Appendix

EC.1. Proof of Theorem 1

For each $\nu > 0$, consider the following problem:

$$\begin{aligned} \max_{X} \quad & V_\lambda(X|B_0) - (1+\alpha)\nu\left(\mathbb{E}[\rho X] - x_0\right) \\ \text{subject to } & X \geq 0. \end{aligned} \tag{EC.1}$$

For each state of the world $\omega \in \Omega$, $F_{\nu,\omega}(x) := (1+\alpha)^{-1}\big[u(x) + \alpha\mu\big(u(x) - u(B_0(\omega))\big)\big] - \nu\rho(\omega)x$ is strictly concave in $x \geq 0$ and its subdifferential with respect to x is

$$\partial F_{\nu,\omega}(x) = \begin{cases} \{u'(x) - \nu\rho(\omega)\}, & x > B_0(\omega), \\ \Big[u'(B_0(\omega)) - \nu\rho(\omega), \frac{1+\alpha\lambda}{1+\alpha}u'(x) - \nu\rho(\omega)\Big], & x = B_0(\omega), \\ \Big\{\frac{1+\alpha\lambda}{1+\alpha}u'(x) - \nu\rho(\omega)\Big\}, & x \in (0, B_0(\omega)). \end{cases}$$

Because of the Inada condition, the maximizer of $F_{\nu,\omega}(x)$ with respect to $x \geq 0$ must be positive. We denote it by x^*. Then, we must have $0 \in \partial F_{\nu,\omega}(x^*)$, which immediately yields that

$$x^* = \begin{cases} (u')^{-1}\big(\nu\rho(\omega)\big), & \nu\rho(\omega) < u'\big(B_0(\omega)\big), \\ B_0(\omega), & \frac{1+\alpha}{1+\alpha\lambda}\nu\rho(\omega) \leq u'\big(B_0(\omega)\big) \leq \nu\rho(\omega), \\ (u')^{-1}\big(\frac{1+\alpha}{1+\alpha\lambda}\nu\rho(\omega)\big), & \frac{1+\alpha}{1+\alpha\lambda}\nu\rho(\omega) > u'\big(B_0(\omega)\big). \end{cases}$$

Moreover, x^* is continuous, decreasing in $\nu > 0$. As a result, the optimal solution to (EC.1) is given by the right-hand side of (5). We denote it by $\hat{X}_{\nu,\lambda}$. We already showed that $\hat{X}_{\nu,\lambda}(\omega)$ is continuous and decreasing in $\nu > 0$ for each $\omega \in \Omega$. Assumption 1 then yields that $\mathbb{E}[\rho\hat{X}_{\nu,\lambda}]$ is continuous and decreasing in $\nu > 0$ and $\lim_{\nu\downarrow 0}\mathbb{E}[\rho\hat{X}_{\nu,\lambda}] = +\infty$ and $\lim_{\nu\uparrow +\infty}\mathbb{E}[\rho\hat{X}_{\nu,\lambda}] = 0$. As a result, there exists $\nu > 0$ such that $\mathbb{E}[\rho\hat{X}_{\nu,\lambda}] = x_0$, and by Lagrange dual theory, $\hat{X}_{\nu,\lambda}$ with this particular ν is the optimal solution to (4). Thanks to Assumption 1, the optimal value of (4) is finite. Then, uniqueness of the optimal solution follows immediately from the strict concavity of $u(x)$ in x and thus of $V(X|B_0)$ in X.

Next, we show that $X^*_{\text{ex}}(\lambda)$ and $V_\lambda\left(X^*_{\text{ex}}(\lambda)|B_0\right)$ are continuous in (λ,α). For notational simplicity, denote $Z:=u'(B_0)/\rho$, $\bar\lambda:=(1+\alpha\lambda)/(1+\alpha)$, and write $\hat X_{\nu,\lambda}$ as $\bar X_{\nu,\bar\lambda}$. It is straightforward to see that for each fixed $\omega\in\Omega$, $\bar X_{\nu,\bar\lambda}(\omega)$ is strictly decreasing in $\nu\in(0,Z(\omega))\cup(\bar\lambda Z(\omega),+\infty)$ and constant in $\nu\in[Z(\omega),\bar\lambda Z(\omega)]$. Thus, $\mathbb{E}[\rho\bar X_{\nu,\bar\lambda}]$ is strictly decreasing in $\nu\in(0,\bar z)\cup(\bar\lambda\underline{z},+\infty)$, where $\underline{z}:=\operatorname{essinf} Z$ and $\bar z:=\operatorname{esssup} Z$, and constant in $\nu\in[\bar z,\bar\lambda\underline{z}]$ if this interval is nonempty. Moreover, when $\nu\in[\bar z,\bar\lambda\underline{z}]$, we have $\bar X_{\nu,\bar\lambda}=B_0$ and thus $\mathbb{E}[\rho\bar X_{\nu,\bar\lambda}]=\mathbb{E}[\rho B_0]$.

On the other hand, it is straightforward to see that for each $\omega\in\Omega$, $\bar X_{\nu,\bar\lambda}(\omega)$ is continuous in $(\nu,\bar\lambda)$, strictly increasing in $\bar\lambda\in[1,\nu/Z(\omega))$, and constant in $\bar\lambda\in[\nu/Z(\omega),+\infty)$, so $\mathbb{E}[\rho\bar X_{\nu,\bar\lambda}]$ is continuous in $(\nu,\bar\lambda)$, strictly increasing in $\bar\lambda\in[1,\nu/\underline{z})$, and constant in $\bar\lambda\in[\nu/\underline{z},+\infty)$.

Now, consider the equation $\mathbb{E}[\rho\bar X_{\nu,\bar\lambda}]=x_0$. When $\mathbb{E}[\rho B_0]\neq x_0$, for each fixed $\bar\lambda$, ν that solves $\mathbb{E}[\rho\bar X_{\nu,\bar\lambda}]=x_0$, denoted as $\nu(\bar\lambda)$, must lie in $(0,\bar z)\cup(\bar\lambda\underline{z},+\infty)$ in which $\mathbb{E}[\rho\bar X_{\nu,\bar\lambda}]$ is strictly decreasing in ν. By the implicit function theorem, $\nu(\bar\lambda)$ is continuous and increasing and thus $\bar X_{\nu(\bar\lambda),\bar\lambda}$ is continuous in $\bar\lambda\ge1$. When $\mathbb{E}[\rho B_0]=x_0$, for $\bar\lambda<\bar z/\underline{z}$, $\mathbb{E}[\rho\bar X_{\nu,\bar\lambda}]$ is strictly decreasing in $\nu\in(0,+\infty)$, so the implicit function theory implies that $\nu(\bar\lambda)$ is continuous and increasing in $\bar\lambda<\bar z/\underline{z}$ and thus $\bar X_{\nu(\bar\lambda),\bar\lambda}$ is continuous in $\bar\lambda<\bar z/\underline{z}$. Moreover, in the case $\bar z/\underline{z}<+\infty$, we claim that $\nu_0:=\lim_{\bar\lambda\uparrow\bar z/\underline{z}}\nu(\bar\lambda)=\bar z$. Indeed, we must have $\lim_{\bar\lambda\uparrow\bar z/\underline{z}}\bar X_{\nu(\bar\lambda),\bar\lambda}=\bar X_{\nu_0,\bar z/\underline{z}}$. Because $\mathbb{E}[\rho\bar X_{\nu_0,\bar z/\underline{z}}]=\lim_{\bar\lambda\uparrow\bar z/\underline{z}}\mathbb{E}[\rho\bar X_{\nu(\bar\lambda),\bar\lambda}]=x_0$, because $\mathbb{E}[\rho\bar X_{\nu,\bar z/\underline{z}}]$ is continuous in $\nu>0$ and strictly increasing in $\nu\in(0,\bar z)$, and because $\mathbb{E}[\rho\bar X_{\bar z,\bar z/\underline{z}}]=x_0$, we immediately conclude that $\nu_0=\bar z$. Consequently, $\lim_{\bar\lambda\uparrow\bar z/\underline{z}}\bar X_{\nu(\bar\lambda),\bar\lambda}=B_0$. When $\mathbb{E}[\rho B_0]=x_0$, for $\bar\lambda\ge\bar z/\underline{z}$, any $\nu(\bar\lambda)\in[\bar z,\bar\lambda\underline{z}]$ solves the equation $\mathbb{E}[\rho\bar X_{\nu,\bar\lambda}]=x_0$ and $\bar X_{\nu(\bar\lambda),\bar\lambda}=B_0$. Combining the above, we conclude that $\bar X_{\nu(\bar\lambda),\bar\lambda}$ is continuous in $\bar\lambda$.

