# Proofs of Technical Lemmas

## EC.1. Proof of Proposition 2

The proof of Proposition 2 makes use of the following lemmas.

LEMMA EC.1 **(Berry-Esseen Theorem; see, e.g., (Korolev and Shevtsova 2010))**. *Let $X_1, X_2, \dots$ be independent and identically distributed random variables such that $\mathbb{E}[X_1] = 0$, $\mathbb{E}[X_1^2] = \sigma^2 > 0$ and $\mathbb{E}[|X_1|^3] < \beta^3 < \infty$. For $n \geq 1$, define*

$$S_n = \frac{X_1 + \ldots + X_n}{\sigma\sqrt{n}}$$

*and let $F_n(x) = \mathbb{P}[S_n \leq x]$ be its cdf. Let $\phi(x)$ be the cdf of a standard normal distribution. For any $x$ and any $n \geq 1$, it holds*

$$|F_n(x) - \phi(x)| \leq \frac{0.34445\beta^3 + 0.16844}{\sqrt{n}}.$$

LEMMA EC.2. *Let $\mu > 0$ and recall the notation $\psi_j(\mu) = \mathbb{P}(Po(\mu) \geq j)$, where $Po(\mu)$ is a Poisson random variable of parameter $\mu$ and $j$ is a nonnegative integer. Denote by $\phi$ the cdf of a standard normal. We have*

$$\psi_j(\mu) \geq 1 - \phi\left(\frac{j-\mu}{\sqrt{\mu}}\right) - \frac{0.55}{\sqrt{\lceil\mu\rceil}}.$$

*In particular, for $\lambda \geq 20$,*

$$\psi_{\lambda/2-1}(2\lambda/3) \geq 0.7$$

*Proof.* We can write $Po(\mu) = \sum_{i=1}^{\lceil\mu\rceil} P_i$ where $P_i, i = 1, \dots, \mu$ are independent Poisson random variables with parameter $\mu/\lceil\mu\rceil \leq 1$. Let $X_i = P_i - \frac{\mu}{\lceil\mu\rceil}$: we have that $\mathbb{E}[X_i] = 0$, $\mathbb{E}[X_i^2] = \mu/\lceil\mu\rceil$, and $\mathbb{E}[X_i^3] = \mu/\lceil\mu\rceil$. From this, we have that

$$\begin{aligned}
\psi_j(\mu) = \mathbb{P}(Po(\mu) \geq j) &= \mathbb{P}\left(\sum_{i=1}^{\lceil\mu\rceil}(Pi - \mathbb{E}[P_i]) \geq j - \mu\right) \\
&= \mathbb{P}\left(\frac{\sum_{i=1}^{\lceil\mu\rceil} X_i}{\sqrt{\mu/\lceil\mu\rceil}\sqrt{\lceil\mu\rceil}} \geq \frac{j-\mu}{\sqrt{\mu/\lceil\mu\rceil}\sqrt{\lceil\mu\rceil}}\right) \\
&= \mathbb{P}\left(\frac{\sum_{i=1}^{\lceil\mu\rceil} X_i}{\sqrt{\mu}} \geq \frac{j-\mu}{\sqrt{\mu}}\right) \\
&= 1 - F_{\lceil\mu\rceil}\left(\frac{j-\mu}{\sqrt{\mu}}\right).
\end{aligned}$$

Using Lemma EC.1 and the fact that $\mathbb{E}[|X_i|^3] = \frac{\mu}{\lceil \mu \rceil} \leq 1$, we have that for any $x$,

$$||1 - F_{\lceil \mu \rceil}(x)| - |1 - \phi(x)|| \leq |F_{\lceil \mu \rceil}(x) - \phi(x)| \leq \frac{0.55}{\sqrt{\lceil \mu \rceil}},$$

which implies $1 - F_{\lceil \mu \rceil}(x) \geq 1 - \phi(x) - \frac{0.55}{\sqrt{\lceil \mu \rceil}}$ and so,

$$\psi_j(\mu) \geq 1 - \phi\left(\frac{j - \mu}{\sqrt{\mu}}\right) - \frac{0.55}{\sqrt{\lceil \mu \rceil}}.$$

