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Supplementary material of "Differential Privacy in Personalized Pricing with Nonparametric Demand Models" by Xi Chen, Sentao Miao, Yining Wang

EC.1. Proof of Theorem 1

We first establish the ε-CDP property of Algorithm 1. For each t and $k = k_t$ define $\boldsymbol{u}_t := (u_{j,t,k_t})_{j\in J} \in \mathbb{R}^J$ and $\boldsymbol{v}_t := (v_{j,t,k_t})_{j\in J}$. It is clear that both $\boldsymbol{u}_t, \boldsymbol{v}_t$ are one-hot vectors (i.e., having zero values except for one entry $j = j_t$) and hence their ℓ_1 sensitivity is upper bounded by 2. Note also that each time TreeBasedAggregation is invoked, the resulting $\widehat{S}_t$ consists of at most $L+1 = \lfloor \log_2 T\rfloor + 1$ partial sums. Therefore, by simple composition of Laplace mechanisms we have that all $\{r_{j,k}(t), \mu_{j,k}(t)\}_{t=1}^T$ are ε-differentially private. Since other parts of Algorithm 1 (e.g., the updates of $\boldsymbol{\rho}_j$ and ς_j) no longer access the sensitive data $\{s_t = (\boldsymbol{x}_t, y_t, p_t)\}_{t=1}^T$ except for the offering price p_t, we conclude that Algorithm 1 satisfies ε-CDP thanks to the closedness-to-post-processing property of differential privacy.

In the rest of the proof we upper bound the regret of Theorem 1. For simplicity we will use the notation const. to denote any universal numerical constant that does not depend on any problem parameters. For $j\in[J]$, $t\in[T]$ and $k\in[5]$ recall the definitions that $u_{j,k,t} = \mathbf{1}\{j = j_t \wedge k = k_t\}p_t y_t$ and $v_{j,k,t} = \mathbf{1}\{j = j_t \wedge k = k_t\}$. We first state a technical lemma that upper bounds (with high probability) the deviation of $r_{j,k}(t), \mu_{j,k}(t)$ from $\sum_{s\le t} u_{s,j,k}$ and $\sum_{s\le t} v_{s,j,k}$, respectively.

Lemma EC.1. *With probability $1 - O(T^{-1})$ the following hold uniformly over all $t\in[T]$, $j\in[J]$ and $k\in[5]$:*

$$\max\left\{\left|r_{j,k}(t) - \sum_{s\le t} u_{s,j,k}\right|, \left|\mu_{j,k}(t) - \sum_{s\le t} v_{s,j,k}\right|\right\} \le 19\varepsilon^{-1}\ln^2(2T^3).$$

Proof of Lemma EC.1. We focus on the $|r_{j,k}(t) - \sum_{s\le t} u_{s,j,k}|$ term only as the proof for the other term is exactly the same. Let $t = \sum_{\ell=0}^L b_\ell(t)2^\ell$ where $L = \lfloor \log_2 T\rfloor$ and $b_\ell(t)\in\{0,1\}$ be the binary expression of t. By the definition of the TreeBasedAggregation, we have that

$$r_{j,k}(t) - \sum_{s\le t} u_{s,j,k} = \sum_{\ell=0}^L b_\ell(t) w_\ell, \tag{EC.1}$$

where $\{w_\ell\}_{\ell=0}^L \overset{i.i.d.}{\sim} \mathrm{Lap}(2/\varepsilon')$ with $\varepsilon'=\varepsilon/(L+1)$. Invoke the concentration inequality for sums of i.i.d. centered Laplace random variables (Chan et al. 2011, Corollary 2.9). We have with probability $1-O(T^{-3})$ that

$$\left|\sum_{\ell=0}^L b_\ell(t)w_\ell\right| \le 2\varepsilon^{-1}(L+1)\left(\sqrt{L+1}+\sqrt{\ln(2T^3)}\right)\sqrt{8\ln(2T^3)}$$
$$\le 19\varepsilon^{-1}\ln^2(2T^3). \tag{EC.2}$$

Combine Eqs. (EC.1,EC.2) and apply a union bound over all $t\in[T]$, $j\in[J]$ and $k\in[5]$. Lemma EC.1 is thus proved. □

In the rest of the proof, define $n_j=t-\varsigma_j$, $\widehat{r}_{jk}=r_{jk}(t)-r_{jk}(\varsigma_j)$ and $\widehat{\mu}_{jk}=\mu_{jk}(t)-\mu_{jk}(t)(\varsigma_j)$. Define $\widetilde{r}_{jk}:=\sum_{\tau=\varsigma_j+1}^t u_{j,\tau,k}$ and $\widetilde{\mu}_{jk}:=\sum_{\tau=\varsigma_j+1}^t v_{j,\tau,k}$. Lemma EC.1 implies that with probability $1-O(T^{-1})$,

$$\max\{|\widetilde{r}_{jk}-\widehat{r}_{jk}|,|\widetilde{\mu}_{jk}-\widehat{\mu}_{jk}|\}\le 38\varepsilon^{-1}\ln^2(2T^3). \tag{EC.3}$$

On the other hand, observe that both $\widetilde{\mu}_{jk}$ and $\widetilde{r}_{jk}$ are sums of i.i.d. random variables. The following technical lemma then upper bounds the concentration of $\widetilde{\mu}_{jk}$ and $\widetilde{r}_{jk}$ towards their expected values.