The above argument shows that the optimal solution to problem (4) is continuous in (λ,α), so the dominated convergence theorem, with the help of Assumption 1, implies that the optimal value is also continuous in (λ,α). ■

EC.2. Proof of Proposition 1

As in the proof of Theorem 1, we again use the notation $\bar\lambda:=(1+\alpha\lambda)/(1+\alpha)$. When $\mathbb{E}[\rho B_0]<x_0$, $\mathbb{E}\big[\rho\big((u')^{-1}(\nu\rho)\vee B_0\big)\big]$ is decreasing in ν and satisfies

$$\lim_{\nu\downarrow0}\mathbb{E}\big[\rho\big((u')^{-1}(\nu\rho)\vee B_0\big)\big]=+\infty,\qquad\lim_{\nu\uparrow+\infty}\mathbb{E}\big[\rho\big((u')^{-1}(\nu\rho)\vee B_0\big)\big]=\mathbb{E}[\rho B_0]<x_0.$$

Moreover, for any ν such that $\mathbb{E}\big[\rho\big((u')^{-1}(\nu\rho)\vee B_0\big)\big] > \mathbb{E}[\rho B_0]$, we have $\mathbb{P}(((u')^{-1}(\nu\rho) > B_0)) > 0$, so $\mathbb{E}\big[\rho\big((u')^{-1}(\nu\rho)\vee B_0\big)\big]$ is strictly decreasing at this particular ν. Therefore, ν_G is well defined. If $\bar\lambda \ge \nu_G/\underline{z}$, straightforward examination shows that the right-hand side of (5) with $\nu=\nu_G$ is equal to $(u')^{-1}(\nu_G\rho)\vee B_0$. Because $\mathbb{E}\big[\rho\big((u')^{-1}(\nu_G\rho)\vee B_0\big)\big]=x_0$, we conclude that $X^*_{\text{ex}}(\lambda)=(u')^{-1}(\nu_G\rho)\vee B_0$. Conversely, if $X^*_{\text{ex}}(\lambda)\ge B_0$, (5) shows that $\rho\le\bar\lambda\left(\frac{1}{\nu}u'(B_0)\right)$ for some $\nu>0$, which implies that $\nu\le\bar\lambda/\underline{z}$ and $X^*_{\text{ex}}(\lambda)=(u')^{-1}(\nu\rho)\vee B_0$. Because $\mathbb{E}[\rho X^*_{\text{ex}}(\lambda)]=x_0$, the definition of ν_G yields that $\nu=\nu_G$ and thus $\bar\lambda\ge\nu_G/\underline{z}$. The case in which $\mathbb{E}[\rho B_0]>x_0$ can be treated similarly. Finally, when $\mathbb{E}[\rho B_0]=x_0$, careful examination of (5) shows that $X^*_{\text{ex}}(\lambda)=B_0$ if and only if $\frac{1}{\nu}u'(B_0)\le\rho\le\bar\lambda\left(\frac{1}{\nu}u'(B_0)\right)$ for some $\nu>0$, which is the case if and only if $\bar\lambda\ge\bar z/\underline{z}$. Moreover, by definition $\bar z\ge\underline{z}$ and thus $\lambda_{\max}\in[1,+\infty]$.

It is straightforward to derive from the above that for $\lambda\ge\lambda_{\max}$, $X^*_{\text{ex}}(\lambda)$ does not depend on λ. Now, consider any $1\le\lambda_1<\lambda_2<\lambda_{\max}$. When $\mathbb{E}[\rho B_0]<x_0$, we have $\mathbb{P}(X^*_{\text{ex}}(\lambda_i)<B_0)>0$ and $\mathbb{P}(X^*_{\text{ex}}(\lambda_i)>B_0)>0$, $i=1,2$. Denote $\bar\lambda_i:=(1+\alpha\lambda_i)/(1+\alpha)$ and ν_i as the Lagrange multiplier in $X^*_{\text{ex}}(\lambda_i)$, $i=1,2$. For the sake of contradiction, suppose $X^*_{\text{ex}}(\lambda_1)=X^*_{\text{ex}}(\lambda_2)$. Then, because of (5) and because $\mathbb{P}(X^*_{\text{ex}}(\lambda_i)<B_0)>0$, $i=1,2$, there exists a $\mathbb{P}$-nonnegligible set of scenarios in which $X^*_{\text{ex}}(\lambda_i)<B_0$, $i=1,2$, so we conclude from (5) that $\nu_1/\bar\lambda_1=\nu_2/\bar\lambda_2$. Because of (5) and $\mathbb{P}(X^*_{\text{ex}}(\lambda_i)>B_0)>0$, $i=1,2$, there exists a $\mathbb{P}$-nonnegligible set of scenarios in which $X^*_{\text{ex}}(\lambda_i)>B_0$, $i=1,2$, so we conclude from (5) that $\nu_1=\nu_2$. This contradicts $\bar\lambda_1<\bar\lambda_2$. Thus, we must have $X^*_{\text{ex}}(\lambda_1)\ne X^*_{\text{ex}}(\lambda_2)$. When $\mathbb{E}[\rho B_0]\ge x_0$, a similar argument to the one as above also shows that $X^*_{\text{ex}}(\lambda_1)\ne X^*_{\text{ex}}(\lambda_2)$. ■

EC.3. Proof of Theorem 2

Let $X\in\mathcal{X}(x_0)$ be fixed. X is a PE if and only if it solves problem (4) with respect to the reference point X. In the light of Theorem 1, this is the case if and only if (7) holds for some $\nu>0$, because $(u')^{-1}\left(\frac{\nu\rho}{1+\alpha}\right)>X$ on $\left\{\rho<\frac{1+\alpha}{\nu}u'(X)\right\}$ and $(u')^{-1}\left(\frac{\nu\rho}{1+\alpha\lambda}\right)<X$ on $\left\{\rho>\frac{1+\alpha\lambda}{\nu}u'(X)\right\}$. The PPE is then given by (8) because $V(X|X)=\mathbb{E}\left[u(X)\right]$ for any $X\in\mathcal{X}(x_0)$. ■

EC.4. Proof of Theorem 3

According to Proposition 6 in De Giorgi and Post (2011), solving (10) is equivalent to solving the following problem:

$$\begin{aligned}\max_X \quad & V_\lambda(X|X)\\ \text{subject to } & \mathbb{E}[\rho X] \le x_0, \;\; X\ge 0,\\ & f_{B_0}\big(V_\lambda(X|B_0)\big) \le c_0.\end{aligned} \tag{EC.2}$$

When $c_0 = \underline{c}$, $\mathcal{Y}(B_0, c_0) = \{X^*_{\text{ex}}(\lambda)\}$, so case (i) of the theorem follows immediately. When $c_0 \ge \overline{c}$, $X^*_{\text{ex}}(1)$, which maximizes the consumption utility, satisfies $f_{B_0}\big(V_\lambda(X^*_{\text{ex}}(1)|B_0)\big) \le c_0$. Noting that $V_\lambda(X|X) = \mathbb{E}[u(X)]$, we immediately conclude that $X^*_{\text{ex}}(1)$ is the optimal solution to (EC.2) in this case.