Now, assume that $\lambda \geq 20$. We have that

$$\psi_{\lambda/2-1}(2\lambda/3) \geq 1 - \phi\left(\frac{\lambda/2 - 1 - 2\lambda/3}{\sqrt{2\lambda/3}}\right) - \frac{0.55}{\sqrt{\lceil 2\lambda/3 \rceil}} = 1 - \phi\left(\frac{-\lambda/6 - 1}{\sqrt{2\lambda/3}}\right) - \frac{0.55}{\lceil \sqrt{2\lambda/3} \rceil}.$$

When $\lambda \geq 20$, we have that $\lambda \mapsto -\frac{\lambda/6+1}{\sqrt{2\lambda/3}}$ is decreasing and that $\lambda \mapsto -\frac{0.55}{\lceil \sqrt{2\lambda/3} \rceil}$ is increasing. From this, we get

$$\psi_{\lambda/2-1}(2\lambda/3) \geq 1 - \phi\left(\frac{-20/6 - 1}{\sqrt{2 \cdot 20/3}}\right) - \frac{0.55}{\sqrt{\lceil 2 \cdot 20/3 \rceil}} \geq 0.7.$$

*Proof of Proposition 2.* The permutation $\pi^*$ is good if the intersection graph of $G_A$ and $G_{B'}$ has at least $n \cdot \frac{1+\alpha}{2}$ nodes that have degree greater than or equal to $\frac{nqs}{2}$. By definition of a $k$-core, if this intersection graph has an $nqs/2$-core of size greater than or equal to $n(1+\alpha)/2$ whp, then $\pi^*$ is good whp.

The intersection graph of $G_A$ and $G_{B'}$ is a graph over $n$ nodes such that there is an edge between nodes $i$ and $j$ if and only if there is an edge in $G_A$ and $G_{B'}$ between $i$ and $j$. As $\mathbb{P}((A_{ij}, B'_{ij}) = (1,1)) = qs$ and $(A_{ij}, B'_{ij})$ is independent of $(A_{kl}, B'_{kl})$ for any $(k,l) \neq (i,j)$, it follows that the intersection graph of $G_A$ and $G_{B'}$ is an Erdős-Rényi graph of parameters $(n, qs)$. If we are able to show that under the conditions of Theorem 2,

(i) $nqs > c_{nqs/2}$

(ii) $\psi_{nqs/2}(\mu_{nqs/2}(nqs)) \geq n \cdot \frac{1+\alpha}{2}$

then from Lemma 1, the $nqs/2$-core of the intersection graph of $G_A$ and $G_{B'}$ contains a fraction greater than or equal to $\frac{1+\alpha}{2}$ of the nodes in the graph whp and Proposition 2 follows.

We start by proving (i). Recall that $c_{nqs/2} = \inf_{\mu > 0} \frac{\mu}{\psi_{nqs/2-1}(\mu)}$. Hence, for any $\mu > 0$, in particular for $\mu = 2nqs/3 > 0$,

$$c_{nqs/2} \leq \frac{2nqs/3}{\psi_{nqs/2-1}(2nqs/3)} \leq \frac{2nqs/3}{0.7} < nqs,$$

where we have used Lemma EC.2 for the second inequality as the conditions of Theorem 2 imply that $\lambda \geq 20$.

We now prove (ii) by showing that under the conditions of Theorem 2, $\frac{2nqs}{3} \leq \mu_{nqs/2}(nqs) \leq nqs$ where $\mu_{nqs/2}(nqs)$ is the largest solution to $\frac{\mu}{\psi_{nqs/2-1}(\mu)} = nqs$. This implies that

$$\begin{aligned}
\psi_{nqs/2}(\mu_{nqs/2}(nqs)) &= \mathbb{P}(Po(\mu_{nqs/2}(nqs)) \geq nqs/2) \\
&= 1 - \mathbb{P}(Po(\mu_{nqs/2}(nqs))) \leq \mu_{nqs/2}(nqs) - (\mu_{nqs/2}(nqs) - nqs/2)) \\
&\geq 1 - \mathbb{P}(Po(\mu_{nqs/2}(nqs)) \leq \mu_{nqs/2}(nqs) - nqs/6) \\
&\geq 1 - e^{-(nqs/6)^2/2(\mu_{nqs/2}(nqs)+nqs/6)} \\
&\geq 1 - e^{-(nqs)^2/(36\cdot 7/3\cdot nqs)} = 1 - e^{-nqs/84},
\end{aligned}$$

where we have used a standard Chernoff bound for Poisson random variables for the second inequality, see, e.g. (Boucheron et al. 2013). As the conditions of Theorem 2 require that $nqs \geq 84 \cdot \log\left(\frac{2}{1-\alpha}\right)$, we get that $\psi_{nqs/2}(\mu_{nqs/2}(nqs)) \geq \frac{1+\alpha}{2}$.

