

Endogenous inverse demand functions

Online appendix

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A Construction of the (modified) Bühlmann equilibrium

In Section 2, we introduced the basic details of the Bühlmann equilibrium. For completeness, in this section we wish to present the general construction of the Bühlmann equilibrium (Y, \mathbb{Q}) . The arguments presented here follow directly from [5].

First, recall the setting of the modified Bühlmann equilibrium problem. That is, consider a market of n participants with utility functions u_i and endowments X_i into which some *external* portfolio $Z \in L^\infty$ is sold. The modified Bühlmann equilibrium (Y, \mathbb{Q}) satisfies:

1. **Utility maximizing:** $Y_i \in \arg \max_{\hat{Y}_i} \mathbb{E} \left[u_i \left(X_i + \hat{Y}_i - \mathbb{E}^{\mathbb{Q}}[\hat{Y}_i] \right) \right]$ with $\mathbb{E} \left[u_i \left(X_i + Y_i - \mathbb{E}^{\mathbb{Q}}[Y_i] \right) \right] \in \mathbb{R}$ for every $i \in \{1, 2, \dots, n\}$; and
2. **Equilibrium transfers:** $\sum_{i=1}^n Y_i = Z$ for *externally* sold position $Z \in L^\infty$.

If $Z \equiv 0$ then this modified equilibrium coincides exactly with the typical Bühlmann equilibrium. Recall that for simplicity of exposition, we have assumed that $\mathcal{X} + Z - \mathbb{E}^{\mathbb{Q}}[Z] \in \mathbb{D}$, and as an ansatz let

$$X_i + Y_i - \mathbb{E}^{\mathbb{Q}}[Y_i] \in \mathbb{D}, \quad i \in \{1, 2, \dots, n\}. \tag{A.1}$$

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The first order condition for utility maximizing implies

$$u'_i(X_i + Y_i - \mathbb{E}^{\mathbb{Q}}[Y_i]) = \underbrace{\mathbb{E} \left[u'_i(X_i + Y_i - \mathbb{E}^{\mathbb{Q}}[Y_i]) \right]}_{=: C_i \in \mathbb{R}_{++}} \frac{d\mathbb{Q}}{d\mathbb{P}} \quad \text{a.s.} \quad (\text{A.2})$$

By [3, 4], any equilibrium must, additionally, depend on $\omega \in \Omega$ only through $\gamma := \mathcal{X} + Z - \mathbb{E}^{\mathbb{Q}}[Z] = \sum_{i=1}^n (X_i + Y_i - \mathbb{E}^{\mathbb{Q}}[Y_i])$. We must also have that $X_i + Y_i - \mathbb{E}^{\mathbb{Q}}[Y_i]$ depends on ω only through γ . Letting $\mathcal{Y}_i(\gamma) := X_i + Y_i - \mathbb{E}^{\mathbb{Q}}[Y_i]$ for every i with $\sum_{i=1}^n \mathcal{Y}_i(\gamma) = \gamma$ and $\phi(\gamma)$ is the associated Radon-Nikodym derivative of \mathbb{Q} w.r.t. \mathbb{P} . Therefore (A.2) can be rewritten as $u'_i(\mathcal{Y}_i(\gamma)) = C_i \phi(\gamma)$. Taking the derivative (w.r.t. γ) of the logarithm of both sides leads to the relation

$$\underbrace{\frac{u''_i(\mathcal{Y}_i(\gamma))}{u'_i(\mathcal{Y}_i(\gamma))}}_{=-\rho_i(\mathcal{Y}_i(\gamma))} \mathcal{Y}'_i(\gamma) = \frac{\phi'(\gamma)}{\phi(\gamma)} \quad \forall i = 1, \dots, n \quad (\text{A.3})$$

where ρ_i denotes the risk aversion of bank i . As $\sum_{i=1}^n \mathcal{Y}'_i(\gamma) = 1$ by construction, we recover the relation

$$1 = -\frac{\phi'(\gamma)}{\phi(\gamma)} \underbrace{\sum_{i=1}^n \frac{1}{\rho_i(\mathcal{Y}_i(\gamma))}}_{=: n\rho(\gamma)^{-1}} \quad (\text{A.4})$$

for harmonic average ρ of risk aversions ρ_i , $1 \leq i \leq n$.

Directly from (A.4), the Bühlmann equilibrium (Y, \mathbb{Q}) can be constructed. As we are primarily concerned with the pricing measure \mathbb{Q} within this work, we will first focus on that measure through the construction of $\phi(\gamma)$. Specifically, $\phi'(\gamma) = -\frac{1}{n}\rho(\gamma)\phi(\gamma)$ or, equivalently (noting that $\mathbb{E}[\phi(\gamma)] = 1$)

$$\phi(\gamma) = \frac{\exp\left(-\frac{1}{n} \int_c^\gamma \rho(\xi) d\xi\right)}{\mathbb{E}\left[\exp\left(-\frac{1}{n} \int_c^\gamma \rho(\xi) d\xi\right)\right]}$$

for arbitrary $c \in \mathbb{D}$. The pricing measure \mathbb{Q} is then constructed as in (2.1) as $\frac{d\mathbb{Q}}{d\mathbb{P}} = \phi(\mathcal{X} + Z - \mathbb{E}^{\mathbb{Q}}[Z])$. For this construction, implicitly we require $\mathcal{Y}_i(\gamma)$ so as to define the risk aversion ρ of the harmonic representative agent. Utilizing both (A.3) and (A.4), we recover the differential system

$$\mathcal{Y}'_i(\gamma) = \frac{1}{n} \frac{\rho(\gamma)}{\rho_i(\mathcal{Y}_i(\gamma))} \quad (\text{A.5})$$

with initial condition $\mathcal{Y}_i(c) \in \mathbb{R}$ such that $\mathbb{E}^{\mathbb{Q}}[\mathcal{Y}_i(\mathcal{X} + Z - \mathbb{E}^{\mathbb{Q}}[Z])] = \mathbb{E}^{\mathbb{Q}}[X_i]$ for every bank i . We

wish to note that [5] studies the existence and uniqueness of the equilibrium by considering the existence and uniqueness of such an initial condition (with $Z \equiv 0$); such results are replicated within Theorem 2.2.

If the individual investments Y_i were desired, then this can be found directly from the construction of $\mathcal{Y}_i(\mathcal{X} + Z - \mathbb{E}^{\mathbb{Q}}[Z])$ given the pricing measure \mathbb{Q} . Specifically,

$$Y_i = -X_i + \mathcal{Y}_i(\mathcal{X} + Z - \mathbb{E}^{\mathbb{Q}}[Z]) + \lambda_i \mathbb{E}^{\mathbb{Q}}[Z]$$

for $\lambda_i \in \mathbb{R}$ arbitrary such that $\sum_{i=1}^n \lambda_i = 1$. By construction of $\mathcal{Y}(\mathcal{X} + Z - \mathbb{E}^{\mathbb{Q}}[Z])$, it immediately follows that equilibrium transfers $\sum_{i=1}^n Y_i = Z$, and if $\mathbb{D} = \mathbb{R}_{++}$ the ansatz (A.1) can be checked as in [1].

B Proof from Section 2

B.1 Proof of Proposition 2.3

Proof. Note that $\rho_i : \mathbb{D} \rightarrow \mathbb{R}_+$ is nonnegative for every i by the assumed properties of $u_i : \mathbb{D} \rightarrow \mathbb{R}$. This implies $\rho : \mathbb{D} \rightarrow \mathbb{R}_+$ is nonnegative as well by construction in (2.2). Therefore $\mathcal{Y}_i : \mathbb{D} \rightarrow \mathbb{R}$ is nondecreasing as its derivative is nonnegative (see (A.5)). By construction, it immediately follows that

$$\rho(\gamma_1) = n \left(\sum_{i=1}^n \frac{1}{\rho_i(\mathcal{Y}_i(\gamma_1))} \right)^{-1} \geq n \left(\sum_{i=1}^n \frac{1}{\rho_i(\mathcal{Y}_i(\gamma_2))} \right)^{-1} = \rho(\gamma_2)$$

for $\gamma_1, \gamma_2 \in \mathbb{D}$ such that $\gamma_1 \leq \gamma_2$. □

C Example of nonunique pricing

The extra conditions of Theorem 3.3 are sufficient (but not necessary) for the uniqueness of the equilibrium price. We present the following counterexample to demonstrate that existence of an equilibrium price does not guarantee uniqueness and, thus, demonstrating some conditions are needed for such a property. Notably, the following counterexample is under the setting $\mathbb{D} = \mathbb{R}$ which, by Theorem 3.2, the existence of a fixed point for H_Z is guaranteed.

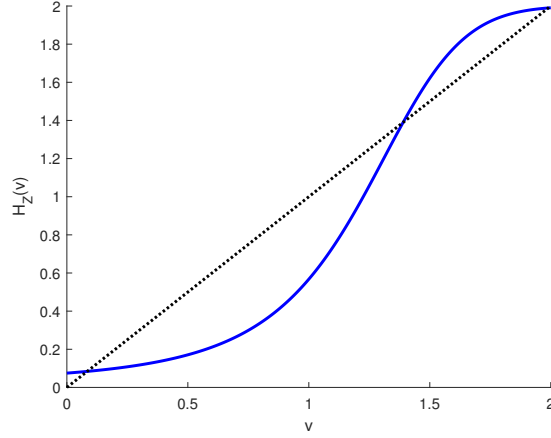


Figure 3: Plot of the price $H_Z(v)$ resulting from initial valuation $v \in [10^{-5}, 2]$ for Example C.1.