Lemma EC.2. *For each hypercube B_j define $\overline{\chi}(B_j):=J\times\Pr[\boldsymbol{x}\in B_j]\in[0,C_X]$. Let $n_j=t-\varsigma_j$. The following holds with probability $1-O(T^{-1})$ uniformly over all $t\in[T]$, $j\in[J]$ and $k\in[5]$:*

$$\max\left\{\left|\frac{\widetilde{r}_{jk}}{n_j}-\frac{\overline{\chi}(B_j)h^d}{5n_j}f_{B_j}(\rho_{jk})\right|,\left|\frac{\widetilde{\mu}_{jk}}{n_j}-\frac{\overline{\chi}(B_j)h^d}{5n_j}\right|\right\}\le\sqrt{\frac{C_Xh^d\ln(2T^3)}{n_j}}+\frac{1.5\ln(2T^3)}{n_j}. \tag{EC.4}$$

Furthermore,

$$\left|\frac{\widetilde{r}_{jk}}{\widetilde{\mu}_{jk}}-f_{B_j}(\rho_{jk})\right|\le\sqrt{\frac{\ln(2T^3)}{2\widetilde{\mu}_{jk}}}. \tag{EC.5}$$

Proof of Lemma EC.2. For Eq. (EC.4) we will prove the concentration inequality involving $\widetilde{\mu}_{jk}$ only, as the other inequality can be proved using the exact same argument. Note that for each $j\in[J]$, $s\in[\varsigma_j+1,t]$ with $s\equiv k \mod 5$, we have that $v_{j,s,k}\in\{0,1\}$ almost surely and furthermore $\Pr[v_{s,j,k}=1]=\Pr[\boldsymbol{x}_s\in B_j, k\equiv s \mod 5]=\overline{\chi}(B_j)h^d/5$. By Bernstein's inequality (Bennett 1962), for any particular t, j and k it holds with probability $1-O(T^{-3})$ that

$$\left|\frac{\widetilde{\mu}_{jk}}{n_j}-\frac{\overline{\chi}(B_j)h^d}{5n_j}\right|\le 2\sqrt{\frac{\overline{\chi}(B_j)h^d\ln(2T^3)}{5n_j}}+\frac{1.5\ln(2T^3)}{n_j}\le\sqrt{\frac{C_Xh^d\ln(2T^3)}{n_j}}+\frac{1.5\ln(2T^3)}{n_j}.$$

Apply the union bound over all $t\in[T]$, $j\in[J]$ and $k\in[5]$. We complete the proof of Eq. (EC.4).

We next prove Eq. (EC.5). Note that $\widetilde{r}_{jk}$ is a sum of $\widetilde{\mu}_{jk}$ i.i.d. random variables each with expectation $f_{B_j}(\rho_{jk})$ and supported on $[0,1]$ almost surely. By Hoeffding's inequality (Hoeffding 1963), it holds with probability $1-O(T^{-3})$ that

$$\left|\frac{\widetilde{r}_{jk}}{\widetilde{\mu}_{jk}} - f_{B_j}(\rho_{jk})\right| \leq \sqrt{\frac{\ln(2T^3)}{2\widetilde{\mu}_{jk}}}.$$

Apply the union bound over all $t\in[T]$, $j\in[J]$ and $k\in[5]$. We complete the proof of Eq. (EC.5). □

Eq. (EC.3) and Lemma EC.2 together yield the following lemma:

Lemma EC.3. *With probability $1-O(T^{-1})$ the following holds uniformly over all $t\in[T]$, $j\in[J]$, $k\in[4]$ that satisfies $\widehat{\mu}_{jk}\geq c_2 = 76\varepsilon^{-1}\ln^2(2T^3)$: $2\widehat{\mu}_{jk}\geq\widetilde{\mu}_{jk}\geq\widehat{\mu}_{jk}/2$, and furthermore*

$$\left|\frac{\widehat{r}_{jk}}{\widehat{\mu}_{jk}} - f_{B_j}(\rho_{jk})\right| \leq \frac{2c_2}{\widetilde{\mu}_{jk}} + \sqrt{\frac{\ln(2T^3)}{2\widetilde{\mu}_{jk}}} \leq \frac{4c_2}{\widehat{\mu}_{jk}} + \sqrt{\frac{\ln(2T^3)}{\widehat{\mu}_{jk}}}.$$

Proof of Lemma EC.3. Fix $t\in[T]$, $j\in[J]$ and $k\in[5]$. For notational simplicity define $\beta := \overline{\chi}(B_j)/(5h^d)$, $\Delta_r := \widehat{r}_{jk}-\widetilde{r}_{jk}$ and $\Delta_\mu := \widehat{\mu}_{jk}-\widetilde{\mu}_{jk}$. By Eq. (EC.3), the condition that $\widehat{\mu}_{jk}\geq c_2 = 76\varepsilon^{-1}\ln^2(2T^3)$ implies that $2\widehat{\mu}_{jk}\geq\widetilde{\mu}_{jk}\geq\widehat{\mu}_{jk}/2$. Subsequently,

$$\begin{aligned}\left|\frac{\widehat{r}_{jk}}{\widehat{\mu}_{jk}} - \frac{\widetilde{r}_{jk}}{\widetilde{\mu}_{jk}}\right| &\leq \left|\frac{\widetilde{r}_{jk}+\Delta_r}{\widehat{\mu}_{jk}} - \frac{\widetilde{r}_{jk}}{\widetilde{\mu}_{jk}}\right| \leq \left|\frac{\widetilde{r}_{jk}}{\widehat{\mu}_{jk}} - \frac{\widetilde{r}_{jk}}{\widetilde{\mu}_{jk}}\right| + \frac{2|\Delta_r|}{\widetilde{\mu}_{jk}} = \left|\frac{\widetilde{r}_{jk}}{\widetilde{\mu}_{jk}+\Delta_\mu} - \frac{\widetilde{r}_{jk}}{\widetilde{\mu}_{jk}}\right| + \frac{2|\Delta_r|}{\widetilde{\mu}_{jk}}\\ &= \frac{\widetilde{r}_{jk}}{\widetilde{\mu}_{jk}}\left|\frac{\widetilde{\mu}_{jk}}{\widetilde{\mu}_{jk}+\Delta_\mu} - 1\right| + \frac{2|\Delta_r|}{\widetilde{\mu}_{jk}} = \frac{\widetilde{r}_{jk}}{\widetilde{\mu}_{jk}}\frac{|\Delta_\mu|}{\widehat{\mu}_{jk}} + \frac{2|\Delta_r|}{\widetilde{\mu}_{jk}} \leq \frac{2(|\Delta_\mu|+|\Delta_r|)}{\widetilde{\mu}_{jk}},\end{aligned}$$