To show that $\frac{2nqs}{3} \leq \mu_{nqs/2}(nqs) \leq nqs$, let $f_{nqs}(\mu) = \mu - nqs \cdot \psi_{nqs/2-1}(\mu)$, which is a continuous function of $\mu$ for fixed $nqs$. Here, $\mu_{nqs/2}(nqs)$ is the largest solution to $f_{nqs}(\mu) = 0$. It is straightforward to see that for $\mu > nqs$, $f_{nqs}(\mu) > 0$ as $\psi_{nqs/2-1} \leq 1$ for any value that $nqs$ takes. Furthermore, from Lemma EC.2,

$$f_{nqs}(2nqs/3) = \frac{2nqs}{3} - nqs \cdot \psi_{nqs/2-1}(2nqs/3) \leq \frac{2nqs}{3} - nqs \cdot 0.7 < 0.$$

Combining these results, we get that $\mu_{nqs/2}(nqs)$ must be less than or equal to $nqs$ and that there is a solution $f_{nqs}(\mu) = 0$ in the interval $[2nqs/3, nqs]$. Our result immediately follows.

## EC.2. Proofs of auxiliary results for Proposition 3

Lemma EC.3. *Let $n \in \mathbb{N}$ and let $2 \leq k \leq n$. Furthermore, let $\alpha, \beta$ be positive real numbers. We have*

$$(\alpha^k + \beta^k)^{n/k} \leq (\alpha^2 + \beta^2)^{n/2}.$$

*Proof.* We have

$$\frac{(\alpha^2+\beta^2)^{n/2}}{(\alpha^k+\beta^k)^{n/k}} = \frac{(\alpha^2+\beta^2)^{k/2\cdot n/k}}{(\alpha^k+\beta^k)^{n/k}} = \left(\frac{(\alpha^2+\beta^2)^{k/2}}{(\alpha^k+\beta^k)}\right)^{n/k}.$$

As $n/k>1$, we simply need to check whether $(a^2+\beta^2)^{k/2}\geq(\alpha^k+\beta^k)$. If $k$ is divisible by 2, this is straightforward from the Binomial theorem as $\alpha>0$ and $\beta>0$. If $k$ is not divisible by 2, then $k-1$ is divisible by 2 and $(\alpha^2+\beta^2)^k=(\alpha^2+\beta^2)^{(k-1)/2}\cdot\sqrt{\alpha^2+\beta^2}$. By applying the Binomial theorem to the first part of the sum and noting that $\sqrt{\alpha^2+\beta^2}\geq\alpha$ and $\beta$, the result follows.

Lemma EC.4. *Let $(A_0,B_0)$, $(A_1,B_1),\dots,(A_n,B_n)$ be $n+1$ pairs of random variables taking values in $\{0,1\}^2$. We suppose that $(A_i,B_i)$ are distributed following the probability distribution $P$ given in Section 2.4.1. In particular, the covariance between $A_i$ and $B_i$ is given by $\sigma^2>0$. We further assume that all pairs $\{(A_i,B_i)\}$ are independent of one another. Define*

$$W:=A_0B_1+A_1B_2+\ldots+A_{n-1}B_n+A_nB_0.$$

*For any $t>0$, we have that*

$$\mathbb{E}[e^{tW}]=\left(\frac{T+\sqrt{T^2-4D}}{2}\right)^{n+1}+\left(\frac{T-\sqrt{T^2-4D}}{2}\right)^{n+1}$$

*where $T=p_{11}(e^t-1)+1$ and $D=\sigma^2(e^t-1)$.*

Remark EC.1. Note that if $\sigma^2=0$, i.e., $A_i$ and $B_i$ are uncorrelated, this moment generating function simplifies to that of a Binomial random variable of parameters $(n,qs)$, as it should.

Our proof will make use of the following result from (Williams 1992).