Example C.1. Let $R : \mathbb{R} \rightarrow \mathbb{R}$ (i.e., $\mathbb{D} = \mathbb{R}$) be such that $R(z) = 1 - e^{-z+2.3}$ for any $z \in \mathbb{R}$. Consider the discrete probability space $(\Omega := \{\omega_1, \omega_2\}, 2^\Omega, \mathbb{P})$ such that $\mathbb{P}(\omega_1) = 0.01$ and $\mathbb{P}(\omega_2) = 0.99$. Further, consider the setting in which the system-wide assets are provided by $\mathcal{X}(\omega_1) = 10^{-5}$ and $\mathcal{X}(\omega_2) = 100$, i.e., the financial system is shocked in scenario ω_1 . Trivially \mathcal{X} satisfies Assumption 2.1 and R satisfies assumption 3.1, therefore this financial system has full domain ($\text{dom } V = L^\infty$) by Theorem 3.2. Consider the specific position to be liquidated: $Z \in L^\infty$ such that $Z(\omega_1) = 2$ and $Z(\omega_2) = 10^{-5}$. Notably, this setting does not satisfy any of the uniqueness conditions of Theorem 3.3. In fact, there exist three equilibrium prices for Z all between $\text{ess inf } Z = 10^{-5}$ and $\text{ess sup } Z = 2$; up to rounding errors, the set of equilibrium prices of Z is given by:

$$V(Z) = \{0.08403, 1.38977, 1.98985\}.$$

The multiplicity of equilibria is visually clear in Figure 3 in which $H_Z(v)$ is plotted as a function of the initial valuation $v \in [10^{-5}, 2]$.

D Proofs from Section 3.1

D.1 Proof of Theorem 3.2

Proof. We first note that R is continuous, and therefore so is $v \mapsto H_Z(v)$.

1. If $\mathbb{D} = \mathbb{R}$, the existence of the fixed point now follows from Brouwer's fixed-point theorem,

because $\text{ess inf } Z \leq H_Z(v) \leq \text{ess sup } Z$ for all $v \in \text{dom } H_Z = \mathbb{R}$. In particular, these bounds hold for $v \in [\text{ess inf } Z, \text{ess sup } Z]$.

2. If $\mathbb{D} = \mathbb{R}_{++}$, we still have that $\text{ess inf } Z \leq H_Z(\text{ess inf } Z)$. If $\text{ess inf}[\mathcal{X} + Z] \notin \text{dom } H_Z$, and if, additionally, $\liminf_{v \nearrow \text{ess inf}[\mathcal{X} + Z]} H_Z(v) =: L < \text{ess inf}[\mathcal{X} + Z]$, then fix $v^* \in (L, \text{ess inf}[\mathcal{X} + Z])$. By assumption, there exists some $\bar{v}^* \in [v^*, \text{ess inf}[\mathcal{X} + Z])$ such that $H_Z(\bar{v}^*) < v^* \leq \bar{v}^*$. Thus $H_Z(v) - v$ must have a root on $[\text{ess inf } Z, \bar{v}^*]$, which is a fixed point for $H_Z(v)$. Alternatively, if $\text{ess inf}[\mathcal{X} + Z] \in \text{dom } H_Z$ and if $H_Z(\text{ess inf}[\mathcal{X} + Z]) \leq \text{ess inf}[\mathcal{X} + Z]$, then $H_Z(v) - v$ again must have a root on $[\text{ess inf } Z, \text{ess inf}[\mathcal{X} + Z]]$, which is again a fixed point for $H_Z(v)$.

In any case, compactness of $V(Z)$ follows from: boundedness as $\text{ess inf } Z \leq H_Z(v) \leq \text{ess sup } Z \vee \text{ess inf}[\mathcal{X} + Z]$ for any $v \in \text{dom } H_Z$; and closedness as the limit of fixed points of a continuous mapping is also a fixed point. \square

D.2 Proof of Theorem 3.3

Proof. We first consider conditions (a) and (b). Consider the derivative of H_Z for a fixed external liquidation Z . Let $v \in \text{dom } H_Z$. For simplicity of notation, define $R := R(\mathcal{X} + Z - v)$ and $R' := R'(\mathcal{X} + Z - v)$.

$$H'_Z(v) = \frac{\mathbb{E}[\exp(-R)]\mathbb{E}[ZR' \exp(-R)] - \mathbb{E}[Z \exp(-R)]\mathbb{E}[R' \exp(-R)]}{\mathbb{E}[\exp(-R)]^2}.$$

Recall, by construction, $R' > 0$. Therefore, $H'_Z(v) \leq 0$ (the derivative is understood to be the right derivative, if v is on the boundary of $\text{dom } H_Z$) if

$$\mathbb{E}[Z \exp(-R(\hat{Z}))]\mathbb{E}[R'(\hat{Z}) \exp(-R(\hat{Z}))] - \mathbb{E}[\exp(-R(\hat{Z}))]\mathbb{E}[ZR'(\hat{Z}) \exp(-R(\hat{Z}))] \geq 0$$

where $\hat{Z} := \mathcal{X} + Z - v$. Indeed, let $A(\omega) := \{\bar{\omega} \in \Omega \mid \hat{Z}(\bar{\omega}) \leq \hat{Z}(\omega)\}$. Then

$$\begin{aligned} -H'_Z(v)\mathbb{E}[\exp(-R)]^2 &= \mathbb{E}[Z \exp(-R(\hat{Z}))]\mathbb{E}[R'(\hat{Z}) \exp(-R(\hat{Z}))] - \mathbb{E}[\exp(-R(\hat{Z}))]\mathbb{E}[ZR'(\hat{Z}) \exp(-R(\hat{Z}))] \\ &= \int_{\Omega} Z(\omega) \exp(-R(\hat{Z}(\omega)))\mathbb{P}(d\omega) \int_{\Omega} R'(\hat{Z}(\bar{\omega})) \exp(-R(\hat{Z}(\bar{\omega})))\mathbb{P}(d\bar{\omega}) \\ &\quad - \int_{\Omega} \exp(-R(\hat{Z}(\omega)))\mathbb{P}(d\omega) \int_{\Omega} Z(\bar{\omega})R'(\hat{Z}(\bar{\omega})) \exp(-R(\hat{Z}(\bar{\omega})))\mathbb{P}(d\bar{\omega}) \end{aligned}$$

$$\begin{aligned}
&= \int_{\Omega} \int_{\Omega} \exp(-R(\hat{Z}(\omega))) \exp(-R(\hat{Z}(\bar{\omega}))) [Z(\omega) - Z(\bar{\omega})] R'(\hat{Z}(\bar{\omega})) \mathbb{P}(d\bar{\omega}) \mathbb{P}(d\omega) \\
&= \int_{\Omega} \int_{A(\omega)} \exp(-R(\hat{Z}(\omega))) \exp(-R(\hat{Z}(\bar{\omega}))) [Z(\omega) - Z(\bar{\omega})] R'(\hat{Z}(\bar{\omega})) \mathbb{P}(d\bar{\omega}) \mathbb{P}(d\omega) \\
&\quad + \int_{\Omega} \int_{A(\omega)^c} \exp(-R(\hat{Z}(\omega))) \exp(-R(\hat{Z}(\bar{\omega}))) [Z(\omega) - Z(\bar{\omega})] R'(\hat{Z}(\bar{\omega})) \mathbb{P}(d\bar{\omega}) \mathbb{P}(d\omega) \\
&= \int_{\Omega} \int_{A(\omega)} \exp(-R(\hat{Z}(\omega))) \exp(-R(\hat{Z}(\bar{\omega}))) [Z(\omega) - Z(\bar{\omega})] R'(\hat{Z}(\bar{\omega})) \mathbb{P}(d\bar{\omega}) \mathbb{P}(d\omega) \\
&\quad + \int_{\Omega} \int_{A(\omega)} \exp(-R(\hat{Z}(\omega))) \exp(-R(\hat{Z}(\bar{\omega}))) [Z(\bar{\omega}) - Z(\omega)] R'(\hat{Z}(\omega)) \mathbb{P}(d\bar{\omega}) \mathbb{P}(d\omega) \\
&= \int_{\Omega} \int_{A(\omega)} \underbrace{\exp(-R(\hat{Z}(\omega))) \exp(-R(\hat{Z}(\bar{\omega})))}_{>0} [Z(\omega) - Z(\bar{\omega})] \left[R'(\hat{Z}(\bar{\omega})) - R'(\hat{Z}(\omega)) \right] \mathbb{P}(d\bar{\omega}) \mathbb{P}(d\omega).
\end{aligned}$$

Therefore, for $v \in \text{dom } H_Z$, $H'_Z(v) \leq 0$ if $[Z(\omega) - Z(\bar{\omega})] \left[R'(\hat{Z}(\bar{\omega})) - R'(\hat{Z}(\omega)) \right] \geq 0$. This holds if either:

- (a) Z and $\mathcal{X} + Z$ are comonotonic because R' is non-increasing;
- (b) R is linear as R' is a constant.

In either case, we have that $\frac{\partial}{\partial v}(H_Z(v) - v) \leq -1 < 0$, which guarantees at most one root exists.

For cases (c) and (d), we note that $v = H_Z(v)$ if and only if $\Theta_Z(v) := \mathbb{E}[(Z - v) \exp(-R(\mathcal{X} + Z - v))] = 0$. Additionally, by Proposition D.1, $z \mapsto z \exp(-R(\mathcal{X} + z))$ is almost surely non-decreasing or concave (under (c) and (d) respectively) where it is well-defined. First assume $\mathbb{P}(Z = \text{ess inf } Z) = 1$, i.e., Z is a constant a.s.; then trivially the only root is given by $V(Z) = \text{ess inf } Z$. Otherwise $\mathbb{P}(Z > \text{ess inf } Z) > 0$. We will focus on $\frac{\partial}{\partial v} \Theta_Z(v)$ over feasible $v \in \mathbb{R}$, i.e.,

$$\frac{\partial}{\partial v} \Theta_Z(v) = -\mathbb{E} \left[(1 - (Z - v) R'(\mathcal{X} + Z - v)) \exp(-R(\mathcal{X} + Z - v)) \right].$$

- (c) If $z \mapsto z \exp(-R(\mathcal{X} + z))$ is almost surely non-decreasing, then $z R'(\mathcal{X} + z) \leq 1$, $z \in \mathbb{D}$ a.s.

This guarantees $(Z - v) R'(\mathcal{X} + Z - v) \leq 1$ a.s., because $(Z(\omega) - v) R'(\mathcal{X}(\omega) + Z(\omega) - v) \leq 0$ on $\{\omega \in \Omega \mid Z(\omega) \leq v\}$ for any feasible $v \in \mathbb{R}$. As $\Theta_Z(\text{ess inf } Z) > 0$, then $\mathbb{P}(Z \leq v^*) > 0$ for any (feasible) price $v^* \in \mathbb{R}$ such that $\Theta_Z(v^*) = 0$. As a direct consequence $\frac{\partial}{\partial v} \Theta_Z(v) < 0$ for any $v \geq v^*$ feasible which contradicts the existence of a multiplicity of equilibria.

