Electronic Companion

A An Equivalence of Ex post IR Notions

We describe a mechanism in Section 2 by identifying the marginal probability a_i of allocation of products and the expected transfer t, without discussing how the mechanism possibly correlates these decisions. Since the buyer maximizes her expected utility, the correlation is irrelevant for incentive constraints. However, to make sure that the ex post utility of the buyer is non-negative even after the random choices of the mechanism, we here explicitly discuss a way to correlate the allocation of the products and the transfer, assuming that the mechanism satisfies the ex post IR constraint. Since the correlation does not affect incentive constraints, we here fix an ex post type v.

Thus fix v and let $u = v \cdot a - t \geq 0$ be the expected utility (over the random choices of the mechanism). Let $n \leq k$ be the number of products i with positive probability of allocation $a_i > 0$. Assume $n \geq 1$, since otherwise if $n = 0$ the allocation is zero and no randomization is required. Allocate each product i independently with probability a_i. For each product i with $a_i > 0$, the buyer transfers $v_i - u/(a_i n)$ if she receives the product, and zero otherwise. Note that the expected total transfer is indeed t, since $\sum_{i:a_i>0}(v_i - u/(a_i n))a_i = v \cdot a - u = t$. Note also that the ex post utility (after the randomization of the mechanism) is non-negative, since for each product i that the buyer receives, she pays $v_i - u/(a_i n)$ which is lower than the value of the product.

B Proofs from Section 3

B.1 Proof of Proposition 2

Proof. By Proposition 1 a mechanism is optimal if and only if its induced separable mechanism is optimal among all separable mechanisms. To establish the result, we show that a separable mechanism is optimal among all separable mechanisms if and only if it is a promised utility mechanism. We will therefore restrict to separable mechanisms for the rest of the proof.

We start by defining an alternative representation of a mechanism via a change of variables. Consider a mechanism (a, t) with utility function $u(v) = \left(\sum_{i \leq k} v_i a_i(v_1, \ldots, v_k)\right) - t(v)$. For each

i and $v_1, \ldots, v_i$, define

$$\tilde{t}_i(v_1, \ldots, v_i) = v_i a_i(v_1, \ldots, v_i) - \mathbf{E}_{v_{i+1}, \ldots, v_k}\left[u(v)\right], \tag{10}$$

(where for $i = k$ we define $\mathbf{E}_{v_{i+1}, \ldots, v_k}[u(v)] := u(v)$). Notice that $(a, \tilde{t})$ must satisfy a martingale property, which is that for all $i < k$,

$$\begin{aligned} v_i a_i(v_1, \ldots, v_i) - \tilde{t}_i(v_1, \ldots, v_i) &= \mathbf{E}_{v_{i+1}, \ldots, v_k}\left[u(v)\right] \\ &= \mathbf{E}_{v_{i+1}}\left[\mathbf{E}_{v_{i+2}, \ldots, v_k}\left[u(v)\right]\right] \\ &= \mathbf{E}_{v_{i+1}}\left[v_{i+1} a_{i+1}(v_1, \ldots, v_{i+1}) - \tilde{t}_{i+1}(v_1, \ldots, v_{i+1})\right]. \end{aligned} \tag{11}$$

Conversely, for any $(a, \tilde{t})$ satisfying the martingale property, we can construct a mechanism (a, t) satisfying (10). In particular, let

$$t(v) = \left(\sum_{i<k} v_i a_i(v_1, \ldots, v_i)\right) + \tilde{t}_k(v). \tag{12}$$

This definition implies that

$$\begin{aligned} u(v) &= \left(\sum_{i} v_i a_i(v_1, \ldots, v_i)\right) - t(v) \\ &= \left(\sum_{i} v_i a_i(v_1, \ldots, v_i)\right) - \left(\sum_{i<k} v_i a_i(v_1, \ldots, v_i)\right) - \tilde{t}_k(v) \\ &= v_k a_k(v) - \tilde{t}_k(v). \end{aligned}$$

Then for each i and $v_1, \ldots, v_i$, iterative application of the martingale property (11) implies that

$$\mathbf{E}_{v_{i+1}, \ldots, v_k}\left[u(v)\right] = \mathbf{E}_{v_{i+1}, \ldots, v_k}\left[v_k a_k(v) - \tilde{t}_k(v)\right] = v_i a_i(v_1, \ldots, v_i) - \tilde{t}_i(v_1, \ldots, v_i), \tag{13}$$

and therefore the mechanism (a, t) satisfies (10). As a result, we can use (10) to uniquely represent a mechanism (a, t) with $(a, \tilde{t})$ satisfying the martingale property (11). We will henceforth refer to $(a, \tilde{t})$ satisfying the martingale property (11) as a mechanism.

We next reformulate the problem with the new notation. Using (12), the revenue of a mechanism $(a, \tilde{t})$ is

$$\mathbf{E}_{v_1,\ldots,v_k}\left[t(v)\right] = \mathbf{E}_{v_1,\ldots,v_k}\left[\left(\sum_{i<k} v_i a_i(v_1,\ldots,v_i)\right) + \tilde{t}_k(v)\right]. \tag{14}$$

Recall that the PIC constraint (5) requires that for each i and $v_1, \ldots, v_i$, $\hat{v}_i = v_i$ maximizes the following expression over all $\hat{v}_i$,

$$\begin{aligned}
&v_i a_i(v_1,\ldots,\hat{v}_i) + CU_i(v_1,\ldots,\hat{v}_i)\\
=&\hat{v}_i a_i(v_1,\ldots,\hat{v}_i) + CU_i(v_1,\ldots,\hat{v}_i) + \sum_{j<i} v_j a_j(v_1,\ldots,v_j) + \Big((v_i-\hat{v}_i)a_i(v_1,\ldots,\hat{v}_i) - \sum_{j<i} v_j a_j(v_1,\ldots,v_j)\Big)\\
=&\,\mathbf{E}_{v_{i+1},\ldots,v_k}\left[u(v_{-i},\hat{v}_i)\right] + \Big((v_i-\hat{v}_i)a_i(v_1,\ldots,\hat{v}_i) - \sum_{j<i} v_j a_j(v_1,\ldots,v_j)\Big),
\end{aligned}$$

(where v_{-i} denotes a vector of values for products other than i). Using the definition of $\tilde{t}$ in (10), this expression becomes

$$\begin{aligned}
=&\hat{v}_i a_i(v_1,\ldots,\hat{v}_i) - \tilde{t}_i(v_1,\ldots,\hat{v}_i) + \Big((v_i-\hat{v}_i)a_i(v_1,\ldots,\hat{v}_i) - \sum_{j<i} v_j a_j(v_1,\ldots,v_j)\Big)\\
=&v_i a_i(v_1,\ldots,\hat{v}_i) - \tilde{t}_i(v_1,\ldots,\hat{v}_i) - \sum_{j<i} v_j a_j(v_1,\ldots,v_j).
\end{aligned}$$

Notice that the last term does not depend on $\hat{v}_i$. Therefore, the PIC constraint holds if and only if for all $\hat{v}_i$,

$$v_i a_i(v_1,\ldots,v_i) - \tilde{t}_i(v_1,\ldots,v_i) \geq v_i a_i(v_1,\ldots,\hat{v}_i) - \tilde{t}_i(v_1,\ldots,\hat{v}_i). \tag{15}$$

The ex post IR constraint is,

$$u(v) = v_k a_k(v_1,\ldots,v_k) - \tilde{t}_k(v_1,\ldots,v_k) \geq 0.$$

Notice that the ex post IR constraint together with (13) implies that for all i and $v_1, \ldots, v_i$,

$$v_i a_i(v_1, \ldots, v_i) - \tilde{t}_i(v_1, \ldots, v_i) \geq 0. \tag{16}$$

Therefore without loss of generality we can replace the ex post IR constraint with its generalized form (16). To summarize, the problem is to maximize revenue (14) over all mechanisms $(a, \tilde{t})$ subject to the martingale property (11), PIC (15), and (generalized) ex post IR (16).