where the last inequality holds because $\widetilde{r}_{jk}/\widetilde{\mu}_{jk}\in[0,1]$ almost surely. By Eq. (EC.3) we have that $|\Delta_\mu|+|\Delta_r|\leq 76\varepsilon^{-1}\ln^2(2T^3)$. Subsequently,

$$\left|\frac{\widehat{r}_{jk}}{\widehat{\mu}_{jk}} - \frac{\widetilde{r}_{jk}}{\widetilde{\mu}_{jk}}\right| \leq \frac{152\varepsilon^{-1}\ln^2(2T^3)}{\widetilde{\mu}_{jk}} = \frac{2c_2}{\widetilde{\mu}_{jk}}. \tag{EC.6}$$

Combining Eq. (EC.6) and Eq. (EC.4) in Lemma EC.2, we have with probability $1-O(T^{-3})$ that

$$\left|\frac{\widehat{r}_{jk}}{\widehat{\mu}_{jk}} - f_{B_j}(\rho_{jk})\right| \leq \frac{2c_2}{\widetilde{\mu}_{jk}} + \sqrt{\frac{\ln(2T^3)}{2\widetilde{\mu}_{jk}}}.$$

With a union bound over t, j and k we prove the first inequality in Lemma EC.3. The second inequality in Lemma EC.3 then holds by noting that $\widetilde{\mu}_{jk}\geq\widehat{\mu}_{jk}/2$. □

Lemma EC.3 and Assumption (A2-a) immediately yield the following corollary:

Corollary EC.1. *Conditioned on the success event in Lemma EC.3, it holds for all $t\in[T]$ and $j\in[J]$ that $p^*(B_j)\in[\rho_{j1},\rho_{j4}]$, where $p^*(B_j)=\arg\max_p f_{B_j}(p)$.*

We are now ready to prove Theorem 1.

Proof of Theorem 1. We will upper bound the regret incurred in each hypercube B_j separately. The proof is also conditioned on the success events in Lemmas EC.1, EC.2, EC.3 and Corollary EC.1 which occurs with probability $1-O(T^{-1})$.

Fix a particular hypercube B_j. We partition the entire T selling periods into *epochs* denoted as $\tau = 1,2,3,\cdots$, with each epoch starting with a time period at which ς_j is reset (at the start of T time periods or as a result of the execution of Line 13 or 15 in Algorithm 1), and ending when either Line 13 or Line 15 is executed again to reset the ς_j pointer. Let $\mathcal{T}_j(\tau)$ be the collection of time periods during epoch τ for hypercube j, and define $n_j(\tau) := |\mathcal{T}_j(\tau)|$. Note that $\mathcal{T}_j(\tau)$ includes selling periods during which customers with $\boldsymbol{x}_t \notin B_j$ arrive as well. Moreover, define $\delta_\tau := \rho_{j5} - \rho_{j1} = (\overline{p} - \underline{p})(3/4)^{\tau-1}$ where the equality is by update rule of price range.

Let us fix a particular epoch τ and let $\mathcal{T}_j(\tau)$ be the set of all time periods in epoch τ. Recall the definition that $\widetilde{\mu}_{jk}$ is the number of time periods in this epoch during which price ρ_{jk} is offered to an incoming customer with feature vector $\boldsymbol{x} \in B_j$. By Assumption (A3-a) and Corollary EC.1, the total regret incurred in the particular epoch and hypercube compared against $f_{B_j}(p^*(B_j))$ is upper bounded by

$$\sum_{t\in\mathcal{T}_j(\tau)} \mathbf{1}\{\boldsymbol{x}_t \in B_j\}[f_{B_j}(p^*(B_j)) - f_{B_j}(p_t)] \le 5\max_k\{\widetilde{\mu}_{jk}\} \times \frac{C_H^2}{2}\max_k\{|\rho_{jk} - p^*(B_j)|^2\}$$
$$\le 2.5 C_H^2 \delta_\tau^2 \times \max_k\{\widetilde{\mu}_k\}. \tag{EC.7}$$

To upper bound Eq. (EC.7) we need to upper bound $\widetilde{\mu}_k$. By the description of the CPPQ policy in Algorithm 1, the epoch τ will terminate when either one of the two following conditions are met:

$$\underline{\mu}_{1\to 3} \ge c_2 \quad \text{and} \quad \min\left\{\frac{\widehat{r}_{j2}}{\widehat{\mu}_{j2}} - \frac{\widehat{r}_{j1}}{\widehat{\mu}_{j1}}, \frac{\widehat{r}_{j3}}{\widehat{\mu}_{j3}} - \frac{\widehat{r}_{j2}}{\widehat{\mu}_{j2}}\right\} \ge \frac{3c_1}{\sqrt{\underline{\mu}_{1\to 3}}} + \frac{3c_1'}{\underline{\mu}_{1\to 3}}, \tag{EC.8}$$

$$\underline{\mu}_{3\to 5} \ge c_2 \quad \text{and} \quad \min\left\{\frac{\widehat{r}_{j3}}{\widehat{\mu}_{j3}} - \frac{\widehat{r}_{j4}}{\widehat{\mu}_{j4}}, \frac{\widehat{r}_{j4}}{\widehat{\mu}_{j4}} - \frac{\widehat{r}_{j5}}{\widehat{\mu}_{j5}}\right\} \ge \frac{3c_1}{\sqrt{\underline{\mu}_{3\to 5}}} + \frac{3c_1'}{\underline{\mu}_{3\to 5}}. \tag{EC.9}$$