- (d) If $z \mapsto z \exp(-R(\mathcal{X} + z))$ is almost surely concave, then $v \mapsto \Theta_Z(v)$ is concave as well. As

$\Theta_Z(\text{ess inf } Z) > 0$, the minimal (feasible) price $v^* \in \mathbb{R}$ such that $\Theta_Z(v^*) = 0$ (if it exists) must satisfy $\frac{\partial}{\partial v}\Theta_Z(v^*) < 0$. By concavity, for any $v \geq v^*$ feasible must therefore also satisfy $\frac{\partial}{\partial v}\Theta_Z(v) < 0$ which contradicts the existence of a multiplicity of equilibria.

□

Proposition D.1. *Assume R satisfies Assumption 3.1. If $z \in \mathbb{R}_{++} \mapsto z \exp(-R(\mathcal{X} + z))$ is non-decreasing (concave) for fixed $\mathcal{X} \in \mathbb{D}$ then this mapping is non-decreasing (concave) over its entire domain $\mathbb{D} - \mathcal{X}$.*

Proof. Consider the derivative of $z \exp(-R(\mathcal{X} + z))$ w.r.t. z , i.e.,

$$\frac{\partial}{\partial z} z \exp(-R(\mathcal{X} + z)) = [1 - zR'(\mathcal{X} + z)] \exp(-R(\mathcal{X} + z)).$$

Therefore, $\frac{\partial}{\partial z} z \exp(-R(\mathcal{X} + z)) > 0$ for any $z \leq 0$; and thus monotonicity holds over the entire domain if it holds for $z \in \mathbb{R}_{++}$. Further, by Assumption 3.1, this derivative is strictly decreasing on $z \leq 0$; therefore concavity holds over the entire domain if it holds for $z \in \mathbb{R}_{++}$. □

D.3 Proof of Corollary 3.4

Proof. We have already shown in Theorem 3.2 that under the condition $H_Z(\text{ess inf}[\mathcal{X} + Z]) \leq \text{ess inf}[\mathcal{X} + Z]$, when $\text{ess inf}[\mathcal{X} + Z] \in \text{dom } H_Z$, or $\liminf_{v \nearrow \text{ess inf}[\mathcal{X} + Z]} H_Z(v) < \text{ess inf}[\mathcal{X} + Z]$ if otherwise $\text{ess inf}[\mathcal{X} + Z] \notin \text{dom } H_Z$, we have existence of the fixed point. Given any of the conditions (a)-(d) of Theorem 3.3 we immediately recover uniqueness as well.

We will show that this condition it is also necessary for existence under any of the conditions of Theorem 3.3. Assume first that $\text{ess inf}[\mathcal{X} + Z] \in \text{dom } H_Z$, and assume that $H_Z(\text{ess inf}[\mathcal{X} + Z]) > \text{ess inf}[\mathcal{X} + Z]$. We will demonstrate that this implies $H_Z(v) - v$ (equivalently $\Theta_Z(v) := \mathbb{E}[(Z - v) \exp(-R(\mathcal{X} + Z - v))]$) has no roots on $[\text{ess inf } Z, \text{ess inf}[\mathcal{X} + Z]]$, and therefore no fixed point for $H_Z(v)$ exists. Indeed, if either (a) or (b) of Theorem 3.3 hold then we have shown in the proof of Theorem 3.3 that $H'_Z(v) \leq 0$, and therefore $H_Z(v) > \text{ess inf}[\mathcal{X} + Z] \geq v$ for $v \in [\text{ess inf } Z, \text{ess inf}[\mathcal{X} + Z]]$. Similarly, if (c) or (d) of Theorem 3.3 hold then we have shown in the proof of Theorem 3.3 that $\Theta_Z(v) \leq 0$ for every $v \geq V(Z)$. As $H_Z(\text{ess inf}[\mathcal{X} + Z]) - \text{ess inf}[\mathcal{X} + Z] > 0$, $\Theta_Z(v) > 0$ for every $v \in [\text{ess inf } Z, \text{ess inf}[\mathcal{X} + Z]]$.

The proof is similar if $\text{ess inf}[\mathcal{X} + Z] \notin \text{dom } H_Z$. By either (a) or (b) of Theorem 3.3, we have that $H'_Z(v) \leq 0$. Therefore, $H_Z(v) \geq \liminf_{v \nearrow \text{ess inf}[\mathcal{X} + Z]} H_Z(v) \geq \text{ess inf}[\mathcal{X} + Z] > v$ for all $v < \text{ess inf}[\mathcal{X} + Z]$, $v \in \text{dom } H_Z$. Similarly and as above, under either (c) or (d) of Theorem 3.3, we have that $\Theta_Z(v) > 0$ for all $v < \text{ess inf}[\mathcal{X} + Z]$. In either setting we get that $H_Z(v) > v$ for all $v < \text{ess inf}[\mathcal{X} + Z]$, $v \in \text{dom } H_Z$, and thus no fixed point exists. \square

D.4 Proof of Corollary 3.5

Proof. Recall $\text{ess inf } \mathcal{X} \in \mathbb{D}$ from Assumption 2.1. To simplify notation throughout this proof, define $\hat{Z}(v) := \mathcal{X} + Z - v$ and $\tilde{Z} := \hat{Z}(\text{ess inf}[\mathcal{X} + Z]) = \mathcal{X} + Z - \text{ess inf}[\mathcal{X} + Z]$. Fix $0 < \varepsilon < \text{ess inf } \mathcal{X}/2$. By construction, $\mathbb{E}[\exp(-R(\hat{Z}(v)))] < \infty$ for $v < \text{ess inf}[\mathcal{X} + Z]$ and $\mathbb{E}[\exp(-R(\tilde{Z}))\mathbb{I}_{\{\tilde{Z} \geq \varepsilon\}}] < \infty$; therefore $\mathbb{E}[\exp(-R(\tilde{Z}))\mathbb{I}_{\{0 \leq \tilde{Z} < \varepsilon\}}] = \infty$ following from the assumption of the corollary. Note that $Z - \text{ess inf}[\mathcal{X} + Z] < \varepsilon - \mathcal{X} < -\varepsilon$ a.s. on $\{0 \leq \tilde{Z} < \varepsilon\}$. Therefore,

$$\begin{aligned}
\liminf_{v \nearrow \text{ess inf}[\mathcal{X} + Z]} H_Z(v) &= \liminf_{v \nearrow \text{ess inf}[\mathcal{X} + Z]} \frac{\mathbb{E}[Z \exp(-R(\hat{Z}(v)))]}{\mathbb{E}[\exp(-R(\hat{Z}(v)))]} \\
&= \liminf_{v \nearrow \text{ess inf}[\mathcal{X} + Z]} \frac{\mathbb{E}[(Z - \text{ess inf}[\mathcal{X} + Z]) \exp(-R(\hat{Z}(v)))]}{\mathbb{E}[\exp(-R(\hat{Z}(v)))]} + \text{ess inf}[\mathcal{X} + Z] \\
&= \liminf_{v \nearrow \text{ess inf}[\mathcal{X} + Z]} \frac{\mathbb{E} \left[(Z - \text{ess inf}[\mathcal{X} + Z]) \exp(-R(\hat{Z}(v))) \left(\mathbb{I}_{\{0 \leq \tilde{Z} < \varepsilon\}} + \mathbb{I}_{\{\tilde{Z} \geq \varepsilon\}} \right) \right]}{\mathbb{E} \left[\exp(-R(\hat{Z}(v))) \right]} + \text{ess inf}[\mathcal{X} + Z] \\
&= \liminf_{v \nearrow \text{ess inf}[\mathcal{X} + Z]} \frac{\mathbb{E} \left[(Z - \text{ess inf}[\mathcal{X} + Z]) \exp(-R(\hat{Z}(v))) \mathbb{I}_{\{0 \leq \tilde{Z} < \varepsilon\}} \right]}{\mathbb{E} \left[\exp(-R(\hat{Z}(v))) \right]} \\
&\quad + \liminf_{v \nearrow \text{ess inf}[\mathcal{X} + Z]} \frac{\mathbb{E} \left[(Z - \text{ess inf}[\mathcal{X} + Z]) \exp(-R(\hat{Z}(v))) \mathbb{I}_{\{\tilde{Z} \geq \varepsilon\}} \right]}{\mathbb{E} \left[\exp(-R(\hat{Z}(v))) \right]} + \text{ess inf}[\mathcal{X} + Z] \\
&= \liminf_{v \nearrow \text{ess inf}[\mathcal{X} + Z]} \frac{\mathbb{E} \left[(Z - \text{ess inf}[\mathcal{X} + Z]) \exp(-R(\hat{Z}(v))) \mathbb{I}_{\{0 \leq \tilde{Z} < \varepsilon\}} \right]}{\mathbb{E} \left[\exp(-R(\hat{Z}(v))) \right]} + \text{ess inf}[\mathcal{X} + Z] \tag{D.1} \\
&\leq \liminf_{v \nearrow \text{ess inf}[\mathcal{X} + Z]} \frac{-\frac{\varepsilon}{2} \mathbb{E} \left[\exp(-R(\hat{Z}(v))) \mathbb{I}_{\{0 \leq \tilde{Z} < \varepsilon\}} \right]}{\mathbb{E} \left[\exp(-R(\hat{Z}(v))) \mathbb{I}_{\{0 \leq \tilde{Z} < \varepsilon\}} \right]} + \text{ess inf}[\mathcal{X} + Z] \\
&= -\frac{\varepsilon}{2} + \text{ess inf}[\mathcal{X} + Z] < \text{ess inf}[\mathcal{X} + Z].
\end{aligned}$$

We recover the equality in (D.1) as

$$\lim_{v \nearrow \text{ess inf}[\mathcal{X}+Z]} \frac{\mathbb{E} \left[(Z - \text{ess inf}[\mathcal{X} + Z]) \exp \left(-R(\hat{Z}(v)) \right) \mathbb{I}_{\{\tilde{Z} \geq \varepsilon\}} \right]}{\mathbb{E} \left[\exp \left(-R(\hat{Z}(v)) \right) \right]} = 0$$

by assumption of this corollary. □

E Proofs from Section 3.2

Within the proof of Lemma 3.6 and later, we frequently make use of an optimization based representation for \bar{V} .