We now characterize optimal solutions to this problem recursively. Consider the sub-problem of designing a mechanism $(a, \tilde{t})$ for only selling products i to n. Such a mechanism maps $v_i, \ldots, v_j$ in each period $j \geq i$ to an allocation and a transfer for that period. Let $CR_i(\mathsf{EU})$ be the optimal revenue of this problem subject to an extra constraint that

$$\mathsf{EU} = \mathbf{E}_{v_i}\left[v_i a_i(v_i) - \tilde{t}_i(v_i)\right].$$

Now consider the optimal solution $(a, \tilde{t})$ to the problem for selling product 1 to n. Consider any history $v_1, \ldots, v_i$, and let

$$\mathsf{PU}_{i+1} = v_i a_i(v_1, \ldots, v_i) - \tilde{t}_i(v_i).$$

The continuation revenue of the optimal mechanism following this history,

$$\mathbf{E}_{v_{i+1},\ldots,v_k}\left[\left(\sum_{i<j<k} v_j a_j(v_1, \ldots, v_j)\right) + \tilde{t}_k(v)\right]$$

must be $CR_{i+1}(\mathsf{PU}_{i+1})$. To see this, notice that this continuation revenue cannot be higher than $CR_{i+1}(\mathsf{PU}_{i+1})$ because the martingale constraint (11) requires that this continuation mechanism must satisfy the constraint that

$$\mathsf{PU}_{i+1} = \mathbf{E}_{v_{i+1}}\left[v_{i+1} a_{i+1}(v_1, \ldots, v_{i+1}) - \tilde{t}_{i+1}(v_1, \ldots, v_{i+1})\right].$$

And if this continuation revenue is lower than $CR_{i+1}(\mathsf{PU}_{i+1})$, then we improve revenue by replacing the continuation mechanism with one that obtains continuation revenue $CR_{i+1}(\mathsf{PU}_{i+1})$. This property implies the recursive characterization of the continuation revenue function as specified in Definition 2. Further, the initial promised utility PU_1 must satisfy

$$CR_1(\mathsf{PU}_1) = \max_{\mathsf{PU}'_1} CR_1(\mathsf{PU}'_1).$$

This is because by definition we must have $CR_1(\mathsf{PU}_1) \leq \max_{\mathsf{PU}'_1} CR_1(\mathsf{PU}'_1)$, and if this inequality holds strictly, then we can replace the mechanism with another one that obtains a higher revenue.

To summarize, any optimal mechanism $(a, \tilde{t})$ is parameterized by solutions $(a_1, t_1), \ldots, (a_k, t_k)$ to the continuation revenue problem as follows. Set the initial promised utility PU_1 equal to any maximizer of $CR_1(\mathsf{PU}'_1)$. At each period i, given the current promised utility PU_i and report v_i,

- the buyer gets product i with probability $a_i^{\mathsf{PU}_i}(v_i)$,
- the buyer pays $v_i a_i^{\mathsf{PU}_i}(v_i)$,
- the promised utility is updated by setting $\mathsf{PU}_{i+1} := v_i a_i^{\mathsf{PU}_i}(v_i) - t_i^{\mathsf{PU}_i}(v_i)$.

At the end of the last period, the buyer additionally pays $-\mathsf{PU}_{k+1}$. This ensures that the payment in the last period is $t_k^{\mathsf{PU}_k}(v_k)$, as specified in the objective (14).

To complete the proof, we transform the mechanism $(a, \tilde{t})$ discussed above to a mechanism (a, t) via the conversion (12). The allocation remains the same. Notice that the transfer in the last period $\tilde{t}_k(v)$ is the surplus $v_k a_k(v)$ minus the promised utility at the end of the last period PU_{k+1}. Therefore, (12) implies that

$$t(v) = \left(\sum_{i<k} v_i a_i(v_1, \ldots, v_i)\right) + \tilde{t}_k(v) = \left(\sum_{i\leq k} v_i a_i(v_1, \ldots, v_i)\right) - \mathsf{PU}_{k+1},$$

which is exactly the transfer T in the promised utility mechanism. ∎

C Proofs from Section 4

C.1 Proof of Proposition 3

Proof. Suppose first that the two conditions of the proposition hold. We show that any static mechanism is sub-optimal.

Assume for contradiction that a static mechanism is optimal. We first show that there must exist an optimal static mechanism (a^{ST}, t^{ST}) that further satisfies the following property: If changing v_1 does not affect the allocation of product 1, then it also does not affect the allocation of product 2. That is,

$$\text{if } a_1^{ST}(v_1, v_2) = a_1^{ST}(v_1', v_2) \text{ then } a_2^{ST}(v_1, v_2) = a_2^{ST}(v_1', v_2). \tag{17}$$

To see this, assume that $a_1(v_1, v_2) = a_1(v_1', v_2)$ and consider the incentive constraint of type $v = (v_1, v_2)$ for reporting (v_1', v_2),

$$v \cdot a(v) - t(v) \geq v \cdot a(v_1', v_2) - t(v_1', v_2).$$

Since $a_1(v_1, v_2) = a_1(v_1', v_2)$, the above inequality simplifies to

$$v_2 a_2(v) - t(v) \geq v_2 a_2(v_1', v_2) - t(v_1', v_2).$$

Similarly, the incentive constraint of type (v_1', v_2) for reporting v is

$$v_2 a_2(v_1', v_2) - t(v_1', v_2) \geq v_2 a_2(v) - t(v).$$

Thus, both incentive constraints must hold with equality. That is, each type is indifferent between her own allocation and the allocation of the other type. Therefore, we can assign both types the allocation and transfer of the type that pays more without violating incentive compatibility or decreasing revenue.

By Proposition 1, the induced separable mechanism of the static mechanism (a^{ST}, t^{ST}) is optimal. By Proposition 4, the induced separable mechanism (a^{ISP}, t^{ISP}) must be equal to the fifth mechanism identified in Proposition 4, that is, $a_1^{ISP}(v_1) = 1$ for all v_1, $a_2^{ISP}(v) = 1$ if $v \neq (\underline{v}, \underline{v})$, and $a_2^{ISP}(\underline{v}, \underline{v}) = 0$. Therefore, the allocation rule of the static mechanism must be as shown below.

v_1	v_2	a_1	a_2
$\underline{v}$	$\underline{v}$	1	0
$\underline{v}$	$\overline{v}$	1	1
$\overline{v}$	$\underline{v}$	1	1
$\overline{v}$	$\overline{v}$	1	1

Thus the property in (17) is violated since $a_1^{ST}(\underline{v},\underline{v}) = a_1^{ST}(\overline{v},\underline{v})$ but $a_2^{ST}(\underline{v},\underline{v}) \neq a_2^{ST}(\overline{v},\underline{v})$.