Note that when $p^*(B_j) \in [\rho_{j1}, \rho_{j5}]$, Assumption (A2-a) implies that either $\min\{f_{B_j}(\rho_{j2}) - f_{B_j}(\rho_{j1}), f_{B_j}(\rho_{j3}) - f_{B_j}(\rho_{j2})\} \ge \frac{\sigma_H^2}{32}\delta_\tau^2$ or $\min\{f_{B_j}(\rho_{j3}) - f_{B_j}(\rho_{j4}), f_{B_j}(\rho_{j4}) - f_{B_j}(\rho_{j5})\} \ge \frac{\sigma_H^2}{32}\delta_\tau^2$ holds. Assume without loss of generality that $\min\{f_{B_j}(\rho_{j2}) - f_{B_j}(\rho_{j1}), f_{B_j}(\rho_{j3}) - f_{B_j}(\rho_{j2})\} \ge \frac{\sigma_H^2}{32}\delta_\tau^2$. Then by Lemma EC.3, the condition (EC.8) is satisfied when

$$\underline{\mu}_{1\to 3} = c_2 + \text{const.} \times \left[\frac{c_1^2}{\sigma_H^4 \delta_\tau^4} + \frac{c_1'}{\sigma_H^2 \delta_\tau^2}\right]. \tag{EC.10}$$

Note that $\widehat{\mu}_{jk} \geq \widetilde{\mu}_{jk}/2$ thanks to Lemma EC.2. Eq. (EC.10) and the symmetric case of $\min\{f_{B_j}(\rho_{j3}) - f_{B_j}(\rho_{j4}), f_{B_j}(\rho_{j4}) - f_{B_j}(\rho_{j5})\} \geq \frac{\sigma_H^2}{32}\delta_\tau^2$ are then implied by

$$\min_k\{\widetilde{\mu}_{jk}\} \leq 2c_2 + \text{const.} \times \left[\frac{c_1^2}{\sigma_H^4 \delta_\tau^4} + \frac{c_1'}{\sigma_H^2 \delta_\tau^2}\right]. \tag{EC.11}$$

On the other hand, note that $|f_{B_j}(\rho_{j,k+1}) - f_{B_j}(\rho_{j,k})| \leq \frac{C_H^2}{2}\delta_\tau^2$ for all $k \in \{1,2,3,4,5\}$. With Lemma EC.3 and the stopping condition in Eqs. (EC.8,EC.9), we have at the end of epoch τ that

$$\max_k\{\widetilde{\mu}_{jk}\} \geq \max_k\{\widehat{\mu}_{jk}/2\} \geq \frac{c_2}{2} + \frac{2c_1^2}{C_H^2 \delta_\tau^4} + \frac{c_1'}{C_H \delta_\tau^2}. \tag{EC.12}$$

Contrasting Eq. (EC.11) with Eq. (EC.7), we need to upper bound the differences between $\widetilde{\mu}_{jk}$ for $k \in [5]$. This can be done by using Eq. (EC.4) in Lemma EC.2 and the triangle inequality, which yield for every $k, k' \in \{1,2,3,4,5\}$ that

$$\left|\widetilde{\mu}_{jk} - \widetilde{\mu}_{jk'}\right| \leq 2\sqrt{C_X h^d n_j \ln(2T^3)} + 3\ln(2T^3). \tag{EC.13}$$

Combine Eqs. (EC.7,EC.11,EC.12,EC.13). The total regret incurred in hypercube B_j and epoch τ (such that $\delta_\tau = \rho_{j4} - \rho_{j1}$) is upper bounded by

$$\begin{aligned}
&2C_H^2\delta_\tau^2 \times \text{const.} \times \left[c_2 + \frac{c_1^2}{\sigma_H^4\delta_\tau^4} + \frac{c_1'}{\sigma_H^2\delta_\tau^2} + \sqrt{C_X h^d n_j \ln(2T^3)} + \ln(2T^3)\right] \\
&\leq \frac{\text{const.} \times c_1^2 C_H^2}{\sigma_H^4 \delta_\tau^2} + \text{const.} \times C_H^2 \left[c_2 + \sigma_H^{-2} c_1' + \sqrt{C_X h^d n_j \ln(2T^3)} + \ln(2T^3)\right] \\
&\leq \frac{\text{const.} \times c_1^2 C_H^2}{\sigma_H^4} \sqrt{\frac{C_H^2 \max_k \widetilde{\mu}_{jk}}{2c_1^2}} + \text{const.} \times C_H^2 \left[c_2 + \sigma_H^{-2}c_1' + \sqrt{C_X h^d T \ln(2T^3)} + \ln(2T^3)\right] \qquad \text{(EC.14)} \\
&\leq \text{const.} \times c_1 C_H^3 \sigma_H^{-4} \max_k\{\sqrt{\widetilde{\mu}_{jk}}\} + \text{const.} \times C_H^2 \left[c_2 + \sigma_H^{-2}c_1' + \sqrt{C_X h^d T \ln(2T^3)} + \ln(2T^3)\right]. \qquad \text{(EC.15)}
\end{aligned}$$