Lemma EC.5. *Let $M$ be a $2\times 2$ matrix with eigenvalues $\alpha$ and $\beta$. Then, if $\alpha\neq\beta$, for all $n\geq 1$,*

$$M^n=\alpha^n\left(\frac{M-\beta I}{\alpha-\beta}\right)+\beta^n\left(\frac{M-\alpha I}{\beta-\alpha}\right).$$

*Proof of Theorem EC.4.* Let

$$u_n^{00}=\mathbb{E}[e^{t(A_1B_2+A_2B_3+\ldots+A_{n-2}B_{n-1}+A_{n-1}B_n)}],$$

$$u_n^{01}=\mathbb{E}[e^{t(A_1B_2+A_2B_3+\ldots+A_{n-2}B_{n-1}+A_{n-1}B_n+A_n)}],$$

$$u_n^{10}=\mathbb{E}[e^{t(B_1+A_1B_2+A_2B_3+\ldots+A_{n-2}B_{n-1}+\ldots+A_{n-1}B_n)}],$$

$$u_n^{11}=\mathbb{E}[e^{t(B_1+A_1B_2+A_2B_3+\ldots+A_{n-2}B_{n-1}+\ldots+A_{n-1}B_n+A_n)}].$$

Note that by conditioning on $(A_0, B_0)$, we have that $\mathbb{E}[e^{tW}] = p_{00}u_n^{00} + p_{01}u_n^{01} + p_{10}u_n^{10} + p_{11}u_n^{11}$. We thus study these four sequences. By conditioning on $(A_1, B_1)$, it is easy to see that the following formulas hold true:

$$u_n^{00} = (p_{00} + p_{01})u_{n-1}^{00} + (p_{10} + p_{11})u_{n-1}^{10}$$

$$u_n^{01} = (p_{00} + p_{01})u_{n-1}^{01} + (p_{10} + p_{11})u_{n-1}^{11}$$

$$u_n^{10} = (p_{00} + p_{01}e^t)u_{n-1}^{00} + (p_{10} + p_{11}e^t)u_{n-1}^{10}$$

$$u_n^{11} = (p_{00} + p_{01}e^t)u_{n-1}^{01} + (p_{10} + p_{11}e^t)u_{n-1}^{11}.$$

Successive conditioning on $(A_k, B_k)$ for $k \geq 2$ leads to these formulas holding for all $n \geq 2$. Hence, if we let

$$M := \begin{bmatrix} p_{00} + p_{01} & p_{10} + p_{11} \\ p_{00} + p_{01}e^t & p_{10} + p_{11}e^t \end{bmatrix}$$

then

$$\begin{bmatrix} u_n^{00} \\ u_n^{10} \end{bmatrix} = M \cdot \begin{bmatrix} u_{n-1}^{00} \\ u_{n-1}^{10} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} u_n^{01} \\ u_n^{11} \end{bmatrix} = M \cdot \begin{bmatrix} u_{n-1}^{01} \\ u_{n-1}^{11} \end{bmatrix}.$$

We further have the following initial conditions

$$u_1^{00} = \mathbb{E}[e^{t(0\cdot B_n + A_n \cdot 0)}] = 1$$

$$u_1^{01} = \mathbb{E}[e^{t(0\cdot B_n + A_n)}] = 1 + (p_{01} + p_{11})(e^t - 1)$$

$$u_1^{10} = \mathbb{E}[e^{t(B_n + A_n \cdot 0)}] = 1 + (p_{01} + p_{11})(e^t - 1)$$

$$u_1^{11} = \mathbb{E}[e^{-t(B_n + A_n)}] = 1 + 2(p_{11} + p_{01})(e^t - 1) + p_{11}(e^t - 1)^2$$

Using these conditions, we get that

$$\begin{bmatrix} u_n^{00} \\ u_n^{10} \end{bmatrix} = M^{n-1} \cdot \begin{bmatrix} u_1^{00} \\ u_1^{10} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} u_n^{01} \\ u_n^{11} \end{bmatrix} = M^{n-1} \cdot \begin{bmatrix} u_1^{01} \\ u_1^{11} \end{bmatrix}.$$

The difficulty now is in computing $M^{n-1}$, which we will do via Lemma EC.5. To this effect, we denote by $T$ the trace of $M$ and by $D$ its determinant. One can easily compute their values:

$$T = \mathrm{tr}(M) = p_{11}(e^t - 1) + 1 \text{ and } D = \det(M) = \sigma^2(e^t - 1).$$