Finally, we consider the case in which $\underline{c} < c_0 < \overline{c}$. Because $f_{B_0}(x)$ is strictly decreasing in x, (EC.2) is equivalent to

$$\begin{aligned}\max_X \quad & V_\lambda(X|X)\\ \text{subject to } & \mathbb{E}[\rho X] \le x_0, \;\; X\ge 0,\\ & V_\lambda(X|B_0) \ge f^{-1}_{B_0}(c_0).\end{aligned} \tag{EC.3}$$

For each $\eta > 0$, consider the following problem:

$$\begin{aligned}\max_X \quad & V_\lambda(X|X) + \eta\big(V_\lambda(X|B_0) - f^{-1}_{B_0}(c_0)\big)\\ \text{subject to } & \mathbb{E}[\rho X] \le x_0, \;\; X\ge 0.\end{aligned} \tag{EC.4}$$

We can rewrite the objective function of the above problem as

$$\frac{1+(1+\alpha)\eta}{1+\alpha} V_\Lambda(X|B_0) - (1+\eta)u(B_0) - \eta f^{-1}_{B_0}(c_0),$$

where

$$\Lambda = \frac{1}{\alpha}\left((1+\alpha)\frac{1+(1+\alpha\lambda)\eta}{1+(1+\alpha)\eta} - 1\right). \tag{EC.5}$$

Then, by Theorem 1, the unique optimal solution to (EC.4) is $X^*_{\text{ex}}(\Lambda)$.

By Theorem 1, $X^*_{\rm ex}(\Lambda)$ is continuous in $\Lambda \geq 1$, so by Assumption 1 and the pre-assumption $\mathbb{E}[|u(B_0)|] < +\infty$, we can show by the dominated convergence theorem that $V_\lambda(X^*_{\rm ex}(\Lambda)|B_0)$ is continuous in $\Lambda \geq 1$. In particular,

$$\lim_{\Lambda \uparrow \lambda\wedge\lambda_{\max}} V_\lambda(X^*_{\rm ex}(\Lambda)|B_0) = V_\lambda(X^*_{\rm ex}(\lambda\wedge\lambda_{\max})|B_0) = V_\lambda(X^*_{\rm ex}(\lambda)|B_0) = f^{-1}_{B_0}(\underline{c}),$$
$$\lim_{\Lambda\downarrow 1} V_\lambda(X^*_{\rm ex}(\Lambda)|B_0) = V_\lambda(X^*_{\rm ex}(1)|B_0) = f^{-1}_{B_0}(\bar{c}),$$

where the second equality in the above is the case because $X^*_{\rm ex}(\Lambda)$ does not depend on $\Lambda \in [\lambda_{\max}, +\infty)$ (as shown in Theorem 1).

On the other hand, for any $1 \leq \Lambda_1 < \Lambda_2 < \lambda\wedge\lambda_{\max}$ and corresponding $\eta_1 < \eta_2$ through the relation (EC.5), we have

$$\begin{aligned}
&V_\lambda(X^*_{\rm ex}(\Lambda_1)|X^*_{\rm ex}(\Lambda_1)) + \eta_1 V_\lambda(X^*_{\rm ex}(\Lambda_1)|B_0) + (\eta_2-\eta_1)V_\lambda(X^*_{\rm ex}(\Lambda_2)|B_0)\\
&> V_\lambda(X^*_{\rm ex}(\Lambda_2)|X^*_{\rm ex}(\Lambda_2)) + \eta_2 V_\lambda(X^*_{\rm ex}(\Lambda_2)|B_0)\\
&> V_\lambda(X^*_{\rm ex}(\Lambda_1)|X^*_{\rm ex}(\Lambda_1)) + \eta_2 V_\lambda(X^*_{\rm ex}(\Lambda_1)|B_0),
\end{aligned}$$

where the two inequalities are the case because $X^*_{\rm ex}(\Lambda_i)$ is the unique optimal solution to (EC.4) with $\eta = \eta_i$, $i = 1,2$ and because $X^*_{\rm ex}(\Lambda_1) \neq X^*_{\rm ex}(\Lambda_2)$ as shown in Theorem 1. We then conclude from the above inequalities and from $\eta_1 < \eta_2$ that $V_\lambda(X^*_{\rm ex}(\Lambda_2)|B_0) > V_\lambda(X^*_{\rm ex}(\Lambda_1)|B_0)$.

Therefore, $V_\lambda(X^*_{\rm ex}(\Lambda)|B_0)$ is strictly increasing in $\Lambda \in [1, \lambda\wedge\lambda_{\max})$. Then, for any $c_0 \in (\underline{c}, \bar{c})$, there exists unique $\Lambda_{\rm pe}(c_0) \in (1, \lambda\wedge\lambda_{\max})$ such that $V_\lambda(X^*_{\rm ex}(\Lambda(c_0))|B_0) = f^{-1}_{B_0}(c_0)$. Lagrange dual theory yields that $X^*_{\rm ex}(\Lambda(c_0))$ is optimal to (EC.3) and thus is the partially endogenous PPE. Moreover, $\lim_{c_0\downarrow\underline{c}} \Lambda_{\rm pe}(c_0) = \lambda\wedge\lambda_{\max}$ and $\lim_{c_0\uparrow\bar{c}} \Lambda_{\rm pe}(c_0) = 1$.■

EC.5. Proof of Theorem 4

The following theorem implies Theorem 4 and gives a full characterization of the optimal prospect and optimal reference point depending on the model parameters.