Here, Eq. (EC.14) holds because $\delta_\tau^{-4} \leq C_H^2 \max_k\{\widetilde{\mu}_{jk}\}/(2c_1^2)$ thanks to Eq. (EC.12). Note also that Eq. (EC.12) implies an upper bound of $\ln(C_H^2 T)$ on the total number of epochs for each j. Define $C_1' :=$ const. $\times\, c_1 C_H^3 \sigma_H^{-4} \ln(C_H^2 T) \leq$ const. $\times\, C_H^3 \sigma_H^{-4} \ln^2(2C_H^2 T^3)$, $C_2' :=$ const. $\times\, C_H^2 \varepsilon(c_2 + \sigma_H^{-2} c_1') \ln(C_H^2 T) \leq$ const. $\times\, C_H^2 \sigma_H^{-2} \ln^3(2C_H^2 T^3)$ and $C_3' =$ const. $\times\, C_H^2(\sqrt{C_X \ln(2T^3)} + \ln(2T^3)) \ln(C_H^2 T) \leq$ const. $\times\, C_H^2 \sqrt{C_X} \ln^2(2C_H^2 T^3)$. Summing Eq. (EC.15) over $j \in [J]$ and $k \in [5]$, we have that

$$\sum_{j=1}^{J} \sum_{t=1}^{T} \mathbf{1}\{\boldsymbol{x}_t \in B_j\}[f_{B_j}(p^*(B_j)) - f_{B_j}(p_t)] \leq C_1' \sum_{j,k} \sqrt{\widetilde{\mu}_{jk}} + C_2' J/\varepsilon + C_3' J \times (\sqrt{h^d T} + 1)$$

$$\le C_1'\sqrt{5J}\sqrt{\sum_{j,k}\widetilde{\mu}_{jk}}+\frac{C_2'J}{\varepsilon}+2C_3'J\sqrt{h^dT}\le\frac{2(C_1'+C_3')\sqrt{T}}{h^{d/2}}+\frac{C_2'}{\varepsilon h^d}. \tag{EC.16}$$

Additionally, Assumption (A3-a) with $B=\{\boldsymbol{x}_t\}$ and (A3-b), (A3-c) imply that $|f(p^*(\boldsymbol{x}),\boldsymbol{x})-f(p^*(B_j),\boldsymbol{x})|\le\frac{C_H^2C_p^2}{2}\sup_{\boldsymbol{x},\boldsymbol{x}'\in B_j}\|\boldsymbol{x}-\boldsymbol{x}'\|_2^2\le\frac{C_H^2C_p^2d}{2}h^2$. This together with Eq. (EC.16) yields with probability $1-O(T^{-1})$ that

$$\sum_{t=1}^T f(p^*(\boldsymbol{x}_t),\boldsymbol{x}_t)-f(p_t,\boldsymbol{x}_t)\le\frac{2(C_1'+C_3')\sqrt{T}}{h^{d/2}}+\frac{C_2'}{\varepsilon h^d}+\frac{C_H^2C_p^2d}{2}h^2T. \tag{EC.17}$$

With the choice $J=\lceil T^{d/(d+4)}\rceil$ corresponding to $h=J^{-1/d}\approx T^{-1/(d+4)}$, the above inequality proves Theorem 1 with $2(C_1'+C_3')+C_H^2C_p^2d/2\le\text{const.}\times C_H^2(\sigma_H^{-4}+C_H\sqrt{C_X})\ln^2(2C_H^2T^3)+C_H^2C_p^2d/2$ and $\overline{C}_1'=2C_2'\le\text{const.}\times C_H^2\sigma_H^{-2}\ln^3(2C_H^2T^3)$. □

EC.2. Proofs of technical lemmas in Section 6

EC.2.1. Proof of Lemma 3

Let $\eta:=(1-2^{-1/d})/2$ and define $H_j:=\{\boldsymbol{x}\in B_j:\mathfrak{d}(\boldsymbol{x},\partial B_j)\ge\eta h\}$. Since P_X is the uniform distribution on $[0,1]^d$, we have $\Pr[\boldsymbol{x}\in H_j]=0.5\Pr[\boldsymbol{x}\in B_j]=0.5h^d$. For any $\boldsymbol{x}\in H_j$, by simple algebra we have that

$$p_{\boldsymbol{\nu}}^*(\boldsymbol{x})=\frac{2}{3}, \qquad \text{if } \nu_j=0; \tag{EC.18}$$

$$p_{\boldsymbol{\nu}}^*(\boldsymbol{x})=\frac{2}{3}-\frac{\mathfrak{d}(\boldsymbol{x},\partial B_j)}{3(1+\mathfrak{d}(\boldsymbol{x},\partial B_j))}\le\frac{2}{3}-\frac{1}{6}\eta h, \qquad \text{if } \nu_j=1. \tag{EC.19}$$

Define $\mathcal{A}:=\mathcal{S}_0\{p:p\ge\frac{2}{3}-\frac{\eta h}{12}\}$ and $\mathcal{B}:=\mathcal{S}_1=\{p:p\le\frac{2}{3}-\frac{\eta h}{12}\}$. Since $f_{\boldsymbol{\nu}}(p,\boldsymbol{x})$ is strongly concave in p, we have that

$$f_{\boldsymbol{\nu}}(p_{\boldsymbol{\nu}}^*(\boldsymbol{x}),\boldsymbol{x})-f_{\boldsymbol{\nu}}(p,\boldsymbol{x})\ge\frac{\eta^2h^2}{144}, \qquad \forall p\in\mathcal{B},\ \text{if } \nu_j=0; \tag{EC.20}$$

$$f_{\boldsymbol{\nu}}(p_{\boldsymbol{\nu}}^*(\boldsymbol{x}),\boldsymbol{x})-f_{\boldsymbol{\nu}}(p,\boldsymbol{x})\ge\frac{\eta^2h^2}{144}, \qquad \forall p\in\mathcal{A},\ \text{if } \nu_j=1. \tag{EC.21}$$