Proposition E.1. *Assume R satisfies Assumption 3.1. If any of the conditions of Theorem 3.3 holds, then the extension \bar{V} , can equivalently be formulated as:*

$$\begin{aligned} \bar{V}(Z) &= \sup \{ v \geq \text{ess inf } Z \mid H_Z(v) \geq v, \text{ess inf}[\mathcal{X} + Z] - v \in \mathbb{D} \} \\ &= \sup \{ v \geq \text{ess inf } Z \mid \mathbb{E}[(Z - v) \exp(-R(\mathcal{X} + Z - v))] \geq 0, \text{ess inf}[\mathcal{X} + Z] - v \in \mathbb{D} \} \end{aligned}$$

for any $Z \in L^\infty$ for any choice of \mathbb{D} .

Proof. We wish to note that the two optimization problems are trivially equivalent by construction of $H_Z(v)$. Further, the conditions of Theorem 3.3 imply: $v \mapsto H_Z(v)$ is monotonic under the first two conditions, $v \mapsto \mathbb{E}[(Z - v) \exp(-R(\mathcal{X} + Z - v))]$ is monotonic under the third condition and monotonic for $v \geq V(Z)$ (for $Z \in \text{dom } V$) in the fourth condition.

Fix $Z \in L^\infty$. If $Z \in \text{dom } V$, i.e., $\#V(Z) = 1$ which we will treat as a scalar value, then we will prove the result by showing that $V(Z)$ is feasible for this optimization problem and $v > V(Z)$ is *not* feasible. As $V(Z) \geq \text{ess inf } Z$ by Lemma 3.6(1), feasibility of $V(Z)$ follows from the construction of the equilibrium pricing problem as $\text{ess inf}[\mathcal{X}+Z] - V(Z) \in \mathbb{D}$ and $H_Z(V(Z)) = V(Z)$. Further, the conditions of Theorem 3.3 imply either $H_Z(v) < v$ (for the first two conditions) or $\mathbb{E}[(Z - v) \exp(-R(\mathcal{X} + Z - v))] < 0$ (for the latter two conditions) for any $v > V(Z)$. This proves the result. If $Z \notin \text{dom } V$ then, noting that this implies $\mathbb{D} = \mathbb{R}_{++}$, $H_Z(v) > v$ for every $v \in [\text{ess inf } Z, \text{ess inf}[\mathcal{X} + Z]]$ by $H_Z(\text{ess inf } Z) > \text{ess inf } Z$ and the nonexistence of an equilibrium

price. Therefore it follows that $\sup\{v \geq \text{ess inf } Z \mid H_Z(v) \geq v, \text{ess inf}[\mathcal{X} + Z] - v \in \mathbb{D}\} = \text{ess inf}[\mathcal{X} + Z] = \bar{V}(Z)$ and the proof is complete. \square

E.1 Proof of Lemma 3.6

Proof. 1. First, if $Z \in \text{dom } V$, then for any $v \in V(Z)$ it follows that $v = \mathbb{E}[Z \exp(-R(\mathcal{X} + Z - v))]/\mathbb{E}[\exp(-R(\mathcal{X} + Z - v))] \in [\text{ess inf } Z, \text{ess sup } Z]$ by construction; as such this holds for the minimal price $\bar{V}(Z)$ as well. Assume, now, $Z \notin \text{dom } V$ and, as such, $\mathbb{D} = \mathbb{R}_{++}$. Immediately the lower bound $\bar{V}(Z) = \text{ess inf}[\mathcal{X} + Z] > \text{ess inf } Z$ holds. If $\text{ess inf}[\mathcal{X} + Z] \leq \text{ess sup } Z$ then the upper bound holds; assume $\text{ess inf}[\mathcal{X} + Z] > \text{ess sup } Z$, then $Z \in \text{dom } V$ because $H_Z(\text{ess sup } Z) \leq \text{ess sup } Z$ by construction and $\text{ess sup } Z$ being a feasible price for Z which forms a contradiction and the proof is complete.

2. This follows immediately by the law invariance of $H_Z(v; \mathcal{X})$ in $(Z, \mathcal{X} + Z)$.

3. Note that $Z + z \in \text{dom } V$ if and only if $Z \in \text{dom } V$. Furthermore, note that by construction $H_{Z+z}(v) = H_Z(v - z) + z$. First, let $Z \in \text{dom } V$ and $v \in V(Z + z)$. Therefore $v = H_{Z+z}(v) = H_Z(v - z) + z$. Take the ansatz that $v - z =: v^* \in V(Z)$, then $v^* = H_Z(v^*)$. As this satisfies the fixed point problem and $Z \in \text{dom } V$ then $V(Z + z) \subseteq V(Z) + z$. The converse relation follows comparably, which immediately leads to the conclusion that $\bar{V}(Z + z) = \bar{V}(Z) + z$. If $Z \notin \text{dom } V$ then $\bar{V}(Z + z) = \text{ess inf}[\mathcal{X} + Z + z] = \text{ess inf}[\mathcal{X} + Z] + z = \bar{V}(Z) + z$.

4. (a) Consider $\mathbb{D} = \mathbb{R}$ where $\text{dom } V = L^\infty$ by Theorem 3.2. Fix $\bar{Z} \in L^\infty$ and let $\mathcal{N} \subseteq \{Z \in L^\infty \mid \|Z - \bar{Z}\|_\infty \leq \delta\}$ for some $\delta > 0$ be a closed neighborhood of \bar{Z} . Define $\bar{H} : \mathcal{N} \times [\text{ess inf } \bar{Z} - \delta, \text{ess sup } \bar{Z} + \delta] \rightarrow [\text{ess inf } \bar{Z} - \delta, \text{ess sup } \bar{Z} + \delta]$ be defined as the restriction of H , i.e., $\bar{H}(Z, v) := H_Z(v)$ for any $(Z, v) \in \mathcal{N} \times [\text{ess inf } \bar{Z} - \delta, \text{ess sup } \bar{Z} + \delta]$. (Note that $H_Z(v) \in [\text{ess inf } Z, \text{ess sup } Z] \subseteq [\text{ess inf } \bar{Z} - \delta, \text{ess sup } \bar{Z} + \delta]$ by construction of H and \mathcal{N} .) Therefore, by construction, $V(Z) = \text{FIX}_v \bar{H}(Z, v)$ for every $Z \in \mathcal{N}$. By [6, Lemma C.1], $Z \in \mathcal{N} \rightarrow V(Z)$ is a set-valued upper continuous mapping (i.e., continuous in the upper Vietoris topology). As a direct consequence of [2, Lemma 17.30], $Z \in \mathcal{N} \mapsto \bar{V}(Z) = \min V(Z)$ is lower semicontinuous. As this is true for any (closed) neighborhood around any $\bar{Z} \in L^\infty$, \bar{V} must be lower semicontinuous on the entire space L^∞ . Furthermore, if any of the conditions of Theorem 3.3 hold, then uniqueness of the

equilibrium price guarantees (scalar) continuity of V which completes the proof in this setting.

- (b) Consider $\mathbb{D} = \mathbb{R}_{++}$. Fix $\bar{Z} \in \text{dom } V$ and let $\mathcal{N} \subseteq \{Z \in L^\infty \mid \|Z - \bar{Z}\|_\infty \leq \delta\}$ for some $\delta > 0$ be a closed neighborhood of \bar{Z} . Define $\bar{H} : \mathcal{N} \times [\text{ess inf } \bar{Z} - \delta, \text{ess sup } \bar{Z} + \delta] \rightarrow \mathfrak{2}^{[\text{ess inf } \bar{Z} - \delta, \text{ess sup } \bar{Z} + \delta]}$ be the set valued mapping with graph

$$\text{graph } \bar{H} := \text{cl} \{(Z, v, H_Z(v)) \mid Z \in \mathcal{N}, v \in [\text{ess inf } \bar{Z} - \delta, \text{ess sup } \bar{Z} + \delta], \text{ess inf}[\mathcal{X} + Z] - v \in \mathbb{D}\}.$$

(Note that $\bar{H}(Z, v) = \{H_Z(v)\}$ and $H_Z(v) \in [\text{ess inf } Z, \text{ess sup } Z] \subseteq [\text{ess inf } \bar{Z} - \delta, \text{ess sup } \bar{Z} + \delta]$ for any $Z \in \mathcal{N}$ and $v \in [\text{ess inf } \bar{Z} - \delta, \text{ess sup } \bar{Z} + \delta]$ such that $\text{ess inf}[\mathcal{X} + Z] - v \in \mathbb{D}$ by construction and continuity of H and using the fact that \mathbb{D} is open.) Define $Z \in \mathcal{N} \mapsto \tilde{V}(Z) := \text{FIX}_v \bar{H}(Z, v)$; this is a set-valued upper continuous mapping by [6, Lemma C.1]. As a direct consequence of [2, Lemma 17.30], $Z \in \mathcal{N} \cap \text{dom } \tilde{V} \mapsto \min \tilde{V}(Z)$ is lower semicontinuous. With the convention $\min \emptyset = \infty$ and noting that $\mathcal{N} \cap \text{dom } \tilde{V}$ is closed (by noting that the graph of \bar{V} is closed due to upper continuity), we can extend this result insofar as $\min \tilde{V}(\cdot)$ is lower semicontinuous on \mathcal{N} . Importantly, by construction, $\bar{V}(Z) = \min\{\min \tilde{V}(Z), \text{ess inf}[\mathcal{X} + Z]\}$ for any $Z \in \mathcal{N}$ since $\bar{V}(Z) = \min \tilde{V}(Z)$ on $Z \in \mathcal{N} \cap \text{dom } V$ and $\bar{V}(Z) \leq \min \tilde{V}(Z)$ on $Z \in \mathcal{N} \cap (\text{dom } V)^c$ (with the convention $\min \emptyset = \infty$). Therefore, $Z \in \mathcal{N} \mapsto \bar{V}(Z)$ is lower semicontinuous as the minimum of a lower semicontinuous and a continuous mapping. As this is true for any (closed) neighborhood around any $\bar{Z} \in L^\infty$, \bar{V} must be lower semicontinuous on the entire space L^∞ .