Theorem EC.1. *Suppose $\lambda > 1$, denote*

$$\Lambda_1 := \frac{1+\alpha\lambda}{1+\alpha}, \quad \Lambda_2 := \frac{1+\alpha\lambda+\alpha(1+\alpha)\lambda^2}{1+\alpha\lambda+\alpha(1+\alpha)\lambda}, \tag{EC.6}$$

$$\bar{\Lambda}_1 := \Lambda_1 \wedge \lambda_{\max}, \quad \bar{\Lambda}_2 := \Lambda_2 \wedge \lambda_{\max}, \tag{EC.7}$$

where $\lambda_{\max}$ is as defined in (6), and denote

$$\begin{aligned}
&\Theta_0 := f_{B_0}\left(V_\lambda(0|B_0)\right), &&\Theta_1 := f_{B_0}\left(V_\lambda\left(X^*_{\mathrm{ex}}(1)\wedge B_0|B_0\right)\right),\\
&\Theta_2 := f_{B_0}\left(V_\lambda\left(X^*_{\mathrm{ex}}(\bar{\Lambda}_1)\wedge B_0\big|B_0\right)\right), &&\Theta_3 := f_{B_0}\left(V_\lambda\left(X^*_{\mathrm{ex}}(\bar{\Lambda}_1)|B_0\right)\right),\\
&\Theta_4 := f_{B_0}\left(V_\lambda\left(X^*_{\mathrm{ex}}(\bar{\Lambda}_2)|B_0\right)\right), &&\Theta_5 := f_{B_0}\left(V_\lambda\left(X^*_{\mathrm{ex}}(\bar{\Lambda}_2)\vee B_0|B_0\right)\right),\\
&\Theta_6 := f_{B_0}\left(V_\lambda\left(X^*_{\mathrm{EU}}\vee B_0|B_0\right)\right), &&\Theta_7 := f_{B_0}\left(\lim_{x\uparrow+\infty} V_\lambda(x|B_0)\right).
\end{aligned} \tag{EC.8}$$

Then, $1 < \Lambda_1 < \Lambda_2 < \lambda$, $\Theta_0 \geq \Theta_1 \geq \Theta_2 \geq \Theta_3 \geq \Theta_4 \geq \Theta_5 \geq \Theta_6 \geq \Theta_7$, and the optimal prospect and optimal reference point of (11), denoted as $X^*_{\mathrm{oe}}(\lambda, c_0)$ and $B^*_{\mathrm{oe}}(\lambda, c_0)$, respectively, are given as follows:

(i) If $u(0) > -\infty$ and $c_0 \geq \Theta_0$, then $X^*_{\mathrm{oe}}(\lambda, c_0) = X^*_{\mathrm{ex}}(1)$ and $B^*_{\mathrm{oe}}(\lambda, c_0) = 0$.

(ii) If $\Theta_1 \leq c_0 < \Theta_0$, then $X^*_{\mathrm{oe}}(\lambda, c_0) = X^*_{\mathrm{ex}}(1)$ and $B^*_{\mathrm{oe}}(\lambda, c_0)$ is any prospect with $0 \leq B^*_{\mathrm{oe}}(\lambda, c_0) \leq X^*_{\mathrm{ex}}(1)\wedge B_0$, $f_{B_0}(V_\lambda(B^*_{\mathrm{oe}}(\lambda, c_0)|B_0)) = c_0$, and $\mathbb{E}[|u(B^*_{\mathrm{oe}}(\lambda, c_0))|] < +\infty$.

(iii) If $\Theta_2 < c_0 < \Theta_1$, then $X^*_{\mathrm{oe}}(\lambda, c_0) = X^*_{\mathrm{ex}}(\Lambda_{\mathrm{oe}}(c_0))$ and $B^*_{\mathrm{oe}}(\lambda, c_0) = X^*_{\mathrm{ex}}(\Lambda_{\mathrm{oe}}(c_0)) \wedge B_0$, where $\Lambda_{\mathrm{oe}}(c_0) \in (1, \bar{\Lambda}_1)$ is uniquely determined by $f_{B_0}(V_\lambda(B^*_{\mathrm{oe}}(\lambda, c_0)|B_0)) = c_0$. Moreover, $\Lambda_{\mathrm{oe}}(c_0)$ is continuous and strictly decreasing in $c_0 \in (\Theta_2, \Theta_1)$ and satisfies

$$\lim_{c_0\downarrow\Theta_2} \Lambda_{\mathrm{oe}}(c_0) = \bar{\Lambda}_1, \quad \lim_{c_0\uparrow\Theta_1} \Lambda_{\mathrm{oe}}(c_0) = 1.$$

(iv) If $\Theta_3 \leq c_0 \leq \Theta_2$, then $X^*_{\mathrm{oe}}(\lambda, c_0) = X^*_{\mathrm{ex}}(\bar{\Lambda}_1)$ and $B^*_{\mathrm{oe}}(\lambda, c_0)$ is any prospect with $X^*_{\mathrm{ex}}(\bar{\Lambda}_1)\wedge B_0 \leq B^*_{\mathrm{oe}}(\lambda, c_0) \leq X^*_{\mathrm{ex}}(\bar{\Lambda}_1)$ and $f_{B_0}\big(V_\lambda(B^*_{\mathrm{oe}}(\lambda, c_0)|B_0)\big) = c_0$.

(v) If $\Theta_4 < c_0 < \Theta_3$, then $X^*_{\mathrm{oe}}(\lambda, c_0) = B^*_{\mathrm{oe}}(\lambda, c_0) = X^*_{\mathrm{ex}}(\Lambda_{\mathrm{oe}}(c_0))$, where $\Lambda_{\mathrm{oe}}(c_0) \in (\bar{\Lambda}_1, \bar{\Lambda}_2)$ is uniquely determined by $f_{B_0}(V_\lambda(B^*_{\mathrm{oe}}(\lambda, c_0)|B_0)) = c_0$. Moreover, $\Lambda_{\mathrm{oe}}(c_0)$ is continuous and strictly decreasing in $c_0 \in (\Theta_4, \Theta_3)$ and satisfies

$$\lim_{c_0\downarrow\Theta_4} \Lambda_{\mathrm{oe}}(c_0) = \bar{\Lambda}_2, \quad \lim_{c_0\uparrow\Theta_3} \Lambda_{\mathrm{oe}}(c_0) = \bar{\Lambda}_1.$$

(vi) *If* $\Theta_5 \le c_0 \le \Theta_4$, *then* $X^*_{\mathrm{oe}}(\lambda, c_0) = X^*_{\mathrm{ex}}(\bar{\Lambda}_2)$ *and* $B^*_{\mathrm{oe}}(\lambda, c_0)$ *is any prospect with* $X^*_{\mathrm{ex}}(\bar{\Lambda}_2) \le B^*_{\mathrm{oe}}(\lambda, c_0) \le X^*_{\mathrm{ex}}(\bar{\Lambda}_2) \vee B_0$ *and* $f_{B_0}\big(V_\lambda(B^*_{\mathrm{oe}}(\lambda, c_0)|B_0)\big) = c_0$.

(vii) *If* $\Theta_6 < c_0 < \Theta_5$, *then* $X^*_{\mathrm{oe}}(\lambda, c_0) = X^*_{\mathrm{ex}}(\Lambda_{\mathrm{oe}}(c_0))$ *and* $B^*_{\mathrm{oe}}(\lambda, c_0) = X^*_{\mathrm{ex}}(\Lambda_{\mathrm{oe}}(c_0)) \vee B_0$, *where* $\Lambda_{\mathrm{oe}}(c_0) \in (1, \bar{\Lambda}_2)$ *is uniquely determined by* $f_{B_0}(V_\lambda(B^*_{\mathrm{oe}}(\lambda, c_0)|B_0)) = c_0$. *Moreover,* $\Lambda_{\mathrm{oe}}(c_0)$ *is continuous and strictly increasing in* $c_0 \in (\Theta_6, \Theta_5)$ *and satisfies*

$$\lim_{c_0 \downarrow \Theta_6} \Lambda_{\mathrm{oe}}(c_0) = 1, \quad \lim_{c_0 \uparrow \Theta_5} \Lambda_{\mathrm{oe}}(c_0) = \bar{\Lambda}_2.$$

(viii) *If* $\Theta_7 < c_0 \le \Theta_6$, *then* $X^*_{\mathrm{oe}}(\lambda, c_0) = X^*_{\mathrm{ex}}(1)$ *and* $B^*_{\mathrm{oe}}(\lambda, c_0)$ *is any prospect with* $B^*_{\mathrm{oe}}(\lambda, c_0) \ge X^*_{\mathrm{ex}}(1) \vee B_0$, $f_{B_0}\big(V_\lambda(B^*_{\mathrm{oe}}(\lambda, c_0)|B_0)\big) = c_0$, *and* $\mathbb{E}[|u(B^*_{\mathrm{oe}}(\lambda, c_0))|] < +\infty$.