Recall that the eigenvalues of a $2\times 2$ matrix satisfy the equation $x^2-Tx+D=0$. The discriminant here is $T^2-4D$, which is positive:

$$\Delta = T^2-4D = p_{11}^2(e^t-1)^2+2(p_{11}-2\sigma^2)(e^t-1)+1 = \left(p_{11}(e^t-1)+\frac{p_{11}-2\sigma^2}{p_{11}}\right)^2+\frac{4\sigma^2(p_{11}-\sigma)}{p_{11}^2}>0$$

Hence solutions to this equation exist. We write:

$$\alpha = \frac{T+\sqrt{T^2-4D}}{2} \text{ and } \beta = \frac{T-\sqrt{T^2-4D}}{2}.$$

In accordance with Lemma EC.5, we now work on computing $M-\beta I$ and $M-\alpha I$. We have:

$$M-\beta I = R+\frac{\sqrt{\Delta}}{2}I,$$

where

$$R := \begin{bmatrix} \frac{p_{00}-p_{11}}{2}-\frac{p_{11}}{2}(e^t-1) & p_{10}+p_{11} \\ p_{00}+p_{01}e^t & -(\frac{p_{00}-p_{11}}{2})+\frac{p_{11}}{2}(e^t-1) \end{bmatrix}.$$

We denote by $r_{ij}$ its entries. Note that $r_{11}=-r_{22}$. Likewise, we have

$$M-\alpha I = R-\frac{\sqrt{\Delta}}{2}I.$$

It follows that

$$M^n = \frac{\alpha^n}{\sqrt{\Delta}}(R+\frac{\sqrt{\Delta}}{2}I)-\frac{\beta^n}{\sqrt{\Delta}}(R-\frac{\sqrt{\Delta}}{2}I) = \left(\frac{\alpha^n-\beta^n}{\sqrt{\Delta}}\right)R+\frac{\alpha^n+\beta^n}{2}\cdot I$$

and from this, for any $n\geq 1$,

$$\begin{aligned}
u_n^{00} &= \left(\frac{\alpha^{n-1}-\beta^{n-1}}{\sqrt{\Delta}}\right)(r_{11}u_1^{00}+r_{12}u_1^{10})+\frac{\alpha^{n-1}+\beta^{n-1}}{2}u_1^{00}\\
u_n^{10} &= \left(\frac{\alpha^{n-1}-\beta^{n-1}}{\sqrt{\Delta}}\right)(r_{21}u_1^{00}+r_{22}u_1^{10})+\frac{\alpha^{n-1}+\beta^{n-1}}{2}u_1^{10}\\
u_n^{01} &= \left(\frac{\alpha^{n-1}-\beta^{n-1}}{\sqrt{\Delta}}\right)(r_{11}u_1^{01}+r_{12}u_1^{11})+\frac{\alpha^{n-1}+\beta^{n-1}}{2}u_1^{01}\\
u_n^{11} &= \left(\frac{\alpha^{n-1}-\beta^{n-1}}{\sqrt{\Delta}}\right)(r_{21}u_1^{01}+r_{22}u_1^{11})+\frac{\alpha^{n-1}+\beta^{n-1}}{2}u_1^{11}.
\end{aligned}$$

This can now be used to compute $\mathbb{E}[e^{tW}]$. We have

$$\begin{aligned}\mathbb{E}[e^{tW}] &= p_{00}u_n^{00} + p_{01}u_n^{01} + p_{10}u_n^{10} + p_{11}u_n^{11} \\ &= \left(r_{11}(p_{00}u_1^{00} + p_{01}u_1^{01} - p_{10}u_1^{10} - p_{11}u_1^{11}) + r_{12}(p_{00}u_1^{10} + p_{01}u_1^{11}) + r_{21}(p_{10}u_1^{00} + p_{11}u_1^{01})\right) \\ &\cdot \left(\frac{\alpha^{n-1} - \beta^{n-1}}{\sqrt{\Delta}}\right) + (p_{00}u_1^{00} + p_{10}u_1^{10} + p_{01}u_1^{01} + p_{11}u_1^{11}) \cdot \left(\frac{\alpha^{n-1} + \beta^{n-1}}{2}\right).\end{aligned}$$