Subsequently,

$$\mathfrak{R}(\pi)=\sup_{\boldsymbol{\nu}\in\{0,1\}^d}\mathbb{E}_{\boldsymbol{\nu}}^{\pi}\left[\sum_{t=1}^T f_{\boldsymbol{\nu}}(p_{\boldsymbol{\nu}}^*(\boldsymbol{x}_t),\boldsymbol{x}_t)-f_{\boldsymbol{\nu}}(p_t,\boldsymbol{x}_t)\right]\ge\frac{1}{2^d}\sum_{\boldsymbol{\nu}\in\{0,1\}^d}\mathbb{E}_{\boldsymbol{\nu}}^{\pi}\left[\sum_{t=1}^T f_{\boldsymbol{\nu}}(p_{\boldsymbol{\nu}}^*(\boldsymbol{x}_t),\boldsymbol{x}_t)-f_{\boldsymbol{\nu}}(p_t,\boldsymbol{x}_t)\right]$$

$$\ge\sum_{t=1}^T\frac{1}{2^d}\sum_{\boldsymbol{\nu}\in\{0,1\}^d}\frac{1}{2J}\sum_{j=1}^J\mathbb{E}_{\boldsymbol{\nu}}^{\pi}\left[f_{\boldsymbol{\nu}}(p_{\boldsymbol{\nu}}^*(\boldsymbol{x}_t),\boldsymbol{x}_t)-f_{\boldsymbol{\nu}}(p_t,\boldsymbol{x}_t)|\boldsymbol{x}_t\in H_j\right]$$

$$\geq \sum_{t=1}^{T} \frac{1}{2^d} \sum_{\boldsymbol{\nu}\in\{0,1\}^d} \frac{1}{2J} \sum_{j=1}^{J} \mathbb{E}_{\boldsymbol{\nu}}^{\pi}\left[\frac{\eta^2 h^2}{144}\mathbf{1}\{p_t \in \mathcal{S}_{\nu_j}\}\middle| x_t \in H_j\right],$$

which is to be proved.

EC.2.2. Proof of Lemma 4

Fix $j \in [J]$. Let $M^{\pi}_{\pm j,t}(\cdot)$ be the distributions of z_t, which is measurable conditioned on $\boldsymbol{z}_{<t} = (z_1, \cdots, z_{t-1})$ thanks to the definition in Eqs. (4,5). More specifically, $M^{\pi}_{\pm j,t}(Z|\boldsymbol{z}_{<t}) = \int_{\mathcal{S}} Q_t(Z|s_t, \boldsymbol{z}_{<t})\mathrm{d}P^{\pi}_{\pm j,t}(s_t|\boldsymbol{z}_{<t})$. By the chain rule of KL-divergence, we have that

$$D^{\mathsf{sy}}_{\mathsf{KL}}(M^{\pi}_{+j}, M^{\pi}_{-j}) \leq 2\sum_{t=1}^{T}\int_{\mathcal{Z}^{t-1}} D^{\mathsf{sy}}_{\mathsf{KL}}(M^{\pi}_{+j,t}(\cdot|\boldsymbol{z}_{<t}), M^{\pi}_{-j,t}(\cdot|\boldsymbol{z}_{<t}))\mathrm{d}\overline{M}^{\pi}_{<t}(\mathrm{d}z_{<t}). \tag{EC.22}$$

Now fix $t \in [T]$ and define $\varsigma_{jt}(\boldsymbol{z}_{<t}) := D^{\mathsf{sy}}_{\mathsf{KL}}(M^{\pi}_{+j,t}(\cdot|\boldsymbol{z}_{<t}), M^{\pi}_{-j,t}(\cdot|\boldsymbol{z}_{<t}))$. Let $m_{+j}(\cdot|\boldsymbol{z}_{<t}), m_{-j}(\cdot|\boldsymbol{z}_{<t})$ be the PDFs of $M^{\pi}_{+j,t}(\cdot|\boldsymbol{z}_{<t})$ and $M^{\pi}_{-j,t}(\cdot|\boldsymbol{z}_{<t})$. Define also $m^0(z|\boldsymbol{z}_{<t}) := \inf_{s_t\in\mathcal{S}} q_t(z|s_t, \boldsymbol{z}_{<t})$, where q_t is the PDF of Q_t defined in Eq. (4). We then have

$$\begin{aligned}
\varsigma_{jt}(\boldsymbol{z}_{<t}) &= \int_{\mathcal{Z}} [m_+(z|\boldsymbol{z}_{<t}) - m_-(z|\boldsymbol{z}_{<t})] \ln\frac{m_+(z|\boldsymbol{z}_{<t})}{m_-(z|\boldsymbol{z}_{<t})}\mathrm{d}z \\
&\leq \int_{\mathcal{Z}} [m_+(z|\boldsymbol{z}_{<t}) - m_-(z|\boldsymbol{z}_{<t})]^2 \cdot \frac{\mathrm{d}z}{\min\{m_+(z|\boldsymbol{z}_{<t}), m_-(z|\boldsymbol{z}_{<t})\}} \\
&\leq \int_{\mathcal{Z}} [m_+(z|\boldsymbol{z}_{<t}) - m_-(z|\boldsymbol{z}_{<t})]^2 \cdot \frac{\mathrm{d}z}{m^0(z|\boldsymbol{z}_{<t})},
\end{aligned} \tag{EC.23}$$

where the first inequality holds because $|\ln(a/b)| \leq |a-b|/\min\{a,b\}$ for all $a, b \in \mathbb{R}_+$ (see, e.g., Duchi et al. 2018, Lemma 4). Note that $m_{\pm}(z|\boldsymbol{z}_{<t}) = \int_{\mathcal{S}} q_t(z|s_t, \boldsymbol{z}_{<t})dP^{\pi}_{\pm j,t}(s_t|\boldsymbol{z}_{<t})$, and $\int_{\mathcal{S}} \mathrm{d}P^{\pi}_{+j,t}(s_t|\boldsymbol{z}_{<t}) - \mathrm{d}P^{\pi}_{-j,t}(s_t|\boldsymbol{z}_{<t}) = 1 - 1 = 0$. We then have