Now assume that all static mechanisms are sub-optimal. Sub-optimality of all static mechanisms imply that in particular, the four static mechanisms of Proposition 4 must be sub-optimal. By Proposition 4, the two conditions of the proposition must hold. $\square$

C.2 Proof of Proposition 4

Before proving Proposition 4, we first prove three auxiliary results. The first result is a characterization of incentive compatibility for the continuation revenue problem (Definition 2). The second result is a characterization of the solution to the continuation revenue problem in the last period. The third result is a partial characterization of the the solution to the continuation revenue problem in the first period. We state these results in full generality (for any number of values) and use them also in the proof of Proposition 5.

C.2.1 A Characterization of Incentive Compatibility

We use a characterization of incentive compatible mechanisms that consists of two parts, formalized in the lemma below. The first part is standard. Namely, for the incentive constraint (7) to be satisfied, the allocation rule a must be monotone non-decreasing. Second, in an optimal mechanism, the transfer rule minimizes the utility to the buyer, which is the area under a. Even though this part parallels the standard characterization, it involves a subtlety. Without the promise-keeping constraint, and if the goal were to maximize the expected transfer t for a given allocation rule, it is clearly optimal to minimize the utility to the buyer. However, in the continuation revenue problem, simply lowering the buyer's utility may cause two complications. First, the promise-keeping constraint may be violated. Second, the continuation revenue may decrease (the continuation revenue function CR is not necessarily monotone). Nevertheless, we show that if the transfer rule t does not minimize the utility of the buyer given an allocation rule a, there exists *another* mechanism that is also feasible and obtains higher revenue than (a, t).

The lemma requires some notation. Throughout this section we drop the index i. For $v \in V$, let $\Delta a(v) = a(v) - a(\max_{v'<v} v')$ denote the change in a at v, where $\Delta a(\underline{v}) = a(\underline{v})$ for the smallest value $\underline{v}$ in V.

Lemma 2. *There exists t such that the mechanism (a, t) satisfies incentive constraints* (7) *if and only if a is monotone non-decreasing. If a mechanism (a, t) is an optimal solution to the continuation revenue problem (Definition 2), then $t(v) = va(v) - \left(\sum_{v' \leq v}(v - v')\Delta a(v')\right) - u(\underline{v})$.*

Proof. Necessity of monotonicity of a and the fact that (a, t) for t defined in the lemma satisfies all incentive constraints is standard. We only verify the optimality of t.

In our model with a finite set of values, the allocation probabilities of types in V does not pin down the transfers. That is, for a given allocation rule a, there may be multiple transfer rules t such that the mechanism (a, t) is incentive compatible. Nevertheless, the transfer rule is pinned down (up to a constant which is the utility of the lowest type) once the mechanism is extended to specify the allocation and transfer of all values in $[\underline{v}, \overline{v}]$, where $\underline{v}$ and $\overline{v}$ are the lowest and highest values in V. For instance, if $V = \{\underline{v}, \overline{v}\}$, then the allocation rule a where $a(\underline{v}) = 0$ and $a(\overline{v}) = 1$ can be implemented by offering the product at any price p between $\underline{v}$ and $\overline{v}$. The extended allocation rule is $a(v) = 0$ if $v < p$ and $a(v) = 1$ if $v \geq p$. The transfer rule that makes the extended allocation rule incentive compatible is $t(v) = 0$ if $v < p$ and $t(v) = p$ if $v \geq p$.

Thus assume without loss of generality that (a, t) is defined over $[\underline{v}, \overline{v}]$. For such a mechanism, the transfer rule is defined from the allocation rule as follows,

$$t(v) = va(v) - (\int_{\underline{v}}^{v} a(z)dz) - u(\underline{v}).$$

We show that for (a, t) to be optimal, a must be constant on $[v', v)$ for any adjacent pair of values $v' < v \in V$. Assume for contradiction that this is not the case. See Figure 6.

Construct a' so that (1) it is constant $[v', v)$, (2) $a'(v) = a(v)$, and (3) the areas between v and

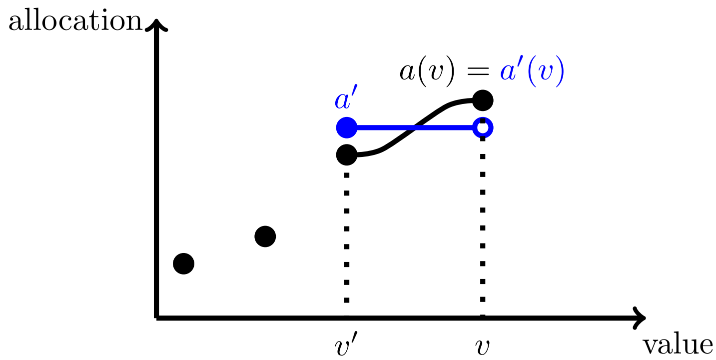

Figure 6: The construction of a' in the proof of Lemma 2.

v' is the same under a and a'. See Figure 6. Define t' as follows

$$t'(v) = va'(v) - (\int_{\underline{v}}^{v} a'(z)dz) - u(\underline{v}).$$

Note that the mechanism (a', t') satisfies the incentive constraints, and gives all types $v \in V$ the same utility as in (a, t). Thus, (a', t') also satisfies the promise-keeping constraint. Finally, because a is not constant on $[v', v)$ but a' is, and because the areas under the two are equal, we have $a'(v') > a(v')$. So because $a'(v) = a(v)$, a' is point-wise higher than a on the set of all possible values V, and strictly so for some value v'. Therefore, a' has higher expected surplus than a, and the mechanism (a, t) cannot be optimal. □

C.2.2 The Last Period

We now characterize the structure of optimal mechanisms in the last period. We show that it is optimal to choose one of at most two prices at random, and sell the product at that price as a take it or leave it offer. At a high level, the set of all monotone allocations is a polytope, and its extreme points correspond to take it or leave it price offers. The set of feasible mechanisms in the last period is the intersection of this polytope and an additional hyperplane that represents the promise keeping constraint. As a result, an optimal mechanism is obtained by randomizing over at most two prices.

We here provide a geometric argument that pins down these prices. To do so, we construct a revenue function that maps the expected utility of any price to the revenue of that price. We then show that the optimal revenue can be found by "concavifying" that revenue function. To be formal, we define some notation.

Recall that $CR_{k+1}(\mathsf{EU}) = -\mathsf{EU}$ for $\mathsf{EU} \geq 0$, that is, any promised utility must be paid back to the buyer immediately in cash. Dropping the index k from the rest of the analysis, the problem becomes to maximize revenue

$$\mathbf{E}_v\left[va(v) + CR_{k+1}(va(v) - t(v))\right] = \mathbf{E}_v\left[t(v)\right],$$

subject to the incentive compatibility constraint (7), the expected utility constraint (9), and non-negative utility constraint,

$$va(v) - t(v) \geq 0.$$

Thus the problem is the standard problem of maximizing revenue of selling a single product, but with an additional expected utility constraint.

