left(\frac{1}{\delta}\right)+\frac{\gamma N}{1-a}\max\left\{\log\left(\frac{1}{c}\right),1\right\}\right]=\tilde{O}\left[\frac{\gamma N}{(1-a)^3c^2}\log\left(\frac{1}{\delta}\right)\right].$$

Proof of Theorem EC.6. If the algorithm terminates before the T-th iteration, then the condition at Line 11 is satisfied for the last iteration point, which we denote as x^t. Let

$$y^t := \operatorname*{arg\,min}_{y\in\mathcal{N}_{x^t}} f(y).$$

Then, by the definition of confidence intervals, it holds

$$\mathbb{E}[H_{x^{t-1}}(y^t,\eta_{y^t})]\ge -3h$$

with probability at least $1-\delta/(2T)\ge 1-\delta$. By inequality (10), we know

$$\min_{y\in\mathcal{N}_{x^t}} f(y)-f(x^t)=f(y^t)-f(x^t)\ge -\frac{3h}{1-a}=-\frac{c}{4}$$

holds wit the same probability. We assume the event happens in the following proof. For any point $y\in\mathcal{X}$ such that $\|y-x^t\|_\infty\le 1$, there exists two disjoint sets $\mathcal{S}_1,\mathcal{S}_2\subset[d]$ such that

$$y=x^t+e_{\mathcal{S}_1}-e_{\mathcal{S}_2},$$

where $e_{\mathcal{S}}:=\sum_{i\in\mathcal{S}}e_i$ is the indicator vector of $\mathcal{S}$. Then, using the $L^\natural$-convexity of $f(x)$, we know

$$f(y)-f(x^t)\ge f(x^t+e_{\mathcal{S}_1})-f(x^t)+f(x^t-e_{\mathcal{S}_2})-f(x^t)\ge -\frac{c}{2}.$$

Let $\tilde{f}(x)$ be the convex extension of $f(x)$ defined in (7). Recalling expression (6), we know

$$\tilde{f}(y) - f(x^t) \geq -\frac{c}{2}, \quad \forall y \in [1,N]^d \quad \text{s.t. } \|y - x^t\|_\infty \leq 1. \tag{EC.27}$$

We assume that x^t is not the minimizer of $f(x)$, which we denote as x^*. Since the indifference zone parameter is c, we know

$$f(y) - f(x^*) \geq c, \quad \forall y \in \mathcal{X} \backslash \{x^*\}. \tag{EC.28}$$

Similarly, using expression (6), we get

$$\tilde{f}(y) - f(x^*) \geq c, \quad \forall y \in [1,N]^d \quad \text{s.t. } \|y - x^*\|_\infty \leq 1.$$

If $\|x^t - x^*\|_\infty \leq 1$, then there exists a point x^* such that $\|x^* - x^t\|_\infty \leq 1$ and

$$f(x^*) - f(x^t) \leq -c,$$

which contradicts with inequality (EC.27). Otherwise if $\|x^t - x^*\|_\infty \geq 2$, we define

$$x^{t,1} := x^t + \frac{x^* - x^t}{\|x^t - x^*\|_\infty}, \quad x^{t,2} := x^* + \frac{x^t - x^*}{\|x^* - x^t\|_\infty}.$$

Then, it holds

$$\|x^t - x^{t,1}\|_\infty = 1, \quad \|x^* - x^{t,2}\|_\infty = 1$$

and $x^{t,1}, x^{t,2}$ are closer to x^t, x^*, respectively. By inequalities (EC.27) and (EC.28), we get

$$\tilde{f}(x^{t,1}) - f(x^t) \geq -\frac{c}{2}, \quad \tilde{f}(x^*) - f(x^{t,2}) \leq -c.$$

However, the convexity of $\tilde{f}(x)$ on the segment $\overline{x^t x^*}$ implies that

$$-\frac{c}{2} \leq \tilde{f}(x^{t,1}) - f(x^t) \leq \tilde{f}(x^*) - f(x^{t,2}) \leq -c,$$

which is a contradiction. Hence, we know $x^t = x^*$ is the minimizer of $f(x)$. This event happens with probability at least $1 - \delta$ and therefore x^t is a (c, δ)-PCS-IZ solution.

Otherwise, we assume the algorithm terminates after T iterations. We use the induction method to prove that

$$f(x^t) - f(x^0) \leq -t \cdot \frac{(1-a)c}{12(1+a)}$$

happens with probability at least $1 - t \cdot \delta/(2T)$. For the initial point x^0, this claim holds trivially. Suppose the induction assumption is true for $x^0, x^1, \dots, x^{t-1}$. For the $(t-1)$-th iteration, by the definition of confidence intervals, it holds

$$\mathbb{E}[H_{x^{t-1}}(x^t, \eta_{x^t})] \leq -h$$

with probability at least $1-\delta/(2T)$. Using inequality (10), we know

$$f(x^t)-f(x^{t-1})\le -\frac{h}{1+a}=-\frac{(1-a)c}{12(1+a)}$$

holds with the same probability. Using the induction assumption for x^{t-1}, we have

$$f(x^t)-f(x^0)\le -(t-1)\cdot\frac{(1-a)c}{12(1+a)}-\frac{(1-a)c}{12(1+a)}=-t\cdot\frac{(1-a)c}{12(1+a)}$$

holds with probability at least $1-(t-1)\delta/(2T)-\delta/(2T)=1-t\cdot\delta/(2T)$. Hence, the induction assumption holds for x^t and, by the induction method, holds for all iterations. Since the algorithm terminates after T iterations, the last point x^T satisfies

$$f(x^T)-f(x^0)\le -T\cdot\frac{(1-a)c}{12(1+a)}=-cN$$

with probability at least $1-T\cdot\delta/(2T)=1-\delta/2$. Recalling the initial point x^0 is a $(cN,\delta/2)$-PGS solution, we know x^T is the optimal point with probability at least $1-\delta$ and therefore is a (c,δ)-PCS-IZ solution.

Finally, we estimate the simulation cost of Algorithm 5. By Theorem 7, the simulation cost for generating the initial point is

$$\tilde{O}\left[\frac{\gamma N}{(1-a)^3c^2}\log\left(\frac{1}{\delta}\right)\right].$$

For each iteration, Hoeffding bound implies that simulating

$$\frac{2\tilde{\sigma}^2}{h^2}\log\left(\frac{4T}{\delta}\right)=\frac{288\tilde{\sigma}^2}{(1-a)^2c^2}\log\left(\frac{4T}{\delta}\right)$$

times is enough for the $1-\delta/(2T)$ confidence half-width to be smaller than h. Hence, the total simulation for iterations is at most

$$T\cdot\gamma\cdot\frac{288\tilde{\sigma}^2}{(1-a)^2c^2}\log\left(\frac{4T}{\delta}\right)=\frac{1152\gamma\tilde{\sigma}^2(1+a)N}{(1-a)^3c^2}\log\left(\frac{4T}{\delta}\right)=O\left[\frac{\gamma N}{(1-a)^3c^2}\log\left(\frac{1}{\delta}\right)\right].$$

Combining the simulation costs for initialization and iterations, we know the asymptotic simulation cost of Algorithm 5 is at most

$$\tilde{O}\left[\frac{\gamma N}{(1-a)^3c^2}\log\left(\frac{1}{\delta}\right)\right].$$

□