Note that if $c_0 \le \Theta_7$, there is no feasible B satisfying the constraint in (11), and that when $u(0) = -\infty$ and $c_0 \ge \Theta_0$, we can choose B that takes arbitrarily small values, making $V(X|B)$ go to $+\infty$, so problem (11) is not meaningful in this case. Thus, Theorem EC.1 is comprehensive in that it covers all cases differentiated by the value of c_0, and it is a bit tedious due to these case distinctions. The main message, however, is clear. Augment the definition of Λ_{oe} by setting $\Lambda_{\mathrm{oe}}(c_0) = 1$ for $c_0 \ge \Theta_1$, $\Lambda_{\mathrm{oe}}(c_0) = \bar{\Lambda}_1$ for $c_0 \in [\Theta_3, \Theta_2]$, $\Lambda_{\mathrm{oe}}(c_0) = \bar{\Lambda}_2$ for $c_0 \in [\Theta_5, \Theta_4]$, and $\Lambda_{\mathrm{oe}}(c_0) = 1$ for $c_0 \in (\Theta_7, \Theta_6]$. Then, the optimal prospect of problem (11) is the same as the optimal prospect with B_0 taken exogenously and a different DLA $\Lambda_{\mathrm{oe}}(c_0)$. Moreover, we have $\Lambda_{\mathrm{oe}}(c_0) \le \bar{\Lambda}_2 = \Lambda_2 \wedge \lambda_{\max}$, and $\Lambda_2 < \lambda$. Therefore, the effect of mentally choosing the reference point is a reduction of the DLA. Because $\Lambda_2 < \lambda$, excluding the case $\lambda_{\max} \le \Lambda_2$, the exhibited DLA $\Lambda_{\mathrm{oe}}(c_0)$ cannot generate the full spectrum of DLA from one to the intrinsic DLA when c_0 varies. This is in contrast to the exhibited DLA for the partially endogenized PPE.

Theorem EC.1 also reveals that the exhibited DLA $\Lambda_{\mathrm{oe}}(c_0)$ is decreasing in $c_0 \ge \Theta_5$; see cases (i)–(vi) of the theorem. This is because with a larger c_0, it is more feasible for the agent to choose a reference point that is smaller than the optimal prospect and consequently loss aversion has a smaller effect. On the other hand, $\Lambda_{\mathrm{oe}}(c_0)$ is increasing in $c_0 \le \Theta_5$; see Theorem EC.1-(vii) and (viii). In this case, the agent is restricted to choose those reference points that are increasingly superior to B_0, i.e., to choose high reference points. Consequently, with a smaller c_0, the agent has to choose higher reference points, and effectively the optimal prospect is more likely to be smaller than the reference point chosen by the agent. As a result, the loss aversion has a smaller effect on the optimal prospect because loss aversion is absent in the evaluation of prospects with pure losses. Note that the case $c_0 \ge \Theta_5$ is more relevant than the case $c_0 < \Theta_5$, because if the agent is able to stick to the exogenous reference point B_0, then his/her c_0 should satisfy $c_0 \ge f_{B_0}\left(V_\lambda\left(B_0|B_0\right)\right) \ge f_{B_0}\left(V_\lambda\left(X^*_{\mathrm{ex}}(\bar{\Lambda}_2) \vee B_0|B_0\right)\right) = \Theta_5$. In other words, if $c_0 < \Theta_5$ then the exogenous reference point would not be admissible.

EC.5.1. Proof of Theorem EC.1

We assume $u(0) > -\infty$ in the following proof, and the case of $u(0) = -\infty$ will be addressed at the end of the proof. For each $\eta > 0$, consider the following problem:

$$\begin{aligned} \max_{X,B} \quad & V_\lambda(X|B) + \eta\left(V_\lambda(B|B_0) - f_{B_0}^{-1}(c_0)\right) \\ \text{subject to } & \mathbb{E}[\rho X] \le x_0, \\ & X, B \ge 0,\ \mathbb{E}[|u(B)|] < +\infty. \end{aligned} \tag{EC.9}$$

For each fixed $X \ge 0$ with $\mathbb{E}[\rho X] \le x_0$, consider the following problem:

$$\max_{B \ge 0} V_\lambda(X|B) + \eta\left(V_\lambda(B|B_0) - f_{B_0}^{-1}(c_0)\right). \tag{EC.10}$$

Denote

$$\eta_1 := \frac{\alpha}{1+\alpha\lambda}, \quad \eta_2 := \frac{\alpha}{1+\alpha}, \quad \eta_3 := \frac{\alpha\lambda}{1+\alpha\lambda}, \quad \eta_4 := \frac{\alpha\lambda}{1+\alpha}.$$

Then, $\eta_1 < \eta_2 < \eta_3 < \eta_4$ because $\lambda > 1$. For each $\omega \in \Omega$,

$$\begin{aligned} & u(X(\omega)) + \alpha\big(u(X(\omega)) - u(B(\omega))\big)^+ - \alpha\lambda\big(u(X(\omega)) - u(B(\omega))\big)^- \\ & + \eta u(B(\omega)) + \eta\alpha\big(u(B(\omega)) - u(B_0(\omega))\big)^+ - \eta\alpha\lambda\big(u(B(\omega)) - u(B_0(\omega))\big)^- \end{aligned}$$

is a piecewise linear function of $B(\omega)$. Straightforward calculation then yields that the optimal solution of (EC.10), denoted as B_X, is as follows:

$$B_X = \begin{cases} 0, & \eta < \eta_1, \\ [0, X \wedge B_0], & \eta = \eta_1, \\ X \wedge B_0, & \eta_1 < \eta < \eta_2, \\ [X \wedge B_0, X], & \eta = \eta_2, \end{cases} \qquad B_X = \begin{cases} X, & \eta_2 < \eta < \eta_3, \\ [X, X \vee B_0], & \eta = \eta_3, \\ X \vee B_0, & \eta_3 < \eta < \eta_4, \\ [X \vee B_0, +\infty), & \eta = \eta_4. \end{cases}$$

In the above, when B_X is an interval, it means that any B_X taking values in this interval is an optimal solution of (EC.10). As a result, the optimal value of (EC.10), denoted as $W_\eta(X)$, is

$$W_\eta(X) = a_\eta V_{\Lambda(\eta)}(X|B_0) + G_\eta(B_0),$$

where a_η is a certain function of η, $G_\eta(B_0)$ is a certain function of η and B_0, and

$$\Lambda(\eta) := \begin{cases} 1, & \eta \le \eta_1, \\ \frac{\eta + \eta\alpha\lambda}{\alpha}, & \eta_1 < \eta < \eta_2, \\ \frac{1 + (\eta + \eta\alpha)\lambda}{1 + \eta + \eta\alpha}, & \eta_2 \le \eta < \eta_3, \\ \frac{\alpha + (1+\alpha)\alpha\lambda - (\eta + \eta\alpha)}{\alpha(1 + \eta + \eta\alpha)}, & \eta_3 \le \eta \le \eta_4. \end{cases}$$