Define a revenue function RU that for each $p \in V$, maps the expected utility (to the buyer) of posting a price p for the product to its revenue. In particular, let $U = \{\mathbf{E}[\max(v-p, 0)] \mid p \in V\}$ be the set of possible expected utilities from posting prices in V, and for each $u \in U$, let $p(u) \in V$ be the price that induces that expected utility (note that such a price is unique since different prices induce different utilities). Notice that the smallest utility in U is zero, and let $\bar{u}$ denote the largest utility. Now define the revenue function $RU : U \to \mathbb{R}$, where $RU(u)$ is the revenue from posing a price that gives the buyer expected utility u,

$$RU(u) = p(u) \sum_{v \geq p(u)} f(v). \tag{18}$$

Define $\hat{RU} : \mathbb{R}^+ \to \mathbb{R}$ as follows. On the interval $[0, \bar{u}]$, $\hat{RU}$ is the concavification of RU, that is, the smallest concave function that is pointwise at least as large as RU. Above $\bar{u}$, the function $\hat{RU}$ continues linearly with slope -1. The lemma below shows that $\hat{RU}$ is the continuation revenue.

To identify the prices in an optimal mechanism, define $\ell(u)$ to be the largest $u' \leq u$ at which the two functions are equal $RU(u') = \hat{RU}(u')$, and similarly $h(u)$ to be the smallest $u' \geq u$ at which $RU(u') = \hat{RU}(u')$.

We now define two mechanisms that are optimal depending on the value of EU. The first mechanism is optimal when EU is large. The mechanism gives the product to all types and pays them $\mathsf{EU} - \bar{u}$,

$$a(v) = 1, t(v) = \bar{u} - \mathsf{EU}, \forall v. \tag{19}$$

The second mechanism is optimal when EU is small. The mechanism randomizes over two prices that induce expected utilities $\ell(\mathsf{EU})$ and $h(\mathsf{EU})$ with probabilities set such that the expected utility constraint is satisfied. That is, the probabilities of $p(\ell(\mathsf{EU}))$ and $p(h(\mathsf{EU}))$ are $1 - \alpha$ and α, respectively, where

$$\alpha = \frac{\mathsf{EU} - \ell(\mathsf{EU})}{h(\mathsf{EU}) - \ell(\mathsf{EU})}.$$

Formally, the allocation and transfer rules are defined as follows (notice that $p(h(\mathsf{EU}))$ is weakly lower than $p(\ell(\mathsf{EU}))$).

$$(a,t)(v) = \begin{cases} (0,0) & \text{if} \quad v < p(h(\mathsf{EU})), \\ (\alpha, \alpha p(h(\mathsf{EU}))) & \text{if} \quad p(h(\mathsf{EU})) \le v < p(\ell(\mathsf{EU})), \\ (1, (1-\alpha)p(\ell(\mathsf{EU})) + \alpha p(h(\mathsf{EU}))) & \text{if} \quad p(\ell(\mathsf{EU})) \le v. \end{cases} \tag{20}$$

Lemma 3. *The solution to the continuation revenue problem (Definition 2) in the last period is as follows. If $\mathsf{EU} \geq \bar{u}$, then the mechanism defined in Equation (19) is optimal. Otherwise, the mechanism defined in Equation (20) is optimal. In addition, $CU_k = \hat{RU}$ and CU_k is concave.*

Proof. First note for future reference that for each $u \in U$,

$$\begin{aligned} u + RU(u) &= \sum_{v \geq p(u)} (v - p(u)) f(v) + \sum_{v \geq p(u)} p(u) f(v) \\ &= \sum_{v \geq p(u)} v f(v). \end{aligned} \tag{21}$$

By Lemma 2, the transfer of a type v in an optimal mechanism is

$$-u(\underline{v}) + va(v) - \sum_{v' \leq v} (v - v')\Delta a(v') = -u(\underline{v}) + \sum_{v' \leq v} v' \Delta a(v').$$

Thus revenue is

$$
\begin{aligned}
& -u(\underline{v})+\sum_v f(v)\Big(\sum_{v'\le v} v'\Delta a(v')\Big)\\
=& -u(\underline{v})+\sum_v \Delta a(v)\Big(v\sum_{v'\ge v} f(v')\Big).
\end{aligned} \tag{22}
$$

And the expected utility is

$$
u(\underline{v})+\sum_v f(v)\sum_{v'\le v}(v-v')\Delta a(v') = u(\underline{v})+\sum_v \Delta a(v)\sum_{v'\ge v}(v'-v)f(v').
$$

Thus the expected utility constraint is

$$
\sum_v \Delta a(v)\sum_{v'\ge v}(v'-v)f(v') = \mathsf{EU}-u(\underline{v}). \tag{23}
$$

As a result, the problem is to maximize (22), subject to (23), the individual rationality constraint $u(\underline{v})\ge 0$, the monotonicity constraint $\Delta a(v)\ge 0$, and $\sum_v \Delta a(v)\le 1$.

Now consider the following change of variables. Let $u=p^{-1}(v)$ and $\mu(u)=\Delta a(v)$. By definition we have $RU(u)=v\sum_{v'\ge v} f(v')$ and $u=\sum_{v'\ge v}(v'-v)f(v')=\sum_{v'>v}(v'-v)f(v')$. The problem becomes to maximize

$$
-u(\underline{v})+\sum_u \mu(u)RU(u), \tag{24}
$$

subject to the expected utility constraint

$$
\sum_u \mu(u)u = \mathsf{EU}-u(\underline{v}), \tag{25}
$$

and the additional constraints that $u(\underline{v})\ge 0$, $\mu(u)\ge 0$, and $\sum_u \mu(u)\le 1$. We next show that the optimal solution must satisfy $\sum_u \mu(u)=1$, and $u(\underline{v})=0$ unless $\mu(\bar{u})=1$.

We first argue that $\sum_u \mu(u)=1$. Otherwise, it is possible to increase $\mu(0)$ without violating feasibility, since such a change does not affect the left hand side of the expected utility constraint (25). Since $RU(0)>0$ (posting the highest value in V as a take it or leave it price gives a strictly positive revenue), this change improves the objective and thus μ is not optimal.

We now argue that $u(\underline{v}) = 0$ unless $\mu(\bar{u}) = 1$. Assume for contradiction that $u(\underline{v}) > 0$ and $\mu(\bar{u}) < 1$. Since $\sum_u \mu(u) = 1$ as argued above, there must exists $u \neq \bar{u}$ such that $\mu(u) > 0$. We show that for δ small enough, it is feasible to decrease $\mu(u)$ by δ and increase $\mu(\bar{u})$ by δ. Notice that this change respects the two constraints $\mu(u) \geq 0$ and $\sum_u \mu(u) \leq 1$. The change in expected utility is $-\delta u + \delta\bar{u}$. Since $u(\underline{v}) > 0$, for small enough δ it is possible to add $\delta u - \delta\bar{u} < 0$ to $u(\underline{v})$ such that the expected utility constraint (25) stays satisfied as well. Now consider the change in objective. It is

$$\delta(\bar{u} - u) + \delta(RU(\bar{u}) - RU(u)) = \delta(\sum_{v \geq p(\bar{u})} vf(v) - \sum_{v \geq p(u)} vf(v)) > 0.$$

where the equality followed from (21), and the inequality followed since $p(\bar{u}) < p(u)$.