It remains to show that \bar{V} is continuous if any of the conditions of Theorem 3.3 hold. By Proposition E.1 (modified by translation by $\text{ess inf } Z$), we can utilize an optimization representation of \bar{V} . By the Berge maximum theorem, if

$$\begin{aligned} Z &\mapsto D(Z) \\ &:= \text{cl}\{v \in [0, \text{ess inf}[\mathcal{X} + Z] - \text{ess inf } Z] \mid \mathbb{E}[(Z - \text{ess inf } Z - v) e^{-R(\mathcal{X} + Z - \text{ess inf } Z - v)}] \geq 0\} \end{aligned}$$

is a set-valued continuous mapping (in the Vietoris topology) then the result holds because $Z \mapsto \text{ess inf } Z$ is continuous in the strong topology. By the closed graph theorem (see, e.g., [2, Theorem 17.11]) and (almost sure) continuity of $z \mapsto z \exp(-R(\mathcal{X} + z))$, D is an upper continuous mapping. Now, let $\mathcal{V} \subseteq \mathbb{R}_+$ be open in the subspace topology and define $D^-[\mathcal{V}] := \{Z \in L^\infty \mid D(Z) \cap \mathcal{V} \neq \emptyset\}$; if $D^-[\mathcal{V}]$ is open then D is lower continuous. Let $Z \in D^-[\mathcal{V}]$ and, in particular, let $v \in \mathcal{V}$ such that $v \in D(Z)$.

i. If $H_{Z-\text{ess inf } Z}(v) - v > 0$ then there exists a neighborhood \mathcal{N}_Z around Z such that $\mathcal{N}_Z \subseteq D^-[\mathcal{V}]$ by continuity of $Z \mapsto H_{Z-\text{ess inf } Z}(v) - v$.

ii. If $H_{Z-\text{ess inf } Z}(v) - v = 0$ then:

A. If $v > 0$ then, by \mathcal{V} open, take $\epsilon > 0$ so that $v - \epsilon \in \mathcal{V}$. If Theorem 3.3(a)-(b) holds then $\hat{v} \mapsto H_{Z-\text{ess inf } Z}(\hat{v}) - \hat{v}$ is strictly decreasing; if Theorem 3.3(c)-(d) holds then $\hat{v} \mapsto \mathbb{E}[(Z - \text{ess inf } Z - \hat{v}) \exp(-R(\mathcal{X} + Z - \text{ess inf } Z - \hat{v}))]$ is strictly decreasing (in a neighborhood of v). Therefore for ϵ small, $H_{Z-\text{ess inf } Z}(v - \epsilon) - (v - \epsilon) > 0$ and the result follows by continuity as in the prior case.

B. If $v = 0$ then, by construction, $D^-[\mathcal{V}] = L^\infty$ and the result follows.

5. First, we wish to note that the condition imposed herein appears also in Theorem 3.3 and, therefore, the optimization representation of Proposition E.1 holds.

(a) Let $Z_1 \geq Z_2$. Note that this implies $\text{ess inf } Z_1 \geq \text{ess inf } Z_2$ and $\text{ess inf}[\mathcal{X} + Z_1] \geq \text{ess inf}[\mathcal{X} + Z_2]$. If $\bar{V}(Z_2) \leq \text{ess inf } Z_1$ then, by the lower bound on $\bar{V}(Z_1)$ proven above, monotonicity trivially follows. Assume, now, that $\bar{V}(Z_2) > \text{ess inf } Z_1$. By a straightforward application of Proposition E.1, $\bar{V}(Z_1) \geq \bar{V}(Z_2)$ if, and only if,

$$\begin{aligned} & \{v \geq \text{ess inf } Z_1 \mid \mathbb{E}[(Z_1 - v) \exp(-R(\mathcal{X} + Z_1 - v))] \geq 0, \text{ess inf}[\mathcal{X} + Z_1] - v \in \mathbb{D}\} \\ & \supseteq \{v \geq \text{ess inf } Z_2 \mid \mathbb{E}[(Z_2 - v) \exp(-R(\mathcal{X} + Z_2 - v))] \geq 0, \text{ess inf}[\mathcal{X} + Z_2] - v \in \mathbb{D}\}. \end{aligned}$$

By Proposition E.1 and the assumption that $\bar{V}(Z_2) > \text{ess inf } Z_1$, these sets are non-empty. By assumption and Proposition D.1, $\mathbb{E}[(Z_1 - v) \exp(-R(\mathcal{X} + Z_1 - v))] \geq \mathbb{E}[(Z_2 - v) \exp(-R(\mathcal{X} + Z_2 - v))]$ and $\text{ess inf}[\mathcal{X} + Z_1] - v \geq \text{ess inf}[\mathcal{X} + Z_2] - v$. Therefore the constraints are more restrictive w.r.t. Z_2 than Z_1 and the result follows.

(b) By translativity and monotonicity, for any $Z_1, Z_2 \in L^\infty$, $\bar{V}(Z_1) \leq \bar{V}(Z_2 + \|Z_1 - Z_2\|_\infty) = \bar{V}(Z_2) + \|Z_1 - Z_2\|_\infty$. Taking the same inequality but switching Z_1 and Z_2 proves $|\bar{V}(Z_1) - \bar{V}(Z_2)| \leq \|Z_1 - Z_2\|_\infty$.

6. First, we wish to note that the condition imposed herein appears also in Theorem 3.3 and, therefore, the optimization representation of Proposition E.1 holds.

(a) \bar{V} is concave if its hypograph

$$\text{hypo } \bar{V} := \{(Z, v) \in L^\infty \times \mathbb{R} \mid \mathbb{E}[(Z - v) \exp(-R(\mathcal{X} + Z - v))] \geq 0, \text{ess inf}[\mathcal{X} + Z] - v \in \mathbb{D}\}$$

is convex. Let $(Z_1, v_1), (Z_2, v_2) \in \text{hypo } \bar{V}$ and $\lambda \in [0, 1]$. Note that $\text{ess inf}[\mathcal{X} + \lambda Z_1 + (1 - \lambda)Z_2] - [\lambda v_1 + (1 - \lambda)v_2] \geq \lambda[\text{ess inf}[\mathcal{X} + Z_1] - v_1] + (1 - \lambda)[\text{ess inf}[\mathcal{X} + Z_2] - v_2]$. Consider now $(Z, v) \mapsto \mathbb{E}[(Z - v) \exp(-R(\mathcal{X} + Z - v))]$. By the concavity assumption on $z \mapsto z \exp(-R(\mathcal{X} + z))$ (and Proposition D.1),

$$\begin{aligned} & \mathbb{E}[(\lambda(Z_1 - v_1) + (1 - \lambda)(Z_2 - v_2)) \exp(-R(\mathcal{X} + \lambda(Z_1 - v_1) + (1 - \lambda)(Z_2 - v_2)))] \\ & \geq \lambda \mathbb{E}[(Z_1 - v_1) \exp(-R(\mathcal{X} + Z_1 - v_1))] + (1 - \lambda) \mathbb{E}[(Z_2 - v_2) \exp(-R(\mathcal{X} + Z_2 - v_2))] \geq 0. \end{aligned}$$

From these properties it is trivial to conclude $\lambda(Z_1, v_1) + (1 - \lambda)(Z_2, v_2) \in \text{hypo } \bar{V}$ and the proof is concluded.

(b) \bar{V} is weak* upper semicontinuous if and only if $\{Z \in L^\infty \mid \bar{V}(Z) \geq v\}$ is weak* closed for every $v \in \mathbb{R}$. By [7, Proposition 5.5.1] and the concavity of \bar{V} , this is true if and only if $\{Z \in L^\infty \mid \bar{V}(Z) \geq v, \|Z\|_\infty \leq k\}$ is closed in probability for every $v \in \mathbb{R}$ and $k \in \mathbb{R}_{++}$. Let $Z_n \rightarrow Z$ in probability so that $Z_n \in \{Z \in L^\infty \mid \bar{V}(Z) \geq v, \|Z\|_\infty \leq k\}$. First, $\|Z\|_\infty \leq k$ trivially. Now we wish to show that $\bar{V}(Z) \geq v$; we will accomplish this separately if $\mathbb{D} = \mathbb{R}$ and if $\mathbb{D} = \mathbb{R}_{++}$.

Let $\mathbb{D} = \mathbb{R}$. If $v \leq \text{ess inf } Z$ then $\bar{V}(Z) \geq v$ trivially by Property (1). Let $v > \text{ess inf } Z$ and define $\mathcal{D}_k := \{Z \in L^\infty \mid \|Z\|_\infty \leq k\}$. By construction, $\bar{V}(Z) \geq v$ if and only if $\mathbb{E}[(Z - v) \exp(-R(\mathcal{X} + Z - v))] \geq 0$. As $Z \in \mathcal{D}_k \mapsto \mathbb{E}[(Z - v) \exp(-R(\mathcal{X} + Z - v))]$ is continuous w.r.t. convergence in probability, the result follows.

Let $\mathbb{D} = \mathbb{R}_{++}$. If $v \leq \text{ess inf } Z$ then $\bar{V}(Z) \geq v$ trivially by Property (1). Let $v > \text{ess inf } Z$. By construction, $\bar{V}(Z) \geq v$ if and only if $\mathbb{E}[(Z - v + \epsilon) \exp(-R(\mathcal{X} + Z - v + \epsilon))] > 0$ for every $\epsilon > 0$ and $\mathcal{X} + Z - v \geq 0$ a.s. (The “if” statement holds immediately. To prove the “only if” claim: if $\mathcal{X} + Z - v > 0$ a.s. then, by continuity, $\mathbb{E}[(Z - v) \exp(-R(\mathcal{X} + Z - v))] \geq 0$ as desired; if $\mathbb{P}(\mathcal{X} + Z - v = 0) > 0$ then $v = \text{ess inf}[\mathcal{X} + Z]$ (with $\mathcal{X} + Z$ attaining its essential infimum with positive probability) and – as a consequence – $\bar{V}(Z) \geq v$ only if $Z \notin \text{dom } V$, i.e., $H_Z(v - \epsilon) > v - \epsilon$ for every $\epsilon > 0$.) As such, the existence of an almost surely converging subsequence implies $\mathcal{X} + Z - v \geq 0$ a.s. It remains to show that $\mathbb{E}[(Z - v + \epsilon) \exp(-R(\mathcal{X} + Z - v + \epsilon))] > 0$ for every $\epsilon > 0$. By continuity of R , $\mathbb{E}[(Z - v + \epsilon) \exp(-R(\mathcal{X} + Z - v + \epsilon))] \geq 0$ for every $\epsilon > 0$. If there exists some $\epsilon^* > 0$ such that $\mathbb{E}[(Z - v + \epsilon^*) \exp(-R(\mathcal{X} + Z - v + \epsilon^*))] = 0$ then, by the concavity assumption, $\mathbb{E}[(Z - v + \epsilon) \exp(-R(\mathcal{X} + Z - v + \epsilon))] < 0$ for every $\epsilon \in (0, \epsilon^*)$ which forms a contradiction and the result follows.