We compute each part separately. We have:

$$\begin{aligned}p_{00}u_1^{00} + p_{10}u_1^{10} + p_{01}u_1^{01} + p_{11}u_1^{11} &= T^2 - 2D \\ r_{11}(p_{00}u_1^{00} + p_{01}u_1^{01} - p_{10}u_1^{10} - p_{11}u_1^{11}) &= \frac{1}{2}\left(\frac{p_{00} - p_{11}}{2} - \frac{p_{11}}{2}(e^t - 1)\right)^2 \cdot T \\ r_{12}(p_{00}u_1^{10} + p_{01}u_1^{11}) + r_{21}(p_{10}u_1^{00} + p_{11}u_1^{01}) &= 2T(p_{01} + p_{11})(1 - p_{11} - p_{01} + p_{01}(e^t - 1))\end{aligned}$$

We combine the results above to obtain:

$$r_{11}(p_{00}u_1^{00} + p_{01}u_1^{01} - p_{10}u_1^{10} - p_{11}u_1^{11}) + r_{12}(p_{00}u_1^{10} + p_{01}u_1^{11}) + r_{21}(p_{10}u_1^{00} + p_{11}u_1^{01}) = \frac{T^3}{2} - 2DT$$

which leads us to finally having

$$\mathbb{E}[e^{tW}] = \left(\frac{\alpha^{n-1} - \beta^{n-1}}{\sqrt{\Delta}}\right) \cdot \left(\frac{T^3}{2} - 2DT\right) + \left(\frac{\alpha^{n-1} + \beta^{n-1}}{2}\right) \cdot (T^2 - 2D).$$

As $\frac{T^3}{2} - 2DT = \frac{T}{2} \cdot (T^2 - 4D) = \frac{T}{2} \cdot \Delta$, we can further simplify

$$\mathbb{E}[e^{tW}] = \left(\alpha^{n-1} - \beta^{n-1}\right) \cdot \frac{T}{2} \cdot \sqrt{\Delta} + \left(\frac{\alpha^{n-1} + \beta^{n-1}}{2}\right) \cdot (T^2 - 2D) = \alpha^{n+1} + \beta^{n+1}$$

## EC.3. Proof of Lemma 2

As mentioned previously, the proof of Lemma 2 is contained in Cullina et al. (2018). We repeat it here for completeness.

*Proof of Lemma 2.* The optimal choice of $z$ is obtained by differentiating $z \mapsto z^{-\tau} e^{q_2(z^2-1)+q_1(z-1)}$ and setting the derivative equal to 0, i.e.,

$$(2q_2 z + q_1) z^{-\tau} e^{q_2(z^2-1)+q_1(z-1)} - \tau z^{-\tau-1} e^{q_2(z^2-1)+q_1(z-1)} = 0,$$

which is equivalent to the quadratic equation

$$2q_2 z^2 + q_1 z - \tau = 0. \tag{EC.1}$$

This equation has one positive root and one negative root: the positive root is $z^*$. From this quadratic equation, we can obtain the expression $z^{*2} = \frac{\tau - q_1 z^*}{2q_2}$. As $q_1 > 0$ and $z^* > 0$, it follows that

$$z^{*2} \le \frac{\tau}{2q_2},$$

which corresponds to (13). One can also give an explicit expression of $z^*$ by solving the quadratic equation:

$$z^* = \frac{-q_1 + \sqrt{q_1^2 + 8\tau q_2}}{4q_2} = \frac{2\tau}{q_1 + \sqrt{q_1^2 + 8\tau q_2}}.$$

Because $q_1 \le \sqrt{q_1^2 + 8\tau q_2}$, we have the bounds

$$\frac{\tau}{\sqrt{q_1^2 + 8\tau q_2}} \le z^* \le \frac{\tau}{q_1}. \tag{EC.2}$$

Starting with one of the factors from the left side of (12), we have

$$e^{q_2(z^{*2}-1)+q_1(z^*-1)} \le e^{\frac{q_1}{2}z^* + \frac{\tau}{2} - q_2 - q_1} \le e^{\tau - q_1 - q_2} \le e^{\tau},$$

where we used (EC.1) to eliminate the $q_2 z^{*2}$ term, applied the upper bound from (EC.2), and used $q_1, q_2 \ge 0$. From the lower bound in (EC.2),

$$z^{*-2} \le \frac{q_1^2}{\tau^2} + \frac{8q_2}{\tau} \le \max\left\{\frac{2q_1^2}{\tau^2}, \frac{16q_2}{\tau}\right\}$$

so $z^{*-\tau} e^{q_2(z^{*2}-1)+q_1(z^*-1)} \le \zeta^\tau$.

## References


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