We now complete the proof. First, consider $\mathsf{EU} \geq \bar{u}$. If $\mu(\bar{u}) < 1$, then $\sum_u \mu(u)u < \mathsf{EU}$ and so we must have $u(\underline{v}) > 0$, which cannot happen as argued above. Therefore, $\mu(\bar{u}) = 1$.

Second, consider $\mathsf{EU} < \bar{u}$. For the expected utility constraint to be satisfied, we must have $\mu(\bar{u}) < 1$, and therefore $u(\underline{v}) = 0$ as argued above. Thus, the problem is to maximize

$$\sum_u \mu(u)RU(u), \tag{26}$$

subject to

$$\sum_u \mu(u)u = \mathsf{EU}, \tag{27}$$

$\mu(u) \geq 0$, and $\sum_u \mu(u) = 1$. In words, the objective is to maximize the expectation of RU over all distributions μ with expectation EU. The optimal value is equal to the concavification of RU at EU. Finally, the concavity of CR follows directly from its definition as concavification of RU. □

We state the following corollary of Lemma 3 for future reference. Increasing the expected utility changes the optimal allocation rule, unless the allocation probability of all types is 1. In particular, as long as $\mathsf{EU} \leq \bar{u}$, increasing the expected utility either changes the prices $\ell(\mathsf{EU})$ and $h(\mathsf{EU})$ or their probabilities. If the allocation rule does not change in EU, then any additional expected utility must be fulfilled with monetary transfers to the buyer. Thus the interpretation of the corollary is that it is never optimal to fulfill promises by paying money, if it is possible to do so by increasing the allocation.

Corollary 1. *Consider the solution to the continuation revenue problem (Definition 2) in the last period. If* $\mathcal{A}^{\mathsf{EU}} = \mathcal{A}^{\mathsf{EU}'}$ *for* $\mathsf{EU} \neq \mathsf{EU}'$*, then* $\mathcal{A}^{\mathsf{EU}}(v) = \mathcal{A}^{\mathsf{EU}'}(v) = 1$ *for all* v*.*

Proof. Consider two expected utility $\mathsf{EU}, \mathsf{EU}' \leq \mathbf{E}[v_k]$. If the two expected utilities are in different concavified regions, that $p(\ell(\mathsf{EU})) \neq p(\ell(\mathsf{EU}'))$ or $p(h(\mathsf{EU}')) \neq p(h(\mathsf{EU}'))$, then the allocation rules are different since they are the results of randomization over different prices. If $p(\ell(\mathsf{EU})) = p(\ell(\mathsf{EU}'))$ and $p(h(\mathsf{EU})) = p(h(\mathsf{EU}'))$, the the allocation rules are different since the probability of $p(\ell(\mathsf{EU}))$ is strictly decreasing in EU, and the probability of $p(\ell(\mathsf{EU}))$ is strictly increasing in EU'. Also if $\mathsf{EU} < \mathbf{E}[v_k] \leq \mathsf{EU}'$, then $\mathcal{A}^{\mathsf{EU}} \neq \mathcal{A}^{\mathsf{EU}'}$. So it must be that $\mathbf{E}[v_k] \leq \mathsf{EU}, \mathsf{EU}'$, which implies that $\mathcal{A}^{\mathsf{EU}}(v) = \mathcal{A}^{\mathsf{EU}'}(v) = 1$ for all v. □

C.2.3 The First Period

We first establish a property of the continuation revenue problem. Namely, increasing the utility promise by δ may decrease the continuation revenue by at most δ. The is because the mechanism has the option to pay the extra promised utility back to the buyer with money.

Lemma 4. *For any* i*, and* $\mathsf{EU} < \mathsf{EU}'$*, the continuation revenue function satisfies* $CR_i(\mathsf{EU}') - CR_i(\mathsf{EU}) \geq \mathsf{EU} - \mathsf{EU}'$*.*

Proof. We prove the lemma inductively. Consider the last period and an optimal mechanism $(\mathcal{A}_k^{\mathsf{EU}}, \mathcal{T}_k^{\mathsf{EU}})$. Consider an alternative mechanism $(\mathcal{A}_k^{\mathsf{EU}}, \mathcal{T}_k^{\mathsf{EU}} - (\mathsf{EU}' - \mathsf{EU}))$. Notice that the alternative mechanism is feasible for the utility promise EU', and obtains a revenue that is equal to the revenue of mechanism $(\mathcal{A}_k^{\mathsf{EU}}, \mathcal{T}_k^{\mathsf{EU}})$ minus $\mathsf{EU}' - \mathsf{EU}$. Thus we must have $CR_k(\mathsf{EU}') - CR_k(\mathsf{EU}) \geq \mathsf{EU} - \mathsf{EU}'$.

Now consider a period $i < k$, and assume that $CR_{i+1}(\mathsf{EU}') - CR_{i+1}(\mathsf{EU}) \geq \mathsf{EU} - \mathsf{EU}'$. We show that the same holds for CR_i. The proof is similar to above. For a given mechanism that satisfies the expected utility constraint for EU, reducing the transfer of all types by $\mathsf{EU}' - \mathsf{EU}$ results in a mechanism that satisfies the expected utility constraint for EU' and a change in objective value that is equal to

$$\begin{aligned}CR_i(\mathsf{EU}') - CR_i(\mathsf{EU}) &= \mathbf{E}\left[CR_{i+1}(v\mathcal{A}_i^{\mathsf{EU}} - \mathcal{T}_i^{\mathsf{EU}}(v) + \mathsf{EU}' - \mathsf{EU}) - CR_{i+1}(v\mathcal{A}_i^{\mathsf{EU}} - \mathcal{T}_i^{\mathsf{EU}}(v))\right] \\ &\geq \mathsf{EU} - \mathsf{EU}',\end{aligned}$$

by the induction hypothesis. □

The following lemma shows that the optimal allocation probabilities are at least as high as what they would be if the goal were to maximize only the stage revenue and to ignore the continuation revenue. In particular, let P be the set of *optimal monopoly prices.* That is, $P = \arg\max_p p \sum_{v \geq p} f(v)$ is the set of prices that maximize revenue of selling only the one product with value distribution f. Let $\underline{p}$ and $\overline{p}$ be the lowest and highest such price. A mechanism is optimal for selling only one product with distribution f if and only if it posts a price that is randomly chosen from P. This observation has two implications. First, in every optimal mechanism, any $v \geq \overline{p}$ must receive the product with probability one. Second, there exists an optimal mechanism (namely the mechanism that posts a price $\underline{p}$) in which any $v \geq \underline{p}$ receives the product with probability one. The lemma below shows that the same must be true for the continuation revenue problem.