Moreover, $\Lambda(\eta)$ is continuous in $\eta \in (0, \eta_4]$, strictly increasing in $\eta \in [\eta_1, \eta_3]$, strictly decreasing in $\eta \in [\eta_3, \eta_4]$, and satisfies

$$\Lambda(\eta_1) = \Lambda(\eta_4) = 1, \quad \Lambda(\eta_2) = \Lambda_1, \quad \Lambda(\eta_3) = \Lambda_2.$$

As a result, the unique optimal solution of the following problem

$$\begin{aligned} \max_X \quad & W_\eta(X) \\ \text{subject to } & \mathbb{E}[\rho X] \le x_0,\ X \ge 0 \end{aligned} \tag{EC.11}$$

is $\hat{X}_\eta = X^*_{\text{ex}}(\Lambda(\eta))$ and, consequently, the optimal solution to (EC.9) is $(\hat{X}_\eta, \hat{B}_\eta)$, where $\hat{B}_\eta = B_{\hat{X}_\eta}$. To find the solution to (11), we only need to find η such that $V_\lambda(\hat{B}_\eta|B_0) = f^{-1}_{B_0}(c_0)$. To this end, we investigate $g(\eta) := V_\lambda(\hat{B}_\eta|B_0)$ in the following. Here, for $\eta \in (0,\eta_4)\backslash\{\eta_1,\eta_2,\eta_3\}$, $\hat{B}_\eta = B_{\hat{X}_\eta}$ is uniquely defined, so $g(\eta)$ is single-valued. For $\eta \in \{\eta_1,\eta_2,\eta_3,\eta_4\}$, $\hat{B}_\eta$ is not uniquely defined, so $g(\eta)$ is set-valued in this case.

Because $\Lambda(\eta)$ is continuous in η and $X^*_{\mathrm{ex}}(\Lambda)$ is continuous in Λ, by Assumption 1, the pre-assumption $\mathbb{E}[|u(B_0)|] < +\infty$, and the dominated convergence theorem, we conclude that

$$g(\eta) = V_\lambda(\hat{B}_\eta|B_0) = V_\lambda(B_{X^*_{\mathrm{ex}}(\Lambda(\eta))}|B_0)$$

is continuous on $(0,\eta_4)\backslash\{\eta_1,\eta_2,\eta_3\}$. For $\eta \in (0,\eta_1)$, we have $g(\eta) = V_\lambda(0|B_0)$. Define

$$\underline{\eta} := \sup\{\eta \in [\eta_1,\eta_3] \mid \Lambda(\eta) \le \lambda_{\max}\},$$
$$\bar{\eta} := \inf\{\eta \in [\eta_3,\eta_4] \mid \Lambda(\eta) \ge \lambda_{\max}\}.$$

Then, for any $\eta \in [\underline{\eta},\bar{\eta}]$, $\Lambda(\eta) \ge \lambda_{\max}$ and thus $\hat{X}_\eta = X^*_{\mathrm{ex}}(\Lambda(\eta)) = X^*_{\mathrm{ex}}(\lambda_{\max})$. Consequently, for any $i = 2,3,4$ and $\eta \in [\underline{\eta},\bar{\eta}] \cap (\eta_{i-1},\eta_i)$, $(\hat{X}_\eta,\hat{B}_\eta)$ does not depend on η and thus $g(\eta)$ is constant. For any $\eta,\eta' \in ((\eta_1,\eta_2)\cup(\eta_2,\eta_3))\backslash[\underline{\eta},\bar{\eta}]$ with $\eta < \eta'$, we have $\Lambda(\eta) < \Lambda(\eta') < \lambda_{\max}$. Consequently, $\hat{X}_\eta \ne \hat{X}_{\eta'}$ and

$$\begin{aligned}
&W_\eta(\hat{X}_\eta) + (\eta'-\eta)V_\lambda(\hat{B}_{\eta'}|B_0) > W_\eta(\hat{X}_{\eta'}) + (\eta'-\eta)V_\lambda(\hat{B}_{\eta'}|B_0)\\
&= W_{\eta'}(\hat{X}_{\eta'}) + (\eta'-\eta)f^{-1}_{B_0}(c_0) > W_{\eta'}(\hat{X}_\eta) + (\eta'-\eta)f^{-1}_{B_0}(c_0)\\
&= W_\eta(\hat{X}_\eta) + (\eta'-\eta)V_\lambda(\hat{B}_\eta|B_0),
\end{aligned}$$

where the two inequalities are the case due to the unique optimality for the problem (EC.11). Because $\eta < \eta'$, we conclude $V_\lambda(\hat{B}_\eta|B_0) < V_\lambda(\hat{B}_{\eta'}|B_0)$, i.e., $g(\eta) < g(\eta')$. Similarly, for any $\eta,\eta' \in (\eta_3,\eta_4)\backslash[\underline{\eta},\bar{\eta}]$ with $\eta < \eta'$, we have $g(\eta) < g(\eta')$.

Straightforward calculation yields that

$$\begin{aligned}
&g(\eta_1-) = V_\lambda(0|B_0) = f^{-1}_{B_0}(\Theta_0),\\
&g(\eta_1+) = V_\lambda(X^*_{\mathrm{ex}}(1)\wedge B_0|B_0) = f^{-1}_{B_0}(\Theta_1),\\
&g(\eta_2-) = V_\lambda(X^*_{\mathrm{ex}}(\Lambda_1)\wedge B_0|B_0) = V_\lambda(X^*_{\mathrm{ex}}(\bar{\Lambda}_1)\wedge B_0|B_0) = f^{-1}_{B_0}(\Theta_2),
\end{aligned}$$

$$g\,(\eta_2+) = V_\lambda(X^*_{\rm ex}(\Lambda_1)|B_0) = V_\lambda(X^*_{\rm ex}(\bar{\Lambda}_1)|B_0) = f^{-1}_{B_0}(\Theta_3),$$

$$g\,(\eta_3-) = V_\lambda(X^*_{\rm ex}(\Lambda_2)|B_0) = V_\lambda(X^*_{\rm ex}(\bar{\Lambda}_2)|B_0) = f^{-1}_{B_0}(\Theta_4),$$

$$g\,(\eta_3+) = V_\lambda(X^*_{\rm ex}(\Lambda_2)\vee B_0|B_0) = V_\lambda(X^*_{\rm ex}(\bar{\Lambda}_2)\vee B_0|B_0) = f^{-1}_{B_0}(\Theta_5),$$

$$g\,(\eta_4-) = V_\lambda(X^*_{\rm ex}(1)\vee B_0|B_0) = f^{-1}_{B_0}(\Theta_6).$$

Note that $g(\eta_i-)\le g(\eta_i+)$ and $g(\eta_i) = [g(\eta_i-), g(\eta_i+)]$, $i = 1,2,3$.

Now, for any c_0 with $f^{-1}_{B_0}(c_0)\le f^{-1}_{B_0}(\Theta_0)$, consider

$$\begin{aligned} &\max_{X,B} \quad V_\lambda(X|B)\\ &\text{subject to } \mathbb{E}[\rho X]\le x_0,\\ &\qquad\qquad X, B\ge 0,\ \mathbb{E}[|u(B)|] < +\infty. \end{aligned}$$

It is straightforward to see that the optimal solution to the above is $(X^*_{\rm oe}(\lambda,c_0), B^*_{\rm oe}(\lambda,c_0)) = (X^*_{\rm ex}(1), 0)$. Because $B^*_{\rm oe}(\lambda,c_0)$ satisfies

$$V_\lambda(B^*_{\rm oe}(\lambda,c_0)|B_0) = V_\lambda(0|B_0) = f^{-1}_{B_0}(\Theta_0)\ge f^{-1}_{B_0}(c_0),$$

$(X^*_{\rm oe}(\lambda,c_0), B^*_{\rm oe}(\lambda,c_0))$ is also optimal to (11).