□

E.2 Proof of Corollary 3.7

Proof. To simplify this proof, we will denote $\mathcal{X}_p = B[p](\mathcal{X} - \text{ess inf } \mathcal{X}) + \text{ess inf } \mathcal{X}$ and $Z_p = B[p](Z - \text{ess inf } Z) + \text{ess inf } Z$ for $p \in [0, 1]$ throughout. First, we may assume that $\mathbb{P}(Z \neq Z_p) > 0$, $p \in [0, 1]$. Otherwise, $Z = c$ for some constant $c \in \mathbb{R}$, and $V(c) = \hat{V}(c) = c$. Second, note that if R satisfies conditions (b)-(d) of Theorem 3.3, the uniqueness result holds for the pair \mathcal{X}_p, Z_p for any $p \in [0, 1]$. If Theorem 3.3(a) holds for \mathcal{X}, Z (i.e., Z and $\mathcal{X} + Z$ are comonotonic), then the same holds also for \mathcal{X}_p, Z_p . Third, note that by our assumption (and to keep R satisfying Assumption 3.1), $R(0) = -\infty$. Finally, note that for $0 \leq p < 1$:

$$\begin{aligned} \text{ess inf}[\mathcal{X}_p + Z_p] &= \text{ess inf}[B[p](\mathcal{X} + Z - \text{ess inf } \mathcal{X} - \text{ess inf } Z) + \text{ess inf } \mathcal{X} + \text{ess inf } Z] \\ &= \text{ess inf}[B[p](\mathcal{X} + Z - \text{ess inf}[\mathcal{X} + Z])] + \text{ess inf}[\mathcal{X} + Z] = \text{ess inf}[\mathcal{X} + Z]. \end{aligned}$$

Therefore, by Corollary 3.5, $Z_p \in \text{dom } V(\cdot; \mathcal{X}_p)$ for any $0 < p < 1$ since

$$\mathbb{E}[\exp(-R(\mathcal{X}_p + Z_p - \text{ess inf}[\mathcal{X}_p + Z_p]))]$$

$$\begin{aligned}
&= \mathbb{E} [\exp(-R(B[p](\mathcal{X} + Z - \text{ess inf}[\mathcal{X} + Z]) + \text{ess inf}[\mathcal{X} + Z] - \text{ess inf}[\mathcal{X} + Z]))] \\
&= \mathbb{E} [\exp(-R(B[p](\mathcal{X} + Z - \text{ess inf}[\mathcal{X} + Z])))] = \infty.
\end{aligned} \tag{E.1}$$

To complete this proof, first, for arbitrary $Z \in L^\infty$, we will prove $\lim_{p \nearrow 1} V(Z_p, \mathcal{X}_p)$ exists. Then we will utilize the definitional representation (3.4) of \bar{V} to prove that $\hat{V}(Z) = \bar{V}(Z)$ for any $Z \in L^\infty$.

We will prove $\lim_{p \nearrow 1} V(Z_p; \mathcal{X}_p)$ exists by showing that $p \in (0, 1) \mapsto V(Z_p; \mathcal{X}_p)$ is monotonic in p . Fix $p \in (0, 1)$ and $B[p]$ independent from Z, \mathcal{X} . To simplify notation, let $V := V(Z_p; \mathcal{X}_p)$ and $V' := \frac{\partial}{\partial p} V(Z_p; \mathcal{X}_p)$. Note that V is the solution to the fixed point problem $V = H_{Z_p}(V; \mathcal{X}_p)$, i.e.,

$$V = \frac{\mathbb{E}[(Z - \text{ess inf } Z) \exp(-R(\mathcal{X} + Z - V))]p}{\mathbb{E}[\exp(-R(\mathcal{X} + Z - V))]p + \exp(-R(\text{ess inf}[\mathcal{X} + Z] - V))(1 - p)} + \text{ess inf } Z.$$

Therefore, assuming the derivative V' exists, it must satisfy:

$$\begin{aligned}
&V' (\mathbb{E}[\exp(-R(\mathcal{X} + Z - V))]p + \exp(-R(\text{ess inf}[\mathcal{X} + Z] - V))(1 - p)) \\
&\quad + (V - \text{ess inf } Z) (V' \mathbb{E}[R'(\mathcal{X} + Z - V) \exp(-R(\mathcal{X} + Z - V))]p) \\
&\quad + (V - \text{ess inf } Z) (\mathbb{E}[\exp(-R(\mathcal{X} + Z - V))] - \exp(-R(\text{ess inf}[\mathcal{X} + Z] - V))) \\
&\quad + (V - \text{ess inf } Z) V' R'(\text{ess inf}[\mathcal{X} + Z] - V) \exp(-R(\text{ess inf}[\mathcal{X} + Z] - V))(1 - p) \\
&= V' \mathbb{E}[(Z - \text{ess inf } Z) R'(\mathcal{X} + Z - V) \exp(-R(\mathcal{X} + Z - V))]p + \mathbb{E}[(Z - \text{ess inf } Z) \exp(-R(\mathcal{X} + Z - V))] \\
&\Rightarrow V' \left(\mathbb{E}[(1 - (Z - V) R'(\mathcal{X} + Z - V)) \exp(-R(\mathcal{X} + Z - V))]p \right. \\
&\quad \left. + (1 + (V - \text{ess inf } Z) R'(\text{ess inf}[\mathcal{X} + Z] - V)) \exp(-R(\text{ess inf}[\mathcal{X} + Z] - V))(1 - p) \right) \\
&= \mathbb{E}[(Z - V) \exp(-R(\mathcal{X} + Z - V))] + (V - \text{ess inf } Z) \exp(-R(\text{ess inf}[\mathcal{X} + Z] - V)) \\
&\Rightarrow V' \mathbb{E}[(1 - (Z_p - V) R'(\mathcal{X}_p + Z_p - V)) \exp(-R(\mathcal{X}_p + Z_p - V))] \\
&= \mathbb{E}[(Z - \text{ess inf } Z) \exp(-R(\mathcal{X} + Z - V))] \\
&\quad + (V - \text{ess inf } Z) (\exp(-R(\text{ess inf}[\mathcal{X} + Z] - V)) - \mathbb{E}[\exp(-R(\mathcal{X} + Z - V))]).
\end{aligned}$$

Therefore V' exists if

$$\mathbb{E}[(1 - (Z_p - V) R'(\mathcal{X}_p + Z_p - V)) \exp(-R(\mathcal{X}_p + Z_p - V))] \neq 0. \tag{E.2}$$

Under conditions (c)-(d) of Theorem 3.3, (E.2) is satisfied by construction as in the proof of Theorem 3.3 as this expression is strictly positive. In cases (a)-(b) of Theorem 3.3, we wish to rewrite (E.2). Specifically, by construction of V ,

$$\begin{aligned} & \mathbb{E}[(1 - (Z_p - V)R'(\mathcal{X}_p + Z_p - V)) \exp(-R(\mathcal{X}_p + Z_p - V))] \\ &= \mathbb{E}[\exp(-R(\mathcal{X}_p + Z_p - V))] \\ & \quad + \frac{\mathbb{E}[Z_p \exp(-R(\mathcal{X}_p + Z_p - V))]\mathbb{E}[R'(\mathcal{X}_p + Z_p - V) \exp(-R(\mathcal{X}_p + Z_p - V))]}{\mathbb{E}[\exp(-R(\mathcal{X}_p + Z_p - V))]} \\ & \quad - \frac{\mathbb{E}[\exp(-R(\mathcal{X}_p + Z_p - V))]\mathbb{E}[Z_p R'(\mathcal{X}_p + Z_p - V) \exp(-R(\mathcal{X}_p + Z_p - V))]}{\mathbb{E}[\exp(-R(\mathcal{X}_p + Z_p - V))]} \end{aligned}$$

Therefore, as shown in the proof of Theorem 3.3, this expression is strictly positive.

Therefore, $V' \geq 0$ if, and only if, $\mathbb{E}[(Z - \text{ess inf } Z) \exp(-R(\mathcal{X} + Z - V))] + (V - \text{ess inf } Z)(\exp(-R(\text{ess inf }[\mathcal{X} + Z] - V)) - \mathbb{E}[\exp(-R(\mathcal{X} + Z - V))]) \geq 0$. Since $Z - \text{ess inf } Z \geq 0$ a.s.: $\mathbb{E}[(Z - \text{ess inf } Z) \exp(-R(\mathcal{X} + Z - V))] \geq 0$ trivially, $V \geq 0$ by Proposition 3.6(1), and $\exp(-R(\text{ess inf }[\mathcal{X} + Z] - V)) \geq \exp(-R(\mathcal{X} + Z - V))$ by monotonicity of R . Therefore $V' \geq 0$. In particular, this implies that $p \in (0, 1) \rightarrow V(Z_p; \mathcal{X}_p)$ is nondecreasing. Additionally, $V(Z_p; \mathcal{X}_p) \leq \text{ess inf }[\mathcal{X} + Z]$ for every $p \in (0, 1)$ by Corollary 3.4. Therefore, by application of the monotone convergence theorem, we can guarantee the existence of $\hat{V}(Z)$.

We will complete this proof by proving that this limit is equivalent to the form (3.4). Fix $Z \in L^\infty$. First, we will show $\hat{V}(Z) = \bar{V}(Z)$ if $Z \in \text{dom } V$. Second, we will consider $Z \notin \text{dom } V$.