$$\begin{aligned}
m_+(z|\boldsymbol{z}_{<t}) - m_-(z|\boldsymbol{z}_{<t}) &= \int_{\mathcal{S}} q_t(z|s_t, \boldsymbol{z}_{<t})\left[\mathrm{d}P^{\pi}_{+j,t}(s_t|\boldsymbol{z}_{<t}) - \mathrm{d}P^{\pi}_{-j,t}(s_t|\boldsymbol{z}_{<t})\right] \\
&= \int_{\mathcal{S}} (q_t(z|s_t, \boldsymbol{z}_{<t}) - m^0(z|\boldsymbol{z}_{<t}))\left[\mathrm{d}P^{\pi}_{+j,t}(s_t|\boldsymbol{z}_{<t}) - \mathrm{d}P^{\pi}_{-j,t}(s_t|\boldsymbol{z}_{<t})\right] \\
&= m^0(z|\boldsymbol{z}_{<t})\int_{\mathcal{S}}\left(\frac{q_t(z|s_t, \boldsymbol{z}_{<t})}{m^0(z|\boldsymbol{z}_{<t})} - 1\right)\left[\mathrm{d}P^{\pi}_{+j,t}(s_t|\boldsymbol{z}_{<t}) - \mathrm{d}P^{\pi}_{-j,t}(s_t|\boldsymbol{z}_{<t})\right].
\end{aligned}$$

Subsequently, Eq. (EC.23) can be upper bounded by

$$\varsigma_{jt}(\boldsymbol{z}_{<t}) \leq \int_{\mathcal{Z}} m^0(z|\boldsymbol{z}_t)\left|\int_{\mathcal{S}}\left(\frac{q_t(z|s_t, \boldsymbol{z}_{<t})}{m^0(z|\boldsymbol{z}_{<t})} - 1\right)\left[\mathrm{d}P^{\pi}_{+j,t}(s_t|\boldsymbol{z}_{<t}) - \mathrm{d}P^{\pi}_{-j,t}(s_t|\boldsymbol{z}_{<t})\right]\right|^2 \mathrm{d}z$$

Note that $q_t(z|s_t, \boldsymbol{z}_{<t})/m^0(z|\boldsymbol{z}_{<t}) \leq e^{2\varepsilon}$ for all $z \in \mathcal{Z}$, because π satisfies 2ε-LDP. Note also that $\int_{\mathcal{Z}} m^0(z|\boldsymbol{z}_{<t})\mathrm{d}z = \int_{\mathcal{Z}} \inf_{s_t\in\mathcal{S}} q_t(z|s_t, \boldsymbol{z}_{<t})\mathrm{d}z \leq \inf_{s_t\in\mathcal{S}}\int_{\mathcal{Z}} q_t(z|s_t, \boldsymbol{z}_{<t})\mathrm{d}z \leq 1$. The above inequality can

then be reduced to

$$\varsigma_{jt}(\boldsymbol{z}_{<t}) \leq (e^{2\varepsilon}-1)^2 \sup_{\|\gamma\|_\infty\leq 1}\left|\int_{\mathcal{S}}\gamma(s_t,\boldsymbol{z}_{<t})\left[\mathrm{d}P^\pi_{+j,t}(s_t|\boldsymbol{z}_{<t})-\mathrm{d}P^\pi_{-j,t}(s_t|\boldsymbol{z}_{<t})\right]\right|^2. \tag{EC.24}$$

Combining Eqs. (EC.22,EC.24) we obtain

$$\begin{aligned}&D^{\mathsf{sy}}_{\mathsf{KL}}(M^\pi_{+j},M^\pi_{-j})\\ \leq& 2(e^{2\varepsilon}-1)^2\sum_{t=1}^T\sup_{\|\gamma\|_\infty\leq 1}\int_{\mathcal{Z}^{t-1}}\left|\int_{\mathcal{S}}\gamma(s_t,\boldsymbol{z}_{<t})\left[\mathrm{d}P^\pi_{+j,t}(s_t|\boldsymbol{z}_{<t})-\mathrm{d}P^\pi_{-j,t}(s_t|\boldsymbol{z}_{<t})\right]\right|^2\mathrm{d}\overline{M}^\pi_{<t}(\mathrm{d}z_{<t}).\end{aligned}$$

Summing both sides of the above inequality over $j=1,2,\cdots,J$ we complete the proof of Lemma 4.

EC.2.3. Proof of Lemma 5

Recall the decomposition that $P^\pi_{\pm j,t}(\boldsymbol{x}_t,y_t,p_t|\boldsymbol{z}_{<t})=\chi(\boldsymbol{x}_t)A_t(p_t|\boldsymbol{x}_t,\boldsymbol{z}_{<t})\,\mathrm{Pr}_{\pm j}(y_t|p_t,\boldsymbol{x}_t)$. Because P^π_{+j} and P^π_{-j} are exactly the same for $\boldsymbol{x}\notin B_j$, it suffices to consider only $\boldsymbol{x}\in B_j$. For any $\boldsymbol{x}\in B_j$, the distribution of $\boldsymbol{x}$ and p is independent of the underlying demand model $f_{\boldsymbol{\nu}}$. Hence,