For any c_0 with $f^{-1}_{B_0}(\Theta_0) < f^{-1}_{B_0}(c_0)\le f^{-1}_{B_0}(\Theta_1)$, we have $f^{-1}_{B_0}(c_0)\in g(\eta_1)$. Thus, we can find $B^*_{\rm oe}(\lambda,c_0)$ such that $0\le B^*_{\rm oe}(\lambda,c_0)\le X^*_{\rm ex}(1)$ and $V_\lambda(B^*_{\rm oe}(\lambda,c_0)|B_0) = f^{-1}_{B_0}(c_0)$ and $B^*_{\rm oe}(\lambda,c_0)$ can be chosen such that $\mathbb{E}[|u(B^*_{\rm oe}(\lambda,c_0))|] < +\infty$. Note that $\Lambda(\eta_1) = 1$, so $(X^*_{\rm ex}(\Lambda(\lambda_1)), B^*_{\rm oe}(\lambda,c_0)) = (X^*_{\rm ex}(1), B^*_{\rm oe}(\lambda,c_0))$ is the optimal solution of (11). Cases (iv), (vi), and (viii) can be proved similarly.

For any c_0 with $f^{-1}_{B_0}(\Theta_1) < f^{-1}_{B_0}(c_0) < f^{-1}_{B_0}(\Theta_2)$, there exists a unique $\eta_{c_0}\in(\eta_1,\eta_2)\backslash[\underline{\eta},\bar{\eta}]$ such that $g(\eta_{c_0}) = f^{-1}_{B_0}(c_0)$, and η_{c_0} is continuous and strictly decreasing in c_0. As a result,

$$(X^*_{\rm oe}(\lambda,c_0), B^*_{\rm oe}(\lambda,c_0)) = (\hat{X}_{\eta_{c_0}}, \hat{B}_{\eta_{c_0}}) = (X^*_{\rm ex}(\Lambda(\eta_{c_0})), X^*_{\rm ex}(\Lambda(\eta_{c_0}))\wedge B_0)$$

is the optimal solution of (11). Defining $\Lambda_{\rm oe}(c_0) := \Lambda(\eta_{c_0})$, then $\Lambda_{\rm oe}(c_0)$ is continuous and strictly decreasing in $c_0\in(\Theta_1,\Theta_0)$. Moreover, because $\eta_{c_0}\in(\eta_1,\eta_2)\backslash[\underline{\eta},\bar{\eta}]$, we have $\eta_{c_0} < \min(\eta_2,\underline{\eta})$ and thus

$\Lambda_{\mathrm{oe}}(c_0)=\Lambda(\eta_{c_0})<\min(\lambda_{\max},\Lambda(\eta_2))=\bar{\Lambda}_1$. On the other hand, because $\eta_{c_0}>\eta_1$, we have $\Lambda_{\mathrm{oe}}(c_0)=\Lambda(\eta_{c_0})>\Lambda(\eta_1)=1$. Next, for any $\Lambda'\in(1,\bar{\Lambda}_1)$ such that $V_\lambda(X^*_{\mathrm{ex}}(\Lambda')\wedge B_0|B_0)=f^{-1}_{B_0}(c_0)$, there exists $\eta'\in(\eta_1,\eta_2)\backslash[\underline{\eta},\bar{\eta}]$ such that $\Lambda(\eta')=\Lambda'$. Consequently, $g(\eta')=V_\lambda(X^*_{\mathrm{ex}}(\Lambda')\wedge B_0|B_0)=f^{-1}_{B_0}(c_0)$, and the strict monotonicity of $g(\eta)$ on $(\eta_1,\eta_2)\backslash[\underline{\eta},\bar{\eta}]$ implies that $\eta'=\eta_{c_0}$ and thus $\Lambda'=\Lambda_{\mathrm{oe}}(c_0)$. In other words, $\Lambda_{\mathrm{oe}}(c_0)$ is the unique number in $(1,\bar{\Lambda}_1)$ such that $V_\lambda(X^*_{\mathrm{ex}}(\Lambda_{\mathrm{oe}}(c_0))\wedge B_0|B_0)=f^{-1}_{B_0}(c_0)$. Finally, because $g((\eta_1\wedge\underline{\eta})+)=g(\eta_1+)=f^{-1}_{B_0}(\Theta_1)$, $g((\eta_2\wedge\underline{\eta})-)=g(\eta_2-)=f^{-1}_{B_0}(\Theta_2)$, and g is continuous and strictly increasing on $(\eta_1,\eta_2)\backslash[\underline{\eta},\bar{\eta}]$, we must have

$$\lim_{c_0\downarrow\Theta_2}\eta_{c_0}=\eta_2\wedge\underline{\eta},\quad \lim_{c_0\uparrow\Theta_1}\eta_{c_0}=\eta_1\wedge\underline{\eta},$$

which yields

$$\begin{aligned}\lim_{c_0\downarrow\Theta_2}\Lambda(\eta_{c_0})&=\Lambda(\eta_2\wedge\underline{\eta})=\Lambda(\eta_2)\wedge\lambda_{\max}=\bar{\Lambda}_1,\\ \lim_{c_0\uparrow\Theta_1}\Lambda(\eta_{c_0})&=\Lambda(\eta_1\wedge\underline{\eta})=\Lambda(\eta_1)\wedge\lambda_{\max}=1.\end{aligned}$$

The proof of case (iii) of the Theorem then completes. Cases (v) and (vii) can be proved similarly.

Finally, the case of $u(0)=-\infty$ can be treated similarly, noting that in this case we only need to consider $c_0<\Theta_0$.■

EC.6. Proof of Proposition 2

By the definition of $\lambda_{\max}$ and Theorem 1, for any $\Lambda<\lambda_{\max}$, $\mathbb{P}(X^*_{\mathrm{ex}}(\Lambda)>B_0)>0$ and $\mathbb{P}(X^*_{\mathrm{ex}}(\Lambda)<B_0)>0$. On the other hand, because of (12), for $\bar{\Lambda}_2$ as defined in (EC.7), we have $\bar{\Lambda}_2<\lambda_{\max}$. Then, it is straightforward to see from the form of $X^*_{\mathrm{oe}}(\lambda,c_0)$ and $B^*_{\mathrm{oe}}(\lambda,c_0)$ in Theorem EC.1 that $B^*_{\mathrm{oe}}(\lambda,c_0)\neq B_0$ for any $c_0\in\mathbb{R}$. In addition, for cases (i)–(iii), (vii), and (viii) of Theorem EC.1, $B^*_{\mathrm{oe}}(\lambda,c_0)\neq X^*_{\mathrm{oe}}(\lambda,c_0)$, for case (iv) $B^*_{\mathrm{oe}}(\lambda,c_0)=X^*_{\mathrm{oe}}(\lambda,c_0)$ if and only if $c_0=\Theta_3$ and for case (vi) $B^*_{\mathrm{oe}}(\lambda,c_0)=X^*_{\mathrm{oe}}(\lambda,c_0)$ if and only if $c_0=\Theta_4$. The proof then completes. ■