1. First, fix $Z \in L^\infty$ such that $V(Z)$ exists. We will prove $\bar{V}(Z) = \lim_{p \nearrow 1} V(Z_p; \mathcal{X}_p)$ by showing that $p \in [0, 1] \mapsto V(Z_p; \mathcal{X}_p)$ is continuous. By Corollary 3.5 and choice of Z , $V(Z_p; \mathcal{X}_p)$ exists for any $p \in [0, 1]$, as we have shown in (E.1), and $V(Z; \mathcal{X})$ exists by our assumption. By construction of \bar{V} , we have that $\bar{V}(Z_p; \mathcal{X}_p) = V(Z_p; \mathcal{X}_p)$, $p \in [0, 1]$. By Lemma 3.6(1), Corollary 3.4 and properties of the pricing function under $\mathbb{D} := \mathbb{R}_{++}$, $V(Z_p; \mathcal{X}_p) \in [\text{ess inf } Z, \min\{\text{ess inf }[\mathcal{X} + Z], \text{ess sup } Z\}]$ for any $p \in [0, 1]$. For any probability $p \in [0, 1]$:

$$V(Z_p; \mathcal{X}_p) = \text{FIX}_v \{ \bar{H}(p, v) \}$$

$$\bar{H}(p, v) := \frac{\mathbb{E}[Z \exp(-R(\mathcal{X} + Z - v))]p + \text{ess inf } Z \exp(-R(\text{ess inf}[\mathcal{X} + Z] - v))}{\mathbb{E}[\exp(-R(\mathcal{X} + Z - v))]p + \exp(-R(\text{ess inf}[\mathcal{X} + Z] - v))(1 - p)} \wedge \text{ess inf}[\mathcal{X} + Z].$$

As $\bar{H} : [0, 1] \times [\text{ess inf } Z, \min\{\text{ess inf}[\mathcal{X} + Z], \text{ess sup } Z\}] \rightarrow [\text{ess inf } Z, \min\{\text{ess inf}[\mathcal{X} + Z], \text{ess sup } Z\}]$ is jointly continuous and $V(Z_p; \mathcal{X}_p)$ is unique for every $p \in [0, 1]$, the fixed point mapping is continuous (see, e.g. [6, Lemma C.1]), i.e., $p \in [0, 1] \rightarrow V(Z_p; \mathcal{X}_p)$ is continuous. The result now follows as $\hat{V}(Z; \mathcal{X}) = \lim_{p \nearrow 1} V(Z_p; \mathcal{X}_p) = V(Z; \mathcal{X})$.

2. Fix $Z \in L^\infty$ such that $Z \notin \text{dom } V$. Consider first the case when $\text{ess inf}[\mathcal{X} + Z] \notin \text{dom } H_Z$. Then we must have that $\liminf_{v \nearrow \text{ess inf}[\mathcal{X} + Z]} H_Z(v) \geq \text{ess inf}[\mathcal{X} + Z]$. Because R is increasing, it also follows that $\mathbb{E}[(Z - \text{ess inf}[\mathcal{X} + Z]) \exp(-R(\mathcal{X} + Z - v))] \geq 0$, for all $\text{ess inf } Z \leq v < \text{ess inf}[\mathcal{X} + Z]$. Therefore, $\mathbb{E}[(Z - v) \exp(-R(\mathcal{X} + Z - v))] > 0$, $\text{ess inf } Z \leq v < \text{ess inf}[\mathcal{X} + Z]$. This also follows in case $\text{ess inf}[\mathcal{X} + Z] \in \text{dom } H_Z$ (in which case we must have $H_Z(\text{ess inf}[\mathcal{X} + Z]) > \text{ess inf}[\mathcal{X} + Z]$, and thus $\mathbb{E}[(Z - v) \exp(-R(\mathcal{X} + Z - v))] > 0$, $\text{ess inf } Z \leq v < \text{ess inf}[\mathcal{X} + Z]$, as otherwise we would reach a contradiction to $Z \notin \text{dom } V$). For any $\epsilon \in (0, \frac{\text{ess inf } \mathcal{X}}{2})$, define

$$p_\epsilon := \frac{(\text{ess inf } \mathcal{X} - \epsilon) \exp(-R(\epsilon))}{(\text{ess inf } \mathcal{X} - \epsilon) \exp(-R(\epsilon)) + \mathbb{E}[(Z - \text{ess inf}[\mathcal{X} + Z] + \epsilon) \exp(-R(\mathcal{X} + Z - \text{ess inf}[\mathcal{X} + Z] + \epsilon))]} \in (0, 1)$$

so that $V(Z_{p_\epsilon}; \mathcal{X}_{p_\epsilon}) = \text{ess inf}[\mathcal{X} + Z] - \epsilon$. Using the fact that $V' \geq 0$, $V(Z_p; \mathcal{X}_p) \geq \text{ess inf}[\mathcal{X} + Z] - \epsilon$ for any $p > p_\epsilon$. Therefore, $p_{\epsilon_2} > p_{\epsilon_1}$ for $\epsilon_2 < \epsilon_1$, and $\lim_{\epsilon \searrow 0} p_\epsilon$ exists and is bounded from above by 1. If $\lim_{\epsilon \searrow 0} p_\epsilon = 1$, then we conclude:

$$\lim_{p \nearrow 1} V(Z_p; \mathcal{X}_p) = \lim_{\epsilon \searrow 0} V(Z_{p_\epsilon}; \mathcal{X}_{p_\epsilon}) = \lim_{\epsilon \searrow 0} (\text{ess inf}[\mathcal{X} + Z] - \epsilon) = \text{ess inf}[\mathcal{X} + Z].$$

Otherwise, if $p_* := \lim_{\epsilon \searrow 0} p_\epsilon < 1$. Then, by the above construction, $V(Z_p; \mathcal{X}_p) = \text{ess inf}[\mathcal{X} + Z]$ for every $p > p_*$ and, as a direct consequence, $\lim_{p \nearrow 1} V(Z_p; \mathcal{X}_p) = \text{ess inf}[\mathcal{X} + Z]$ as well.

□

F Proofs for Section 3.3

F.1 Proof of Lemma 3.9

Proof. 1. First we will show that $f^q(s) \in \mathbb{R}$ for every $s \in \mathbb{R}_+$. This is trivially true if $sq \notin \text{dom } V$. Assume $sq \in \text{dom } V$; $f^q(s)$ exists if and only if the denominator $(\mathbb{E}[(1 - [sq - \bar{V}(sq)]R'(\mathcal{X} + sq - \bar{V}(sq))) \exp(-R(\mathcal{X} + sq - \bar{V}(sq)))]])$ is nonzero. In fact, we will demonstrate that this denominator is strictly positive. Note also that $\bar{V}(sq) = V(sq)$ by Theorem 3.3. As demonstrated in the proof of Corollary 3.7, this denominator is strictly positive under any of the conditions of Theorem 3.3.

Second assume $s \in \text{int}\{s \in \mathbb{R}_+ \mid sq \in \text{dom } V\}$. We now wish to consider $\frac{\partial}{\partial s} V(sq)$. For simplicity of notation, let $V := V(sq)$, $V' := \frac{\partial}{\partial s} V(sq)$, $R := R(\mathcal{X} + sq - V)$, and $R' := R'(\mathcal{X} + sq - V)$:

$$\begin{aligned} V' \mathbb{E}[\exp(-R)] - V \mathbb{E}[(q - V')R' \exp(-R)] &= \mathbb{E}[q \exp(-R)] - \mathbb{E}[sq(q - V')R' \exp(-R)] \\ \Rightarrow V' \mathbb{E}[(1 - [sq - V]R') \exp(-R)] &= \mathbb{E}[q(1 - [sq - V]R') \exp(-R)] \\ \Rightarrow V' &= \frac{\mathbb{E}[q(1 - [sq - V]R') \exp(-R)]}{\mathbb{E}[(1 - [sq - V]R') \exp(-R)]}. \end{aligned}$$

That is, $V' = f^q(s)$ in this case.

Third assume $s \in \text{int}\{s \in \mathbb{R}_+ \mid sq \notin \text{dom } V\}$. By construction, $\bar{V}(sq) = \text{ess inf}[\mathcal{X} + sq]$. Immediately this implies $\frac{\partial}{\partial s} \bar{V}(sq) = f^q(s)$ in this case by Assumption 3.8.

Finally, by continuity of $s \mapsto \bar{V}(sq)$ (see Lemma 3.6(4)) and the fact that $\bar{V}(0) = 0$ the result follows.

2. The assumption implies, for $sq \in \text{dom } V$, $(1 - [sq - V(sq)]R'(\mathcal{X} + sq - V(sq))) \exp(-R(\mathcal{X} + sq - V(sq))) \geq 0$ a.s. (resp. strictly positive if strict monotonicity). Therefore $f^q(s) \geq \text{ess inf } q$ for every $s \in \mathbb{R}_+$ trivially and if $sq \in \text{dom } V$ with $\mathbb{P}(q > \text{ess inf } q) > 0$ then $f^q(s) > \text{ess inf } q$.
3. The result follows trivially by continuity of R' and \bar{V} .
4. By the relation in Property (1), if $s \mapsto \bar{V}(sq)$ is concave then f^q is nonincreasing. Therefore the relation holds by Lemma 3.6(6).

□

F.2 Proof of Lemma 3.10

Proof. 1. $\bar{f}^q(0) = \mathbb{E}[q \exp(-R(\mathcal{X}))]/\mathbb{E}[\exp(-R(\mathcal{X}))] \geq \text{ess inf } q$ and, for $s > 0$, $\bar{f}^q(s) = \frac{\bar{V}(sq)}{s} \geq \text{ess inf } q$ by Lemma 3.6(1). Now consider $\mathbb{P}(q > \text{ess inf } q) > 0$:

(a) If $sq \in \text{dom } V$ then it follows that $\bar{f}^q(s) = \frac{\mathbb{E}[q \exp(-R(\mathcal{X} + sq - V(sq)))]}{\mathbb{E}[\exp(-R(\mathcal{X} + sq - V(sq)))]} > \text{ess inf } q$.

(b) If $sq \notin \text{dom } V$ then, by Theorem 3.2, it must be that $\mathbb{D} = \mathbb{R}_{++}$ and, in particular, $\text{ess inf } \mathcal{X} \in \mathbb{R}_{++}$. Therefore, $\bar{f}^q(s) = \frac{\text{ess inf}[\mathcal{X} + sq]}{s} = \frac{\text{ess inf } \mathcal{X}}{s} + \text{ess inf } q > \text{ess inf } q$.