Lemma 5. *Consider the solutions to the continuation revenue problem (Definition 2) in the first period. There exists an optimal mechanism (a, t) where $a(v) = 1$ for all $v \geq \underline{p}$, where $\underline{p}$ is the lowest optimal monopoly price. Additionally, in every optimal mechanism, $a(v) = 1$ for all $v \geq \overline{p}$, where $\overline{p}$ is the highest optimal monopoly price.*

Proof. Consider any two mechanisms (a, t) and (a', t'), each satisfying the payment equation of Lemma 2 with the same utility for the lowest type $\underline{v}$. Assume further that $a'(v) - a(v) \geq 0$ for all v, which in turn implies that $u'(v) - u(v) \geq 0$, where $u(v) = va(v) - t(v)$ is the utility of v. The difference between the objective values of the two mechanisms are

$$
\begin{aligned}
&\sum_v \Big(f(v)(va'(v) + CR(u'(v))) \Big) - \sum_v \Big(f(v)(va(v) + CR(u(v))) \Big) \\
&\geq \sum_v f(v) \Big(v(a'(v) - a(v)) - (u'(v) - u(v)) \Big) \\
&= \sum_v f(v)(t'(v) - t(v)), \\
&= \sum_v f(v) \Big(\sum_{v' \leq v} (\Delta a(v') - \Delta a'(v'))v' \Big), \\
&= \sum_v \Big(\Delta a'(v) - \Delta a(v) \Big) \Big(v \sum_{v' \geq v} f(v') \Big) \qquad (28)
\end{aligned}
$$

where the inequality followed from Lemma 4, the second equality followed from substituting t using the payment equation of Lemma 2, and the last equality followed from re-arranging summations.

The interpretation of (28) is that we can think of a mechanism (a, t) as a distribution over posted prices, where the probability of price v is $\Delta a(v)$. Revenue-maximizing mechanisms are distributions over the set of optimal monopoly prices. Therefore, there exists a revenue-maximizing mechanism where $a(\underline{p}) = 1$, and in every revenue-maximizing mechanism we must have $a(\overline{p}) = 1$. The challenge is that (28) is valid only if a and a' are ranked. Our formal proof ensures this ranking.

To prove the first statement of the lemma, consider an optimal mechanism (a, t). Consider an alternative mechanism (a', t') where a' is identical to a except that $a'(v) = 1$ for all $v \geq \underline{p}$. Let $R = \max_p p \sum_{v \geq p} f(v)$ and notice that $R = \underline{p} \sum_{v \geq \underline{p}} f(v)$ because $\underline{p}$ is an optimal monopoly price. We have

$$
\begin{aligned}
\sum_{v \geq \underline{p}} \Delta a(v) \Big(v \sum_{v' \geq v} f(v') \Big) &\leq \sum_{v \geq \underline{p}} \Delta a(v) R \\
&\leq \sum_{v \geq \underline{p}} \Delta a'(v) R \\
&= \sum_{v \geq \underline{p}} \Delta a'(v) \Big(v \sum_{v' \geq v} f(v') \Big), \qquad (29)
\end{aligned}
$$

where the second inequality followed because $\sum_{v \geq \underline{p}} \Delta a(v) = a(\bar{v}) - a(\max_{v' < \underline{p}} v') \leq 1 - a'(\max_{v' < \underline{p}} v') = \sum_{v \geq \underline{p}} \Delta a'(v)$. Now notice that $a'(v) - a(v) \geq 0$ for all v. Thus, by (28) and (29), the objective value of mechanism (a', t') minus the objective value of the mechanism (a, t) is at least

$$\sum_{v} \Big(\Delta a'(v) - \Delta a(v)\Big)\Big(v \sum_{v' \geq v} f(v')\Big) = \sum_{v \geq \underline{p}} \Big(\Delta a'(v) - \Delta a(v)\Big)\Big(v \sum_{v' \geq v} f(v')\Big) \geq 0.$$

Therefore, the mechanism (a', t') must also be optimal.

To prove the second statement, consider an optimal mechanism (a, t). Assume for contradiction that $a(v) < 1$ for some $v \geq \overline{p}$, i.e., $a(\overline{p}) < 1$. Consider an alternative mechanism (a', t') where a' is identical to a except that $a'(v) = 1$ for all $v \geq \overline{p}$. Notice that $a(\overline{p}) < 1$ implies that either $a(\overline{v}) < 1$ and therefore $\sum_{v \geq \overline{p}} \Delta a(v) < \sum_{v \geq \overline{p}} \Delta a'(v)$, or $\Delta a(v) > 0$ for some $v > \overline{p}$ and therefore $v \sum_{v' \geq v} f(v') < R$ because $\overline{p}$ is the largest optimal monopoly price. In either case, a sequence of inequalities similar to (29) implies that

$$\sum_{v \geq \overline{p}} \Delta a(v)\Big(v \sum_{v' \geq v} f(v')\Big) < \sum_{v \geq \overline{p}} \Delta a(v)\Big(v \sum_{v' \geq v} f(v')\Big). \tag{30}$$

Now notice that $a'(v) - a(v) \geq 0$ for all v. Thus, by (28) and (30), the objective value of mechanism (a', t') minus the objective value of the mechanism (a, t) is at least

$$\sum_{v} \Big(\Delta a'(v) - \Delta a(v)\Big)\Big(v \sum_{v' \geq v} f(v')\Big) = \sum_{v \geq \overline{p}} \Big(\Delta a'(v) - \Delta a(v)\Big)\Big(v \sum_{v' \geq v} f(v')\Big) > 0.$$

Therefore, the mechanism (a, t) cannot be optimal. □

C.2.4 Proof of Proposition 4

Proof. Consider any optimal separable mechanism (a, t). Let $P_1 \in \arg\max_{p \in V_1} p \cdot \mathbf{Pr}[v_1 \geq p]$ be the set of optimal monopoly prices that maximize the stage revenue for selling the first product. There are three possibilities, $P_1 = \{\underline{v}\}, P_1 = \{\overline{v}\}$, or $P_1 = \{\underline{v}, \overline{v}\}$. In any case, $\overline{v} \geq \underline{p}_1$, where $\underline{p}_1$ is the lowest monopoly price. Thus, by Lemma 5, the allocation probability of $\overline{v}$ in the first period is equal to one, $a_1(\overline{v}) = 1$. Thus, to specify the mechanism in the first period, we need to only specify the allocation probability of the low value, $a_1(\underline{v})$, and transfers. By Lemma 2, the transfers satisfy $t_1(\overline{v}) = \overline{v} - a_1(\underline{v})(\overline{v} - \underline{v}) - u_1(\underline{v})$. Substituting $a = a_1(\underline{v})$ and $u = u_1(\underline{v})$ to simplify notation, the

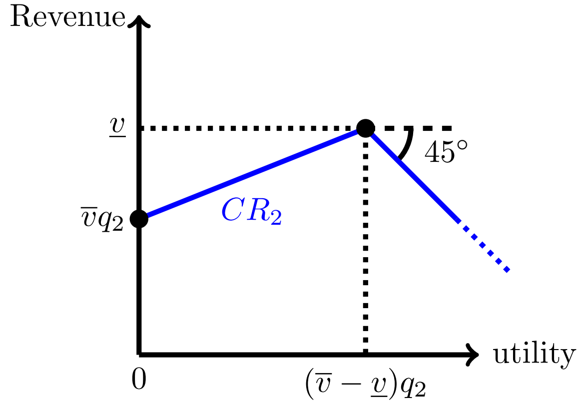

Figure 7: The continuation revenue function CR_2.

problem is

$$\max_{a,u}(1-q_1)\Big(a\underline{v}+CR_2(u)\Big)+q_1\Big(\overline{v}+CR_2(u+a(\overline{v}-\underline{v}))\Big) \tag{31}$$

subject to $0 \le a \le 1$ and $u \ge 0$.