$$\begin{aligned}\|P^\pi_{+j,t}(\cdot|\boldsymbol{z}_{<t})-P^\pi_{-j,t}(\cdot|\boldsymbol{z}_{<t})\|_{\mathsf{TV}} &\leq \frac{1}{2}\int_{B_j}\mathbb{E}_{p\sim A_t(\cdot|\boldsymbol{x},\boldsymbol{z}_{<t})}\left[\sup_{\boldsymbol{\nu}_j\neq\boldsymbol{\nu}'_j}\left|\lambda_{\boldsymbol{\nu}}(p,\boldsymbol{x})-\lambda_{\boldsymbol{\nu}'}(p,\boldsymbol{x})\right|\right]\mathrm{d}P_X(\boldsymbol{x})\\ &\leq \frac{1}{2}\int_{B_j}\mathbb{E}_{p\sim A_t(\cdot|\boldsymbol{x},\boldsymbol{z}_{<t})}\left[\left|\frac{1}{3}-\frac{p}{2}\right|\sup_{\boldsymbol{x}\in B_j}\mathfrak{d}(\boldsymbol{x},\partial B_j)\right]\mathrm{d}P_X(\boldsymbol{x})\\ &\leq \frac{\sqrt{d}h}{4}\int_{B_j}\mathbb{E}_{p\sim A_t(\cdot|\boldsymbol{x},\boldsymbol{z}_{<t})}[|p-p_0|]\mathrm{d}P_X(\boldsymbol{x}),\end{aligned}$$

where $p_0=2/3$. Note that $\int_{B_j}\mathrm{d}P_X(\boldsymbol{x})=\Pr[\boldsymbol{x}\in B_j]=1/J$. Subsequently, by Jensen's inequality that $(\mathbb{E}[\cdot])^2\leq\mathbb{E}[\cdot^2]$, we have

$$\|P^\pi_{+j,t}(\cdot|\boldsymbol{z}_{<t})-P^\pi_{-j,t}(\cdot|\boldsymbol{z}_{<t})\|^2_{\mathsf{TV}}\leq\frac{dh^2}{16J^2}\mathbb{E}_{\boldsymbol{x}\sim U(B_j)}\mathbb{E}_{p\sim A_t(\cdot|\boldsymbol{x},\boldsymbol{z}_{<t})}[(p-p_0)^2],$$

where $U(B_j)$ is the uniform distribution on B_j. This proves Lemma 5.

EC.2.4. Proof of Lemma 6

Let $\lambda_{\boldsymbol{\nu}},f_{\boldsymbol{\nu}}$ be the demand model corresponding to an arbitrary $\boldsymbol{\nu}\in\{0,1\}^J$, and π be a personalized pricing policy. Let $\boldsymbol{x}\in B_j$ be an arbitrary context vector belonging to hypercube B_j, and $p^*(\boldsymbol{x})=\arg\max_p f_{\boldsymbol{\nu}}(p,\boldsymbol{x})$. Eqs. (EC.18,EC.19) in the proof of Lemma 3 show that $|p^*(\boldsymbol{x})-p_0|\leq\eta h/6\leq$

$h/12$, where $p_0 = 2/3$, regardless of the value of ν_j. Note also that $f_{\boldsymbol{\nu}}(p, \boldsymbol{x})$ is strongly concave with respect to p. Subsequently, we have for all $\boldsymbol{x} \in [0,1]^d$ and $p \in [0,1]$ that

$$f(\boldsymbol{p}^*(\boldsymbol{x}), \boldsymbol{x}) - f(p, \boldsymbol{x}) \geq (p - p^*(\boldsymbol{x}))^2 \geq \inf_{|p'-p_0| \leq h/12} (p - p')^2 \geq \frac{1}{4}\phi(|p - p_0|^2, h^2). \tag{EC.25}$$

Consequently,

$$\begin{aligned}
\mathfrak{R}(\pi) &= \sup_{\boldsymbol{\nu} \in \{0,1\}^J} \mathbb{E}^{\pi}_{\boldsymbol{\nu}} \left[\sum_{t=1}^{T} f_{\boldsymbol{\nu}}(p^*(\boldsymbol{x}_t), \boldsymbol{x}_t) - f_{\boldsymbol{\nu}}(p_t, \boldsymbol{x}_t) \right] \geq \frac{1}{2^J} \sum_{\boldsymbol{\nu} \in \{0,1\}^J} \mathbb{E}^{\pi}_{\boldsymbol{\nu}} \left[\sum_{t=1}^{T} f_{\boldsymbol{\nu}}(p^*(\boldsymbol{x}_t), \boldsymbol{x}_t) - f_{\boldsymbol{\nu}}(p_t, \boldsymbol{x}_t) \right] \\
&\geq \frac{1}{2^J} \sum_{\boldsymbol{\nu} \in \{0,1\}^J} \mathbb{E}^{\pi}_{\boldsymbol{\nu}} \left[\sum_{t=1}^{T} \frac{1}{4} \phi(|p_t - p_0|^2, h^2) \right] \\
&= \frac{1}{2^J} \sum_{\boldsymbol{\nu} \in \{0,1\}^J} \sum_{t=1}^{T} \mathbb{E} \left[\frac{1}{4} \phi(|p_t - p_0|^2, h^2) \middle| p_t \sim A_t(\cdot | \boldsymbol{x}_t, \boldsymbol{z}_{<t}), \boldsymbol{x}_t \sim P_X, \boldsymbol{z}_{<t} \sim M^{\pi}_{\boldsymbol{\nu}, <t} \right] \\
&= \frac{1}{4} \sum_{t=1}^{T} \mathbb{E} \left[\phi(|p_t - p_0|^2, h^2) \middle| p_t \sim A_t(\cdot | \boldsymbol{x}_t, \boldsymbol{z}_{<t}), \boldsymbol{x}_t \sim P_X, \boldsymbol{z}_{<t} \sim \overline{M}^{\pi}_{<t} \right],
\end{aligned}$$

which is to be proved.