EC.7. Proof of Theorem 5

We first consider the case in which the reference point is updated as a probabilistic mixture. Then, B_1 is as given in (13). For notational simplicity, denote $X^{(0)} := X^{*,(0)}_{\mathrm{mu}}(\lambda)$. Denote

$$\bar{\lambda} := \frac{1+\alpha\lambda}{1+\alpha}, \quad \bar{\lambda}_{p,\omega} = \begin{cases} \frac{1+p\alpha\lambda+(1-p)\alpha}{1+\alpha}, & X^{(0)}(\omega) < B_0(\omega), \\ \frac{1+p\alpha+(1-p)\alpha\lambda}{1+\alpha}, & X^{(0)}(\omega) > B_0(\omega) \end{cases}$$

and for each $\nu > 0$, define

$$\bar{F}_{\nu,\omega}(x) := \frac{u(x) + p\alpha\mu\big(u(x) - u(B_0(\omega))\big) + (1-p)\alpha\mu\big(u(x) - u(B_0(\omega))\big)}{1+\alpha} - \nu\rho(\omega)x.$$

With B_1 as the reference point, using the argument in the proof of Theorem 1, we can show that a prospect X with $\mathbb{E}[\rho X] = x_0$ is optimal if there exists $\nu > 0$ such that $0 \in \partial\bar{F}_{\nu,\omega}(X(\omega))$ for $\mathbb{P}$-almost all $\omega \in \Omega$. Straightforward calculation shows that for ω with $X^{(0)}(\omega) \neq B_0(\omega)$, $\partial\bar{F}_{\nu,\omega}(x)$ is equal to

$$\begin{cases} \{u'(x) - \nu\rho(\omega)\}, & x > X^{(0)}(\omega) \vee B_0(\omega), \\ \left[u'(x) - \nu\rho(\omega), \bar{\lambda}_{p,\omega}u'(x) - \nu\rho(\omega)\right], & x = X^{(0)}(\omega) \vee B_0(\omega), \\ \left\{\bar{\lambda}_{p,\omega}u'(x) - \nu\rho(\omega)\right\}, & x \in \big(X^{(0)}(\omega) \wedge B_0(\omega), X^{(0)}(\omega) \vee B_0(\omega)\big), \\ \left[\bar{\lambda}_{p,\omega}u'(x) - \nu\rho(\omega), \bar{\lambda}u'(x) - \nu\rho(\omega)\right], & x = X^{(0)}(\omega) \wedge B_0(\omega), \\ \left\{\bar{\lambda}u'(x) - \nu\rho(\omega)\right\}, & x < X^{(0)}(\omega) \wedge B_0(\omega) \end{cases}$$

and for ω with $X^{(0)}(\omega) = B_0(\omega)$,

$$\partial\bar{F}_{\nu',\omega}(x) = \begin{cases} \{u'(x) - \nu\rho(\omega)\}, & x > B_0(\omega), \\ \left[u'(x) - \nu\rho(\omega), \bar{\lambda}u'(x) - \nu\rho(\omega)\right], & x = B_0(\omega), \\ \left\{\bar{\lambda}u'(x) - \nu\rho(\omega)\right\}, & x < B_0(\omega). \end{cases}$$

Recall that $X^{(0)} = X^*_{\mathrm{ex}}(\lambda)$ is given by (5). Straightforward calculation then shows that $0 \in \partial\bar{F}_{\nu',\omega}(X^{(0)}(\omega))$ for all ω, implying that $X^{(0)}$ is the optimal prospect with B_1 as the reference point.

We just proved that $X^{*,(1)}(\lambda)=X^{*,(0)}(\lambda)=X^*_{\mathrm{ex}}(\lambda)$. Then,

$$B_2=\begin{cases} B_1, & \text{with probability } p,\\ X^{*,(1)}_{\mathrm{mu}}(\lambda), & \text{with probability } 1-p,\end{cases}$$
$$=\begin{cases} B_0, & \text{with probability } p^2,\\ X^*_{\mathrm{ex}}(\lambda), & \text{with probability } 1-p^2.\end{cases}$$

Repeating the above argument, we conclude that $X^{*,(n)}(\lambda)=X^*_{\mathrm{ex}}(\lambda)$ and

$$B_n=\begin{cases} B_0, & \text{with probability } p^n,\\ X^*_{\mathrm{ex}}(\lambda), & \text{with probability } 1-p^n,\end{cases}$$

for all $n=0,1,\ldots$. The Borel-Cantelli Lemma then shows that B_n converges to $X^*_{\mathrm{ex}}(\lambda)$ almost surely as n goes to infinity.

Next, we consider the case in which the reference point is updated as a state-wise convex combination. Then, following the same argument as in the proof of Theorem 1, a prospect X with $\mathbb{E}[\rho X]=x_0$ is optimal with reference point B_1 if there exists $\nu>0$ such that for $\mathbb{P}$-almost surely ω,

$$0\in\begin{cases} \{u'(X(\omega))-\nu\rho(\omega)\}, & X(\omega)>B_1\omega),\\ \left\{\frac{1+\alpha\lambda}{1+\alpha}u'(X(\omega))-\nu\rho(\omega)\right\}, & X(\omega)\in(0,B_1(\omega)),\\ \left[u'(X(\omega))-\nu\rho(\omega),\frac{1+\alpha\lambda}{1+\alpha}u'(X(\omega))-\nu\rho(\omega)\right], & X(\omega)=B_1(\omega).\end{cases}\tag{EC.12}$$

Because $B_1=wB_0+(1-w)X^{(0)}$, $X^{(0)}>B_1$ ($X^{(0)}<B_1$ and $X^{(0)}=B_1$, respectively) if and only if $X^{(0)}>B_0$ ($X^{(0)}<B_0$ and $X^{(0)}=B_0$, respectively). Recall that $X^{(0)}=X^*_{\mathrm{ex}}(\lambda)$ is given by (5). Straightforward calculation then shows that (EC.12) holds with X replaced by $X^{(0)}$. As a result, $X^{(0)}$ is the optimal prospect with B_1 as the reference point.

We just proved that $X^{*,(1)}(\lambda)=X^{*,(0)}(\lambda)=X^*_{\mathrm{ex}}(\lambda)$. Then,

$$B_2=wB_1+(1-w)X^{*,(1)}_{\mathrm{mu}}(\lambda)=w^2B_0+(1-w^2)X^*_{\mathrm{ex}}(\lambda).$$

Repeating the above argument, we conclude that $X^{*,(n)}(\lambda) = X^*_{\mathrm{ex}}(\lambda)$ and

$$B_n = w^n B_0 + (1 - w^n) X^*_{\mathrm{ex}}(\lambda)$$

for all $n = 0, 1, \ldots$. Then, B_n converges to $X^*_{\mathrm{ex}}(\lambda)$ almost surely as n goes to infinity. ■

EC.8. Proof of Corollary 1

Suppose X is a PE. Then, by definition, X is the optimal prospect with the reference point to be X.

On the other hand, suppose X is an optimal prospect with certain reference point B_0. Then, by Theorem 5, for any Y with $\mathbb{E}[\rho Y] \le x_0$ and any $n \in \mathbb{N}$, $V_\lambda(X|B_n) \ge V_\lambda(Y|B_n)$. Sending n to infinity, recalling that B_n converges to X, and applying the dominated convergence theorem, we conclude that $V_\lambda(X|X) \ge V_\lambda(Y|X)$. Thus, X is a PE. ■