2. Recall, $\bar{f}^q(s) = \bar{V}(sq)/s$ for $s > 0$ and $\bar{f}^q(0) = \mathbb{E}[q \exp(-R(\mathcal{X}))]/\mathbb{E}[\exp(-R(\mathcal{X}))]$. By continuity of \bar{V} (see Lemma 3.6(4)), continuity of the inverse demand function holds so long as $\lim_{s \rightarrow 0} \bar{f}^q(s) = \mathbb{E}[q \exp(-R(\mathcal{X}))]/\mathbb{E}[\exp(-R(\mathcal{X}))]$. We will consider two cases: $\mathbb{D} = \mathbb{R}$ and $\mathbb{D} = \mathbb{R}_{++}$.

(a) Let $\mathbb{D} = \mathbb{R}$. Then $\bar{V} = V$ by construction as $\text{dom } V = L^\infty$. Note that, here, $\bar{f}^q(s) = V(sq)/s = \mathbb{E}[q \exp(-R(\mathcal{X} + sq - V(sq)))]/\mathbb{E}[\exp(-R(\mathcal{X} + sq - V(sq)))]$ for every $s > 0$. Noting $V(0) = 0$ implies $\lim_{s \searrow 0} \bar{f}^q(s) = \mathbb{E}[q \exp(-R(\mathcal{X}))]/\mathbb{E}[\exp(-R(\mathcal{X}))]$ and the result is proven.

(b) Let $\mathbb{D} = \mathbb{R}_{++}$. First we want to consider a small remark on the domain of V ; if $Z \in L^\infty$ such that $\|Z - \text{ess inf } Z\|_\infty < \text{ess inf } \mathcal{X}/2$ then $Z \in \text{dom } V$ since $H_Z(v) \leq \text{ess sup } Z < \text{ess inf } \mathcal{X}/2 + \text{ess inf } Z < \text{ess inf}[\mathcal{X} + Z]$ for any $v \in [\text{ess inf } Z, \text{ess inf}[\mathcal{X} + Z]]$.

If q is deterministic, then by Lemma 3.6(1) $\bar{V}(sq) = sq$, and therefore $\bar{f}^q(s) = q$. Otherwise, if $\text{ess sup } q - \text{ess inf } q > 0$, we have that $\bar{f}^q(s) = \frac{\bar{V}(sq)}{s} = \frac{V(s(q - \text{ess inf } q)) + s \text{ess inf } q}{s}$ for $s < \text{ess inf } \mathcal{X}/[2(\text{ess sup } q - \text{ess inf } q)]$. As with the prior case, for this small s ,

$$\bar{f}^q(s) = \frac{\mathbb{E}[(q - \text{ess inf } q) \exp(-R(\mathcal{X} + s[q - \text{ess inf } q] - V(s[q - \text{ess inf } q])))]}{\mathbb{E}[\exp(-R(\mathcal{X} + s[q - \text{ess inf } q] - V(s[q - \text{ess inf } q])))]} + \text{ess inf } q.$$

Again, noting $V(0) = 0$ then trivially $\lim_{s \searrow 0} \bar{f}^q(s) = \mathbb{E}[q \exp(-R(\mathcal{X}))]/\mathbb{E}[\exp(-R(\mathcal{X}))]$.

3. Note that $\bar{f}^q(s) = \sup\{p \in \mathbb{R}_+ \mid \mathbb{E}[(q - \text{ess inf } q - p) \exp(-R(\mathcal{X} + s(q - \text{ess inf } q - p)))] \geq 0, \text{ess inf } \mathcal{X} - sp \in \mathbb{D}\} + \text{ess inf } q$ by using the optimization representation of \bar{V} provided in

Proposition E.1. By construction, $\bar{f}^q(s) = \bar{f}^{q-\text{ess inf } q}(s) + \text{ess inf } q$. Therefore, monotonicity holds in general if it is true for every random variable q such that $\text{ess inf } q = 0$. Assume $\text{ess inf } q = 0$ and let $s_1 \geq s_2$. Fix $p \in [0, \bar{f}^q(s_1)]$. Immediately $\text{ess inf } \mathcal{X} - s_1 p \leq \text{ess inf } \mathcal{X} - s_2 p$; by $\text{ess inf } \mathcal{X} - s_1 p \in \text{cl } \mathbb{D}$ the same must be true for $\text{ess inf } \mathcal{X} - s_2 p$. Therefore monotonicity follows if $\mathbb{E}[(q-p) \exp(-R(\mathcal{X} + s_2(q-p)))] \geq \mathbb{E}[(q-p) \exp(-R(\mathcal{X} + s_1(q-p)))]$. This holds because

$$\frac{\partial}{\partial s} \mathbb{E}[(q-p) \exp(-R(\mathcal{X} + s(q-p)))] = -\mathbb{E}[(q-p)^2 R'(\mathcal{X} + s(q-p)) \exp(-R(\mathcal{X} + s(q-p)))] \leq 0.$$

□

G Proofs for Section 4

G.1 Proof of Proposition 4.2

Proof. Recall that $\alpha = \left(\sum_{i=1}^n \frac{1}{\alpha_i}\right)^{-1}$. Let $\alpha^* := \left(\sum_{i=1}^{n+1} \frac{1}{\alpha_i}\right)^{-1} = \left(\frac{1}{\alpha} + \frac{1}{\alpha_{n+1}}\right)^{-1} < \alpha$. Consider now $\frac{\partial}{\partial \alpha} V(Z; \alpha)$. We will show that $\frac{\partial}{\partial \alpha} V(Z; \alpha) \leq 0$ and therefore, it follows that $V(Z; \alpha) \leq V(Z; \alpha^*)$ for every $Z \in L^\infty$.

$$\begin{aligned} \frac{\partial}{\partial \alpha} V(Z; \alpha) &= \frac{\mathbb{E}[\exp(-\alpha Z)] \frac{\partial}{\partial \alpha} \mathbb{E}[Z \exp(-\alpha Z)] - \mathbb{E}[Z \exp(-\alpha Z)] \frac{\partial}{\partial \alpha} \mathbb{E}[\exp(-\alpha Z)]}{\mathbb{E}[\exp(-\alpha Z)]^2} \\ &= \frac{\mathbb{E}[\exp(-\alpha Z)] \mathbb{E}[-Z^2 \exp(-\alpha Z)] - \mathbb{E}[Z \exp(-\alpha Z)] \mathbb{E}[-Z \exp(-\alpha Z)]}{\mathbb{E}[\exp(-\alpha Z)]^2}. \end{aligned}$$

Let $x = \alpha Z$. Then $\frac{\partial}{\partial \alpha} V(Z; \alpha) \leq 0$ if, and only if, $\mathbb{E}[\exp(-x)] \mathbb{E}[x^2 \exp(-x)] - \mathbb{E}[x \exp(-x)]^2 \geq 0$.

$$\begin{aligned} &\mathbb{E}[\exp(-x)] \mathbb{E}[x^2 \exp(-x)] - \mathbb{E}[x \exp(-x)]^2 \\ &= \mathbb{E}[\exp(-x)] \mathbb{E}[x^2 \exp(-x)] - \mathbb{E}\left[\exp\left(-\frac{x}{2}\right) x \exp\left(-\frac{x}{2}\right)\right]^2 \\ &\geq \mathbb{E}[\exp(-x)] \mathbb{E}[x^2 \exp(-x)] - \mathbb{E}\left[\exp\left(-\frac{x}{2}\right)\right]^2 \mathbb{E}\left[\left(x \exp\left(-\frac{x}{2}\right)\right)^2\right] \\ &= \mathbb{E}[\exp(-x)] \mathbb{E}[x^2 \exp(-x)] - \mathbb{E}[\exp(-x)] \mathbb{E}[x^2 \exp(-x)] = 0, \end{aligned}$$

where the inequality above follows from the Cauchy-Schwartz inequality. □

G.2 Proof of Proposition 4.5

- Proof.* 1. By Theorem 3.3, $Z \in \text{dom } V$ if and only if $H_Z(\mathcal{X} + \text{ess inf } Z) \leq \mathcal{X} + \text{ess inf } Z$ provided $\mathcal{X} + \text{ess inf } Z \in \text{dom } H_Z$ or $\liminf_{v \nearrow \mathcal{X} + \text{ess inf } Z} H_Z(v) < \mathcal{X} + \text{ess inf } Z$ if $\mathcal{X} + \text{ess inf } Z \notin \text{dom } H_Z$. First assume $\mathcal{X} + \text{ess inf } Z \in \text{dom } H_Z$, then $H_Z(\mathcal{X} + \text{ess inf } Z) \leq \mathcal{X} + \text{ess inf } Z$ if and only if $\mathbb{E}[Z(Z - \text{ess inf } Z)^{-\eta}] \leq (\mathcal{X} + \text{ess inf } Z)\mathbb{E}[(Z - \text{ess inf } Z)^{-\eta}]$. Rearranging terms completes the proof. Now assume $\mathcal{X} + \text{ess inf } Z \notin \text{dom } H_Z$. This can only happen if $\mathbb{E}[(Z - \text{ess inf } Z)^{-\eta}] = \infty$. However, by $\eta \leq 1$, $\mathbb{E}[(Z - \text{ess inf } Z)^{1-\eta}] \leq \|Z - \text{ess inf } Z\|_\infty^{1-\eta} < \infty$. Therefore $\liminf_{v \nearrow \mathcal{X} + \text{ess inf } Z} H_Z(v) = \text{ess inf } Z < \mathcal{X} + \text{ess inf } Z$ and, thus, $Z \in \text{dom } V$. Notably, in this case $\mathbb{E}[(Z - \text{ess inf } Z)^{1-\eta}] \leq \mathcal{X}\mathbb{E}[(Z - \text{ess inf } Z)^{-\eta}]$ as well.
2. This follows immediately by Lemma 3.6 as $z \exp(-R(\mathcal{X} + z)) = z(\mathcal{X} + z)^{-\eta}$.
3. This follows by the same logic as Lemma 3.6 because $z \exp(-R(\mathcal{X} + z)) = z(\mathcal{X} + z)^{-\eta}$.

□

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