The solution to (31) identifies the optimal separable mechanism, which by Proposition 1 is optimal among all mechanisms. However, it is possible that there exist other mechanisms, and in particular static mechanisms, that are optimal as well. To study this possibility, once the solution to (31) is obtained, we verify whether there exists a static mechanism that obtains the same revenue. If so, the static mechanism is optimal as well.

We solve (31) by considering how the objective value changes in a and u. By Lemma 3, the continuation revenue CR_2 is obtained by concavifying the function RU that maps the expected utility of different prices to their revenue. In particular, in the second period, $RU(0) = \overline{v}q_2$ and $RU((\overline{v}-\underline{v})q_2) = \underline{v}$, as shown in Figure 7.

Thus let

$$R' := \frac{RU((\overline{v}-\underline{v})q_2) - RU(0)}{(\overline{v}-\underline{v})q_2}$$

be the slope of the first linear piece of the continuation revenue function. Note that

$$R' = \frac{\underline{v}-\overline{v}q_2}{(\overline{v}-\underline{v})q_2} \ge \frac{\underline{v}q_2-\overline{v}q_2}{(\overline{v}-\underline{v})q_2} = -1.$$

The subderivatives of CR_2 are either R' or -1. Let CR_2' be the derivative of CR_2 whenever the derivative exists, and the subderivatives of CR_2 otherwise.

We now calculate the derivative (or subderivatives) of the objective with respect to a and u. The subderivative with respect to a is

$$(1-q_1)\underline{v} + q_1(\overline{v}-\underline{v})CR_2'(u + a(\overline{v}_1 - \underline{v}_1)). \tag{32}$$

and the subderivative with respect to u is

$$(1-q_1)CR_2'(u) + q_1 CR_2'(u + a(\overline{v}_1 - \underline{v}_1)). \tag{33}$$

The right derivative of CR_2 at any value at or above $q_2(\overline{v}-\underline{v})$ is -1. Thus, if $u \geq q_2(\overline{v}-\underline{v})$, the right derivative of the objective with respect to u is negative. Therefore, any optimal solution must satisfy $u \leq q_2(\overline{v}-\underline{v})$.

Since $R' \geq -1$, the two possible subderivatives of the objective with respect to a satisfy $(1-q_1)\underline{v} - q_1(\overline{v}-\underline{v}) \leq (1-q_1)\underline{v} + q_1(\overline{v}-\underline{v})R'$. Consider the three possible cases for the signs of these two subderivatives.

1. $(1-q_1)\underline{v} + q_1(\overline{v}-\underline{v})R' \leq 0$. Both subderivatives of the objective with respect to a are non-positive. As a result, it is optimal to set a as small as possible, $a = 0$. Notice also that in this case, $R' \leq 0$ and thus the subdrivatives of the objective with respect to u are also non-positive. Thus it is optimal to set $u = 0$. The optimal revenue is

$$(1-q_1)(CR_2(0)) + q_1(\overline{v} + CR_2(0)) = q_1\overline{v} + q_2\overline{v}.$$

   The optimal revenue is equal to the revenue of selling each product separately at price $\overline{v}$.

2. $(1-q_1)\underline{v} - q_1(\overline{v}-\underline{v}) \leq 0$ and $(1-q_1)\underline{v} + q_1(\overline{v}-\underline{v})R' \geq 0$. In this case, it is optimal to set $u + a(\overline{v}-\underline{v}) = q_2(\overline{v}-\underline{v})$. Otherwise, if $u + a(\overline{v}-\underline{v}) > q_2(\overline{v}-\underline{v})$, a can be decreased without decreasing the objective. Similarly, if $u + a(\overline{v}-\underline{v}) < q_2(\overline{v}-\underline{v})$, a can be increased without decreasing the objective. Now consider increasing u by ϵ, and decreasing a by $\epsilon/(\overline{v}-\underline{v})$. The

change in the objective value is

$$\epsilon(1 - q_1)(\frac{-\underline{v}}{\overline{v} - \underline{v}} + R').$$

If the expression above is positive, it is optimal to set $a = 0$ and $u = q_2(\overline{v} - \underline{v})$. Otherwise, it is optimal to set $a = q_2$ and $u = 0$. In the first case, the optimal revenue is

$$(1 - q_1)(0 + CR_2(q_2(\overline{v} - \underline{v}))) + q_1(\overline{v} + CR_2(q_2(\overline{v} - \underline{v}))) = q_1\overline{v} + \underline{v}.$$

Thus, the optimal revenue is equal to the revenue of selling the first product at price $\overline{v}$ and the second product at price $\underline{v}$. In the second case, the optimal revenue is

$$\begin{aligned}(1 - q_1)(q_2\underline{v} + CR_2(0)) + q_1(\overline{v} + CR_2(q_2(\overline{v} - \underline{v}))) &= (1 - q_1)(q_2\underline{v} + q_2\overline{v}) + q_1(\overline{v} + \underline{v}) \\ &= (\underline{v} + \overline{v})(q_1 + q_2 - q_1q_2).\end{aligned}$$

The optimal revenue is equal to the revenue of selling only the bundle of products at price $\underline{v} + \overline{v}$.

3. $(1 - q_1)\underline{v} - q_1(\overline{v} - \underline{v}) \geq 0$. Both subderivatives of the objective with respect to a are non-negative. Thus it is optimal to set a as large as possible, $a = 1$. To identify u, consider two cases. This corresponds to the fifth mechanism in the proposition. If $(1 - q_1)R' - q_1 \leq 0$, then it is optimal to set u as small as possible, $u = 0$. If $(1 - q_1)R' - q_1 \geq 0$, it is optimal to set u as large as possible, $u = q_2(\overline{v} - \underline{v})$. In the second case, the optimal revenue is

$$\begin{aligned}&(1 - q_1)(\underline{v} + CR_2(q_2(\overline{v} - \underline{v}))) + q_1(\overline{v} + CR_2((\overline{v} - \underline{v})(1 + q_2))) \\ &= (1 - q_1)(\underline{v} + \underline{v}) + q_1(\overline{v} + \underline{v} - (\overline{v} - \underline{v})) \\ &= 2\underline{v}.\end{aligned}$$

The optimal revenue is equal to the revenue of selling each product separately at price $\underline{v}$.

To prove the second and third statements, notice that $q_1 < \underline{v}/\overline{v}$ is equivalent to $(1-q_1)\underline{v}-q_1(\overline{v}-\underline{v}) > 0$ and $q_2 > \underline{v}(1 - q_1)/(\overline{v} - q_1\underline{v})$ is equivalent to $(1 - q_1)R' - q_1 < 0$. If $(1 - q_1)\underline{v} - q_1(\overline{v} - \underline{v}) > 0$

and $(1-q_1)R' - q_1 < 0$, then following the analysis of the third case, the unique optimal separable mechanism satisfies $a = 1$ and $u = 0$. Conversely, if either $(1-q_1)\underline{v} - q_1(\overline{v} - \underline{v}) \leq 0$ or $(1-q_1)R' - q_1 \geq 0$, then the analysis above shows that at least one of the four static mechanisms discussed above are optimal. □

C.3 Proof of Proposition 5

Proof. Assume for contradiction that there exists a static IC mechanism (a, t) that is optimal. For this proof let v_{-i}, a_{-i} denote values and allocations of products other than i, and similarly $v_{-i,-j}, a_{-i,-j}$ denote values and allocations of products other than i, j. Also $\underline{v}_{-i}$ denote a vector where values of all products other than i are $\underline{v}$.

We can assume without loss of generality that if changing v_1 does not affect the allocation of product 1, then it also does not affect the allocation of other products. That is, if $a_1(v_1, v_{-1}) = a_1(v_1', v_{-1})$ for some v_1, v_1', v_{-1}, then $a(v_1, v_{-1}) = a(v_1', v_{-1})$. To see this, consider the incentive constraint of type $v = (v_1, v_{-1})$ for reporting (v_1', v_{-1}),

$$v \cdot a(v) - t(v) \geq v \cdot a(v_1', v_{-1}) - t(v_1', v_{-1}).$$

Since $a_1(v_1, v_{-1}) = a_1(v_1', v_{-1})$, the above inequality is equivalent to

$$v_{-1} \cdot a_{-1}(v) - t(v) \geq v_{-1} \cdot a_{-1}(v_1', v_{-1}) - t(v_1', v_{-1}).$$

Similarly, the incentive constraint of type (v_1', v_{-1}) for reporting v is

$$v_{-1} \cdot a_{-1}(v_1', v_{-1}) - t(v_1', v_{-1}) \geq v_{-1} \cdot a_{-1}(v) - t(v).$$

Thus, both incentive constraints must hold with equality. That is, each type is indifferent between her own allocation and the allocation of the other type. Therefore, we can assign both types the allocation and the transfer of the type that pays more (or either allocation and transfer, if the transfers are the same) without decreasing revenue or violating incentive compatibility. Thus, without loss of generality we assume that both types have the same allocation (and transfer).

We show in the following two paragraphs that optimality of static mechanism (a,t) leads to two contradictory conclusions.

First, for any, we must have $u(\underline{v}_{-k}, \underline{p}_k) > 0$. By Proposition 1, the induced separable mechanism (a^{ISP}, t^{ISP}) of (a,p) must be optimal. By definition,

$$a_i^{ISP}(v_1,\ldots,v_i) := \mathbf{E}_{v_{i+1},\ldots,v_k}\left[a_i(v)\right]$$

By Lemma 5, we must have $a_1^{ISP}(v_1) = 1$ for all $v_1 \geq \overline{p}_1$. Since $a_1^{ISP}(v_1)$ is the expectation of $a_1(v_1, v_{-1})$ over v_{-1}, we have

$$a_1(v_1, v_{-1}) = 1, \forall v_1 \geq \overline{p}_1, v_{-1}. \tag{34}$$

Now consider any pair of values $v_1 < v_1'$ that are at least as large as $\overline{p}_1$. Such a pair exists since $\overline{p}_1 < \underline{p}_k$ implies that $\overline{p}_1 < \overline{v}$ and therefore there exists two distinct values in V_1 that are at least as large as $\overline{p}_1$.

The fact that $a_1(v_1, v_{-1}) = a_1(v_1', v_{-1}) = 1$ for all v_{-1} implies that the ex post utility of the first type is strictly lower than the ex post utility of the second type. To see this, first note that by our discussion above, the allocations and payments must be identical following v_1 and v_1', that is, $a(v_1, v_{-1}) = a(v_1', v_{-1}), t(v_1, v_{-1}) = t(v_1', v_{-1})$ for all v_{-1}. So the ex post utility of type v is strictly smaller than the ex post utility of type (v_1', v_{-1}) because these types have the same allocation and price and the probability of allocation of product one is strictly positive. Therefore, in the separable mechanism, for any $v_{-1,-k}$ (the vector of values of products other than 1 and k), the expected utility of the interim type $(v_1, v_{-1,-k})$ is strictly smaller than the expected utility to type $(v_1', v_{-1,-k})$. Corollary 1 implies that for the allocations in the last period to be equal following two different promised utilities, the allocation probability in the last period must be 1. That is,

$$a_k(v_1, v_{-1}) = a_k^{ISP}(v_1, v_{-1}) = 1, \forall v_1 \geq \overline{p}_1, v_{-1}.$$

Together with (34), the above equality implies that in particular, $a_1(\overline{p}_1, \underline{v}_{-1}) = a_k(\overline{p}_1, \underline{v}_{-1}) = 1$. By individual rationality, $t(\overline{p}_1, \underline{v}_{-1}) \leq \overline{p}_1 + \underline{v} + \underline{v}_{-1,-k} \cdot a_{-1,-k}(\overline{p}_1, \underline{v}_{-1})$. Incentive compatibility of the static mechanism requires that the utility of type $(\underline{v}_{-k}, \underline{p}_k)$ must be at least the utility that this type would get from reporting $(\overline{p}_1, \underline{v}_{-1})$. Therefore, we have

$$
\begin{aligned}
u(\underline{v}_{-k}, \underline{p}_k) &\geq \underline{v} + \underline{p}_k + \underline{v}_{-1,-k} \cdot a_{-1,-k}(\overline{p}_1, \underline{v}_{-1}) - t(\overline{p}_1, \underline{v}_{-1}) \\
&\geq \underline{v} + \underline{p}_k - (\overline{p}_1 + \underline{v}) \\
&= \underline{p}_k - \overline{p}_1 \\
&> 0.
\end{aligned}
$$

We now show $u(\underline{v}_{-k}, \underline{p}_k) \leq 0$, arriving at a contradiction. Notice because $u(\underline{v}_{-k}, v_k) \leq u(v_{-k}, v_k)$ for all v_{-k} and v_k, the continuation utility to interim type $\underline{v}_{-k}$ is smaller than any other type. Therefore, the promised utility to type $\underline{v}_{-k}$ in the induced separable mechanism cannot be larger than the level that maximizes CR_k, as otherwise, since CR_k is concave by Lemma 3, we can decrease the utility of all types and increase the continuation revenue. As a result, by Lemma 3, following a history of report $\underline{v}_{-k}$, product k is sold by randomizing over at most two prices that are both at least as large as $\underline{p}_k$. Therefore, the ex post utility of type $(\underline{v}_{-k}, \underline{p}_k)$ in the separable mechanism is zero. Since ex post utilities are equal in mechanisms (a, t) and its induced separable mechanism, we conclude that $u(\underline{v}_{-k}, \underline{p}_k) \leq 0$. $\square$